Definable fixed points in modal and temporal logics — a survey

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ABSTRACT. *The paper presents a survey of author's results on definable fixed points in modal, temporal, and intuitionistic propositional logics. The well-known Fixed Point Theorem considers the modalized case, but here we investigate the positive case. We give a classification of fixed point theorems, describe some classes of models with definable least fixed points of positive operators, special positive operators, and give some examples of undefinable least fixed points. Some other interesting phenomena are discovered – definability by formulas that do not preserve positivity of parameters and definability by finite sets of formulas. We also consider negative operators, graded modalities, construct undefinable inflationary fixed points, and put some problems.*

KEYWORDS: *fixed point, modal logic, temporal logic, definability.*

1. Introduction

It is well-known that many predicates can be defined as the least fixed points of positive operators. Such a definition is implicit, but if the least fixed point is formula-definable, an explicit definition also exists. Usually an implicit definition is much shorter than an explicit one, and to obtain an explicit definition, we need an algorithm for its construction. This paper identifies some classes of modal and temporal models, in which the least fixed points of positive operators are definable. We also consider special positive operators and classes of modal models, where the least fixed points of negative operators are definable. For every definability theorem stated in this paper, there exists an algorithm constructing the corresponding formula (or a finite set of formulas). All algorithms and detailed proofs can be found in the cited papers. Thesis (Mardaev, 2001b) and the papers (Mardaev, 2006; Mardaev, 2004) also contain proofs and numerous examples.

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Let us first recall some standard definitions. *Modal propositional formulas* are constructed from propositional variables p, q, r ... and the constant $\bot$ (falsity) using the binary connectives $\wedge$, $\vee$, and the unary connectives $\neg$ and $\Box$. We introduce the following abbreviations:

$$\Diamond = \neg\Box\neg, \qquad \top = \neg\bot,$$
$$\boxdot\alpha = \alpha \wedge \Box\alpha, \qquad \Diamondblack\alpha = \alpha \vee \Diamond\alpha,$$
$$\alpha \to \beta = \neg\alpha \vee \beta, \qquad \alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha).$$

A *modal Kripke frame* $\langle W, R \rangle$ consists of a non-empty set W with a binary relation R. A *Kripke model based* on a frame $\langle W, R \rangle$ is a triple $\langle W, R, v \rangle$, where v is a *valuation function* assigning a subset of W (the value) to every propositional variable. This function is extended to formulas in the usual way as follows: the value of the constant $\bot$ is always the empty set; the connectives $\neg$, $\wedge$, $\vee$ respectively correspond to the complement, the intersection, and the union; $\Box$ and $\Diamond$ correspond to the following operations on sets:

$$\Box A = \{x \,|\, \forall y\,(xRy \Rightarrow y \in A)\}, \ \Diamond A = \{x \,|\, \exists y\,(xRy \wedge y \in A)\}.$$

The value of a variable q is denoted by the corresponding uppercase letter Q. The notation $\alpha(q_1, \ldots, q_n)$ means that all variables occurring in the formula α are in the list $q_1, \ldots, q_n$. Henceforth $\alpha(Q_1, \ldots, Q_n)$ denotes the value of the formula $\alpha(q_1, \ldots, q_n)$.

A formula $\alpha(q_1, \ldots, q_n)$ is *true* at a point $x \in W$ in a model $\langle W, R, v \rangle$ if $x \in \alpha(Q_1, \ldots, Q_n)$. A formula α is *true* in a model $\langle W, R, v \rangle$ if α is true at each $x \in W$ in the model. A formula α is *valid* in a frame $\langle W, R \rangle$ if α is true in all models based on $\langle W, R \rangle$.

DEFINITION 1. — *Consider a formula $\varphi(p, q_1, \ldots, q_n)$ in variables $p, q_1, \ldots, q_n$ and a Kripke model $\langle W, R, v \rangle$ with values $Q_1, \ldots, Q_n$ of variables $q_1, \ldots, q_n$. Define the operator F_φ by the equality $F_\varphi(P) = \varphi(P, Q_1, \ldots, Q_n)$. This operator sends every $P \subseteq W$ to the value $\varphi(P, Q_1, \ldots, Q_n)$ of the formula φ. A set P is called a* fixed point *(of F_φ) if $P = F_\varphi(P)$. A fixed point P is called the* least fixed point *if $P \subseteq P'$ holds for every fixed point P'. A formula $\omega(q_1, \ldots, q_n)$* defines *a fixed point P if $P = \omega(Q_1, \ldots, Q_n)$.*

DEFINITION 2. — *A modal formula $\varphi(p, q_1, \ldots, q_n)$ is called modalized in p if every occurrence of p in φ is within the scope of $\Box$ or $\Diamond$.*

DEFINITION 3. — *A frame $\langle W, R \rangle$ satisfies the* ascending chain condition *if there is no infinite sequence $x_1 R x_2 R x_3 \ldots$ such that $x_i \neq x_j$ for all $i \neq j$.*

The well-known Fixed Point Theorem (Bernardi, 1975; Bernardi, 1976; Smoryński, 1975; Sambin, 1976; Sambin *et al.*, 1982; Smoryński, 1985; Reidhaar-Olson, 1990) states that fixed points are unique and definable:

FIXED POINT THEOREM (DE JONGH, SAMBIN, BERNARDI, SMORYŃSKI). — *For any formula $\varphi(p, q_1, ..., q_n)$ modalized in p, there exists a unique fixed point of the operator F_φ in every strictly partially ordered (i.e., irreflexive and transitive) model with the ascending chain condition and there is a formula $\omega(q_1, ..., q_n)$, which defines the fixed point in every model of this kind.*

Recall that modal propositional calculus K has the following axioms:

(1) classical tautologies,

(2) $\Box(q \to s) \to (\Box q \to \Box s)$

and inference rules:

(1) *Modus ponens*: $\alpha, \alpha \to \beta / \beta$,

(2) substitution of modal formulas for variables,

(3) the rule of necessitation: $\alpha / \Box\alpha$.

Modal logic $\mathbf{K}$ consists of all formulas derivable in K. By a (normal) modal logic we mean an arbitrary set $\mathbf{L}$ of modal formulas containing $\mathbf{K}$ and closed under the rules (1)–(3).

A modal logic $\mathbf{L}$ is *characterized* by a class $\mathcal{C}$ of frames if $\mathbf{L}$ consists of all formulas α valid in every frame $\langle W, R\rangle \in \mathcal{C}$. It is well-known (Chagrov *et al.*, 1997; Rybakov, 1997) that $\mathbf{K}$ is characterized by the class of all frames.

If $\mathbf{L}$ is a normal modal logic and α is a modal formula, then $\mathbf{L} \oplus \alpha$ denotes the least normal modal logic containing $\mathbf{L}$ and α. The following notations of formulas and logics are standard:

$$\boldsymbol{4} : \Box p \to \Box\Box p, \qquad \boldsymbol{w} : \Box(\Box p \to p) \to \Box p, \qquad \boldsymbol{t} : \Box p \to p,$$
$$\mathbf{K4} = \mathbf{K} \oplus \boldsymbol{4}, \qquad \mathbf{GL} = \mathbf{K4} \oplus \boldsymbol{w}, \qquad \mathbf{S4} = \mathbf{K4} \oplus \boldsymbol{t}.$$

It is well-known (Segerberg, 1971; Chagrov *et al.*, 1997; Rybakov, 1997) that modal logic $\mathbf{GL}$ is characterized by the class of all strictly partially ordered frames with the ascending chain condition. So we can reformulate Fixed Point Theorem.

FIXED POINT THEOREM. — *For any formula $\varphi(p, q_1, ..., q_n)$ modalized in p, there exists a formula $\omega(q_1, ..., q_n)$ such that logic $\mathbf{GL}$ contains the formula*

$$\Box(p \leftrightarrow \varphi) \leftrightarrow \Box(p \leftrightarrow \omega).$$

2. Modal positive operators

2.1. *The least fixed points*

An occurrence of a variable in a formula is called *positive* if it is within the scope of an even number of negations. A formula $\varphi(p, q_1, \ldots, q_n)$ is called *positive in* p if all occurrences of p are positive.

The positive case can be reduced to the modalized one. In fact, let a formula φ be positive in p. Transform the formula φ as follows: replace all occurrences of p that are not within the scope of $\Box$ or $\Diamond$, with $\bot$. The resulting formula ζ is positive, modalized in p and has the following property.

PROPOSITION 4. — *In every Kripke model the least fixed points of operators F_φ and F_ζ coincide.*

Fixed Point Theorem does not hold for strictly partially ordered Kripke models without the ascending chain condition or for partially ordered (i.e., reflexive, transitive, and antisymmetric) models. For example, consider a Kripke model based on the frame $\langle N, < \rangle$, where N is the set of natural numbers. If $\varphi = \Box p$, then the operator F_φ has two fixed points, $\varnothing$ and N.

If $\varphi = \Box\neg p$, then the operator F_φ does not have fixed points. So theorems from this paper (except for Theorem 28) do not follow from Fixed Point Theorem.

For a positive operator F_φ in a Kripke model, consider the ordinal sequence of sets:

$P^0 = \varnothing$, $P^{\alpha+1} = F_\varphi(P^\alpha)$, $P^\alpha = \bigcup\limits_{\beta<\alpha} P^\beta$ if α is a limit ordinal.

Then obviously, F_φ is *monotonic*, i.e.,

$$\forall P_1, P_2 \subseteq W\, (P_1 \subseteq P_2 \Rightarrow F_\varphi(P_1) \subseteq F_\varphi(P_2)).$$

From the general theory of monotonic operators (Aczel, 1977) it is known that the sequence P^α reaches the least fixed point of F_φ.

Let us give some examples of undefinable least fixed points of positive operators in partially ordered models.

