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OBJECT-ORIENTED DATA VIA PREFIX REWRITING. PART II: EFFECTIVE ANALYSIS OF REWRITING

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Abstract—For an arbitrary deterministic longest-prefix rewriting system, a criterion is obtained for the infinite rewritability of a word, on the basis of which an effective procedure is developed for analyzing the rewriting sequence.

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Introduction

This paper is part of a planned series of publications devoted to the systematic study, announced in [1], of prefix rewriting as a tool for representing and analyzing object-oriented data systems.

In the overview [1], the central object of study is a deterministic longest-prefix rewriting system $\langle\rightarrow, \omega\rangle$, where a rewriting function $\rightarrow$ acting on a set of words is considered together with a distinguished letter ω, called the generic object. Interpreting the relations $X \rightarrow A$ and $XY \rightarrow B$ as "X is A" and, respectively, "the Y of X is B" makes it possible to use prefix rewriting to define and analyze concepts related to object-oriented data systems—such as inheritance of classes and objects, instances of classes, class and instance attributes, conceptual dependence and consistency, conceptual scheme, types and subtypes, etc. Since this paper deals only with those aspects of prefix rewriting in which the generic object ω is not involved, the consideration is restricted to the rewriting function.

The material of the paper is grouped into three sections. Section 1 contains the basic definitions and notation concerning the rewriting function, Section 2 gives some auxiliary assertions, and the main Section 3 includes a criterion for infinite rewritability of a word, an algorithm for finding a word in a rewriting sequence, and an effective procedure for verifying finite rewritability of all words.

1. Definitions and Notation

1.1. Throughout the paper, $\mathbb{A}$ is a nonempty finite alphabet, and $\mathbb{A}^*$ (respectively, $\mathbb{A}^+$) is the set of all words (respectively, all nonempty words) over $\mathbb{A}$. The elements of $\mathbb{A}$ are called *letters*. We conventionally identify letters with the corresponding single-letter words. The length of a word X is denoted by $|X|$. For $X \in \mathbb{A}^+$ and $n \geqslant 0$, the symbol X^n denotes the concatenation $XX\cdots X$ consisting of n occurrences of the word X; moreover, X^0 is assumed to be the empty word $\varnothing$.

For brevity, in the sequel we use "*word*" as shorthand for "nonempty word over $\mathbb{A}$."

The symbol $\mathbb{N}$ denotes the set of naturals $\{1, 2, \dots\}$. We agree to write ordered tuples and sequences with angle brackets: $\langle x_1, \dots, x_n\rangle$, $\langle x_n\rangle_{n\in\mathbb{N}}$.

1.2. Say that X is a *prefix* of $Y \in \mathbb{A}^+$ and write $X \sqsubseteq Y$, if $X \in \mathbb{A}^+$ and $Y = XS$ for some $S \in \mathbb{A}^*$. The suffix S of the word $Y = XS$ determined by the prefix $X \sqsubseteq Y$ will be denoted by $\frac{Y}{X}$. Thus,

$$X\tfrac{Y}{X} = Y \quad \text{for all } X, Y \in \mathbb{A}^+,\ X \sqsubseteq Y;$$
$$\tfrac{XS}{X} = S \quad \text{for all } X \in \mathbb{A}^+,\ S \in \mathbb{A}^*.$$

If $n \in \mathbb{N}$, $X \in \mathbb{A}^+$, and $|X| \geqslant n$, then $X{\restriction_n}$ denotes the prefix of X having length n.

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1.3. By a *rewriting function* we mean an arbitrary function $\to$ defined on a finite nonempty subset of $\mathbb{A}^+$ and acting into $\mathbb{A}^+$. We treat the function $\to$ as a binary relation on $\mathbb{A}^+$ (i.e., as a subset $\to \subseteq \mathbb{A}^+ \times \mathbb{A}^+$) and use the infix notation $X \to Y$ instead of the membership $\langle X, Y\rangle \in \to$ or the equality $\to(X) = Y$.

In what follows, when giving definitions and formulating statements, we assume by default that an arbitrary rewriting function $\to$ is fixed, and all introduced notions, as well as the terms and notation used, are interpreted with respect to this function.

