

A Brief Proof of the Existence of an α -Core for Generalized Strategy Games *

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Abstract

This paper studies the problem of whether an α -core exists (is non-empty) for generalized normal-form games where player payoffs are described by point-to-set convex-valued, irreflexive, and weakly continuous preferences. We propose a short effective proof of the non-emptiness of the core; it based on a contradiction with Brouwer's theorem on the existence of a fixed point for a continuous mapping of a convex compact set into itself.

Keywords and Phrases: generalized strategic game, α -core, Brouwer's fixed point theorem

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1 Introduction

In game theory and its applications, the concept of core is one of the key notions of the “solution” for any type of game model: in cooperative games specified via a characteristic function and also in a strategic (normal) form. The core includes such admissible outcomes (profile of individual strategies) that are not blocked by (admissible) coalitions of players—taking into account possible opposition from the members of a complementary coalition.

In *Aumann* (1961) there was first proposed the concepts of the α -core and β -core for strategic games, with the former subsequently receiving extensive analysis in the literature. Essentially, if any coalition tries to change the game’s current social state, a complementary coalition can counter this attempt with a move that harms at least one member of the original coalition, thereby preventing the change because that member would lose. Thus, the counter-coalition destroys the cooperation of the opponents and counters the threat of changing the outcome of the game. The stability of outcomes from the β -core is much stronger—when repelling an attempt to change the outcome, the complementary coalition must have a counter-strategy that destroys the aggressor’s cooperation, common to all kinds of change options. Accordingly, the β -core is a subset of the α -core and has (much) less chance of existence.

The existence of α -core solutions for n -person normal-form games was proved by *Scarf* (1971) for games with quasi-concave payoff functions and a convex compact set of possible outcomes (strategy profiles). In *Kajii* (1992) Scarf’s existence theorem was generalized to encompass n -person normal-form games with general preferences. Moreover, these were games in which the strategy sets were convex compacta of a normed vector space, potentially infinite-dimensional. But this is also the main weakness of *Kajii* (1992)’s work — unlike finite-dimensional spaces, where a compactum is characterized as norm-bounded and closed, compactness in an infinite-dimensional normed space does not have such a simple characterization.¹ However, *Kajii*’s proof relies heavily on compactness, without which the method employed is ineffective. This shortcoming and a number of other weaknesses were overcome in *Martins-da-Rocha, Yannelis* (2011), where the transition to an infinite-dimensional Hausdorff space is accomplished using Bewley’s method: first, a finite-dimensional result is proved, and then the limit passage is performed over a network of finite-dimensional subspaces. The latter can be implemented using weak topologies using Alaoglu’s theorem, in which weakly star-compact sets are characterized in exactly the same way as in a finite-dimensional space: closed and bounded in the original topology of the space (with respect to the norm). In addition, *Martins-da-Rocha, Yannelis* (2011) defines the notion of the α -core itself, in which the requirements for coalition dominance are weakened in the context of the generalized game by restricting the ability of an additional coalition to influence the outcome of coalition blocking. The existence of an element of the α -core itself in the finite-dimensional setting is proved using the construction proposed by

¹For example, one can recall the Arzela-Ascoli theorem, which characterizes compacta in spaces of type $C(X)$, where $X \subset \mathbb{R}$ is compact.

Florenzano in *Florenzano* (1989) in the context of the Arrow—Debreu model of the economy, supplemented by one essential conclusion. In general, the question reduces to Kakutani’s theorem on the existence of a fixed point for the constructed (nontrivial) point-set mapping.

2 Generalized Strategic Games

We consider a cooperative game environment in the context of a generalized normal-form game with a finite set of players (agents) $\mathcal{I} = \{1, 2, \dots, n\}$. Here, any nonempty subset $S \subseteq \mathcal{I}$ is called a coalition of players; further, we assume that in the game there are defined a set of admissible coalitions $\mathbb{I}$, such that $\mathcal{I}$ and each single-element coalition $\{i\}$ belong to $\mathbb{I}$. Agent $i \in \mathcal{I}$ chooses an individual strategy $x_i \in X_i$ within its own set of admissible strategies X_i .

Assumption A1. *For each agent $i \in \mathcal{I}$, the set of strategies X_i is a nonempty and convex subset of the Hausdorff topological vector space L .*

We call an element $x = (x_i)_{i \in \mathcal{I}} \in X = \prod_{i \in \mathcal{I}} X_i$ a joint strategy (or strategic profile), and we consider it as a social state. If $S \subseteq \mathcal{I}$, then X^S denotes the Cartesian product $\prod_{i \in S} X_i$. For simplicity, we denote the set $\mathcal{I} \setminus S = -S$ in the standard way. We denote elements of joint strategies of coalition $S \subseteq \mathcal{I}$ as $y^S \in X^S$.

