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WEAK LATERAL CONVERGENCE IN BANACH SPACES

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Abstract—Lateral convergence of sequences in Banach spaces endowed with the weak topology is described.

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In [1], the concept of lateral convergence in a topological space was introduced and studied. In particular, it was shown that the lateral convergence of a sequence ensures the existence of continuous mappings (including vector-valued functions, sections, and homomorphisms of Banach bundles) that take preassigned values at the points of the sequence. In this connection, the problem arises of describing lateral convergence in specific classes of topological spaces.

Lateral convergence implies a certain separation of the terms of a sequence from each other and from the limit point. In regular topological spaces with countable pseudocharacter (in particular, in metric spaces), lateral convergence admits a very simple description: a convergent sequence converges laterally if and only if it converges injectively, i.e., when its terms are distinct from each other and from the limit point (see [1, 1.15]). A less obvious problem is to describe lateral convergence in Banach spaces endowed with the weak topology, as well as in closed balls of such spaces. The present note is devoted to the solution of this problem.

The main results are as follows. Let X_{w} and B_{w} be a Banach space X and its closed unit ball endowed with the weak topology, and let X'_{w} and B'_{w} be the topological dual of X and its closed unit ball endowed with the weak* topology. If X is separable, then in X'_{w} and B'_{w} lateral convergence is equivalent to injective convergence (Corollary 16). If, on the other hand, X is nonseparable, then there are no laterally convergent sequences in the space X'_{w} (Theorem 18), whereas in the ball B'_{w} the lateral convergence of a sequence $(f_n)_{n\in\mathbb{N}}$ is closely related to the convergence $\|f_n\| \to 1$ (Theorems 17 and 23 and Corollary 24). Dual statements are also valid, describing the lateral convergence of sequences $(x_n)_{n\in\mathbb{N}}$ in X_{w} and B_{w} in connection with the separability of X'_{w} and the convergence $\|x_n\| \to 1$. In addition, examples are presented in which there are no laterally convergent sequences in B_{w} and B'_{w} (Theorem 25).

Definition 1 [1, Section 1]. Let X be an arbitrary topological space.

A *covering of a sequence* of points $x_n \in X$ is an arbitrary sequence of sets $U_n \subseteq X$, each of which is a neighborhood of the corresponding point x_n.

A sequence of sets $U_n \subseteq X$ *converges laterally* to a point $x \in X$ if

$$\operatorname{cl} U_m \cap \operatorname{cl} \bigcup_{n>m} U_n = \varnothing$$

for all $m \in \mathbb{N}$, and, moreover, x is the unique proper limit point of the union $\bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n$, i.e.,

$$\Bigl(\operatorname{cl} \bigcup_{n\in\mathbb{N}} U_n\Bigr) \setminus \bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n = \{x\}.$$

A sequence of points $x_n \in X$ *converges laterally* to a point $x \in X$ if $x_n \to x$ and the sequence $(x_n)_{n\in\mathbb{N}}$ admits a covering that converges laterally to x. A sequence $(x_n)_{n\in\mathbb{N}}$ *converges injectively* to x if $x_n \to x$, $x_n \neq x_m$ for $n \neq m$, and, moreover, $x_n \neq x$ for all $n \in \mathbb{N}$. (Obviously, every laterally convergent sequence converges injectively.)

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Lemma 2 [1, 1.2]. *Let X be a topological space. A sequence of sets $U_n \subseteq X$ converges laterally to a point $x \in X$ if and only if the following three conditions hold:*

(a) $\operatorname{cl} U_m \cap \operatorname{cl} U_n = \varnothing$ *for* $m \neq n$;
(b) $x \in \operatorname{cl} \bigcup_{n\in\mathbb{N}} U_n \setminus \bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n$;
(c) *if* $x \neq y \in X$, *then* $y \notin \operatorname{cl} \bigcup_{n\geqslant m} U_n$ *for some* $m \in \mathbb{N}$.