1.4. Denote by $\mathbb{E}$ the nonempty finite subset of $\mathbb{A}^+$ which is the domain of the rewriting function $\to$. The elements of $\mathbb{E}$ will be called *explicit words*. Say that E is an *explicit prefix* of a word X if $E \sqsubseteq X$ and $E \in \mathbb{E}$. A word $S \in \mathbb{A}^+$ is called an *explicit complement* of a word X if $XS \in \mathbb{E}$. In addition, define μ as the length of the longest explicit word:

$$\mu := \max\{|E| : E \in \mathbb{E}\}.$$

1.5. The value of the function $\to$ on an explicit word $E \in \mathbb{E}$ will be denoted by E' and called the *explicit rewrite* of E. Thus,

$$E \to E'$$

for all $E \in \mathbb{E}$. A word X is called *rewritable* if X has at least one explicit prefix. Given a rewritable word X, consider its longest explicit prefix E, determine the corresponding suffix $S \in \mathbb{A}^*$ such that $X = ES$, and introduce the notation

$$X' := E'S.$$

We call X' the *rewrite* of X. The longest explicit prefix of a rewritable word X will be denoted by $\mathrm{lep}(X)$. Thus, if $E = \mathrm{lep}(X)$, then, taking into account the notation introduced in 1.2, we have

$$\left(E\tfrac{X}{E}\right)' = E'\tfrac{X}{E}.$$

1.6. Introduce the binary relation $\Rightarrow$ on $\mathbb{A}^+$ by setting

$$X \Rightarrow Y \ \text{ if and only if } \ X \text{ is rewritable and } X' = Y.$$

Denote by $\overset{*}{\Rightarrow}$ and $\overset{+}{\Rightarrow}$ the reflexive transitive closure of $\Rightarrow$ and the transitive closure of $\Rightarrow$, respectively. Thus, $x \overset{*}{\Rightarrow} y$ (respectively, $x \overset{+}{\Rightarrow} y$) means that

$$x = z_0 \Rightarrow z_1 \Rightarrow \cdots \Rightarrow z_n = y$$

for some $z_1, \ldots, z_n$, where $n \geqslant 0$ (respectively, $n \geqslant 1$). In particular, $x \overset{*}{\Rightarrow} y$ if and only if $x = y$ or $x \overset{+}{\Rightarrow} y$.

1.7. For all integers $n \geqslant 0$, define the sets $\mathbb{W}_n$ of words rewritable n times and the nth rewrites $X^{(n)}$ of words $X \in \mathbb{W}_n$ by the following joint recursion:

$$\begin{aligned} &\mathbb{W}_0 := \mathbb{A}^+, && X^{(0)} := X \ \text{ for } X \in \mathbb{W}_0; \\ &\mathbb{W}_n := \{X \in \mathbb{W}_{n-1} : X^{(n-1)} \text{ is rewritable}\}, && X^{(n)} := (X^{(n-1)})' \ \text{ for } X \in \mathbb{W}_n, \quad n \geqslant 1. \end{aligned}$$

Put

$$\mathbb{W}_\infty := \bigcap_{n=1}^{\infty} \mathbb{W}_n, \quad \mathbb{W}_{\text{fin}} := \mathbb{A}^+ \setminus \mathbb{W}_\infty.$$

The word $X^{(n)}$ is called the *nth rewrite* of X. Thus, $\mathbb{W}_n$ is the set of all words X that can be rewritten n times, and, for each word $X \in \mathbb{W}_n$, the following relations hold:

$$X = X^{(0)} \Rightarrow X^{(1)} \Rightarrow \cdots \Rightarrow X^{(n)};$$

$$X \overset{*}{\Rightarrow} X^{(n)} \text{ for } n \geqslant 0, \quad X \overset{+}{\Rightarrow} X^{(n)} \text{ for } n > 0.$$

We have the chain of inclusions

$$\mathbb{W}_0 \supseteq \mathbb{W}_1 \supseteq \mathbb{W}_2 \supseteq \cdots \supseteq \mathbb{W}_n \supseteq \mathbb{W}_{n+1} \supseteq \cdots \supseteq \mathbb{W}_\infty.$$

The words belonging to $\mathbb{W}_\infty$ are called *infinitely rewritable*, and the other words, i.e., the elements of $\mathbb{W}_{\text{fin}}$, are called *finitely rewritable*. Given a word X, call the maximal (finite or infinite) sequence of the form

$$\langle X^{(0)}, X^{(1)}, X^{(2)}, \ldots\rangle$$

the *rewriting sequence* of X. Thus, a word X is finitely (infinitely) rewritable if and only if the rewriting sequence of X is finite (infinite).