EXAMPLE 5. — Let $\varphi = \Box(p \vee q) \vee \Box(p \vee \neg q)$. Consider a linearly ordered model of type $\omega + {}^*\omega$; its frame consists of two copies N and N' of the natural numbers $0 \leqslant 1 \leqslant 2 \leqslant \cdots \leqslant 2' \leqslant 1' \leqslant 0'$ (see Fig. 1, left; the models in Fig. 1 are ordered "upwards"). Let Q be the set of even numbers (marked by bullets). Then the set $N' = P^\omega$ (marked by double circles) is the least fixed point of the operator F_φ. This set is undefinable. □

EXAMPLE 6. — The next model is an infinite partially ordered tree (Fig. 1, middle). The set Q is marked by bullets. It is easy to show that the least fixed point P^ω (marked

by double circles) of the operator F_φ is the tree without its trunk. This point is undefinable. Analogous examples can be constructed for strictly partially ordered models. □

EXAMPLE 7. — In Fig. 1 (right) irreflexive elements $\cdots < 2' < 1' < 0'$ are above the reflexive element a (marked by a bullet). Let $\varphi = \Box p$. The set $N' = P^\omega$ (marked by double circles) is the least fixed point of the operator F_φ. This point is also undefinable. □

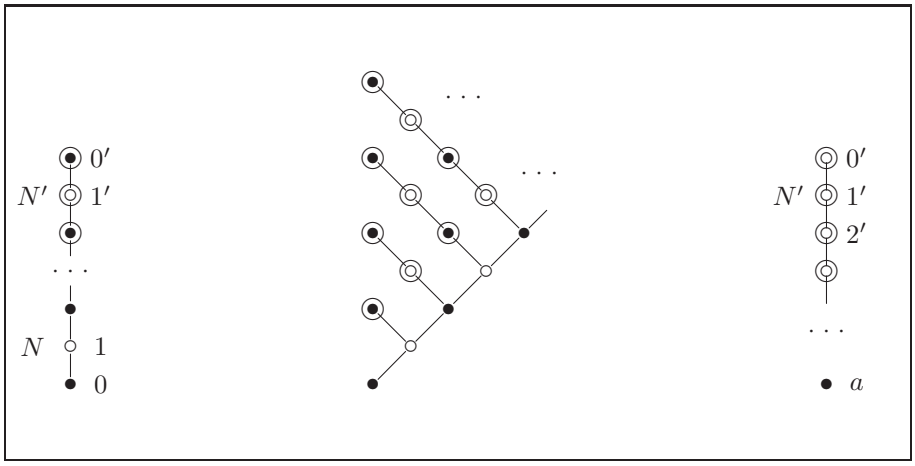

Figure 1. *The least fixed points are undefinable*

2.2. *E-frames*

A subset $A \subseteq W$ is called *cofinal* in a frame $\langle W, R \rangle$ if for any $x \in W$ there is $y \in A$ such that xRy.

For a frame $\langle W, R \rangle$ the *weak cofinality condition for infinite ascending chains* is the following:

Let $x_1 R x_2 R \ldots$ be an infinite chain such that for all $i \neq j$, $x_i \neq x_j$; then for some i, the chain $x_{i+1} R x_{i+2} R x_{i+3} \ldots$ is cofinal in the subframe $\{y \in W \mid x_i R y\}$ (Fig. 2).

An *E-frame* is a partially ordered Kripke frame with the weak cofinality condition for infinite ascending chains. Here are some examples: every partially ordered Kripke frame with the ascending chain condition; the natural numbers $\langle N, \leqslant \rangle$; the integers $\langle Z, \leqslant \rangle$.

Here is another example. Take a finite partially ordered frame and attach ω-chains to some its elements. Then we obtain an E-frame (Fig. 3).

A Kripke model based on an E-frame is called an *E-model*.

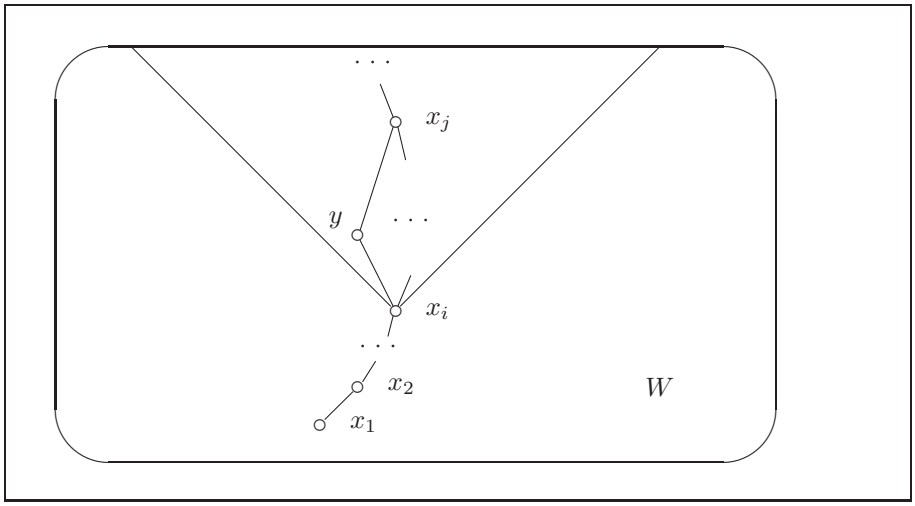

Figure 2. *Weak cofinality condition for infinite ascending chains*

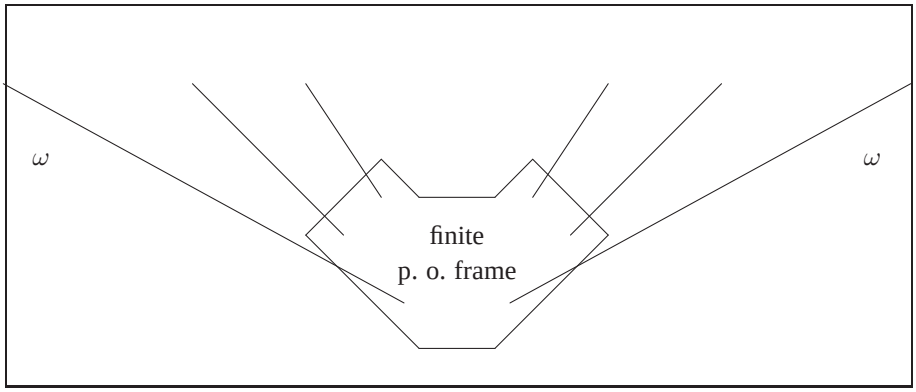

Figure 3. *Example of an E-frame*

DEFINITION 8. — *A formula $\omega(q_1, \ldots, q_n)$ (defining the least fixed point) preserves the positivity of parameters for $\varphi(p, q_1, \ldots, q_n)$ if for every variable q_i, the following condition holds: if $\varphi(p, q_1, \ldots, q_n)$ is positive in q_i, then ω is also positive in q_i.*

THEOREM 9 (MARDAEV, 2002B). — *For any formula $\varphi(p, q_1, ..., q_n)$ positive in p, there is a formula $\omega(q_1, ..., q_n)$ preserving the positivity of parameters for $\varphi(p, q_1, ..., q_n)$ and defining the least fixed point of the operator F_φ in every E-model.*

Examples: if $\varphi = \Box(p \vee q) \vee \Box(p \vee \neg q)$, then ω is equivalent to $\Box\Diamond(\Box q \vee \Box\neg q)$ in E-models. If $\varphi = \Box(p \vee q)$, then ω is equivalent to $\Box q$ in E-models.

Let us introduce the following notation:

$$\begin{aligned} \boldsymbol{e} &: \Box(\Box(p \to \Box p) \to p) \to (\Box\Diamond\Box p \to p), \\ \mathbf{S4E} &= \mathbf{S4} \oplus \boldsymbol{e}. \end{aligned}$$

Let $\langle W, R\rangle$ be a transitive frame. Define an equivalence relation $\sim$ on W by putting $x \sim y$ iff either $x = y$ or xRy and yRx. The equivalence classes with respect to $\sim$ are called *clusters*. Then the quotient relation

$$C_1 \ R/\!\sim \ C_2 \ \text{ iff } \exists x \in C_1 \ \exists y \in C_2 \ \ xRy$$

is transitive and antisymmetric.

PROPOSITION 10. — *The logic* $\mathbf{S4E}$ *is characterized by the class of all* E*-frames.* $\mathbf{S4E}$ *is also characterized by the class of all finite preordered (i.e., reflexive and transitive) frames, in which every non-maximal cluster (with respect to* $R/\sim$*) consists of a single element.*

COROLLARY 11. — *For any formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a formula* $\omega(q_1, ..., q_n)$ *preserving the positivity of parameters such that the logic* $\mathbf{S4E}$ *contains the formulas*

$$\begin{aligned} &\omega \leftrightarrow \varphi(\omega, q_1, \ldots, q_n), \\ &\Box(\varphi \to p) \to (\omega \to p). \end{aligned}$$

Recall that the language of μ-calculus (Kozen, 1982; Dawar *et al.*, 2001; Dawar *et al.*, 2004) extends the modal language by μ-operator, with the following rule for building formulas: if formula $\psi(q, q_1, \ldots, q_n)$ is positive in q, then $\mu q.\psi\ (q_1, \ldots, q_n)$ is a formula. The value of this formula in a Kripke model $\langle\, W, R, v\,\rangle$ with values $Q_1, \ldots, Q_n$ of variables $q_1, \ldots, q_n$ is the least fixed point of the operator $F_\psi(Q) = \psi(Q, Q_1, \ldots, Q_n)$.

If we have the preservation of positivity of parameters within some class of models, then every formula of μ-calculus is equivalent to a modal formula (i.e., to a formula without occurrences of μ-operator) within this class of models.

The preservation of positivity also allows us to find the least fixed points of the positive system

$$\begin{cases} P_1 = \varphi_1(P_1, \ldots, P_m, Q_1, \ldots, Q_n) \\ \quad \ldots \\ P_m = \varphi_m(P_1, \ldots, P_m, Q_1, \ldots, Q_n) \end{cases}$$

using the method of eliminating unknowns.

2.3. *SE-frames*

A strictly partially ordered Kripke frame with the weak cofinality condition for infinite ascending chains is called an SE-*frame*. Here are some examples: a strictly partially ordered Kripke frame with the ascending chain condition, the natural numbers $\langle N, < \rangle$; the integers $\langle Z, < \rangle$. If $\langle W, \leqslant \rangle$ is an E-frame, then $\langle W, < \rangle$ is an SE-frame.

THEOREM 12 (MARDAEV, 2002B). — *For any formula $\varphi(p, q_1, ..., q_n)$ positive in p, there is a formula $\omega(q_1, ..., q_n)$ preserving the positivity of parameters and defining the least fixed point of the operator F_φ in every SE-model.*

EXAMPLE. — if $\varphi = \Box(p \vee q)$, then ω is equivalent to $\Box\Diamond\Box q$ in SE-models. □

Let us introduce the following notation:

$$\begin{aligned} \boldsymbol{se} &: \Box(\Box p \to p) \to (\Box\Diamond\Box p \to \Box p), \\ \mathbf{K4Se} &= \mathbf{K4} \oplus \boldsymbol{se}. \end{aligned}$$

PROPOSITION 13. — *The logic* $\mathbf{K4Se}$ *is characterized by the class of all SE-frames. It is also characterized by the class of all finite transitive frames satisfying the condition: if a cluster is non-maximal, then it consists of a single irreflexive element.*

COROLLARY 14. — *For any formula $\varphi(p, q_1, ..., q_n)$ positive in p, there is a formula $\omega(q_1, ..., q_n)$ preserving the positivity of parameters such that the logic* $\mathbf{K4Se}$ *contains the formulas*

$$\begin{aligned} &\omega \leftrightarrow \varphi(\omega, q_1, \ldots, q_n), \\ &\boxdot(\varphi \to p) \to (\omega \to p). \end{aligned}$$

In some cases we cannot construct a defining formula preserving the positivity of parameters that is suitable both for E- and SE-models. For example, consider $\varphi = \Box(p \vee q)$ and two models: an E-model $\langle Z, \leqslant \rangle$ and an SE-model $\langle Z, < \rangle$, where Z is the set of integers. Let $Q = \{n \in Z \mid n \geqslant 0\}$ in both models. There does not exist a formula positive in q defining the least fixed points of the operator F_φ in both models. Note that although the formula $\boxdot(\Box q \to q) \to q$ defines the least fixed points in these two models, it contains a negative occurrence of q.