Lemma 3 [1, 1.13]. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a regular topological space that converges injectively to a point x and admits a covering $(U_n)_{n\in\mathbb{N}}$ with the following property: for every neighborhood V of x there exists $m \in \mathbb{N}$ such that $U_n \subseteq V$ for all $n \geqslant m$. Then $(x_n)_{n\in\mathbb{N}}$ converges laterally to x.*

Definition 4. The *pseudocharacter* of a T_1 space X *at a point* $x \in X$ is denoted by $\psi(x, X)$ and is defined as the least cardinality $|\mathcal{U}|$ of families $\mathcal{U}$ consisting of open subsets of X and satisfying the equality $\cap\mathcal{U} = \{x\}$:

$$\psi(x, X) := \min\bigl\{|\mathcal{U}| : \mathcal{U} \subseteq \mathit{Open}(X),\ \cap\mathcal{U} = \{x\}\bigr\}.$$

The *pseudocharacter of the space* X is the cardinal $\sup\{\psi(x, X) : x \in X\}$.

Remark 5. Obviously, the pseudocharacter of a space X is countable if and only if X has countable pseudocharacter at each point. Examples of spaces with countable pseudocharacter include spaces satisfying the first axiom of countability. In particular, all metric spaces have this property.

Proposition 6 [1, 1.15]. *Let X be a regular topological space and let $x \in X$. If the pseudocharacter $\psi(x, X)$ is countable, then every sequence in X that converges injectively to x converges laterally.*

Corollary 7. *In a regular topological space with countable pseudocharacter, injective convergence and lateral convergence of sequences are equivalent.*

Remark 8. As is easy to see, given a Hausdorff topological vector space X, the following are equivalent:

(a) X has countable pseudocharacter;
(b) X has countable pseudocharacter at each point;
(c) X has countable pseudocharacter at some point;
(d) X has countable pseudocharacter at zero, i.e., $\bigcap_{n\in\mathbb{N}} U_n = \{0\}$ for some sequence $(U_n)_{n\in\mathbb{N}}$ of neighborhoods of zero in X.

Definition 9. Let X be a Hausdorff locally convex space. (All vector spaces considered here are assumed to be real.) The topological dual of X (i.e., the vector space of continuous linear functionals on X) is denoted by X'. We agree to denote by X_{w} and X'_{w} the spaces X and X', endowed with the weak and weak* topologies $\sigma(X, X')$ and $\sigma(X', X)$, respectively. As is well known, in locally convex spaces X_{w} and X'_{w} basic neighborhoods of arbitrary points $x \in X$ and $f \in X'$, respectively, are sets of the form $x + (X\langle f_1\rangle \cap \cdots \cap X\langle f_m\rangle)$ and $f + (X'\langle x_1\rangle \cap \cdots \cap X'\langle x_m\rangle)$, where $m \in \mathbb{N}$, $f_i \in X'$, and $x_i \in X$. Here and below we use the notation

$$X\langle f\rangle := \bigl\{x \in X : |f(x)| < 1\bigr\}, \quad X'\langle x\rangle := \bigl\{f \in X' : |f(x)| < 1\bigr\},$$

with $x \in X$ and $f \in X'$.

Definition 10. Let X be a Hausdorff locally convex space.

(i) As is well known, the following properties of a subset $T \subseteq X$ are equivalent:
  (a) the linear span of T is dense in X;
  (b) if $f \in X'$ and $f = 0$ on T, then $f = 0$ on X;
  (c) if $f, g \in X'$ and $f \neq g$, then $f(t) \neq g(t)$ for some $t \in T$.

(ii) Assertion (i) implies the equivalence of the following properties of a subset $F \subseteq X'_{\mathrm{w}}$:
  (a) the linear span of F is dense in X'_{w};
  (b) if $x \in X$ and $f(x) = 0$ for all $f \in F$, then $x = 0$;
  (c) if $x, y \in X$ and $x \neq y$, then $f(x) \neq f(y)$ for some $f \in F$.

Subsets $T \subseteq X$ and $F \subseteq X'_{\mathrm{w}}$ possessing the above equivalent properties are called *total* (also fundamental or separating; see [2]).

Lemma 11. *Let X be a Hausdorff locally convex space.*

(i) *The following assertions are equivalent:*
  (a) *X is separable;*
  (b) *X_{w} is separable;*
  (c) *X includes a countable total subset;*
  (d) *X_{w} includes a countable total subset;*
  (e) *X'_{w} has countable pseudocharacter.*

  Moreover, if X is nonseparable, then for every sequence $(U_n)_{n\in\mathbb{N}}$ of neighborhoods of zero in X'_{w} the intersection $\bigcap_{n\in\mathbb{N}} U_n$ includes a nonzero vector subspace.