2. Auxiliary Assertions

2.1. Proposition. *Let $X \in \mathbb{A}^+$, $S \in \mathbb{A}^*$, and $\alpha \in \mathbb{A}$.*
(1) *If $X\alpha \notin \mathbb{E}$ and $X\alpha \in \mathbb{W}_1$, then $X \in \mathbb{W}_1$ and $(X\alpha)' = X'\alpha$.*
(2) *If X has no explicit complements and $XS \in \mathbb{W}_1$, then $X \in \mathbb{W}_1$ and $(XS)' = X'S$.*
(3) *If $|X| \geqslant \mu$ and $XS \in \mathbb{W}_1$, then $X \in \mathbb{W}_1$ and $(XS)' = X'S$.*

Proof. Put $S := \alpha$ in case (1). In each of cases (1) and (2)—when $X\alpha \notin \mathbb{E}$ or when X has no explicit complements—the following condition holds: every explicit prefix $E \sqsubseteq XS$ is a prefix of X and, in particular, satisfies the equality

$$\tfrac{XS}{E} = \tfrac{X}{E}S.$$

Now assume that $XS \in \mathbb{W}_1$ and consider $E := \operatorname{lep}(XS)$. Then E is a prefix of X, whence $E = \operatorname{lep}(X)$. Therefore, $X \in \mathbb{W}_1$ and

$$(XS)' = \big(E\tfrac{XS}{E}\big)' = E'\tfrac{XS}{E} = E'\tfrac{X}{E}S = \big(E\tfrac{X}{E}\big)'S = X'S.$$

Assertion (3) follows from (2), since, in the case $|X| \geqslant \mu$, the word X has no explicit complements. □

2.2. Remark. Every word having a rewritable prefix is itself rewritable. In particular, under the assumptions of each assertion of Proposition 2.1, the inclusions $X \in \mathbb{W}_1$ and $XS \in \mathbb{W}_1$ are equivalent.

2.3. Lemma. *Let $r \in \mathbb{N}$, $X \in \mathbb{W}_r$, $\big|X^{(1)}\big|, \dots, \big|X^{(r)}\big| \geqslant |X|$, and let $Y \in \mathbb{A}^+$ and $S \in \mathbb{A}^*$ be such that $X = YS$ and $|Y| \geqslant \mu$. Then*

$$Y \in \mathbb{W}_i, \ X^{(i)} = Y^{(i)}S \ \text{ and } \ \big|Y^{(i)}\big| \geqslant \mu \tag{C$_i$}$$

for all $i = 0, \dots, r$.

Proof. We prove (C$_i$) for all $i = 0, \dots, r$ by induction on i. Assertion (C$_0$) holds by the assumptions of the lemma. Suppose that (C$_i$) holds for some $0 \leqslant i < r$, and establish (C$_{i+1}$). Indeed, applying Proposition 2.1(3) with $Y^{(i)}$ in the role of X, we conclude that $Y^{(i)} \in \mathbb{W}_1$ (i.e., $Y \in \mathbb{W}_{i+1}$) and

$$X^{(i+1)} = (X^{(i)})' = (Y^{(i)}S)' = (Y^{(i)})'S = Y^{(i+1)}S.$$

The equality obtained implies that

$$\big|Y^{(i+1)}\big| + |S| = \big|Y^{(i+1)}S\big| = \big|X^{(i+1)}\big| \geqslant |X| = |YS| = |Y| + |S| \geqslant \mu + |S|,$$

and hence $\big|Y^{(i+1)}\big| \geqslant \mu$. □

2.4. Lemma. *For every sequence of naturals $\langle x_n \rangle_{n\in\mathbb{N}}$, there exists a strictly increasing sequence of naturals $\langle n_i \rangle_{i\in\mathbb{N}}$ such that, for all $n, i \in \mathbb{N}$, the inequality $n \geqslant n_i$ implies $x_n \geqslant x_{n_i}$.*

Proof. The sequence $\langle n_i \rangle_{i\in\mathbb{N}}$ can be defined recursively by choosing n_1 so that $x_{n_1} = \min\{x_n : n \in \mathbb{N}\}$ and then, for each $i \in \mathbb{N}$, choosing $n_{i+1} > n_i$ so that $x_{n_{i+1}} = \min\{x_n : n > n_i\}$. □

3. Main Results

3.1. The following assertion was announced in [1, Theorem 5.11].

Theorem. *A word X is infinitely rewritable if and only if one of the following two mutually exclusive conditions holds:*