Assumption A2. *For each feasible coalition $S \in \mathbb{I}$, a nonempty compact set $F^S \subseteq X^S$ of feasible joint strategies exists; furthermore, the set $F = F^{\mathcal{I}}$ is convex.*

We consider the case where agents have (possibly unordered) preferences, in the presence of externalities (i.e., preference interdependence). Formally, each player i has a preference relation on X , which is described by a correspondence $\mathcal{P}_i$ from X to X . For any outcome $x \in X$, the set $\mathcal{P}_i(x)$ is the set of allocations $y \in X$ that are strictly preferred by player i to profile x . We make the standard assumption of convex irreflexivity of preferences.

Assumption A3. *For each $x \in X$ and $i \in \mathcal{I}$, $x \notin \text{co}\mathcal{P}_i(x)$ holds.*

Thus, the generalized game under study is represented by the following set of parameters

$$G = \langle L, (X_i, \mathcal{P}_i)_{i \in \mathcal{I}}, F, (F^S)_{S \in \mathbb{I}} \rangle.$$

Let F_i be the set of all individual strategies $z_i \in X_i$ of player i that can be applied in $z = (z_j) \in F^T$, where $T \in \mathbb{I}(i)$ is a feasible coalition containing i and distinct from the coalition of all players $\mathcal{I}$, i.e.

$$F_i = \{\pi_i(z) \mid z \in F^T, i \in T \in \mathbb{I}, T \neq \mathcal{I}\}.$$

Here $\pi_i(z)$ denotes a natural projection of $z = (z_j)_{j \in T} \in F^T$ into X_i , where $\pi_i(z) = z_i$. Thus, the set F_i includes all strategies that person i is capable of applying

in the case where he joins any coalition distinct from the total one.

Definition 1 A coalition $S \in \mathbb{I}$ **α -blocks** a strategic profile $x \in F$ if S has a joint feasible strategy $y^S \in F^S$ such that each member of S strictly prefers the outcome (state) (y^S, w^{-S}) to the outcome x , regardless of the reaction $w^{-S} = (w_j)_{j \notin S}$ of the complementary coalition, where each w_j belongs to the set $W_j = \text{co}F_j$.

The bundle $x \in F$ such that there is no coalition capable of α -blocking x is called the **α -core** of the game G .

The above definition can now be reformulated in the following way:

If S is a feasible coalition, then let P^S be a correspondence from X to X^S , defined as

$$P^S(x) = \{y^S \in X^S \mid y^S \times W^{-S} \subset \bigcap_{i \in S} \mathcal{P}_i(x)\},$$

where $y^S \times W^{-S} = \{(y^S, w) \mid w \in W^{-S}\}$. Following this notation, a feasible joint strategy $x \in F$ belongs to the α -core if and only if there is no feasible coalition $S \in \mathbb{I}$ such that $F^S \cap P^S(x) \neq \emptyset$.

Note that for every feasible coalition $S \in \mathcal{I}$ we have

$$F^{-S} \subseteq W^{-S} \subseteq X^{-S}.$$

Now we recall the notion of a “balanced” n -person game. It first appears in *Bondareva* (1962, 1963) and *Shapley* (1965) later discussed it in the context of a game with transferable utility (a game with side-payments). Scarf applied the notion of balancedness to a game in characteristic function form in *Scarf* (1967) and in *Scarf* (1971) he applied it to a game in normal form.

Let Δ be the set of sets $\lambda = (\lambda_S)_{S \in \mathbb{I}} \in \mathbb{R}_+^{\mathbb{I}}$ satisfying the condition

$$\forall i \in \mathcal{I}, \quad \sum_{S \in \mathbb{I}, i \in S} \lambda_S = 1 \quad \Longleftrightarrow \quad \sum_{S \in \mathbb{I}} \lambda_S \mathbf{1}_S = \mathbf{1}_{\mathcal{I}},$$

where $\mathbf{1}_S(i) = 1$ for $i \in S$ and $\mathbf{1}_S(i) = 0$ otherwise. The element $\lambda \in \Delta$ is called a balanced covering (or weights), and the family of coalitions $\{S \in \mathbb{I} \mid \lambda_S > 0\}$ associated with it is called balanced.

Definition 2 A generalized normal form game $G = \langle \mathcal{I}, \{X_i, \mathcal{P}_i(\cdot)\}_{i \in \mathcal{I}}, F, (F^S)_{S \in \mathbb{I}} \rangle$ is called **balanced** if for every balanced set $\lambda \in \Delta$, and every set of feasible strategies $y^S \in F^S$ with $S \in \mathbb{I}$ and $\lambda_S > 0$, the state $z = (z_i)_{i \in \mathcal{I}}$ defined as

$$\forall i \in \mathcal{I} \quad z_i = \sum_{S \in \mathbb{I}: i \in S} \lambda_S y_i^S, \tag{1}$$

is **feasible**, i.e., $z \in F$.