(ii) *The following assertions are equivalent:*
  (a) *X'_{w} is separable;*
  (b) *X'_{w} includes a countable total subset;*
  (c) *X_{w} has countable pseudocharacter.*

  Moreover, if X'_{w} is nonseparable, then for every sequence $(U_n)_{n\in\mathbb{N}}$ of neighborhoods of zero in X_{w} the intersection $\bigcap_{n\in\mathbb{N}} U_n$ includes a nonzero vector subspace.

Proof. (i): The equivalence of conditions (a)–(d) is well known and is easily verified, for instance, by approximating arbitrary linear combinations by combinations with rational coefficients, and also by using the fact that the closure of a convex set does not depend on the choice of a topology compatible with the duality.

(c)⇒(e): If T is a total subset of X, then for every nonzero $f \in X'$ there exist $t \in T$ and $n \in \mathbb{N}$ such that $|f(nt)| \geqslant 1$, and hence $\bigcap_{(t,n)\in T\times\mathbb{N}} X'\langle nt\rangle = \{0\}$.

(e)⇒(c): Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of neighborhoods of zero in X'_{w}. Each of the sets U_n includes some basic neighborhood $X'\langle x_1^n\rangle \cap \cdots \cap X'\langle x_{m_n}^n\rangle$, where $m_n \in \mathbb{N}$ and $x_i^n \in X$. If the countable subset $T := \bigcup_{n\in\mathbb{N}}\{x_1^n, \ldots, x_{m_n}^n\} \subseteq X$ is not total, then there exists a nonzero functional $f \in X'$ that vanishes on T, and then

$$\{\lambda f : \lambda \in \mathbb{R}\} \subseteq \bigcap_{n\in\mathbb{N}} X'\langle x_1^n\rangle \cap \cdots \cap X'\langle x_{m_n}^n\rangle \subseteq \bigcap_{n\in\mathbb{N}} U_n.$$

Assertion (ii) follows immediately from (i). □

Definition 12. In what follows, X is an arbitrary real Banach space, and X' is the topological dual of X. We agree to use the following notation:

$B := \{x \in X : \|x\| \leqslant 1\}$;
$S := \{x \in X : \|x\| = 1\}$;
B_{w} is the ball B as a topological subspace of X_{w};
$B(x,\varepsilon) := \{y \in B : \|x-y\| < \varepsilon\}$ for $x \in B$ and $\varepsilon > 0$;
$B\langle x,f\rangle := (x + X\langle f\rangle) \cap B = \{y \in B : |f(x-y)| < 1\}$ for $x \in B$ and $f \in X'$

and similarly

$B' := \{f \in X' : \|f\| \leqslant 1\}$;
$S' := \{f \in X' : \|f\| = 1\}$;
B'_{w} is the ball B' as a topological subspace of X'_{w};
$B'(f,\varepsilon) := \{g \in B' : \|f-g\| < \varepsilon\}$ for $f \in B'$ and $\varepsilon > 0$;
$B'\langle f,x\rangle := (f + X'\langle x\rangle) \cap B' = \{g \in B' : |f(x)-g(x)| < 1\}$ for $f \in B'$ and $x \in X$.

Using the introduced notation, basic neighborhoods of arbitrary points $x \in B$ and $f \in B'$ in the topological spaces B_{w} and B'_{w}, respectively, take the form $B\langle x,f_1\rangle \cap \cdots \cap B\langle x,f_m\rangle$ and $B'\langle f,x_1\rangle \cap \cdots \cap B'\langle f,x_m\rangle$, where $m \in \mathbb{N}$, $f_i \in X'$, and $x_i \in X$.

Lemma 13. *Let X be a Banach space.*

(i) *If X is nonseparable and $(U_n)_{n\in\mathbb{N}}$ is a covering of a sequence $(f_n)_{n\in\mathbb{N}}$ in $X'_{\rm w}$, then for every $\lambda \geqslant 0$ there exists $g \in X'$ such that $\|g\| = \lambda$ and $f_n + g \in U_n$ for all $n \in \mathbb{N}$.*

(ii) *If $X'_{\rm w}$ is nonseparable and $(U_n)_{n\in\mathbb{N}}$ is a covering of a sequence $(x_n)_{n\in\mathbb{N}}$ in $X_{\rm w}$, then for every $\lambda \geqslant 0$ there exists $y \in X$ such that $\|y\| = \lambda$ and $x_n + y \in U_n$ for all $n \in \mathbb{N}$.*

PROOF. (i): For each $n \in \mathbb{N}$, the difference $U_n - f_n$ is a neighborhood of zero in $X'_{\rm w}$. According to Lemma 11(i), the intersection $\bigcap_{n\in\mathbb{N}}(U_n - f_n)$ includes a nonzero vector subspace, which obviously contains the desired element g.