But under the ascending chain condition these two cases can be joined:

THEOREM 15. — *For any formula $\varphi(p, q_1, ..., q_n)$ positive in p, there is a formula $\omega(q_1, ..., q_n)$ preserving the positivity of parameters and defining the least fixed point of the operator F_φ in every partially ordered or strictly partially ordered model with the ascending chain condition.*

2.4. *IRE-frames*

A subset $K \subseteq W$ is called an *upper cone* in a frame $\langle W, R \rangle$ if $\forall x, y\,((xRy \ \& \ x \in K) \Rightarrow y \in K)$.

DEFINITION 16. — *A transitive antisymmetric frame with the weak cofinality condition for infinite ascending chains is called an* IRE-frame *if the set of its reflexive elements is an upper cone.*

Clearly, E-frames are reflexive IRE-frames, and SE-frames are irreflexive IRE-frames. It is easy to construct other IRE-frames containing irreflexive and reflexive elements.

THEOREM 17 (MARDAEV, 2002A). — *For any formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a formula* $\omega(q_1, ..., q_n)$ *defining the least fixed point of the operator* F_φ *in every* IRE*-model.*

The positivity of parameters is not always preserved in the IRE-case. Here is a counterexample: if $\varphi = \Box(p \vee q)$, then ω is equivalent to $\Box(\boxdot(\Box q \to q) \to q)$ in IRE-models. In IRE-models the formula $\omega = \Box(\boxdot(\Box q \to q) \to q)$ is monotonic in q (i.e., if $Q_1 \subseteq Q_2$, then $\omega(Q_1) \subseteq \omega(Q_2)$), but non-equivalent to any formula positive in q.

Let us introduce the following notation:

$$\begin{aligned} \boldsymbol{tae} &: \Box(\Box(p \to \Box p) \to p) \to (\Box\diamondsuit\Box p \to \Box p), \\ \boldsymbol{ir} &: \boxdot(\Box(p \vee q) \to p) \to (\Box(\boxdot(\Box q \to q) \to q) \to p), \\ \mathbf{K4Ire} &= \mathbf{K4} \oplus \boldsymbol{tae} \oplus \boldsymbol{ir}. \end{aligned}$$

PROPOSITION 18. — *The logic* $\mathbf{K4Ire}$ *is characterized by the class of all* IRE*-frames. It is also characterized by the class of all finite transitive frames satisfying two conditions:*

1) if a cluster is non-maximal, then it is a singleton;
2) the set of reflexive elements is an upper cone.

COROLLARY 19. — *For any formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a formula* $\omega(q_1, ..., q_n)$ *such that* $\mathbf{K4Ire}$ *contains the formulas*

$$\omega \leftrightarrow \varphi(\omega, q_1, \ldots, q_n),$$
$$\boxdot(\varphi \to p) \to (\omega \to p).$$

From Proposition 18 it follows that $\mathbf{K4Ire}$ is decidable, therefore monotonicity is decidable within the class of all IRE-models. But monotonicity is not equivalent to positivity over IRE-models.

EXAMPLE 20. — In some cases we cannot construct a defining formula preserving the positivity of parameters in strictly partially ordered models (and partially ordered models). For example, let $\varphi = \Box(p \vee q)$. Consider a strictly partially ordered model in Fig. 4. This is not an SE-model. Let Q be marked by bullets. Then the set marked by double circles is the least fixed point of the operator F_φ. The defining formula in this model is $\Box(\boxdot(\Box q \to q) \to q)$. Are the least fixed points of positive operators definable in models based on the frame in Fig. 4? □

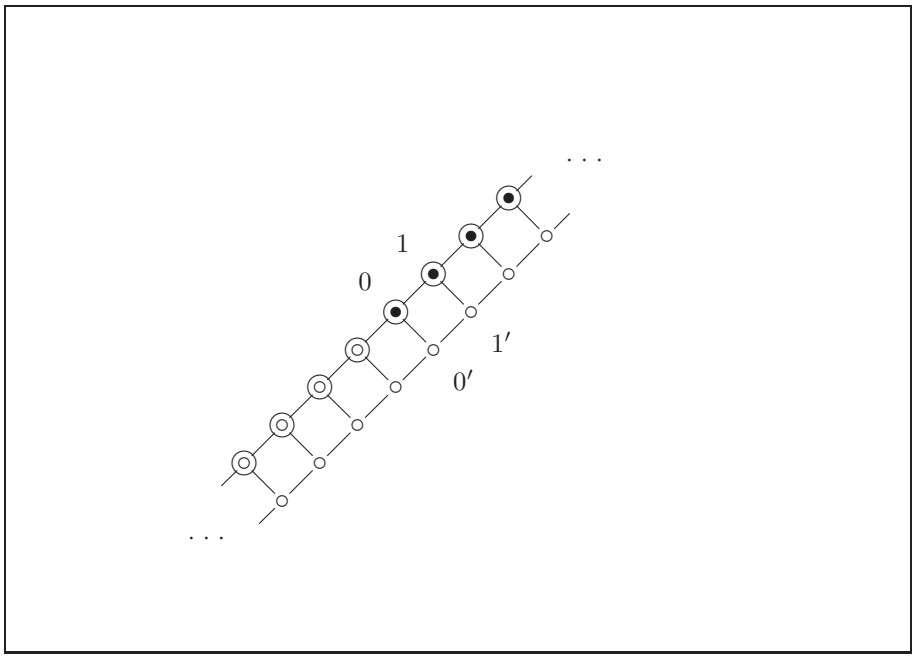

Figure 4. *Defining formula does not preserve the positivity of parameters*

We can also find defining formulas for the least fixed points of the positive system

$$\begin{cases} P_1 = \varphi_1(P_1, \ldots, P_m, Q_1, \ldots, Q_n) \\ \quad \ldots \\ P_m = \varphi_m(P_1, \ldots, P_m, Q_1, \ldots, Q_n) \end{cases}$$

in the class of IRE-models, although the positivity of parameters is not preserved.

Note, that E-, SE-, and IRE-models differ only in the way how reflexive and irreflexive elements are arranged. As for definability, there may be four options.

(1) If all elements are reflexive (i.e., in the case of E-models), we can construct a defining formula preserving the positivity of parameters.

(2) If all elements are irreflexive (i.e., in the case of SE-models), we can also construct a defining formula preserving the positivity of parameters. However it is not always possible to construct a single formula for cases (1) and (2).

(3) For the case of IRE-models (i.e. if reflexive elements are above irreflexive), a defining formula exists, but the positivity of parameters is not always preserved.

(4) If irreflexive elements may be above reflexive, a defining formula does not always exist (Example 7).

2.5. *Switches of truth values*

Consider a model $\langle\, W, R, v \,\rangle$ and the value of a formula α in this model. For a finite chain $x_1 R x_2 R \ldots R x_n$ consider the number of those i, for which the truth values of α are different at x_i and x_{i+1}. The maximum of these numbers over all finite chains is called the *number of switches (of the truth values) for* α. This maximum may be equal to infinity. If the chains are taken from a certain model M, the corresponding maximum is called the *number of switches for* α *in* M.

For a formula $\varphi(p, q_1, \ldots, q_n)$. Let us introduce the following notation:

$$\varphi^0(q_1, \ldots, q_n) = \bot,$$
$$\varphi^{m+1}(q_1, \ldots, q_n) = \varphi(\varphi^m(q_1, \ldots, q_n), q_1, \ldots, q_n).$$

THEOREM 21 (MARDAEV, 2001A). — *For any natural number k and a formula $\varphi(p, q_1, ..., q_n)$ positive in p, there is a natural number m such that φ^m defines the least fixed point of the operator F_φ in every preordered Kripke model with the number of switches for φ not greater than k.*

An upper bound for m (depending on k and φ) can be extracted from the proof.

Note that $\varphi^m(q_1, \ldots, q_n)$ preserves the positivity of parameters.

Since in a transitive model $\Box q$ changes its truth value at most once, we obtain the following

COROLLARY 22. — *For any formula $\varphi(p, \Box q_1, ..., \Box q_n)$ positive in p (in which every occurrence of q_i is within a subformula $\Box q_i$) there is a natural number m such that the formula $\varphi^m(\Box q_1, ..., \Box q_n)$ defines the least fixed point of the operator F_φ in every preordered Kripke model.*

It is well known (Segerberg, 1971; Chagrov *et al.*, 1997; Rybakov, 1997) that the logic $\mathbf{S4}$ is characterized by the class of all preordered frames.

COROLLARY 23 (MARDAEV, 1994). — *For any formula $\varphi(p, \Box q_1, \ldots, \Box q_n)$ positive in p, there is a number m such that $\mathbf{S4}$ contains the formula*

$$\varphi^{m+1}(\Box q_1, \ldots, \Box q_n) \leftrightarrow \varphi^m(\Box q_1, \ldots, \Box q_n).$$

The next corollary is proved by applying a translation from intuitionistic to modal formulas.

COROLLARY 24 (MARDAEV, 1994). — *For any intuitionistic propositional formula* $\varphi(p, q_1, \ldots, q_n)$ *positive in* p*, there is a number* m *such that intuitionistic propositional logic* **Int** *contains the formula* $\varphi^{m+1} \leftrightarrow \varphi^m$.

This corollary also follows from Ruitenburg's theorem (Ruitenburg, 1984). Namely, for a formula $\varphi(p, q_1, \ldots, q_n)$ let us introduce the following notation:

$$\varphi^{(0)}(p, q_1, \ldots, q_n) = p,$$
$$\varphi^{(k+1)}(p, q_1, \ldots, q_n) = \varphi(\varphi^{(k)}(p, q_1, \ldots, q_n), q_1, \ldots, q_n).$$

Note that $\varphi^{(k)}$ differs from φ^k defined above.

THEOREM 25 (RUITENBURG, 1984). — *For any intuitionistic propositional formula* $\varphi(p, q_1, \ldots, q_n)$*, there is a number* m *such that* **Int** *contains the formula*

$$\varphi^{(m+2)}(p, q_1, \ldots, q_n) \leftrightarrow \varphi^{(m)}(p, q_1, \ldots, q_n).$$

3. Subclasses of positive operators

3.1. Σ- *and* Π-*formulas*

Modal Σ-*formulas* are constructed from propositional variables, their negations, the constants $\bot$, $\top$ using the connectives $\wedge$, $\vee$, and $\Diamond$. Examples: p, $\bot$, $\Diamond\Diamond p$, $\Diamond(q \vee \Diamond(p \vee \neg q))$.