Finally, we will also assume a specific type of continuity of preferences.

Definition 3 A generalized game G is called α -continuous if for every feasible coalition $S \in \mathbb{I}$ the correspondence P^S has open lower sections in X , that is, for every $i \in S$ and every feasible strategy $y^S \in F^S$, the set $\{x \in X \mid y^S \times W^{-S} \subset \mathcal{P}_i(x)\}$ is open in X .

3 Theorem on Non-emptiness of α -core

Theorem 1 *Under assumptions A1-A3 if the game $G = \langle \mathcal{I}, \{X_i, \mathcal{P}_i(\cdot)\}_{i \in \mathcal{I}}, F, (F^S)_{S \in \mathbb{I}} \rangle$ is α -continuous and balanced, then α -core is nonempty, i.e. $\mathcal{C}_\alpha(G) \neq \emptyset$.*

Proof. By contradiction. Assuming $\mathcal{C}_\alpha(G) = \emptyset$ we show that does exist a continuous mapping $f : F \rightarrow F$ that has no fixed point, which contradicts Brouwer's theorem. We describe the construction of this mapping below.

The assumption $\mathcal{C}_\alpha(G) = \emptyset$ means that for each $x = (x_i)_{i \in \mathcal{I}} \in F$ there exists a coalition $S \in \mathbb{I}$ that blocks x , i.e.

$$\exists y^S \in \mathcal{P}^S(x) \cap F^S \neq \emptyset.$$

Moreover, by the assumption of α -continuity of G , for some $V(x) \subset X$ open in X , one has

$$x \in V(x) \ \& \ \forall x' \in V(x) \ y^S \in \mathcal{P}^S(x') \cap F^S.$$

Here $V(x)$ is an open neighborhood of $x \in F$. Since $V(x)$ form a covering of the compact set F , there is a finite subcovering $V_k = V(x^k)$, and the corresponding coalition $S_k \subseteq \mathcal{I}$, α -blocking every $x \in V_k$, $k = 1, 2, \dots, m$. Next, we consider the continuous decomposition of the identity corresponding to the family V_k , i.e., we find a set of continuous functions $\eta_k : F \rightarrow [0, 1]$ such that

$$\forall x \in F \ \sum_{k=1}^m \eta_k(x) = 1 \ \& \ \eta_k(x) > 0 \iff x \in V_k, \ k = 1, 2, \dots, m.$$

Next, for an arbitrary $z = (z^{S_k}, z^{-S_k}) \in \prod_{i \in \mathcal{I}} F^{\{i\}}$, we define on F the function $h(\cdot)$ by the formula

$$h(x) = \sum_{k=1}^m \eta_k(x) (y^{S_k}, z^{-S_k}).$$

The continuity of $h(\cdot)$ follows from the construction. Next, we specify

$$\theta(x) = \min_{i \in \mathcal{I}} \sum_{k: i \notin S_k} \eta_k(x) < 1$$

and consider

$$f(x) = \frac{h(x) - \theta(x)z}{1 - \theta(x)}.$$

The function $f(x) = (f_1(x), \dots, f_n(x))$ is the desired one; we will show that it has the required properties. One has

$$\forall i \in \mathcal{I} \quad f_i(x) = \sum_{k: i \in S_k} \lambda_k(x) y_i^{S_k} + \lambda_{m+i}(x) z_i,$$

where

$$\lambda_k(x) = \frac{\eta_k(x)}{1 - \theta(x)} \quad \& \quad \lambda_{m+i}(x) = \frac{\sum_{k: i \notin S_k} \eta_k(x) - \theta(x)}{1 - \theta(x)}.$$

By construction, the set $\lambda = \{\lambda_\xi\}_{\xi=1}^{m+n}$ forms a balanced set of weights for the family of coalition strategies $y^{S_k} \in F^{S_k}$, $k = 1, \dots, m$ and $z_i \in F^{\{i\}}$, $i = 1, \dots, n$. The assumption of a balanced game leads us to conclude that $f(x) \in F$.