Assertion (ii) is proved analogously to (i), with Lemma 11(ii) used instead of Lemma 11(i). □

Lemma 14. *Let X be a Banach space.*

(i) *The following assertions are equivalent:*

  (a) *X is separable;*

  (b) *$B'_{\rm w}$ has countable pseudocharacter;*

  (c) *$B'_{\rm w}$ has countable pseudocharacter at some point of $B'\backslash S'$.*

(ii) *The following assertions are equivalent:*

  (a) *$X'_{\rm w}$ is separable;*

  (b) *$B_{\rm w}$ has countable pseudocharacter;*

  (c) *$B_{\rm w}$ has countable pseudocharacter at some point of $B\backslash S$.*

PROOF. (i): The implications (a)⇒(b)⇒(c) are trivial in view of Lemma 11(i).

(c)⇒(a): Assume that X is nonseparable. Consider an arbitrary point $f \in B'\backslash S'$ and a sequence of its neighborhoods U_n in $B'_{\rm w}$, and show that $\bigcap_{n\in\mathbb{N}} U_n \neq \{f\}$. Each of the sets U_n is of the form $V_n \cap B'$, where V_n is a neighborhood of f in $X'_{\rm w}$. By Lemma 13(i), there exists $g \in X'$ such that $\|g\| = 1 - \|f\|$ and $f + g \in V_n$ for all $n \in \mathbb{N}$. Then $f + g \in B'$, and hence $f \neq f + g \in \bigcap_{n\in\mathbb{N}} V_n \cap B' = \bigcap_{n\in\mathbb{N}} U_n$.

Assertion (ii) is proved analogously to (i), using Lemmas 11(ii) and 13(ii) instead of Lemmas 11(i) and 13(i). □

REMARK 15. Conditions (a)–(c) of Lemma 14(ii) hold, for instance, if the Banach space X is separable or is the dual of a separable Banach space (see [3, 3.1.5]).

The assertion stated below follows from Corollary 7 and Lemmas 11 and 14.

Corollary 16. *Let X be a Banach space.*

(i) *If X is separable, then in the spaces $X'_{\rm w}$ and $B'_{\rm w}$ injective convergence and lateral convergence of sequences are equivalent.*

(ii) *If $X'_{\rm w}$ is separable, then in the spaces $X_{\rm w}$ and $B_{\rm w}$ injective convergence and lateral convergence of sequences are equivalent.*

Theorem 17. *Let X be a Banach space.*

(i) *If X is nonseparable and $(f_n)_{n\in\mathbb{N}}$ is laterally convergent in $B'_{\rm w}$, then $\|f_n\| \to 1$ as $n \to \infty$.*

(ii) *If $X'_{\rm w}$ is nonseparable and $(x_n)_{n\in\mathbb{N}}$ is laterally convergent in $B_{\rm w}$, then $\|x_n\| \to 1$ as $n \to \infty$.*

PROOF. (i): Let X be nonseparable, let $f_n \to f$ in $B'_{\rm w}$, and let $(U_n)_{n\in\mathbb{N}}$ be a covering of $(f_n)_{n\in\mathbb{N}}$ in $B'_{\rm w}$ that converges laterally to f. Assume to the contrary that there exist an infinite set $I \subseteq \mathbb{N}$ and a real $0 < \lambda < 1$ such that $\|f_i\| \leqslant \lambda$ for all $i \in I$.

