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TWO EXAMPLES OF QUASIDENSE VECTOR SUBSPACES OF $\mathbb{R}^{\mathbb{N}}$

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Abstract—We prove that the classes of exponentially dense, Cartesian dense, and recursively dense vector subspaces of $\mathbb{R}^{\mathbb{N}}$ are pairwise distinct.

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In [1], the authors initiated the study of the question of which Hausdorff locally convex spaces have all Archimedean cones closed. (Such spaces are convenient in that, given any definition of an Archimedean vector order on them, linear inequalities persist under limits of nets.) Finite-dimensional spaces are well known to have this property (see, for instance, [2, 3.4]). In [1], it was shown that spaces of uncountable dimension do not have this property, whereas for countable-dimensional spaces the question was left open.

In [3], the notion of quasidenseness was introduced, and a complete description was obtained for countable-dimensional Hausdorff locally convex spaces in which all Archimedean cones are closed. These turned out to be precisely those countable-dimensional spaces X whose topological dual X' is quasidense in the algebraic dual $X^{\#}$ endowed with the weak* topology.

Owing to [3], quasidense spaces became an object of thorough study, and since in the countable-dimensional case the algebraic dual $X^{\#}$ is linearly and topologically isomorphic to $\mathbb{R}^{\mathbb{N}}$, the class of objects under consideration naturally narrowed to quasidense vector subspaces of $\mathbb{R}^{\mathbb{N}}$. In [4], such subspaces were characterized in terms of their relation to projective parallelotopes and automorphisms. These results clarified the structure of quasidense subspaces of $\mathbb{R}^{\mathbb{N}}$ to a large extent from both geometric and algebraic viewpoints, but did not lead to an immediate discovery of new examples and left open the question of whether the class of such spaces coincides with other classes allowing substantially simpler definitions.

In this note, in Theorems 1 and 2, we construct examples supporting the hypothesis formulated in [3, 9.8] on the noncoincidence of the three types of quasidense vector subspaces of $\mathbb{R}^{\mathbb{N}}$—exponentially dense, Cartesian dense, and recursively dense.

The symbol $\mathbb{N}$ stands for the set of naturals $\{1, 2, \dots\}$. The sets of rationals and reals are denoted by $\mathbb{Q}$ and $\mathbb{R}$. The vector space $\mathbb{R}^{\mathbb{N}}$ of numerical sequences $s : \mathbb{N} \to \mathbb{R}$ is endowed with the Tychonoff topology (also called the topology of pointwise convergence). Tuples $x = \big(x(1), \dots, x(n)\big) \in \mathbb{R}^n$, where $n \in \mathbb{N}$, are regarded as functions $x : \{1, \dots, n\} \to \mathbb{R}$. The linear operator $\pi_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^n$ is defined by

$$\pi_n s = s\big|_{\{1,\dots,n\}} = \big(s(1), \dots, s(n)\big). \tag{1}$$

We agree to use the notation (1) not only for sequences $s \in \mathbb{R}^{\mathbb{N}}$, but also for tuples $s \in \mathbb{R}^m$ with $m \geqslant n$.

Definition 1 [3, 4.1]. The *quasi-interior* $\operatorname{qi} S$ of a subset S of a locally convex space X is defined as the set of those elements $x \in S$ for which the wedge

$$\mathbb{R}^+(S - x) = \{\lambda(s - x) : s \in S,\ \lambda \in \mathbb{R},\ \lambda \geq 0\}$$

is dense in X.

Proposition 1 [3, 4.13]. *For every convex subset $S \subseteq \mathbb{R}^{\mathbb{N}}$, we have*

$$\operatorname{qi} S = \{s \in S : \pi_n s \in \operatorname{int} \pi_n S \text{ for all } n \in \mathbb{N}\},$$

where $\operatorname{int} \pi_n S$ is the interior of $\pi_n S$ in $\mathbb{R}^n$.

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Definition 2 [3]. Consider the following properties of a subset $Y \subseteq \mathbb{R}^{\mathbb{N}}$:

(a) Y includes the power $\Lambda^{\mathbb{N}}$ for some dense subset $\Lambda \subseteq \mathbb{R}$;

(b) Y includes the product $\prod_{n\in\mathbb{N}} \Lambda_n$ for some sequence of dense subsets $\Lambda_n \subseteq \mathbb{R}$ ($n \in \mathbb{N}$);

(c) Y includes a subset $P \subseteq \mathbb{R}^{\mathbb{N}}$ satisfying the following three conditions:

(i) if $s \in \mathbb{R}^{\mathbb{N}}$ and $\pi_n s \in \pi_n P$ for all $n \in \mathbb{N}$, then $s \in P$;

(ii) the set $\{p(1) : p \in P\}$ is dense in $\mathbb{R}$;

(iii) for all $n \in \mathbb{N}$ and $q \in P$, the set $\{p(n+1) : p \in P,\ \pi_n p = \pi_n q\}$ is dense in $\mathbb{R}$;

(d) Y intersects every closed bounded convex subset of $\mathbb{R}^{\mathbb{N}}$ with nonempty quasi-interior.