THEOREM 26 (MARDAEV, 1992). — *For any* Σ*-formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a natural number* m *such that* $\varphi^m(q_1, \ldots, q_n)$ *defines the least fixed point of the operator* F_φ *in every transitive Kripke model.*

An upper bound for m can be extracted from the proof.

It is well known (Chagrov *et al.*, 1997; Rybakov, 1997) that the logic $\mathbf{K4}$ is characterized by the class of all transitive frames.

COROLLARY 27. — *For any* Σ*-formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a natural number* m *such that* $\mathbf{K4}$ *contains the formula* $\varphi^{m+1} \leftrightarrow \varphi^m$.

Modal Π-*formulas* are constructed from propositional variables, their negations, the constants $\bot$, $\top$ using the connectives $\wedge$, $\vee$, and $\Box$. The examples are p, $\bot$, $\Box\Box p$, $\Box(q \vee \Box(p \vee \neg q))$.

The positive case can be reduced to the modalized case (Proposition 4), and there are constructions of defining formulas for Fixed Point Theorem (for example, Sambin's construction (Sambin *et al.*, 1982; Reidhaar-Olson, 1990)) preserving the positivity of parameters and the property of being a Π-formula. Therefore the following theorem is a corollary of Fixed Point Theorem.

THEOREM 28 (MARDAEV, 1993A). — *For any* Π*-formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a* Π*-formula* $\omega(q_1, ..., q_n)$ *preserving the positivity of parameters and defining the least fixed point of the operator* F_φ *in every strictly partially ordered model with the ascending chain condition.*

COROLLARY 29. — *For any* Π*-formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a* Π*-formula* $\omega(q_1, ..., q_n)$ *preserving the positivity of parameters such that the logic* $\mathbf{GL}$ *contains the formulas*

$$\omega \leftrightarrow \varphi(\omega, q_1, \ldots, q_n),$$
$$\boxdot(\varphi \to p) \to (\omega \to p).$$

3.2. *Partial orders with the ascending chain condition*

Fixed Point Theorem does not hold for partially ordered Kripke models, but still there is the following result:

THEOREM 30 (MARDAEV, 1993B). — *For any* Π*-formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a* Π*-formula* $\omega(q_1, ..., q_n)$ *preserving the positivity of parameters and defining the least fixed point of* F_φ *in every partially ordered model with the ascending chain condition.*

The following notation is standard:

$$\begin{aligned} \boldsymbol{grz} &: \Box(\Box(p \to \Box p) \to p) \to p, \\ \mathbf{Grz} &= \mathbf{S4} \oplus \boldsymbol{grz}. \end{aligned}$$

It is well known (Segerberg, 1971; Chagrov *et al.*, 1997; Rybakov, 1997) that the logic $\mathbf{Grz}$ is characterized by the class of all partially ordered frames with the ascending chain condition.

COROLLARY 31. — *For any* Π*-formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a* Π*-formula* $\omega(q_1, ..., q_n)$ *preserving the positivity of parameters such that* $\mathbf{Grz}$ *contains the formulas*

$$\omega \leftrightarrow \varphi(\omega, q_1, \ldots, q_n),$$
$$\Box(\varphi \to p) \to (\omega \to p).$$

Sometimes there does not exist a defining Π-formula common for strictly partially ordered models with the ascending chain condition and partially ordered models with the ascending chain condition. For example, consider $\varphi = \Box p$ and two models on natural numbers with reverse orders, $\langle N, \geqslant \rangle$ and $\langle N, > \rangle$; then there does not exist a Π-formula defining the least fixed point of F_φ in both models.

3.3. C-*frames*

A frame $\langle W, R \rangle$ has *cofinal ascending chains* if every infinite chain $x_1 R x_2 R \ldots$ with different elements is a cofinal subset in $\langle W, R \rangle$.

Clearly, frames with the ascending chain condition have cofinal ascending chains.

A partially ordered frame with cofinal ascending chains is called a C-*frame*. Here are some examples: partially ordered frames with the ascending chain condition, the natural numbers $\langle N, \leqslant \rangle$, and the integers $\langle Z, \leqslant \rangle$. Note that the class of C-frames is contained in the class of E-frames.

THEOREM 32 (MARDAEV, 1997B). — *For any* Π*-formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a finite set of* Π*-formulas such that in every* C*-model the least fixed point of the operator* F_φ *is defined by a formula from this set. All these formulas preserve the positivity of parameters.*

Generally speaking, one formula may be insufficient. For example, consider $\varphi = \Box(p \vee q) \vee \Box(p \vee \neg q)$ and two models on the frame $\langle Z, \leqslant \rangle$, with $Q = \{n \in Z \mid n \,\text{is even or}\, n \geqslant 0\}$ in the first model and $Q = \{n \in Z \mid n \,\text{is even}\}$ in the second model. The least fixed point of F_φ is defined by constant $\top$ in the first model and by $\bot$ in the second model, but there is no Π-formula defining the least fixed points of F_φ in both models. One can be prove that in every C-model the least fixed point of this operator is defined by either $\top$ or $\bot$. Note that the defining formula $\Box\Diamond(\Box q \vee \Box\neg q)$ mentioned earlier, involves $\Diamond$.

Let us now consider the logic $\mathbf{Dum} = \mathbf{S4} \oplus \Box(\Box(p \to \Box p) \to p) \to (\Diamond\Box p \to p)$.

PROPOSITION 33 (SEGERBERG, 1971). — *The logic* $\mathbf{Dum}$ *is characterized by the class of all* C*-frames.*

COROLLARY 34. — *For any* Π*-formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a finite set of* Π*-formulas* $\omega_i(q_1, \ldots, q_n)$ *preserving positivity of parameters such that* $\mathbf{Dum}$ *contains the formula*

$$\bigvee_i (\Box(\omega_i \leftrightarrow \varphi(\omega_i, q_1, \ldots, q_n)) \wedge (\Box(\varphi \to p) \to (\omega_i \to p))).$$

Theorem 2.7 can be extended to positive systems.

THEOREM 35. — *For any positive system*

$$(1) \qquad \begin{cases} P_1 = \varphi_1(P_1, \ldots, P_m, Q_1, \ldots, Q_n) \\ \quad \ldots \\ P_m = \varphi_m(P_1, \ldots, P_m, Q_1, \ldots, Q_n), \end{cases}$$

there is a finite set of collections of Π*-formulas such that in every* C*-model the least fixed point of (1) is defined by a collection from this set. All these formulas preserve the positivity of parameters.*

So Corollary 2.9 also can be generalized.

COROLLARY 36. — *For any* Π*-formulas*

$$\varphi_1(p_1,\dots,p_m,q_1,\dots,q_n),\dots,\varphi_m(p_1,\dots,p_m,q_1,\dots,q_n)$$

positive in $p_1,\dots,p_m$*, there is a finite set of collections*

$$\langle\, \omega_1^1(q_1,\dots,q_n),\dots,\omega_m^1(q_1,\dots,q_n)\,\rangle,$$
$$\dots$$
$$\langle\, \omega_1^k(q_1,\dots,q_n),\dots,\omega_m^k(q_1,\dots,q_n)\,\rangle$$

of Π*-formulas preserving the positivity of parameters such that* $\mathbf{Dum}$ *contains the formula*

$$\bigvee_j(\Box\bigwedge_i(\omega_i^j\leftrightarrow\varphi_i(\omega_1^j,\dots,\omega_m^j,q_1,\dots,q_n))\wedge(\Box\bigwedge_i(\varphi_i\to p_i)\to\bigwedge_i(\omega_i^j\to p_i))).$$

3.4. *SC-frames*

An SC*-frame* is a strictly partially ordered frame with cofinal ascending chains. Here are some examples: strictly partially ordered frames with the ascending chain condition, the natural numbers $\langle\, N,<\,\rangle$, and the integers $\langle\, Z,<\,\rangle$. The class of SC-frames is contained in the class of SE-frames.

THEOREM 37 (MARDAEV, 1997B). — *For any* Π*-formula* $\varphi(p,q_1,...,q_n)$ *positive in* p*, there is a finite set of* Π*-formulas such that in every* SC*-model the least fixed point of the operator* F_φ *is defined by a formula from this set. All these formulas preserve the positivity of parameters.*

In general one formula is not sufficient for definability.

Now consider the logic $\mathbf{K4Z}=\mathbf{K4}\oplus\Box(\Box p\to p)\to(\Diamond\Box p\to\Box p)$ introduced in (Segerberg, 1971).

PROPOSITION 38 (SEGERBERG, 1971). — $\mathbf{K4Z}$ *is characterized by the class of all* SC*-frames.*

COROLLARY 39. — *For any* Π*-formula* $\varphi(p,q_1,...,q_n)$ *positive in* p*, there is a finite set of* Π*-formulas* $\omega_i(q_1,\dots,q_n)$ *preserving the positivity of parameters such that* $\mathbf{K4Z}$ *contains the formula*

$$\bigvee_i(\boxdot(\omega_i\leftrightarrow\varphi(\omega_i,q_1,\dots,q_n))\wedge(\boxdot(\varphi\to p)\to(\omega_i\to p))).$$

Theorem 37 and Corollary 39 can be extended to positive systems.

So we have three types of definability of the least fixed points for positive Π-operators.

(1) Definability by one Π-formula; this happens in strictly partially ordered models with the ascending chain condition (Theorem 28) or partially ordered models with the ascending chain condition (Theorem 30).

(2) Definability by a finite set of Π-formulas; this happens in C-models (Theorem 32) or in SC-models (Theorem 37).

(3) Definability by an infinite set of Π-formulas. For example, consider all finite models. For any Π-formula φ and a finite model, there is k such that Π-formula φ^k defines the least fixed point of F_φ in this model. Obviously, for some formulas φ there is no finite set of defining formulas within the class of all finite models.

Defining Π-formulas preserve positivity of parameters for all mentioned classes of models.

4. Negative operators

DEFINITION 40. — *An occurrence of p is called* negative *if it is within the scope of an odd number of negation. A formula $\varphi(p, q_1, \ldots, q_n)$ is called* negative in p *if all occurrences of p are negative.*

A negative operator may not have fixed points in a given model. The following example shows that a fixed point of a negative operator is not necessarily unique. Consider the formula $\varphi = (q \vee \Box(\neg p \vee \neg q)) \wedge (\neg q \vee \Box(\neg p \vee q))$ and the frame $\langle N, \leqslant \rangle$. The value Q of q is the set of all even numbers. Then Q and $\neg Q$ are the fixed points of F_φ.