(a) *there are integers $n \geqslant 0$ and $r > 0$ such that $X^{(n)} = X^{(n+r)}$;*
(b) *there are integers $n \geqslant 0$ and $r > 0$ such that*

$$\mu \leqslant \big|X^{(n)}\big| \leqslant \big|X^{(n+1)}\big|, \dots, \big|X^{(n+r)}\big|,$$
$$X^{(n)} \neq X^{(n+r)}, \quad X^{(n)}\restriction_{\mu} = X^{(n+r)}\restriction_{\mu}.$$

In case (a), the rewriting of X becomes periodic, starting with the nth term, with period of length r:

$$X \overset{*}{\Rightarrow} \underbrace{X^{(n)} \Rightarrow \cdots \Rightarrow X^{(n+r-1)}}_{\text{period}} \Rightarrow X^{(n)} \Rightarrow \cdots.$$

In case (b), *represent* $X^{(n)}$ *as* YS, *where* $Y \in \mathbb{A}^+$, $S \in \mathbb{A}^*$, *and* $|Y| = \mu$, *i.e., put* $Y := X^{(n)}\restriction_\mu$ *and* $S := \frac{X^{(n)}}{Y}$. *Then* $Y \in \mathbb{W}_r$ *and there exists a word* $R \in \mathbb{A}^+$ *such that* $Y^{(r)} = YR$ *and the rewriting sequence of* X *contains a subsequence formed by the words* $X^{(n+mr)} = YR^mS$, $m \geqslant 0$, *each of which starts a regular "growth period" of length* r*:*

$$\begin{aligned} X \overset{*}{\Rightarrow} X^{(n)} &= YS \Rightarrow Y^{(1)}S \Rightarrow \cdots \Rightarrow Y^{(r-1)}S \\ \Rightarrow X^{(n+r)} &= YRS \Rightarrow Y^{(1)}RS \Rightarrow \cdots \Rightarrow Y^{(r-1)}RS \\ \Rightarrow X^{(n+2r)} &= YR^2S \Rightarrow Y^{(1)}R^2S \Rightarrow \cdots \Rightarrow Y^{(r-1)}R^2S \\ \Rightarrow \cdots & \\ \Rightarrow X^{(n+mr)} &= YR^mS \Rightarrow Y^{(1)}R^mS \Rightarrow \cdots \Rightarrow Y^{(r-1)}R^mS \\ \Rightarrow \cdots. & \end{aligned}$$

In particular, $\{X^{(n)}, X^{(n+1)}, \dots\} = \{Y^{(j)}R^mS : 0 \leqslant j < r,\ m \geqslant 0\}$.

PROOF. NECESSITY. Let $X \in \mathbb{W}_\infty$.

Suppose first that, for every $n \in \mathbb{N}$, there is a natural $m > n$ such that $\left|X^{(m)}\right| < \mu$. We show that condition (a) holds in this case. In the case under consideration, there is a strictly increasing sequence of naturals $\langle n_i \rangle_{i \in \mathbb{N}}$ such that $\left|X^{(n_i)}\right| < \mu$ for all $i \in \mathbb{N}$. Since the set $\{Y \in \mathbb{A}^+ : |Y| < \mu\}$ is finite, the words $X^{(n_i)}$, $i \in \mathbb{N}$, cannot be pairwise distinct. Hence there are naturals $i < j$ such that $X^{(n_i)} = X^{(n_j)}$. Then, putting $n = n_i$ and $r = n_j - n_i$, we obtain $X^{(n)} = X^{(n+r)}$, and so condition (a) holds.

Now suppose that there exists $n_0 \in \mathbb{N}$ such that $\left|X^{(n)}\right| \geqslant \mu$ for all $n > n_0$. We show that one of conditions (a) or (b) holds in this case. By Lemma 2.4, there is a strictly increasing sequence of naturals $\langle n_i \rangle_{i \in \mathbb{N}}$ such that

$$n \geqslant n_i \ \text{ implies } \ \left|X^{(n)}\right| \geqslant \left|X^{(n_i)}\right| \tag{$*$}$$

for all $n, i \in \mathbb{N}$. Without loss of generality, we may assume that $n_1 > n_0$. Then

$$\left|X^{(n_i)}\right| \geqslant \mu \ \text{ for all } \ i \in \mathbb{N}. \tag{$**$}$$

Since the set of words of length μ is finite, the words $X^{(n_i)}\restriction_\mu$, $i \in \mathbb{N}$, cannot be pairwise distinct. Hence there are naturals $i < j$ such that $X^{(n_i)}\restriction_\mu = X^{(n_j)}\restriction_\mu$. Put $n = n_i$ and $r = n_j - n_i$. Then $X^{(n)}\restriction_\mu = X^{(n+r)}\restriction_\mu$. Moreover, by $(*)$ and $(**)$, we have $\mu \leqslant \left|X^{(n)}\right| \leqslant \left|X^{(n+1)}\right|, \dots, \left|X^{(n+r)}\right|$. Thus, if $X^{(n)} = X^{(n+r)}$, then condition (a) holds, while if $X^{(n)} \neq X^{(n+r)}$, then condition (b) holds.