Each of the sets U_i is of the form $V_i \cap B'$, where V_i is a neighborhood of f_i in $X'_{\rm w}$. By Lemma 13(i), there exists $g \in X'$ such that $\|g\| = 1 - \lambda$ and $f_i + g \in V_i$ for all $i \in I$. From the relations

$$\|f_i + g\| \leqslant \|f_i\| + \|g\| \leqslant \lambda + (1 - \lambda) = 1$$

it follows that $f_i + g \in V_i \cap B' = U_i$ for all $i \in I$. In particular, for each $m \in \mathbb{N}$,

$$f_i + g \in U_i \ \subseteq \bigcup_{n\in I,\, n\geqslant m} U_n \subseteq \bigcup_{n\geqslant m} U_n \quad \text{for all } i \in I,\ i \geqslant m.$$

Moreover, the convergence $f_i + g \underset{i\in I}{\longrightarrow} f + g$ in X'_{w} and the closedness of B' in X'_{w} imply $f + g \in B'$. Thus, in the topological space B'_{w} we have $f \neq f + g \in \operatorname{cl} \bigcup_{n \geqslant m} U_n$ for all $m \in \mathbb{N}$, which contradicts condition (c) of Lemma 2.

Assertion (ii) is proved analogously to (i), with Lemma 13(ii) used instead of Lemma 13(i). □

Theorem 18. *Let X be a nonzero Banach space.*

(i) *The following assertions are equivalent:*

- (a) *X is separable;*
- (b) *in X'_{w} injective convergence and lateral convergence of sequences are equivalent;*
- (c) *there exists a laterally convergent sequence in X'_{w}.*

(ii) *The following assertions are equivalent:*

- (a) *X'_{w} is separable;*
- (b) *in X_{w} injective convergence and lateral convergence of sequences are equivalent;*
- (c) *there exists a laterally convergent sequence in X_{w}.*

Proof. (i): The implication (a)⇒(b) is contained in Corollary 16.

The implication (b)⇒(c) is trivial: for instance, given a nonzero $f \in X'$, the convergence $\frac{1}{n} f \to 0$ in X'_{w} is injective and, by virtue of (b), is lateral.

(c)⇒(a): Consider a laterally convergent sequence $g_n \to g$ in X'_{w} and assume to the contrary that X is nonseparable. From the weak* convergence $g_n \to g$ it follows that there exists $\lambda > 0$ such that $\|g\| \leqslant \lambda$ and $\|g_n\| \leqslant \lambda$ for all $n \in \mathbb{N}$. Put $f_n := \frac{1}{2\lambda} g_n$ and $f := \frac{1}{2\lambda} g$. Clearly, the sequence $(f_n)_{n\in\mathbb{N}}$ converges laterally to f in X'_{w}, and the terms of this sequence as well as its limit belong to B'_{w}. Therefore, $(f_n)_{n\in\mathbb{N}}$ converges laterally in B'_{w} (see [1, 1.10(b)]). By Theorem 17(i), this implies that $\|f_n\| \to 1$, which contradicts the relations

$$\|f_n\| = \frac{1}{2\lambda}\|g_n\| \leqslant \frac{1}{2\lambda}\lambda = \frac{1}{2}.$$

Assertion (ii) is proved analogously to (i), using Theorem 17(ii) instead of Theorem 17(i). □

Definition 19. We recall several classical definitions and facts concerning convex and smooth spaces (see, e.g., [4, Chapter II; 5, Part 3]).

A Banach space X is

(a) *strictly convex* if for all $x, y \in X$

$$\|x\| = \|y\| = \left\|\frac{x+y}{2}\right\| = 1 \ \Rightarrow \ x = y;$$

(b) *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$

$$\|x\| = \|y\| = 1, \ \|x - y\| \geqslant \varepsilon \ \Rightarrow \ \left\|\frac{x+y}{2}\right\| \leqslant 1 - \delta;$$

(c) *smooth* if for every nonzero $x \in X$ there exists a unique $f \in X'$ such that $\|f\| = 1$ and $f(x) = \|x\|$;

(d) *uniformly smooth* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$

$$\|x\| = 1, \ \|y\| \leqslant \delta \ \Rightarrow \ \|x+y\| + \|x-y\| \leqslant 2 + \varepsilon\|y\|.$$

Classical examples of spaces with properties (a)–(d) include Hilbert spaces and spaces of the form L^p, $1 < p < \infty$.

Remark 20 (see [4, Chapter II; 5, Part 3]).

(a) Every uniformly convex space is strictly convex, and every uniformly smooth space is smooth.

(b) The smoothness of X' implies the strict convexity of X, and the strict convexity of X' implies the smoothness of X.

(c) The uniform convexity of X is equivalent to the uniform smoothness of X', and the uniform smoothness of X is equivalent to the uniform convexity of X'.

(d) Uniformly convex Banach spaces and uniformly smooth Banach spaces are reflexive.