Now let a formula φ be negative in p. Transform φ by replacing all occurrences of p that are not within the scope of $\Box$ or $\Diamond$, with $\bot$. The resulting formula ζ is negative and modalized in p.

PROPOSITION 41 (CF. PROPOSITION 4). — *In every Kripke model $\langle W, R, v \rangle$ every fixed point of the operator F_φ is a fixed point of the operator F_ζ.*

Now consider the case when a fixed point of a negative operator is unique.

DEFINITION 42. — *A frame $\langle W, R \rangle$ satisfies the* strong ascending chain condition *if there does not exist infinite sequence $x_1 R x_2 R x_3 \ldots$ such that $x_i \neq x_{i+1}$ for all i.*

Let us denote the formula $\varphi(\varphi(p, q_1, \ldots, q_n), q_1, \ldots, q_n)$ by $\psi(p, q_1, \ldots, q_n)$. If φ is negative in p, then ψ is positive in p.

THEOREM 43 (MARDAEV, 1998). — *A negative operator F_φ has a fixed point in a Kripke model with the strong ascending chain condition if and only if the corresponding positive operator F_ψ has a unique fixed point in this model. These two fixed points coincide.*

COROLLARY 44. — *A fixed point of a negative operator in a Kripke model with the strong ascending chain condition is unique if it exists.*

Consider the above mentioned formula $\varphi = (q \vee \Box(\neg p \vee \neg q)) \wedge (\neg q \vee \Box(\neg p \vee q))$ and finite linearly ordered models with the worlds $0 \leqslant 1 \leqslant \cdots \leqslant n$ and with $Q = \{i \mid i \text{ is even}\}$. If n is even, then Q is a fixed point of F_φ. If n is odd, then $\neg Q$ is a fixed point.

Note that uniqueness does not transfer to negative systems. For example, consider the system

$$\begin{cases} P = \neg S \\ S = \neg P. \end{cases}$$

Obviously, it has many solutions, with arbitrary P and $S = \neg P$.

The following notation is standard:

$$\begin{array}{lcl} \mathbf{grz_1} & : & \Box(\Box(p \to \Box p) \to p) \to \Box p, \\ \mathbf{Grz_1} & = & \mathbf{K4} \oplus \mathbf{grz_1}. \end{array}$$

PROPOSITION 45. — *The logic* $\mathbf{Grz_1}$ *is characterized by the class of all transitive frames with the strong ascending chain condition. It is also characterized by the class of all finite transitive and antisymmetric frames.*

If R is transitive, then the strong ascending chain condition is equivalent to two conditions: antisymmetry and the ascending chain condition.

THEOREM 46 (MARDAEV, 1998). — *For any formula* $\varphi(p, q_1, ..., q_n)$ *negative in* p*, there is a formula* $\omega(q_1, ..., q_n)$ *defining the fixed point of the operator* F_φ *in every transitive model with the strong ascending chain condition, where* F_φ *has fixed points.*

Let $\mathbf{L}$ be a modal logic, Γ a set of formulas , α a formula. The notation $\Gamma \vdash_{\mathbf{L}} \alpha$ means that α is derivable from $\Gamma \cup \mathbf{L}$ by rules of necessitation and *modus ponens*.

We can prove Theorem 3.5 using Theorem 1 from (Maksimova, 1992), which asserts that every normal modal logic containing $\mathbf{K4}$ has Beth property. The latter means that for every formula $\alpha(p, q_1, \ldots, q_n)$, if

$$\alpha(p, q_1, \ldots, q_n), \alpha(s, q_1, \ldots, q_n) \vdash_{\mathbf{L}} p \leftrightarrow s,$$

then there is a formula $\beta(q_1, \ldots, q_n)$, for which

$$\alpha(p, q_1, \ldots, q_n) \vdash_{\mathbf{L}} p \leftrightarrow \beta(q_1, \ldots, q_n).$$

But the proof of Theorem 1 from (Maksimova, 1992) does not construct a defining formula in an explicit form. So we give another proof (without applying Beth property). Let us describe Sambin's construction (Sambin *et al.*, 1982; Reidhaar-Olson, 1990) of a defining formula for Fixed Point Theorem.

Let $\varphi(p, q_1, \ldots, q_n) = \alpha(\Box\beta_1(p, q_1, \ldots, q_n), \ldots, \Box\beta_m(p, q_1, \ldots, q_n), q_1, \ldots, q_n)$ where the modal formula $\alpha(s_1, \ldots, s_m, q_1, \ldots, q_n)$ is in variables $q_1, \ldots, q_n$ and new variables $s_1, \ldots, s_m$. For the sake of brevity, we write β_i for $\beta_i(p, q_1, \ldots, q_n)$. We construct a defining formula ω by induction on m.

If $m = 0$, then $\omega(q_1, \ldots, q_n) = \varphi(q_1, \ldots, q_n)$.

At step $(m+1)$ we have $\varphi(p, q_1, \ldots, q_n) = \alpha(\Box\beta_1, \ldots, \Box\beta_{m+1}, q_1, \ldots, q_n)$. For each i with $1 \leqslant i \leqslant m+1$, let

$$\varphi_i(p, q_1, \ldots, q_n) = \alpha(\Box\beta_1, \ldots, \Box\beta_{i-1}, \top, \Box\beta_{i+1}, \ldots, \Box\beta_{m+1}, q_1, \ldots, q_n).$$

By induction hypothesis, for every i we have a formula $\omega_i(q_1, \ldots, q_n)$, which defines the fixed point of F_{φ_i}. Put

$$\omega(q_1, \ldots, q_n) = \alpha(\Box\beta_1(\omega_1, q_1, \ldots, q_n), \ldots, \Box\beta_{m+1}(\omega_{m+1}, q_1, \ldots, q_n), q_1, \ldots, q_n).$$

Then ω defines the fixed point of F_φ.

The proof in (Mardaev, 2003) based on a special version of Sambin's construction, gives a defining formula for Theorem 3.5. The only difference from the original Sambin's construction is the condition that every β_i must contain at least one occurrence of p. For example, if $\varphi = (q \vee \Box(\neg p \vee \neg q)) \wedge (\neg q \vee \Box(\neg p \vee q))$, then ω is equivalent to $(q \vee \Box(\Diamond\neg q \vee \neg q)) \wedge (\neg q \vee \Box(\Diamond q \vee q))$.

(Mardaev, 1998) gives another explicit construction for Theorem 3.5, but it is more complicated.

COROLLARY 47. — *For any formula $\varphi(p, q_1, ..., q_n)$ negative in p, there is a formula $\omega(q_1, ..., q_n)$ such that the logic $\mathbf{Grz_1}$ contains the formula*

$$\boxdot(p \leftrightarrow \varphi) \rightarrow (p \leftrightarrow \omega).$$

Now let us consider subclasses of negative operators.

Let $\psi, \theta_1, \ldots, \theta_m, \tau$ be formulas in variables $p, q_1, \ldots, q_n$, and assume that all occurrences of p in ψ, $\theta_1, \ldots, \theta_m$, τ are negative. Also assume that there are no occurrences of p in ψ within the scope of $\Box$ or $\Diamond$.

We consider the formula $\varphi(p, q_1, \ldots, q_n) = \psi \vee \Box\theta_1 \vee \ldots \vee \Box\theta_m \vee \Diamond\tau$. Let

$$\begin{aligned} \psi'(q_1, \ldots, q_n) &= \psi(\bot, q_1, \ldots, q_n), \\ \varphi'(p, q_1, \ldots, q_n) &= \psi' \vee \Box\theta_1 \vee \ldots \vee \Box\theta_m \vee \Diamond\tau, \\ \omega(q_1, \ldots, q_n) &= \varphi'(\varphi'(\top, q_1, \ldots, q_n), q_1, \ldots, q_n). \end{aligned}$$

THEOREM 48 (MARDAEV, 1997A). — *If a fixed point of the operator F_φ exists in a transitive Kripke model, then a fixed point is unique and defined by ω.*

COROLLARY 49. — *The logic $\mathbf{K4}$ contains the formula*

$$\boxdot(p \leftrightarrow \varphi) \rightarrow (p \leftrightarrow \omega).$$

Finally we consider the formula

$$\eta(p, q_1, \ldots, q_n) = \psi \wedge \Diamond\theta_1 \wedge \ldots \wedge \Diamond\theta_m \wedge \Box\tau.$$

COROLLARY 50. — *If a fixed point of the operator F_η exists in a transitive Kripke model, then a fixed point is unique and definable.*

5. Temporal positive operators

Temporal formulas are constructed from propositional variables p, q, r ... and the constant $\bot$ (falsity) using the connectives $\wedge$, $\vee$, $\neg$, $\Box_L$, and $\Box_R$. We use the following abbreviations:

$$\begin{aligned} \Diamond_L &= \neg\Box_L\neg, & \Diamond_R &= \neg\Box_R\neg, \\ \Box\alpha &= \Box_L\alpha \wedge \Box_R\alpha, & \boxdot\alpha &= \alpha \wedge \Box_L\alpha \wedge \Box_R\alpha, \\ \Diamond\alpha &= \Diamond_L\alpha \vee \Diamond_R\alpha, & \diamondsuit\alpha &= \alpha \vee \Diamond_L\alpha \vee \Diamond_R\alpha. \end{aligned}$$

Recall that in a Kripke model $\langle W, R, v \rangle$, we can extend the valuation v to all temporal formulas in the same way as for the modal case (Section 1), but now $\Box_L$, $\Box_R$, $\Diamond_L$, and $\Diamond_R$ correspond to the following operations on sets:

$$\Box_L A = \{x \mid \forall y\,(yRx \Rightarrow y \in A)\},\ \Box_R A = \{x \mid \forall y\,(xRy \Rightarrow y \in A)\},$$
$$\Diamond_L A = \{x \mid \exists y\,(yRx \wedge y \in A)\},\ \Diamond_R A = \{x \mid \exists y\,(xRy \wedge y \in A)\}.$$

In this section pictures of frames represent accessibility relations R from the left to the right on pictures. So the subscripts L and R (there should be no confusion with the notation of the binary relation) mean “left" and “right" (or in temporal terms, “past" and “future").

The temporal case differs from the modal case. The least fixed points of a temporal positive operators may be undefinable on some classes of linear models. For example, consider the formula $\varphi = q \wedge \Box_L\Box_L(p \vee \Diamond_R p)$ and finite strictly linearly ordered models $0 < 1 < \cdots < 4n$ (the upper model in Fig. 5). The value of q consists of odd numbers from 1 to $2n-1$ and even numbers from $2n+2$ to $4n$ (marked by bullets). The least fixed point (marked by double circles) of the operator F_φ consists of odd numbers from 1 to $2n-1$. These sets are undefinable on the class of these models.