A subset Y possessing property (a), (b), (c), (d) is called, respectively, *exponentially dense*, *Cartesian dense*, *recursively dense*, and *quasidense* (see [3, 6.2, 8.9, 9.8]).

Proposition 2 [3, 8.9]. *For every subset $Y \subseteq \mathbb{R}^{\mathbb{N}}$, the following implications hold:*

$$\text{(a)} \Rightarrow \text{(b)} \Rightarrow \text{(c)} \Rightarrow \text{(d)}.$$

Remark 1. In [3, 6.4], it was also proved that each of the properties (a)–(d) implies that Y is dense in $\mathbb{R}^{\mathbb{N}}$ with respect to the Tychonoff topology. In the same paper, an example is given of a dense vector subspace $Y \subseteq \mathbb{R}^{\mathbb{N}}$ that does not possess properties (a)–(d).

Until recently, the list of known examples of proper quasidense vector subspaces of $\mathbb{R}^{\mathbb{N}}$ included only the exponentially dense spaces $\operatorname{lin}\mathbb{Q}^{\mathbb{N}}$, $\operatorname{lin}\mathbb{N}^{\mathbb{N}}$, and their images under projective automorphisms (see [3, 8.10; 4, 4.2]). (Here and below, $\operatorname{lin} S$ denotes the linear span of a set S in the vector space $\mathbb{R}^{\mathbb{N}}$.) The question of the equivalence of conditions (a)–(d) also remained open for vector subspaces $Y \subseteq \mathbb{R}^{\mathbb{N}}$ (see [3, 9.8]). The theorems stated below shed partial light on this question.

Theorem 1. *Conditions* (a) *and* (b) *are not equivalent for vector subspaces $Y \subseteq \mathbb{R}^{\mathbb{N}}$.*

Proof. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of pairwise distinct reals algebraically independent over $\mathbb{Q}$. For each $n \in \mathbb{N}$, consider the dense subset

$$\mathbb{Q}\, t_n = \{q t_n : q \in \mathbb{Q}\} \subseteq \mathbb{R}$$

and show that the Cartesian dense vector subspace

$$Y = \operatorname{lin} \prod_{n\in\mathbb{N}} \mathbb{Q}\, t_n \subseteq \mathbb{R}^{\mathbb{N}}$$

is not exponentially dense.

It suffices to show that Y contains no nonzero constant sequence. Assume the contrary. Then $(1, 1, \dots) \in Y$, and hence there exist $m \in \mathbb{N}$, $x \in \mathbb{R}^m$, and a sequence $(q_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}^m$ such that for every $n \in \mathbb{N}$ we have

$$\sum_{i=1}^{m} x(i) q_n(i) t_n = 1,$$

or, equivalently, $\langle q_n \,|\, x\rangle = t_n^{-1}$, where $\langle u \,|\, v\rangle = \sum_{i=1}^{m} u(i)v(i)$ for $u, v \in \mathbb{R}^m$.

Write the system of equations

$$\begin{cases} \langle q_1 \,|\, x\rangle = t_1^{-1}, \\ \dots, \\ \langle q_m \,|\, x\rangle = t_m^{-1} \end{cases}$$

in the form $Qx = y$, where the matrix $Q \in \mathbb{Q}^{m\times m}$ has entries $Q_{ij} = q_i(j)$, and the vector $y \in \mathbb{R}^m$ is taken to be $\left(t_1^{-1}, \dots, t_m^{-1}\right)$.

Note that the matrix Q is invertible. Indeed, otherwise the transpose Q^{T} satisfies the equality $Q^{\mathrm{T}}q = 0$ for some nonzero vector $q \in \mathbb{Q}^m$, and then

$$\sum_{i=1}^{m} q(i)t_i^{-1} = \langle q \,|\, y\rangle = \langle q \,|\, Qx\rangle = \langle Q^{\mathrm{T}}q \,|\, x\rangle = 0,$$

which contradicts the algebraic independence of the reals $t_1, \dots, t_m$ over $\mathbb{Q}$.