SUFFICIENCY. Condition (a) clearly implies that $X \in \mathbb{W}_\infty$. Suppose that condition (b) holds. Then there exist integers $n \geqslant 0$ and $r > 0$, a word Y, and $S, T \in \mathbb{A}^*$ such that

$$\left|X^{(n+1)}\right|, \dots, \left|X^{(n+r)}\right| \geqslant \left|X^{(n)}\right|, \quad X^{(n)} \neq X^{(n+r)},$$
$$|Y| = \mu, \quad X^{(n)} = YS, \quad X^{(n+r)} = YT.$$

Applying Lemma 2.3 to the word $X^{(n)}$ in the role of X, we conclude that $Y \in \mathbb{W}_r$ and

$$X^{(n+i)} = Y^{(i)}S \ \text{ and } \ \left|Y^{(i)}\right| \geqslant \mu \ \text{ for all } \ i = 0, \dots, r.$$

In particular, $Y^{(r)}S = X^{(n+r)} = YT$ and $\left|Y^{(r)}\right| \geqslant \mu$. On the other hand, $|Y| = \mu$, and therefore $Y^{(r)} = YR$ for some $R \in \mathbb{A}^*$. Moreover, $R \neq \varnothing$, since otherwise $X^{(n)} = YS = Y^{(r)}S = X^{(n+r)}$. Taking into account the inequalities $\left|Y^{(0)}\right|, \dots, \left|Y^{(r)}\right| \geqslant \mu$ and applying Proposition 2.1(3), we conclude that $(YZ)^{(i)} = Y^{(i)}Z$ for all $Z \in \mathbb{A}^*$ and $i = 0, \dots, r$. Consequently, the rewriting sequence of X, starting with the nth term, has the form

$$\begin{aligned} X^{(n)} &= YS \Rightarrow Y^{(1)}S \Rightarrow \cdots \Rightarrow Y^{(r)}S \\ &= YRS \Rightarrow Y^{(1)}RS \Rightarrow \cdots \Rightarrow Y^{(r)}RS \\ &= YR^2S \Rightarrow \cdots \Rightarrow YR^3S \Rightarrow \cdots \Rightarrow YR^mS \Rightarrow \cdots \end{aligned}$$

and hence $X \in \mathbb{W}_\infty$.

It remains to observe that conditions (a) and (b) are mutually exclusive, since in the first case the sequence of lengths $\left|X^{(n)}\right|$ is bounded, whereas in the second case (as shown above) it tends to infinity. □

3.2. REMARK. In the final formula of Theorem 3.1, the set of words

$$\{Y^{(j)}R^mS \,:\, 0 \leqslant j < r,\ m \geqslant 0\}$$

arises, which outwardly resembles the standard picture underlying the pumping lemma for regular languages: a sufficiently long word is decomposed into parts in such a way that one of its fragments can be repeated an arbitrary number of times while preserving membership in the corresponding language. In its classical form, this lemma goes back to the paper of Rabin and Scott [2], where it appears as one of the basic consequences of finite automata theory. A canonical textbook presentation of this lemma, its proof, and typical applications can be found, for instance, in the monographs [3, 4].

Theorem 3.1 is not a direct application of the pumping lemma. The proof given above relies not on the standard automata-theoretic argument involving a repeated state while reading a sufficiently long word, but rather on the internal structure of the rewriting sequence under consideration. Nevertheless, it would be interesting to find out whether there is a deeper connection behind this outward similarity. In particular, it is natural to ask whether the appearance of words of the form YR^mS can be interpreted in terms of some automata-theoretic or regular mechanism and whether the above description of rewriting sequences can be obtained as a consequence of a suitable version of the pumping lemma. This question is not considered in the present paper.

3.3. Let $\mathcal{P}$ be an arbitrary set of "constructive entities," i.e., a set whose elements are admissible as inputs for algorithms, and let $C(Y,p)$ be a condition that can be imposed on words Y with additional parameters $p \in \mathcal{P}$. Formally, we may assume that C is a subset of $\mathbb{A}^+ \times \mathcal{P}$ and, for all $Y \in \mathbb{A}^+$ and $p \in \mathcal{P}$, the expression $C(Y,p)$ means the inclusion $\langle Y,p\rangle \in C$.