For another example, consider the formula $\varphi = s \wedge \Box_L(\Box_L(\Box_L(\Box_L(\Diamond_R p \vee \neg q) \vee q) \vee \neg q) \vee q)$ and finite linearly ordered models with the worlds $0 \leqslant 1 \leqslant \cdots \leqslant 8n+5$ (the lower model in Fig. 5) such that the value of q consists of all odd numbers (marked by bullets), the value of s consists of odd numbers 3, 7, 11,... from 3 to $4n-1$ and $4n+5$, $4n+9$, $4n+11$,... from $4n+5$ to $4n+5+4n = 8n+5$ (marked by s). The least fixed point (marked by double circles) of the operator F_φ consists of odd

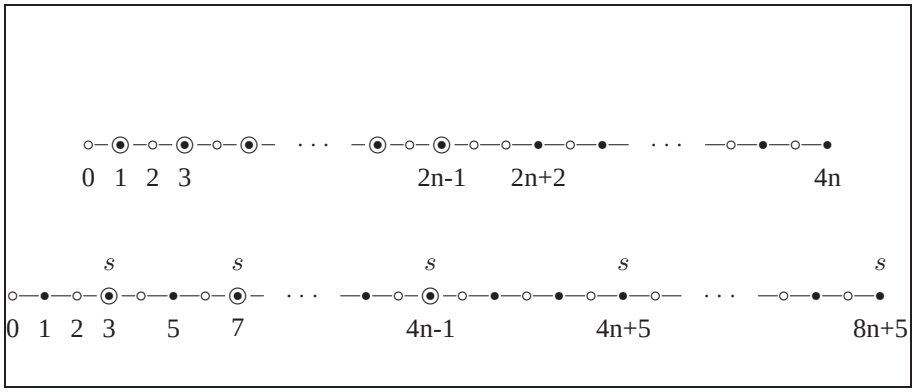

Figure 5. *The least fixed points are undefinable*

numbers 3, 7, 11,... from 3 to $4n-1$. These sets are undefinable on the class of these models.

Nevertheless the linear temporal case is similar to the general modal case as far as Σ- and Π-formulas are concerned. Similarly to the modal case, *temporal* Σ*-formulas* are constructed from propositional variables, their negations, the constants $\bot$, $\top$ and the connectives $\wedge$, $\vee$, $\Diamond_L$, $\Diamond_R$. Temporal Π*-formulas* are constructed from propositional variables, their negations, the constants $\bot$, $\top$ and the connectives $\wedge$, $\vee$, $\Box_L$, and $\Box_R$.

The basic normal bimodal logic is denoted by $\mathbf{K}^2$. The following notation for formulas and logics is standard (Segerberg, 1970):

$$\begin{array}{ll}
\boldsymbol{c}_L : \Diamond_L\Box_R p \to p, & \boldsymbol{c}_R : \Diamond_R\Box_L p \to p,\\
\boldsymbol{4}_L : \Box_L p \to \Box_L\Box_L p, & \boldsymbol{4}_R : \Box_R p \to \Box_R\Box_R p,\\
\boldsymbol{l}_L : \boxdot p \to \Box_L\Box_R p, & \boldsymbol{l}_R : \boxdot p \to \Box_R\Box_L p,\\
\boldsymbol{t}_L : \Box_L p \to p, & \boldsymbol{t}_R : \Box_R p \to p,\\
\boldsymbol{grz}_L : \Box_L(\Box_L(p \to \Box_L p) \to p) \to p, & \boldsymbol{grz}_R : \Box_R(\Box_R(p \to \Box_R p) \to p) \to p,\\
\boldsymbol{m}_L : \Diamond_L(\Diamond_L p \to \Box_L p), & \boldsymbol{m}_R : \Diamond_R(\Diamond_R p \to \Box_R p),\\
\boldsymbol{w}_L : \Box_L(\Box_L p \to p) \to \Box_L p, & \boldsymbol{w}_R : \Box_R(\Box_R p \to p) \to \Box_R p,\\
\boldsymbol{z}_L : \Box_L(\Box_L p \to p) \to (\Diamond_L\Box_L p \to \Box_L p), & \boldsymbol{z}_R : \Box_R(\Box_R p \to p) \to (\Diamond_R\Box_R p \to \Box_R p),\\
\boldsymbol{d}_L : \Diamond_L\top, & \boldsymbol{d}_R : \Diamond_R\top,\\
\boldsymbol{e}_L : \Diamond\Box_L\bot, & \boldsymbol{e}_R : \Diamond\Box_R\bot,
\end{array}$$

$$
\begin{aligned}
\boldsymbol{dum}_L &: \Box_L(\Box_L(p \to \Box_L p) \to p) \to (\Diamond_L \Box_L p \to p),\\
\boldsymbol{dum}_R &: \Box_R(\Box_R(p \to \Box_R p) \to p) \to (\Diamond_R \Box_R p \to p),\\
\mathbf{Lin} &= \mathbf{K}^2 \oplus \boldsymbol{c}_L \oplus \boldsymbol{c}_R \oplus \boldsymbol{4}_L \oplus \boldsymbol{4}_R \oplus \boldsymbol{l}_L \oplus \boldsymbol{l}_R,\\
\mathbf{LinTGrz} &= \mathbf{Lin} \oplus \boldsymbol{t}_L \oplus \boldsymbol{t}_R \oplus \boldsymbol{grz}_L \oplus \boldsymbol{grz}_R,\\
\mathbf{LinTDum} &= \mathbf{Lin} \oplus \boldsymbol{t}_L \oplus \boldsymbol{t}_R \oplus \boldsymbol{dum}_L \oplus \boldsymbol{dum}_R,\\
\mathbf{LinTDumM}_L &= \mathbf{Lin} \oplus \boldsymbol{t}_L \oplus \boldsymbol{t}_R \oplus \boldsymbol{dum}_L \oplus \boldsymbol{dum}_R \oplus \boldsymbol{m}_L,\\
\mathbf{LinW} &= \mathbf{Lin} \oplus \boldsymbol{w}_L \oplus \boldsymbol{w}_R,\\
\mathbf{LinZD} &= \mathbf{Lin} \oplus \boldsymbol{z}_L \oplus \boldsymbol{z}_R \oplus \boldsymbol{d}_L \oplus \boldsymbol{d}_R,\\
\mathbf{LinZD}_R\mathbf{E}_L &= \mathbf{Lin} \oplus \boldsymbol{z}_L \oplus \boldsymbol{z}_R \oplus \boldsymbol{d}_R \oplus \boldsymbol{e}_L.
\end{aligned}
$$

Now let us study the least fixed point in these logics

5.1. Lin

It is well known (Segerberg, 1970) that the logic **Lin** is characterized by the class of all transitive linear frames.

THEOREM 51 (MARDAEV, 2004). — *For any temporal Σ-formula $\varphi(p, q_1, ..., q_n)$ positive in p, there is a natural number m such that the formula $\varphi^m(q_1, \ldots, q_n)$ defines the least fixed point of the operator F_φ in every transitive linear Kripke model.*

An upper bound for m can be extracted from the proof.

COROLLARY 52. — *For any temporal Σ-formula $\varphi(p, q_1, ..., q_n)$ positive in p, there is a natural number m such that* **Lin** *contains the formula $\varphi^{m+1} \leftrightarrow \varphi^m$.*

5.2. LinTGrz

It is well known (Segerberg, 1970) that the logic **LinTGrz** is characterized by the class of all finite linearly ordered frames.

THEOREM 53 (MARDAEV, 1999). — *For any temporal Π-formula $\varphi(p, q_1, ..., q_n)$ positive in p, there is a Π-formula $\omega(q_1, ..., q_n)$ preserving the positivity of parameters and defining the least fixed point of the operator F_φ in every finite linearly ordered model.*

COROLLARY 54. — *For any temporal Π-formula $\varphi(p, q_1, ..., q_n)$ positive in p, there is a Π-formula $\omega(q_1, ..., q_n)$ preserving the positivity of parameters such that* **LinTGrz** *contains the formulas*

$$\omega \leftrightarrow \varphi(\omega, q_1, \ldots, q_n),$$
$$\Box(\varphi \to p) \to (\omega \to p).$$

5.3. LinW

It is well known (Segerberg, 1970) that the logic **LinW** is characterized by the class of all finite strictly linearly ordered frames.

THEOREM 55 (MARDAEV, 1999). — *For any temporal* Π*-formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a* Π*-formula* $\omega(q_1, ..., q_n)$*, which preserves the positivity of parameters and defines the least fixed point of the operator* F_φ *in every finite strictly linearly ordered model.*

COROLLARY 56. — *For any temporal* Π*-formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a* Π*-formula* $\omega(q_1, ..., q_n)$ *preserving the positivity of parameters such that* **LinW** *contains the formulas*

$$\omega \leftrightarrow \varphi(\omega, q_1, \ldots, q_n),$$
$$\boxdot(\varphi \to p) \to (\omega \to p).$$

5.4. LinTDum

It is well known (Segerberg, 1970) that the logic **LinTDum** is characterized by the frame $\langle Z, \leqslant \rangle$.

THEOREM 57 (MARDAEV, 1999). — *For any temporal* Π*-formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a finite set of* Π*-formulas such that in every model* $\langle Z, \leqslant, v \rangle$ *the least fixed point of the operator* F_φ *is defined by a formula from this set. All these formulas preserve the positivity of parameters.*

COROLLARY 58. — *For any temporal* Π*-formula* $\varphi(p, q_1, ..., q_n)$ *positive in* p*, there is a finite set of* Π*-formulas* $\omega_i(q_1, \ldots, q_n)$ *preserving the positivity of parameters such that* **LinTDum** *contains the formula*

$$\bigvee_i (\Box(\omega_i \leftrightarrow \varphi(\omega_i, q_1, \ldots, q_n)) \wedge (\Box(\varphi \to p) \to (\omega_i \to p))).$$

Theorem 57 and Corollary 58 can be extended to positive systems.

COROLLARY 59. — *For any temporal* Π*-formulas*

$$\varphi_1(p_1, \ldots, p_m, q_1, \ldots, q_n), \ldots, \varphi_m(p_1, \ldots, p_m, q_1, \ldots, q_n)$$

positive in $p_1, \ldots, p_m$*, there is a finite set of collections*

$$\langle \omega_1^1(q_1, \ldots, q_n), \ldots, \omega_m^1(q_1, \ldots, q_n) \rangle,$$
$$\ldots$$
$$\langle \omega_1^k(q_1, \ldots, q_n), \ldots, \omega_m^k(q_1, \ldots, q_n) \rangle$$

of Π*-formulas preserving the positivity of parameters such that* $\mathbf{LinTDum}$ *contains the formula*

$$\bigvee_j(\Box\bigwedge_i(\omega_i^j \leftrightarrow \varphi_i(\omega_1^j,\ldots,\omega_m^j,q_1,\ldots,q_n)) \wedge (\Box\bigwedge_i(\varphi_i \to p_i) \to \bigwedge_i(\omega_i^j \to p_i))).$$

5.5. $\mathbf{LinZD}, \mathbf{LinTDumM}_L, \mathbf{LinZD}_R\mathbf{E}_L$

Consider other temporal models based on the integers and the natural numbers.