Finally, using the invertibility of Q and setting

$$p = (Q^{-1})^{\mathrm{T}} q_{m+1} \in \mathbb{Q}^m,$$

we conclude that

$$\sum_{i=1}^{m} p(i)t_i^{-1} = \langle (Q^{-1})^{\mathrm{T}} q_{m+1} \,|\, y\rangle = \langle q_{m+1} \,|\, Q^{-1}y\rangle = \langle q_{m+1} \,|\, x\rangle = t_{m+1}^{-1},$$

contrary to the algebraic independence of the reals $t_1, \dots, t_m, t_{m+1}$ over $\mathbb{Q}$. $\square$

Theorem 2. *Conditions* (b) *and* (c) *are not equivalent for vector subspaces $Y \subseteq \mathbb{R}^{\mathbb{N}}$.*

Proof. Let $\mathbb{Q}_\circ = \mathbb{Q}\backslash\{0\}$, and consider an injective function

$$t : \bigcup_{n\in\mathbb{N}} \mathbb{Q}_\circ^n \to \mathbb{R}$$

whose image is an algebraically independent subset of $\mathbb{R}$ over $\mathbb{Q}$. Given a sequence $\rho \in \mathbb{Q}_\circ^{\mathbb{N}}$, define $\hat\rho \in \mathbb{R}^{\mathbb{N}}$ by

$$\hat\rho = \big(\rho(1),\ \rho(2)\, t(\pi_1\rho),\ \rho(3)\, t(\pi_2\rho),\ \dots,\ \rho(n)\, t(\pi_{n-1}\rho),\ \dots\big)$$

and show that the set

$$P = \big\{\hat\rho :\ \rho \in \mathbb{Q}_\circ^{\mathbb{N}}\big\}$$

satisfies conditions (i)–(iii).

First observe that for all $\rho, \sigma \in \mathbb{Q}_\circ^{\mathbb{N}}$ the equalities $\rho = \sigma$ and $\hat\rho = \hat\sigma$ are equivalent and, moreover,

$$\pi_n\rho = \pi_n\sigma \ \Leftrightarrow\ \pi_n\hat\rho = \pi_n\hat\sigma \tag{2}$$

for each $n \in \mathbb{N}$. Indeed, the implication "$\Rightarrow$" is trivial, since for every $\rho \in \mathbb{Q}_\circ^{\mathbb{N}}$ the tuple $\pi_n\hat\rho$ is uniquely determined by the numbers $\rho(1), \dots, \rho(n)$ and by the values of t on the tuples formed by these numbers. The implication "$\Leftarrow$" can easily be established by induction on n. For $n = 1$ it follows from the equalities $\hat\rho(1) = \rho(1)$ and $\hat\sigma(1) = \sigma(1)$. If this implication holds for n and if $\pi_{n+1}\hat\rho = \pi_{n+1}\hat\sigma$, then by $\pi_n\hat\rho = \pi_n\hat\sigma$ we have $\pi_n\rho = \pi_n\sigma$, and the remaining equality $\rho(n+1) = \sigma(n+1)$ follows from the relations

$$\rho(n+1)\, t(\pi_n\rho) = \hat\rho(n+1) = \hat\sigma(n+1) = \sigma(n+1)\, t(\pi_n\sigma) = \sigma(n+1)\, t(\pi_n\rho)$$

and the fact that the image of t contains no zero.

(i): Let $s \in \mathbb{R}^{\mathbb{N}}$ and assume that $\pi_n s \in \pi_n P$ for all $n \in \mathbb{N}$, i.e., there exists a sequence $(\rho_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{Q}_\circ^{\mathbb{N}}$ with $\pi_n s = \pi_n\widehat{\rho_n}$ for all $n \in \mathbb{N}$. We will show that $s \in P$.

Define $\rho \in \mathbb{Q}_\circ^{\mathbb{N}}$ by setting $\rho(n) = \rho_n(n)$ for all $n \in \mathbb{N}$.

We prove by induction on n that

$$\pi_n\widehat{\rho_n} = \pi_n\hat\rho \tag{3}$$

for all $n \in \mathbb{N}$. Indeed, for $n = 1$ equality (3) follows immediately from (2). If (3) holds for n, then

$$\pi_n\widehat{\rho_{n+1}} = \pi_n\pi_{n+1}\widehat{\rho_{n+1}} = \pi_n\pi_{n+1}s = \pi_n s = \pi_n\widehat{\rho_n} = \pi_n\hat\rho,$$

whence, by (2), we obtain $\pi_n\rho_{n+1} = \pi_n\rho$, which yields the missing equality

$$\widehat{\rho_{n+1}}(n+1) = \rho_{n+1}(n+1)\, t(\pi_n\rho_{n+1}) = \rho(n+1)\, t(\pi_n\rho) = \hat\rho(n+1).$$

By (3), we have

$$s(n) = (\pi_n s)(n) = (\pi_n \widehat{\rho_n})(n) = (\pi_n \hat{\rho})(n) = \hat{\rho}(n)$$

for all $n \in \mathbb{N}$, i.e., $s = \hat{\rho}$, and hence $s \in P$.

(ii): The set

$$\{p(1) : p \in P\} = \{\hat{\rho}(1) : \rho \in \mathbb{Q}_\circ^{\mathbb{N}}\}$$

coincides with $\mathbb{Q}_\circ$ and is therefore dense in $\mathbb{R}$.