A condition $C(Y,p)$ is called *decidable* if there is an algorithm verifying $C(Y,p)$ for arbitrary inputs $Y \in \mathbb{A}^+$ and $p \in \mathcal{P}$. A condition $C(Y,p)$ is called *cyclically decidable* if there is an algorithm verifying the following relation $C'(Y,R,S,p)$ for arbitrary inputs $Y \in \mathbb{A}^+$, $R, S \in \mathbb{A}^*$, and $p \in \mathcal{P}$:

$$C'(Y,R,S,p) \ \Leftrightarrow\ (\exists\, m \in \mathbb{N})\ C(YR^mS,p).$$

3.4. Proposition.
(1) *Every cyclically decidable condition is decidable.*
(2) *For every nonempty alphabet $\mathbb{A}$, there exists a condition $C(Y)$, $Y \in \mathbb{A}^+$, that is decidable but not cyclically decidable.*

PROOF. (1): The relation $C(Y,p)$ is equivalent to $C'(Y,\varnothing,\varnothing,p)$.
(2): Let D be a recursive subset of $\mathbb{N}^2$ for which the set

$$D' := \{n \in \mathbb{N} :\ (\exists\, m \in \mathbb{N})\ \langle n,m\rangle \in D\}$$

is not recursive (see, for instance, [5, Chapter C.1, §6]).

If the alphabet $\mathbb{A}$ contains two distinct letters α and β, consider the condition

$$C(Y) \ \Leftrightarrow\ (\exists\, \langle n,m\rangle \in D)\ Y = \alpha^n\beta^m.$$

The decidability of $C(Y)$ follows immediately from the recursiveness of D. On the other hand, the set

$$\{n \in \mathbb{N} :\ C'(\alpha^n,\beta,\varnothing)\} = \{n \in \mathbb{N} :\ (\exists\, m \in \mathbb{N})\ C(\alpha^n\beta^m)\} = D'$$

is not recursive, and hence the condition $C(Y)$ is not cyclically decidable.

In the case of a one-letter alphabet $\mathbb{A} = \{\alpha\}$, one can consider the condition

$$C(Y) \ \Leftrightarrow\ (\exists\, \langle n,m\rangle \in D)\ Y = \alpha^{2^n(2m+1)},$$

for which

$$\bigl\{n \in \mathbb{N} :\ C'\bigl(\alpha^{2^n},\ \alpha^{2^{n+1}},\ \varnothing\bigr)\bigr\} = \bigl\{n \in \mathbb{N} :\ (\exists\, m \in \mathbb{N})\ C\bigl(\alpha^{2^n(2m+1)}\bigr)\bigr\} = D'. \quad \square$$

3.5. Let $C(Y,p)$ be an arbitrary cyclically decidable condition. The algorithm described below takes as input a rewriting function, an arbitrary word X, and a parameter $p \in \mathcal{P}$, and returns a message on the existence or nonexistence of a word Y satisfying the conditions $X \overset{*}{\Rightarrow} Y$ and $C(Y,p)$, supplementing the answer with an explanatory fragment of the rewriting sequence of X (see [1, Algorithm 5.3]).

The algorithm uses integer variables μ, n, i, and r, variables $R \in \mathbb{A}^+$ and $S \in \mathbb{A}^*$, and also two variable-length arrays whose members X_i and Y_i are words.

Algorithm. Is there a word Y such that $X \overset{*}{\Rightarrow} Y$ and $C(Y,p)$?

(1) Put $\mu := \max\{|E| : E \in \mathbb{E}\}$, $n := 0$, and $X_0 := X$.

(2) If $C(X_n,p)$ holds, then terminate with the answer
- **Yes**, *the word* $Y = X_n$ *satisfies the conditions* $X \overset{*}{\Rightarrow} Y$ *and* $C(Y,p)$*:*
  $X_0 \Rightarrow \cdots \Rightarrow X_n$.

(3) If X_n is not rewritable, then terminate with the answer
- **No**, *there is no word* Y *such that* $X \overset{*}{\Rightarrow} Y$ *and* $C(Y,p)$*:*
  $X_0 \Rightarrow \cdots \Rightarrow X_n$ *is the complete rewriting sequence.*

(4) If $X_i = X_n$ for some $0 \leqslant i < n$, then terminate with the answer
- **No**, *there is no word* Y *such that* $X \overset{*}{\Rightarrow} Y$ *and* $C(Y,p)$*:*
  $X_0 \Rightarrow \cdots \Rightarrow \underbrace{X_i \Rightarrow \cdots \Rightarrow X_{n-1}}_{\text{period}} \Rightarrow X_i \Rightarrow \cdots$.