Lemma 21. *Let X be a Banach space.*
(i) *If X' is strictly convex, then B'_{w} has countable pseudocharacter at each point of S'.*
(ii) *If X is strictly convex, then B_{w} has countable pseudocharacter at each point of S.*

Proof. (i): Let X' be strictly convex, and let $f \in S'$. Since $\|f\| = 1$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in B such that $f(x_n) \to 1$ as $n \to \infty$. We show that $\bigcap_{n\in\mathbb{N}} B'\langle f, nx_n\rangle = \{f\}$. To this end, take an arbitrary $g \in \bigcap_{n\in\mathbb{N}} B'\langle f, nx_n\rangle$ and establish the equality $f = g$. Indeed, for all $n \in \mathbb{N}$ we have $|g(x_n) - f(x_n)| < \frac{1}{n}$, whence

$$1 \geqslant \|g\| \geqslant g(x_n) = f(x_n) + g(x_n) - f(x_n) > f(x_n) - \frac{1}{n} \underset{n\to\infty}{\longrightarrow} 1,$$

$$1 \geqslant \left\|\frac{f+g}{2}\right\| \geqslant \frac{f+g}{2}(x_n) = f(x_n) + \frac{g(x_n) - f(x_n)}{2} > f(x_n) - \frac{1}{2n} \underset{n\to\infty}{\longrightarrow} 1,$$

and hence $f = g$ by the strict convexity of X'.

(ii): The proof can be carried out along the same lines as for (i). In this case the arguments are more elementary: if X is strictly convex, then the countability of the pseudocharacter of B_{w} at a point $x \in S$ is ensured by the equality $\bigcap_{n\in\mathbb{N}} B\langle x, nf\rangle = \{x\}$, where $f \in X'$ is such that $f(x) = \|f\| = 1$. □

Corollary 22. *Let X be a Banach space.*
(i) *If X' is strictly convex, then in B'_{w} the injective convergence and the lateral convergence of a sequence to an element of S' are equivalent.*
(ii) *If X is strictly convex, then in B_{w} the injective convergence and the lateral convergence of a sequence to an element of S are equivalent.*

Theorem 23. *Let X be a Banach space.*
(i) *Let X be uniformly smooth, let $(f_n)_{n\in\mathbb{N}}$ converge injectively to f in B'_{w}, and suppose that $\|f_n\| \to 1$ as $n \to \infty$. Then $(f_n)_{n\in\mathbb{N}}$ converges laterally to f in B'_{w}.*
(ii) *Let X be uniformly convex, let $(x_n)_{n\in\mathbb{N}}$ converge injectively to x in B_{w}, and suppose that $\|x_n\| \to 1$ as $n \to \infty$. Then $(x_n)_{n\in\mathbb{N}}$ converges laterally to x in B_{w}.*

Proof. (i): Let the assumptions of (i) hold. We may assume that $\dim X' > 1$ and that $f_n \neq 0$ for all $n \in \mathbb{N}$. Since X' is uniformly convex (see Remark 20(c)), for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all $g, h \in X'$

$$\|g\| = \|h\| = 1, \quad \left\|\frac{g+h}{2}\right\| > 1 - \delta(\varepsilon) \;\Rightarrow\; \|g - h\| < \varepsilon. \tag{1}$$

We show that for all $x \in X$ and $g, h \in X'$

$$\|x\| = \|g\| = g(x) = 1, \quad \|h\| \leqslant 1, \quad |h(x) - 1| < \delta(\varepsilon) \;\Rightarrow\; \|g - h\| < \varepsilon. \tag{2}$$

Indeed, let $\|x\| = \|g\| = g(x) = 1$ and $|h(x) - 1| < \delta(\varepsilon)$. In the case $\|h\| = 1$ we have

$$\left\|\frac{g+h}{2}\right\| \geqslant \frac{g+h}{2}(x) = 1 + \frac{1}{2}\big(h(x) - 1\big) > 1 - \frac{\delta(\varepsilon)}{2} > 1 - \delta(\varepsilon),$$

whence $\|g - h\| < \varepsilon$ by (1). Now let $\|h\| < 1$. Consider a nonzero functional $z \in X'$ satisfying $z(x) = 0$, and put $h_\lambda := h + \lambda z$ for $\lambda \in \mathbb{R}$. Clearly, there exist reals $\lambda < 0 < \mu$ such that $\|h_\lambda\| = \|h_\mu\| = 1$. Since $h_\lambda(x) = h_\mu(x) = h(x)$ and relation (2) has been proved for functionals h of unit norm, we have

$$\|g - h_\lambda\| < \varepsilon, \quad \|g - h_\mu\| < \varepsilon,$$

and hence $\|g - h\| < \varepsilon$, because h lies in the segment $[h_\lambda, h_\mu]$.