(iii): In view of (2), for all $n \in \mathbb{N}$ and $\sigma \in \mathbb{Q}_\circ^{\mathbb{N}}$ the set

$$\begin{aligned}\{p(n+1) : p \in P,\ \pi_n p = \pi_n \hat{\sigma}\} &= \{\hat{\rho}(n+1) : \rho \in \mathbb{Q}_\circ^{\mathbb{N}},\ \pi_n \hat{\rho} = \pi_n \hat{\sigma}\} \\ &= \{\rho(n+1)\, t(\pi_n \rho) : \rho \in \mathbb{Q}_\circ^{\mathbb{N}},\ \pi_n \rho = \pi_n \sigma\}\end{aligned}$$

coincides with $\mathbb{Q}_\circ\, t(\pi_n \sigma)$ and is therefore dense in $\mathbb{R}$.

Thus the vector subspace

$$Y = \operatorname{lin} P \subseteq \mathbb{R}^{\mathbb{N}}$$

is recursively dense. We will show that it is not Cartesian dense.

Assume, toward a contradiction, that Y satisfies (b). In this case there exist sequences $y, z \in Y$ with $y(1) \neq z(1)$ and $y(n) = z(n)$ for $n > 1$, and hence

$$e_1 = (1, 0, 0, \dots) = \frac{1}{y(1) - z(1)}(y - z) \in Y = \operatorname{lin} P.$$

Therefore, there exist $n \in \mathbb{N}$, nonzero reals $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, and pairwise distinct sequences $\rho_1, \dots, \rho_n \in \mathbb{Q}_\circ^{\mathbb{N}}$ such that

$$\sum_{i=1}^{n} \lambda_i \hat{\rho}_i = e_1.$$

Let I be the set of all pairs (i, j) with $i, j \in \{1, \dots, n\}$ and $i \neq j$. Since $\rho_1, \dots, \rho_n$ are pairwise distinct, we may consider the naturals

$$m_{ij} = \min\{k \in \mathbb{N} : \rho_i(k) \neq \rho_j(k)\}, \quad (i, j) \in I,$$

and put

$$m = \max\{m_{ij} : (i, j) \in I\}.$$

Then $\pi_k \rho_i \neq \pi_k \rho_j$ for $k \geq m$ and $(i, j) \in I$. In particular, the reals $t(\pi_k \rho_i)$ are distinct for different pairs (k, i) such that $k \geq m$ and $i \in \{1, \dots, n\}$.

Since

$$\sum_{i=1}^{n} \lambda_i\, \rho_i(k)\, t(\pi_{k-1} \rho_i) = \sum_{i=1}^{n} \lambda_i\, \hat{\rho}_i(k) = e_1(k) = 0 \quad \text{for } k > m,$$

we obtain the system of equalities

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \lambda_i\, \rho_i(m+1)\, t(\pi_m \rho_i) = 0, \\ \dots, \\ \displaystyle\sum_{i=1}^{n} \lambda_i\, \rho_i(m+n)\, t(\pi_{m+n-1} \rho_i) = 0. \end{cases} \tag{4}$$

Consider the $n \times n$ matrix

$$M = \begin{pmatrix} \rho_1(m+1)\,t(\pi_m\rho_1) & \rho_2(m+1)\,t(\pi_m\rho_2) & \dots & \rho_n(m+1)\,t(\pi_m\rho_n) \\ \rho_1(m+2)\,t(\pi_{m+1}\rho_1) & \rho_2(m+2)\,t(\pi_{m+1}\rho_2) & \dots & \rho_n(m+2)\,t(\pi_{m+1}\rho_n) \\ \dots & \dots & \dots & \dots \\ \rho_1(m+n)\,t(\pi_{m+n-1}\rho_1) & \rho_2(m+n)\,t(\pi_{m+n-1}\rho_2) & \dots & \rho_n(m+n)\,t(\pi_{m+n-1}\rho_n) \end{pmatrix}.$$

Its determinant $|M|$ is the value of a homogeneous polynomial of degree n in the pairwise distinct reals $t(\pi_{m+j-1}\rho_i)$, $i, j \in \{1, \dots, n\}$, which are algebraically independent over $\mathbb{Q}$, and the coefficients of this polynomial are rational and nonzero, since, up to sign, they are products of rationals of the form $\rho_i(m+j) \in \mathbb{Q}_\circ$. Hence $|M| \neq 0$. On the other hand, system (4) means that $Mx = 0$ for the nonzero vector $x = (\lambda_1, \dots, \lambda_n)$. $\square$

Remark 2. The question of the equivalence of conditions (c) and (d) for vector subspaces $Y \subseteq \mathbb{R}^{\mathbb{N}}$ remains open at present.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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