(5) If $n = 0$ or $|X_n| < \mu$, then go to (9).
Otherwise, put $i := n - 1$.

(6) If $|X_i| < \mu$, then go to (9).

(7) If $X_i{\restriction}_\mu = X_n{\restriction}_\mu$ and $|X_i| \leqslant |X_{i+1}|, \dots, |X_n|$, then do the following.

(a) Put $r := n - i$, $Y_0 := X_i{\restriction}_\mu$, $Y_1 := Y_0'$, $\dots$, $Y_r := Y_{r-1}'$
and define $S \in \mathbb{A}^*$ from the equality $X_i = Y_0 S$.
It will then turn out that $Y_r = Y_0 R$ for some word R.

(b) Using the cyclic decidability of the condition $C(Y,p)$,
determine whether there exist $j \in \{1, \dots, r\}$ and $m \in \mathbb{N}$ satisfying $C(Y_j R^m S, p)$.

(c) If the result of test (b) is positive, then (exclusively for constructing an explanatory fragment of the rewriting sequence of X) continue the calculation cycle $n := n + 1$, $X_n := X_{n-1}'$ until the condition $C(X_n,p)$ is fulfilled, and terminate with the answer
- **Yes**, *the word* $Y = X_n$ *satisfies the conditions* $X \overset{*}{\Rightarrow} Y$ *and* $C(Y,p)$*:*
  $X_0 \Rightarrow \cdots \Rightarrow X_n$.

If the result of test (b) is negative, then terminate with the answer
- **No**, *there is no word* Y *such that* $X \overset{*}{\Rightarrow} Y$ *and* $C(Y,p)$*:*
  $X_0 \Rightarrow \cdots \Rightarrow \underbrace{X_i \Rightarrow \cdots \Rightarrow X_{n-1}}_{\text{growth period}} \Rightarrow X_n \Rightarrow \cdots$.

(8) If $i > 0$, then put $i := i - 1$ and go to (6).

(9) Put $n := n + 1$, $X_n := X_{n-1}'$, and go to (2).

3.6. Proposition. *If* $C(Y,p)$ *is a cyclically decidable condition, then, for every rewriting function, every word* X*, and every parameter* $p \in \mathcal{P}$*, Algorithm* 3.5 *returns in finite time the correct answer to the question whether there exists a word* Y *satisfying the conditions* $X \overset{*}{\Rightarrow} Y$ *and* $C(Y,p)$*.*

Proof. It is clear that, at each execution of step 3.5(9), the value X_n is set equal to the nth rewrite of X, and hence $\langle X_0, \dots, X_n \rangle$ is an initial fragment of the rewriting sequence of X.

It follows from the above that step 3.5(2), when $C(X_n,p)$ holds, returns the correct answer.

If, at step 3.5(3), the word X_n turns out to be nonrewritable, then the answer returned at this step is correct, since $\langle X_0, \dots, X_n \rangle$ is the complete rewriting sequence of X and the preceding executions of step 3.5(2) have not detected the condition $C(X_i,p)$ for any $i = 0, \dots, n$.

Similar arguments justify the correctness of the answer returned at step 3.5(4), since the equality $X_i = X_n$ for some $0 \leqslant i < n$ guarantees that, in the rewriting sequence of X, starting with the nth term, only words belonging to the set $\{X_{i+1}, \dots, X_n\}$ will occur.

As is easily seen, step 3.5(5) and the subsequent loop 3.5(6)–(8) verify the existence of an integer $0 \leqslant i < n$ such that $\mu \leqslant |X^{(i)}| \leqslant |X^{(i+1)}|, \dots, |X^{(n)}|$, $X^{(i)}\restriction_\mu = X^{(n)}\restriction_\mu$. Since the algorithm has not terminated at step 3.5(4), the inequalities $X^{(i)} \neq X^{(n)}$ for all $0 \leqslant i < n$ hold during this verification. Thus, steps 3.5(5)–(8) verify condition 3.1(b) for the fragment $\langle X_0, \dots, X_n\rangle$. If this verification gives a positive result, then the existence of the word R mentioned in 3.5(7)(a) and the correctness of the answer returned at step 3.5(7)(c) are justified by Theorem 3.1, according to which

$$\{X^{(n+1)}, X^{(n+2)}, \dots\} = \{Y_j R^m S : 1 \leqslant j \leqslant r,\ m \in \mathbb{N}\}.$$