Since X is reflexive (see Remark 20(d)), by James's theorem there exists a sequence of $x_n \in S$ such that $f_n(x_n) = \|f_n\|$ for all $n \in \mathbb{N}$. Owing to the convergence $\|f_n\| \to 1$, for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for $n \geqslant N(\varepsilon)$

$$\frac{1}{n} + \big|\|f_n\| - 1\big| < \delta(\varepsilon), \tag{3}$$

$$\big|\|f_n\| - 1\big| < \varepsilon. \tag{4}$$

We show that for all $\varepsilon > 0$, $n \geqslant N(\varepsilon)$, and $h \in X'$

$$\|h\| \leqslant 1, \quad |h(x_n) - f_n(x_n)| < \frac{1}{n} \;\Rightarrow\; \|f_n - h\| < 2\varepsilon. \tag{5}$$

Indeed, if $n \geqslant N(\varepsilon)$, $\|h\| \leqslant 1$, and $|h(x_n) - f_n(x_n)| < \frac{1}{n}$, then by (3) we have

$$|h(x_n) - 1| \leqslant |h(x_n) - f_n(x_n)| + |f_n(x_n) - 1| < \frac{1}{n} + \big|\|f_n\| - 1\big| < \delta(\varepsilon). \tag{6}$$

Put $g_n := f_n/\|f_n\|$. Then $\|x_n\| = \|g_n\| = g_n(x_n) = 1$, and hence, in view of (2) and (6), we get $\|g_n - h\| < \varepsilon$. Using (4), we conclude that

$$\|f_n - h\| \leqslant \|f_n - g_n\| + \|g_n - h\| < \left\| f_n - \frac{f_n}{\|f_n\|} \right\| + \varepsilon = \left\| \big(\|f_n\| - 1\big) \frac{f_n}{\|f_n\|} \right\| + \varepsilon = \big|\|f_n\| - 1\big| + \varepsilon < \varepsilon + \varepsilon = 2\varepsilon.$$

By Lemma 3, to justify the lateral convergence $f_n \to f$ in B'_{w} it suffices to show that for every neighborhood V of f in B'_{w}, the inclusion $B'\langle f_n, nx_n\rangle \subseteq V$ holds for all large n. In view of the structure of basic neighborhoods in B'_{w} (see Definition 12), it is enough, in turn, to fix an arbitrary nonzero $x \in X$ and, for all large n, establish the inclusion

$$B'\langle f_n, nx_n\rangle \subseteq B'\langle f, x\rangle. \tag{7}$$

Since $f_n \to f$ weakly, there exists $M \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \frac{1}{2}$ for $n \geqslant M$. We show that inclusion (7) holds for

$$n \geqslant \max\Big\{ N\Big(\frac{1}{4\|x\|}\Big), M \Big\}.$$

Indeed, if $h \in B'\langle f_n, nx_n\rangle$, i.e., $\|h\| \leqslant 1$ and $|h(x_n) - f_n(x_n)| < \frac{1}{n}$, then by (5) we have $\|f_n - h\| < \frac{1}{2\|x\|}$ and hence

$$|f(x) - h(x)| \leqslant |f(x) - f_n(x)| + |f_n(x) - h(x)| < \frac{1}{2} + \|f_n - h\|\,\|x\| < \frac{1}{2} + \frac{1}{2\|x\|}\,\|x\| = 1,$$

i.e., $h \in B'\langle f, x\rangle$.