Moreover, there exists an individual $i \in \mathcal{I}$ such that $\theta(x) = \sum_{k:i \notin S_k} \eta_k(x) \Rightarrow \lambda_{m+i} = 0$. Therefore, we can write the i th component of $f(x)$ as

$$f_i(x) = \sum_{k:i \in S_k} \lambda_k(x) y_i^{S_k},$$

and for other $j \in \mathcal{I}$, $j \neq i$ we have

$$f_j(x) = \sum_{k:j \in S_k} \lambda_k(x) y_j^{S_k} + \sum_{k:i \in S_k, j \notin S_k} \lambda_k(x) w_j(x),$$

where

$$w_j = \frac{\sum_{k:j \in S_k, i \notin S_k} \lambda_k(x) y_j^{S_k} + \lambda_{m+j}(x) z_j}{\sum_{k:j \in S_k, i \notin S_k} \lambda_k(x) + \lambda_{m+j}(x)} \in F_j.$$

Since $\sum_{k:j \in S_k, i \notin S_k} \lambda_k(x) + \lambda_{m+j}(x) = \sum_{k:j \notin S_k, i \in S_k} \lambda_k(x)$, then $f(x)$ is represented as a convex combination of vectors with numbers $k : i \in S_k$ such that for $w^{-S_k} = (w_j)_{j \notin S_k} \in W^{-S_k}$ we have

$$(y^{S_k}, w^{-S_k}) \in \bigcap_{j \in S_k} \mathcal{P}_j(x) \subseteq \mathcal{P}_i(x).$$

Therefore, $f(x) \in \text{co}\mathcal{P}_i(x)$. However, now if we assume that there is a fixed point $\bar{x} = f(\bar{x})$, then for this player, we get a contradiction with the assumed in A3 convex irreflexivity: $\bar{x} \notin \text{co}\mathcal{P}_i(\bar{x})$, which is what we had to prove. $\blacksquare$

The following example illustrates the principle underlying the construction of the function $f(x)$ that yields the desired contradiction.

Example 1 Consider a three-player game, and let $x = (x_1, x_2, x_3) \in F$ be an outcome that is blocked by coalitions $S_1 = \{1, 2\}$, $S_2 = \{1, 3\}$, and $S_3 = \{2, 3\}$. Suppose $\eta_1(x) = 1/2$, $\eta_2(x) = 1/3$ and $\eta_3(x) = 1/6$ are the coefficients of the unity decomposition, defining

$$h(x) = \frac{1}{2}(y_1^{S_1}, y_2^{S_1}, z_3) + \frac{1}{3}(y_1^{S_2}, z_2, y_3^{S_2}) + \frac{1}{6}(z_1, y_2^{S_3}, y_3^{S_3}) =$$

$$\left(\frac{1}{2}y_1^{S_1} + \frac{1}{3}y_1^{S_2} + \frac{1}{6}z_1, \frac{1}{2}y_2^{S_1} + \frac{1}{6}y_2^{S_3} + \frac{1}{3}z_2, \frac{1}{3}y_3^{S_2} + \frac{1}{6}y_3^{S_3} + \frac{1}{2}z_3 \right).$$

Here $z = (z_1, z_2, z_3) \in F^{\{1\}} \times F^{\{2\}} \times F^{\{3\}}$ is an arbitrarily chosen element. Next we find $\theta(x) = 1/6$ and specify

$$f(x) = \frac{h(x) - 1/6z}{1/2 + 1/3} = \left(\frac{3}{5}y_1^{S_1} + \frac{2}{5}y_1^{S_2}, \frac{3}{5}y_2^{S_1} + \frac{1}{5}y_2^{S_3} + \frac{1}{5}z_2, \frac{2}{5}y_3^{S_2} + \frac{1}{5}y_3^{S_3} + \frac{2}{5}z_3 \right) =$$

$$\frac{3}{5} \left(y_1^{S_1}, y_2^{S_1}, \frac{1}{3} y_3^{S_3} + \frac{2}{3} z_3 \right) + \frac{2}{5} \left(y_1^{S_2}, \frac{1}{2} y_2^{S_3} + \frac{1}{2} z_2, y_3^{S_2} \right).$$

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$$\left(y_1^{S_1}, y_2^{S_1}, \frac{1}{3} y_3^{S_3} + \frac{2}{3} z_3 \right) \in \mathcal{P}_1(x) \cap \mathcal{P}_2(x) \quad \& \quad \left(y_1^{S_2}, \frac{1}{2} y_2^{S_3} + \frac{1}{2} z_2, y_3^{S_2} \right) \in \mathcal{P}_1(x) \cap \mathcal{P}_3(x),$$

whence $f(x) \in \text{co}\mathcal{P}_1(x)$. Moreover, $f_1(x) = \frac{3}{5}y_1^{S_1} + \frac{2}{5}y_1^{S_2}$, $f_2(x) = \frac{3}{5}y_2^{S_1} + \frac{1}{5}y_2^{S_3} + \frac{1}{5}z_2$ and $f_3(x) = \frac{2}{5}y_3^{S_2} + \frac{1}{5}y_3^{S_3} + \frac{2}{5}z_3$. It follows that the balanced covering implementing $f(x)$ is the family $\{S_1, S_2, S_3, \{2\}, \{3\}\}$, implemented by the set of weights: $\lambda_{12} = 3/5$, $\lambda_{13} = 2/5$, $\lambda_{23} = 1/5$, $\lambda_2 = 1/5$, $\lambda_3 = 2/5$.

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