It remains to observe that, by the same Theorem 3.1, the algorithm necessarily terminates at one of steps 3.5(2)–(4) or (7)(c). □

3.7. As is easily seen, for a given rewriting function $\to$, the condition

$$\begin{aligned} C(Y, \to) &\Leftrightarrow Y \text{ is not rewritable with respect to } \to \\ &\Leftrightarrow Y \text{ has no explicit prefix} \end{aligned}$$

is cyclically decidable. Therefore, Algorithm 3.5, specialized to this condition, verifies finite rewritability of a given word. A simplified version of this procedure is presented below (see [1, Algorithm 5.4]).

Algorithm. Is the word X finitely rewritable?

- Start calculating the rewriting sequence $X^{(0)}, X^{(1)}, \dots$.
- If a nonrewritable word $X^{(i)}$ occurs, return **Yes**.
- Otherwise, by Theorem 3.1, a fragment $\langle X^{(n)}, X^{(n+1)}, \dots, X^{(n+r)}\rangle$ will occur that satisfies (a) or (b) of Theorem 3.1; in this case, return **No**.

3.8. The following assertion was announced in [1, Theorem 1.12].

Theorem. *If* $\mathbb{E} \subseteq \mathbb{W}_{\text{fin}}$, *then* $\mathbb{W}_{\text{fin}} = \mathbb{A}^+$.

Proof. Suppose that all explicit words are finitely rewritable, and prove finite rewritability of an arbitrary word X by induction on the length $|X|$ of X.

Let $|X| = 1$. If $X \in \mathbb{E}$, then $X \in \mathbb{W}_{\text{fin}}$ by the assumption of the theorem. If $X \notin \mathbb{E}$, then, being single-letter, the word X has no explicit prefixes and hence is not rewritable, i.e., belongs to $\mathbb{A}^+ \setminus \mathbb{W}_1 \subset \mathbb{W}_{\text{fin}}$.

Suppose that all words of length l are finitely rewritable. Consider an arbitrary word X of length $l+1$ and assume, contrary to what we need to prove, that X has an infinite rewriting sequence

$$X^{(0)} \Rightarrow X^{(1)} \Rightarrow \dots \Rightarrow X^{(n)} \Rightarrow X^{(n+1)} \Rightarrow \cdots .$$

Then all the words $X^{(n)}$ are infinitely rewritable and hence, by the assumption of the theorem, are not explicit. Moreover, $|X^{(n)}| \geqslant 2$ for all $n \geqslant 0$ by the induction base proved above.

For every word Y of length $|Y| \geqslant 2$, denote by $\lfloor Y \rfloor$ the prefix of Y of length $|Y| - 1$.

We show that $\lfloor X^{(n)} \rfloor \Rightarrow \lfloor X^{(n+1)} \rfloor$ for every $n \geqslant 0$. Indeed, let α be the last letter of the word $X^{(n)}$. Since the word $X^{(n)} = \lfloor X^{(n)} \rfloor \alpha$ is nonexplicit and rewritable, by Proposition 2.1(1), we have $\lfloor X^{(n)} \rfloor \in \mathbb{W}_1$ and $(\lfloor X^{(n)} \rfloor \alpha)' = \lfloor X^{(n)} \rfloor' \alpha$. Therefore,

$$\lfloor X^{(n)} \rfloor \Rightarrow \lfloor X^{(n)} \rfloor' = \big\lfloor \lfloor X^{(n)} \rfloor' \alpha \big\rfloor = \big\lfloor (\lfloor X^{(n)} \rfloor \alpha)' \big\rfloor = \lfloor (X^{(n)})' \rfloor = \lfloor X^{(n+1)} \rfloor.$$

Thus, the rewriting sequence of the word $\lfloor X \rfloor$, which has length l, is infinite:

$$\lfloor X^{(0)} \rfloor \Rightarrow \lfloor X^{(1)} \rfloor \Rightarrow \dots \Rightarrow \lfloor X^{(n)} \rfloor \Rightarrow \lfloor X^{(n+1)} \rfloor \Rightarrow \cdots .$$

The contradiction obtained with the induction hypothesis completes the proof. □

3.9. Since the set $\mathbb{E}$ of explicit words is finite, Theorem 3.8 implies the existence of an effective verification of finite rewritability of all words with respect to a given rewriting function: for such verification, it suffices to apply Algorithm 3.7 to all explicit words.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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