(ii): A uniformly convex space X is reflexive, and its dual space X' is uniformly smooth (see Remark 20(c),(d)). These observations make it possible to identify B_{w} with the unit ball of the second dual space X'', endowed with the weak* topology, and to apply assertion (i). □

Corollary 24. (i) *Let X be a nonseparable uniformly smooth Banach space. In the space B'_{w}, a sequence $(f_n)_{n\in\mathbb{N}}$ converges laterally to f if and only if $(f_n)_{n\in\mathbb{N}}$ converges injectively to f and $\|f_n\| \to 1$ as $n \to \infty$.*

(ii) *Let X be a nonseparable uniformly convex Banach space. In the space B_{w}, a sequence $(x_n)_{n\in\mathbb{N}}$ converges laterally to x if and only if $(x_n)_{n\in\mathbb{N}}$ converges injectively to x and $\|x_n\| \to 1$ as $n \to \infty$.*

Theorem 25. (i) *There exists a nonzero Banach space X such that B'_{w} contains no laterally convergent sequence.*

(ii) *There exists a nonzero Banach space X such that B_{w} contains no laterally convergent sequence.*

Proof. (i): Consider an uncountable set I and the classical Banach spaces $X := \ell^1(I)$ and $\ell^\infty(I)$, consisting, respectively, of summable functions $x : I \to \mathbb{R}$ and bounded functions $f : I \to \mathbb{R}$, with norms

$$\|x\|_1 = \sum_{i\in I} |x(i)|, \quad \|f\|_\infty = \sup_{i\in I} |f(i)|.$$

As is well known, the mapping $f \mapsto \widehat{f}$, which assigns the functional $\widehat{f} : x \mapsto \sum_{i\in I} x(i)f(i)$ to each function $f \in \ell^\infty(I)$, implements a linear isometry of $\ell^\infty(I)$ onto X'.

Suppose that there exist functions $f_n \in \ell^\infty(I)$ $(n \in \mathbb{N})$ and $f \in \ell^\infty(I)$ such that the sequence $(\widehat{f}_n)_{n\in\mathbb{N}}$ admits a covering $(U_n)_{n\in\mathbb{N}}$ in B'_{w} that converges laterally to $\widehat{f}$.

Each of the sets U_n includes some basic neighborhood $B'\langle \widehat{f}_n, x_1^n\rangle \cap \dots \cap B'\langle \widehat{f}_n, x_{m_n}^n\rangle$, where $m_n \in \mathbb{N}$ and $x_k^n \in X$. Since every function $x \in \ell^1(I)$ has countable support $\operatorname{supp} x := \{i \in I : x(i) \neq 0\}$, all x_k^n vanish outside the countable subset

$$I_0 := \bigcup_{n\in\mathbb{N}} \bigcup_{k=1}^{m_n} \operatorname{supp} x_k^n \subseteq I.$$

As is easy to see, for each $n \in \mathbb{N}$ the set U_n contains every functional $\widehat{g}$ corresponding to a function $g \in \ell^\infty(I)$ such that $\|g\|_\infty \leqslant 1$ and $g = f_n$ on I_0.

Fix an element $j \in I \setminus I_0$ and a real $\lambda \in [-1, 1]$ distinct from $f(j)$. Denote by e_j the indicator function of the singleton $\{j\} \subseteq I$, and define functions $g, g_n \in \ell^\infty(I)$ by setting

$$g := f + \big(\lambda - f(j)\big)\, e_j, \quad g_n := f_n + \big(\lambda - f_n(j)\big)\, e_j.$$

Clearly, $\|g_n\|_\infty \leqslant 1$ and $g_n = f_n$ on $I \setminus \{j\} \supseteq I_0$. Therefore, $\widehat{g}_n \in U_n$ for all $n \in \mathbb{N}$. Moreover, in B'_{w} we have the convergence

$$\widehat{g}_n = \widehat{f}_n + \big(\lambda - f_n(j)\big)\, \widehat{e}_j \to \widehat{f} + \big(\lambda - f(j)\big)\, \widehat{e}_j = \widehat{g}$$

as $n \to \infty$. Thus, $\widehat{f} \neq \widehat{g} \in \operatorname{cl} \bigcup_{n \geqslant m} U_n$ for all $m \in \mathbb{N}$, which contradicts condition (c) of Lemma 2.

(ii): A corresponding example is provided by the space $X := c_0(I)$, endowed with the supremum norm, which consists of real functions defined on an uncountable set I and vanishing at infinity, i.e., functions $x : I \to \mathbb{R}$ such that the set $\{i \in I : |x(i)| > \varepsilon\}$ is finite for every $\varepsilon > 0$. The proof can be obtained by repeating the arguments in (i) with $c_0(I)$ in place of $\ell^\infty(I)$ and using the linear isometry of the Banach spaces $c_0(I)'$ and $\ell^1(I)$. □

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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