THEOREM 60 (MARDAEV, 1999). —

1) For any temporal Π*-formula* $\varphi(p,q_1,...,q_n)$ *positive in* p*, there is a finite set of* Π*-formulas such that in every model* $\langle Z,<,v\rangle$ *the least fixed point of the operator* F_φ *is defined by a formula from this set. All these formulas preserve the positivity of parameters.*

2) The same holds for the models $\langle N,\leqslant,v\rangle$.

3) The same holds for the models $\langle N,<,v\rangle$.

It is well known (Segerberg, 1970) that

(1) the logic $\mathbf{LinZD}$ is characterized by the frame $\langle Z,<\rangle$,
(2) the logic $\mathbf{LinTDumM}_L$ is characterized by the frame $\langle N,\leqslant\rangle$,
(3) the logic $\mathbf{LinZD}_R\mathbf{E}_L$ is characterized by the frame $\langle N,<\rangle$.

Similarly to the above examples, we can obtain appropriate corollaries for these logics and versions for positive systems.

6. Inflationary fixed points

The modal language with inflationary fixed points MIC was investigated in (Dawar *et al.*, 2001; Dawar *et al.*, 2004). Speaking informally, MIC is a propositional multi-modal language, augmented with simultaneous inflationary fixed points.

Fix a set A of *actions*. *Multi-modal propositional formulas* are constructed from propositional variables p, q, r ... and the constant $\bot$ (falsity) using the binary connectives $\wedge$, $\vee$, and the unary connectives $\neg$ and $[a]$ for all $a \in A$. A *multi-modal Kripke frame* $\langle W,\{R_a \mid a \in A\}\rangle$ consists of a non-empty set W and binary relations R_a on W for all $a \in A$. A *Kripke model* consists of a frame and a valuation function v. A connective $[a]$ corresponds to the following operation on sets:

$$[a]A = \{x \mid \forall y\,(xR_ay \Rightarrow y \in A)\}.$$

MIC extends the propositional multi-modal language by the following rule for building formulas: if $\psi_1(q_1, \ldots, q_n, s_1, \ldots, s_m)$, ..., $\psi_n(q_1, \ldots, q_n, s_1, \ldots, s_m)$ are formulas of MIC, then

$$\Psi \ : \ = \begin{cases} q_1 \leftarrow \psi_1 \\ \quad \ldots \\ q_n \leftarrow \psi_n \end{cases}$$

is a system of rules, and $\mathbf{ifp}\, q_i : \Psi\ (s_1, \ldots, s_m)$ is a formula of MIC.

On every Kripke model, the system Ψ of rules defines a tuple $(Q_1^\alpha, \ldots, Q_n^\alpha)$ of sets for each ordinal α, by the following inflationary induction (for $i = 1, \ldots, n$):
$Q_i^0 = \varnothing$,
$Q_i^{\alpha+1} = Q_i^\alpha \cup \psi_i(Q_1^\alpha, \ldots, Q_n^\alpha, S_1, \ldots, S_m)$,
$Q_i^\alpha = \bigcup\limits_{\beta<\alpha} Q_i^\beta$ if α is a limit ordinal.

As the sequence of tuples is increasing (i.e. $Q_i^\alpha \subseteq Q_i^\beta$ for any $\alpha < \beta$), it reaches the *inflationary fixed point* $(Q_1^\infty, \ldots, Q_n^\infty)$. Then we put the value of $\mathbf{ifp}\, q_i : \Psi\ (s_1, \ldots, s_m)$ equal to Q_i^∞.

In this section a *language* is a subset of the set of all finite words over an alphabet. Consider a finite alphabet $q_1, \ldots, q_k$. Let $A = a_1 \ldots a_n$ be a finite word. For our purposes, we regard this word as a Kripke model with $W = \{1, \ldots, n\}$, the binary successor relation $iR(i+1)$, where $1 \leqslant i \leqslant n-1$, and the valuation v: if q_j is a letter of the alphabet, then $v(q_j) = \{i \in W \mid a_i = q_j\}$. A formula α of MIC (respectively, of the μ-calculus) is *true* in a model $A = \langle W, R, v \rangle$ if α is true at each $i \in W$. A language L is *expressible* in MIC (respectively, in the μ-calculus) if there is a formula $\theta(q_1, \ldots, q_k)$ of MIC (respectively, of the μ-calculus) such that $L = \{A \mid \theta \text{ is true in } A\}$.

MIC is more expressive than μ-calculus. In the papers (Dawar *et al.*, 2001; Dawar *et al.*, 2004) (with Martin Otto) a language is constructed, which is expressible in MIC but not in the μ-calculus.

Let us give another proof showing that MIC is more expressive than μ-calculus. Recall examples from (Mardaev, 2004) of inflationary fixed points for the class of all finite strictly linearly ordered modal models and for the class of all finite linearly ordered models that are undefinable (in modal logic). The least fixed points of positive operators are definable within these classes (Theorems 9, 12) by formulas preserving the positivity of parameters. Therefore if we consider the language of μ-calculus, the μ-operator can be eliminated in these classes. So the inflationary fixed points are undefinable in the μ-calculus.

Consider the modal (not multi-modal) language, modal models and non-simultaneous inflationary fixed points. Consider a model with an arbitrary operator F, and a sequence of sets
$P^0 = \varnothing$,

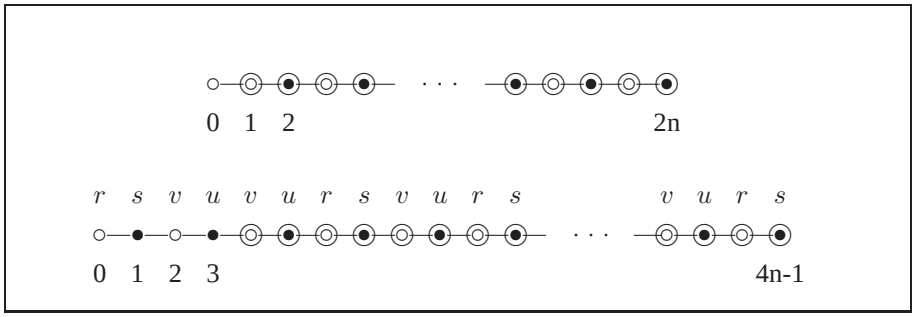

Figure 6. *Undefinable inflationary fixed points*

$P^{\alpha+1} = P^\alpha \cup F(P^\alpha)$,
$P^\alpha = \bigcup\limits_{\beta<\alpha} P^\beta$ if α is a limit ordinal.

This sequence reaches the inflationary fixed point of F. If F is monotonic, the inflationary fixed point coincides with the least fixed point.

EXAMPLE 61. — Consider the formula $\varphi = (q \wedge \Box p) \vee (\Box\Box p \wedge \neg\Box p)$ and finite strictly linearly ordered models over the frame $0 < 1 < \cdots < 2n$ (the upper model in Fig. 6). Note that unlike Fig. 1, modal models in Fig. 6 are shown "from the left to the right". The value of q consists of all nonzero even numbers (marked by bullets). The inflationary fixed point of the operator F_φ (marked by double circles) consists of all nonzero elements. It is undefinable on the class of all these models. □

EXAMPLE 62. — Consider the formula

$$\begin{aligned}\varphi = {} & (s \wedge \Box(p \vee q)) \vee \\ & \vee (r \wedge \Box(\Box(p \vee q) \vee \neg q) \wedge \neg\Box(p \vee q)) \vee \\ & \vee (u \wedge \Box(\Box(\Box(p \vee q) \vee \neg q) \vee q) \wedge \neg\Box(\Box(p \vee q) \vee \neg q)) \vee \\ & \vee (v \wedge \Box(\Box(\Box(\Box(p \vee q) \vee \neg q) \vee q) \vee \neg q) \wedge \neg\Box(\Box(\Box(p \vee q) \vee \neg q) \vee q))\end{aligned}$$

and the models over the finite linearly ordered frame $0 \leqslant 1 \leqslant \cdots \leqslant 4n-1$ (the lower model in Fig. 6). The value of q consists of all odd numbers (marked by bullets). The values of variables s, r, u, and v are indicated in Fig. 6. The corresponding sequence of true variables begins with $rsvu$, then it becomes periodic, with the period $vurs$. The inflationary fixed point of the operator F_φ (marked by double circles) consists of all elements ≥ 4. It is undefinable on the class of our models. □

EXAMPLE 63. — There are simpler examples in the temporal case. Consider the formula $\varphi = p \vee (\Box_L\Box_L(p \vee \Diamond_R p) \wedge \neg\Box_L(p \vee \Diamond_R p))$ and models over the finite strictly linearly ordered frame $0 < 1 < \cdots < n$. The inflationary fixed point of F_φ consists of all odd elements. This point is undefinable. □

EXAMPLE 64. — Consider the formula $\varphi = p \vee (\Box_L(\psi \vee q) \wedge \neg\psi)$, where $\psi = \Box_L(\Box_L(\Box_L(\Diamond_R p \vee \neg q) \vee q) \vee \neg q)$ and models over the finite linearly ordere frame

$0 \leqslant 1 \leqslant \cdots \leqslant n$, in which the value of q consists of all odd numbers. Then the inflationary fixed point of F_φ consists of all numbers of the form $4k+3$. This point is undefinable. □

7. Graded modalities

Modal graded propositional formulas are constructed from propositional variables p, q, r ... and the constant $\bot$ (falsity) using the binary connectives $\wedge$, $\vee$, and the unary connectives $\neg$ and $\Box_k$ for all natural k. We introduce abbreviations $\Diamond_k = \neg\Box_k\neg$. Connectives $\Box_k$ and $\Diamond_k$ (graded modalities) correspond to the following operations on sets:

$$\Box_k A = \{x \mid \text{the number of } y \text{ such that } xRy \text{ and } y \notin A \text{ is less than } k\},$$

$$\Diamond_k A = \{x \mid \text{there are at least } k\ y\text{'s such that } xRy \text{ and } y \in A\}.$$

Clearly, $\Box_0$ and $\Diamond_0$ are trivial, $\Box_1 = \Box$ and $\Diamond_1 = \Diamond$.

The following theorem generalizes the Fixed Point Theorem. The proof uses Sambin's construction.

THEOREM 65 (MARDAEV, 2006). — *For any graded formula $\varphi(p, q_1, ..., q_n)$ modalized in p, there is a unique fixed point of the operator F_φ in every strictly partially ordered model with the ascending chain condition and there is a graded formula $\omega(q_1, ..., q_n)$, which defines the fixed point in every model of this kind. The formula ω contains only those graded modalities, which are contained in φ.*

So Sambin's construction works in two cases:

(1) for modalized operators in strictly partially ordered frames with the ascending chain condition (Fixed Point Theorem, Theorem 65),

(2) for negative operators in transitive antisymmetric frames with the ascending chain condition (Theorem 46).

In models based on the frame $\langle N, <\rangle$, where N is the set of natural numbers, $\Diamond_k\alpha$ is equivalent to $\Diamond(\alpha \wedge \Diamond(\alpha \wedge \ldots \wedge \Diamond(\alpha \wedge \Diamond\alpha)\ldots))$ with k occurrences of $\Diamond$. The frame $\langle N, <\rangle$ is SE-frame. From Theorem 12 we obtain

THEOREM 66 (MARDAEV, 2006). — *For any graded formula $\varphi(p, q_1, ..., q_n)$ positive in p, there is a modal formula $\omega(q_1, ..., q_n)$ preserving the positivity of parameters and defining the least fixed point of the operator F_φ in every model based on the frame $\langle N, <\rangle$.*

8. Problems

PROBLEM 67. — Find lower bounds for complexity of defining formulas (in terms of length or modal depth). □

PROBLEM 68. — Investigate definability of the least fixed points of monotonic operators. □

PROBLEM 69. — When does monotonicity coincide with positivity? □

PROBLEM 70. — Find syntactic proofs. □

PROBLEM 71. — Find proofs using Sambin's construction. □

PROBLEM 72. — Find proofs using automata. □

PROBLEM 73. — Investigate the case when negative operator has finitely (infinitely) many definable fixed points. How many fixed points can a negative operator have? □

PROBLEM 74. — Is the problem of fixed points existence for negative operators in finitely presented models algorithmically decidable? □

PROBLEM 75. — Investigate definability of the least fixed points of squares of negative operators. □

PROBLEM 76. — A transitive antisymmetric frame is called an IRF*-frame* if the following conditions holds: □

1. there does not exist an antichain $y_1, y_2, \ldots$ (finite or infinite) and an infinite chain $x_1 < x_2 < \ldots$ such that for any x_i there is y_j such that $x_i < y_j$;

2. the set of reflexive elements is an upper cone.

The frame in Example 20 is an IRF-frame. Investigate definability of the least fixed points of positive (monotonic) operators in IRF-models.

PROBLEM 77. — Do these logics capture some classes of finite or cellular automata in these models (Janin *et al.*, 1995)? □

PROBLEM 78. — How many defining Π-formulas may be necessary for Π-operators? □

PROBLEM 79. — Investigate definability of inflationary fixed points. □

PROBLEM 80. — Investigate definability of the least fixed points of positive (monotonic) operators in models with finite chains of bounded length attached to some elements. □

PROBLEM 81. — Investigate definability of fixed points in multi-modal languages. □

PROBLEM 82. — Does defining formula exist in theorems 57, 60? □

PROBLEM 83. — Investigate definability of the least fixed points of graded operators. □

PROBLEM 84. — Describe frames and models with definable fixed points. □

PROBLEM 85. — How is Theorem 65 related to arithmetic? □

PROBLEM 86. — Investigate definability of fixed points in non-transitive models. □

PROBLEM 87. — Obtain the reflexive case from the irreflexive one by replacing $\Box$ with $\boxdot$. □

PROBLEM 88. — The μ-operator is trivially definable in the μ-calculus. So there is an analogy between the μ-calculus and logics with definable least fixed points. What properties of μ-calculus can be transferred to these logics and vice versa? For example, how about the uniform interpolation property (D'Agostino *et al.*, 1996)? □

PROBLEM 89. — Adding the least fixed point operator for monotonic formulas to syntax is natural when monotonicity is decidable. But if the complexity of syntax increases, this is not so natural. □

PROBLEM 90. — Does Theorem 66 hold for $\langle N, \leqslant \rangle$ and a graded formula ω? □

And so on ...

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9. References

Aczel P., "An introduction to inductive definitions", *Handbook in Mathematical Logic, edited by J.Barwise*, vol. 90 of *Studies in Logic and the Foundations of Mathematics*, North-Holland Publishing Company, pp. 739-782, 1977.

Bernardi C., "The fixed-point theorem for diagonalizable algebras", *Studia Logica*, vol. 34, num. 3, pp. 239–251, 1975.

Bernardi C., "The uniqueness of the fixed-point in every diagonalizable algebra", *Studia Logica*, vol. 35, num. 4, pp. 335–343, 1976.

Chagrov A., Zakharyaschev M., *Modal Logic, Oxford Logic Guides, vol.35*, Clarendon Press, 1997.

D'Agostino G., Hollenberg M., *Uniform interpolation, automata and the modal μ-calculus, Preprint 165*, Logic Group Preprint Series, May, 1996. http://www.phil.uu.nl/preprints/preprints/PREPRINTS/preprint165.ps.Z.

Dawar A., Grädel E., Kreutzer S., "Inflationary Fixed Points in Modal Logic", *Computer Science Logic, Proceedings of the 15th International Workshop, CSL 2001. 10th Annual Conference of the European Association for Computer Science Logic (EACSL), ed. by Laurent Fribourg*, vol. 2142 of *Lecture Notes in Computer Science*, Springer-Verlag, Paris, France, September 10-13, pp. 277-291, 2001. http://www-mgi.informatik.rwth-aachen.de/Publications/pub/kreutzer/csl01.ps.

Dawar A., Grädel E., Kreutzer S., "Inflationary Fixed Points in Modal Logic", *ACM Transactions on Computational Logic (TOCL)*, vol. 5, num. 2, pp. 282-315, 2004.

Janin D., Walukiewicz I., "Automata for the μ-calculus and related results", *MFCS'95*, vol. 969 of *Lecture Notes in Computer Science*, Springer-Verlag, pp. 552-562, 1995. http://www.labri.fr/perso/igw/Papers/igw-mu-aut.pdf.

Kozen D., "Results on the propositional μ-calculus", *Automata, Languages, and Programming, Ninth Colloquium*, Springer, pp. 348-359, 1982.

Maksimova L., "Analog of Beth's Theorem in normal extensions of the modal logic $K4$", *Siberian Mathematical Journal*, vol. 33, num. 6, pp. 1052-1065, 1992.

Mardaev S., "Fixed points in modal schemes", *Algebra and Logic*, vol. 31, num. 5, pp. 292-295, 1992.

Mardaev S., "The least fixed points in Godel-Lob logic", *Algebra and Logic*, vol. 32, num. 6, pp. 372-375, 1993a.

Mardaev S., "The least fixed points in Grzegorczyk logic and in the intuitionistic propositional logic", *Algebra and Logic*, vol. 32, num. 5, pp. 279-288, 1993b.

Mardaev S., "Convergence of positive schemes in $S4$ and Int", *Algebra and Logic*, vol. 33, num. 2, pp. 95-101, 1994.

Mardaev S., "Fixed points of modal negative operators", *Bulletin of the Section of Logic*, vol. 26, num. 3, pp. 135-138, 1997a.

Mardaev S., "Modal Π-schemes", *Algebra and Model Theory, Proceedings of the Second Summer School "Intermediate problems in Model Theory and Universal Algebra", ed. by A.G. Pinus and K.N. Ponomaryov, Novosibirsk (in Russian)*, Novosibirsk State Technical University, Erlagol, Russia, June 17-21, pp. 99-109, 1997b.

Mardaev S., "Negative modal schemes", *Algebra and Logic*, vol. 37, num. 3, pp. 181-191, 1998.

Mardaev S., "Fixed points of temporal operators", *Algebra and Model Theory, Proceedings of the Third Summer School "Intermediate problems in Model Theory and Universal Algebra", ed. by A.G. Pinus and K.N. Ponomaryov, Novosibirsk (in Russian)*, Novosibirsk State Technical University, Erlagol, Russia, June 21-27, pp. 68-77, 1999.

Mardaev S., "Fixed points in preordered Kripke models", *Algebra and Model Theory, Proceedings of the Fourth Summer School "Intermediate problems in Model Theory and Universal Algebra", ed. by A.G. Pinus and K.N. Ponomaryov, Novosibirsk (in Russian)*, Novosibirsk State Technical University, Erlagol, Russia, June 24-30, pp. 76-82, 2001a.

Mardaev S., *Fixed points of modal operators, Habilitation Thesis*, Institute of Mathematics of Siberian Branch of Russian Academy of Sciences, Novosibirsk (in Russian), 2001b.

Mardaev S., "Definability of the least fixed points", *Algebra and Logic*, vol. 41, num. 4, pp. 237-253, 2002a.

Mardaev S., "The least fixed points in modal logic", *in* S. Goncharov (ed.), *Proceedings of International Conferences in Mathematical logic*, Novosibirsk State University, Novosibirsk, pp. 92-103, 2002b.

Mardaev S., “Sambin’s construction and negative operators”, *Algebra and Model Theory, Proceedings of the Fifth Summer School "Intermediate problems in Model Theory and Universal Algebra", ed. by A.G. Pinus and K.N. Ponomaryov, Novosibirsk (in Russian)*, Novosibirsk State Technical University, Erlagol, Russia, June 17-24, pp. 69-75, 2003.

Mardaev S., “Fixed points in tense models”, *Algebra and Logic*, vol. 43, num. 5, pp. 331-338, 2004.

Mardaev S., “Graded Modal Operators and Fixed Points”, *Vestnik NGU (Bulletin of the Novosibirsk State University), Mathematics*, vol. 6, num. 1, pp. 70-76 (in Russian), 2006.

Reidhaar-Olson L., “A new proof of the fixed-point theorem of provability logic”, *Notre Dame Journal of Formal Logic*, vol. 31, num. 1, pp. 37-43, 1990.

Ruitenburg W., “On the period of sequences $(A^n(p))$ in intuitionistic propositional calculus”, *The Journal of Symbolic Logic*, vol. 49, num. 3, pp. 892-899, 1984.

Rybakov V., *Admissibility of Logical Inference Rules*, vol. 136 of *Studies in Logic and the Foundations of Mathematics*, Elsevier Science Publishers B.V., 1997.

Sambin G., “An effective fixed-point theorem in intuitionistic diagonalizable algebras”, *Studia Logica*, vol. 35, num. 4, pp. 345-361, 1976.

Sambin G., Valentini S., “The modal logic of provability: The sequential approach”, *Journal of Philosophical Logic*, vol. 11, num. 3, pp. 311-342, 1982.

Segerberg K., “Modal logics with linear alternative relations”, *Theoria*, vol. 36, num. 3, pp. 301-322, 1970.

Segerberg K., *An Essay in Classical Modal Logic, Filosofiska Studier, No.13*, Uppsala Universitet, Uppsala, 1971.

Smoryński C., *Consistency and related metamathematical properties, Mathematics Institute Technical Report 75-02*, Univ. of Amsterdam, 1975.

Smoryński C., *Self-Reference and Modal Logic, Universitext*, Springer-Verlag, New York, 1985.