Perfect competition without Slater condition: the equivalence of non-standard and contractual approach

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Abstract

In the neoclassical Arrow–Debreu model under the conditions of perfect competition *every* allocation from the core allows price decentralization, i.e. it is an equilibrium allocation. Moreover, precisely the conditions, under which the core and equilibria coincide, are called perfect competition. However, in all known models of perfect competition the exact theorem on the coincidence of the core and equilibria is proved exclusively under the survival assumption, which implies that Slater's condition for consumer's problem is fulfilled. How important is this additional requirement and what will happen if it is discarded? The paper is addressed to this problem; the classical Debreu–Scarf approach is analyzed and is compared with the contractual model of perfect competition developed by the author. It is shown that the most accurate model of this is provided by the contractual approach; namely, this is a concept of fuzzy contractual allocation which provides stability with respect to the signing of a new contract and an asymmetric partial break of already existing ones. Under weak assumptions, it is proved that these allocations coincide with those with nonstandard prices. Generally, allocations implemented in this case are different from the elements of the classical core in perfect competition conditions (Edgeworth equilibria). However, in the case when model assumptions (irreducibility) provide the survival assumption for non-standard equilibria, the contractual approach coincides with the classical one.

Keywords and Phrases: competitive equilibrium, survival assumption (Slater condition), perfect competition, fuzzy core and fuzzy contractual allocations, Edgeworth equilibria.

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Introduction

In the neoclassical formulation, the concept of competitive equilibrium receives substantial support in the form of the fact that no group of agents (coalition) has incentives for the formation of an autonomous sub-economy (an equilibrium allocation belongs to the core, i.e., it is not dominated by any coalition). Moreover, in the conditions of perfect competition *every* allocation from the core allows price decentralization, i.e. is an equilibrium one (this is the well-known Edgeworth hypothesis). Thus, in the ideal world of the Arrow–Debreu economy, the competitive equilibrium, originally defined in purely descriptive categories, receives a normative justification as an ideally stable allocation (in this sense).

Modern points of view on perfect competition go back to the classical works of Edgeworth[1], which for a particular example of an economy with two products and two agents noticed and demonstrated the important fact that contract curve (core in modern terminology) compressing when the number of identical agents tends to infinite; he suggested that contract curve shrinks to equilibria. Fruitless attempts to develop this approach and give proof (rather than postulate) for the concept of perfect competition continued till the beginning of the second half of the 20th century, when G. Debreu and H. Scarf (*Debreu, Scarf*, 1963) first rigorously substantiated Edgeworth views. They proposed an approach based on the concept of a replicated economy: to the given model, its exact copies are added (the agents from the copy receive new names-indexes), which are combined into a common economy and this copying process rushes to infinity. The resource allocations of the original model replicated in this way are allocations in the replicated one. Debreu and Scarf are interesting in allocations that do not lose the key property of stability after replication — there is no group of agents for which it would be advantageous to abandon this allocation and, having organized an autonomous economy, find a better reallocation of resources. In the framework of standard assumptions (later weakened), it was proved that only the competitive equilibrium allocations have this kind of stability.

Later, in the theory of general equilibrium there are appeared other methods of modeling perfect competition. First of all it is Aumann's approach (*Aumann*, 1964), which later was also developed by a number of other authors; according to Aumann the economy under study has a measurable non-atomic space of economic agents. Here coalitions are represented as measurable subsets of agent space, their resources are defined as an integral over this set of (measurable) mapping specifying as initial endowments. Since the measure of any finite set is zero for a non-atomic measure, then the contribution of an individual to the coalition possibilities will also be zero. With the other standard assumptions Aumann proves that in this economy the core and equilibrium coincide. This is very nontrivial result, in the proofs of which one applies (with this purpose it was developed by Aumann) a complex theory of integration of point-to-set mappings (see also *Hildenbrand* (1986)). Thus, unlike Debreu and Scarf, where the asymptotic result was presented, Aumann works directly with the "limit economy", which presents the model of perfect competition, where a single individual is negligible and is not able to influence the choice of the final allocation.

The third method for modeling of perfect competition appeared in (*Brown, Robinson*, 1974). It is based on the application of methods of non-standard analysis and consists

[1] First of all it is *"Mathematical Psychics", 1881*

in examining of an economic model with a hyper-finite set of economic agents. Here it is postulated that the total number of economic agents is *infinite* natural $n \in {}^*\mathbb{N} \setminus \mathbb{N}$, and the norm of initial endowments $\mathbf{e}_i$ of the individual $i \in \{1, \dots, n\} = \mathcal{I}$ is bounded by a *finite* standard $m(i) \in \mathbb{N}$. Thus, the share of the individual i in the total resources of the economy is infinitesimal: $\mathbf{e}_i/n \approx 0$, $i \in \mathcal{I}$. The authors prove the coincidence of equilibria with allocations from the core. A detailed description of the non-standard approach can be found in (*Rashid*, 1987; *Anderson*, 1992). However, does such a complex and advanced mathematics really necessary to justify perfect competition?

The fourth and key method for this paper was proposed by me in the context of a contractual approach. The barter contract is any permissible exchange realized among the members of some coalition S of economic agents; it is formally represented by the vector $v = (v_i)_\mathcal{I}$, $\sum_S v_i = 0$, $v_i = 0\ \forall i \notin S$. Adding a contract v to a current allocation $x = (x_i)_\mathcal{I}$ forms a new allocation $z = (z_i)_\mathcal{I} = v + x = (v_i + x_i)_\mathcal{I}$. Contracts can be combined, broken off and concluded new ones (usually mutually beneficial). A finite set of contracts forms a web if after the break of any set of previously concluded contracts an acceptable allocation is formed. The concept of coalition domination is transferred to the domination between the webs of contracts, which makes it possible to qualify them according to stability. The possibility to break off contracts plays a key role here: it can be also partial break (replacement $v = (v_i)_\mathcal{I}$ by the contract $w = tv = (tv_i)_\mathcal{I}$, $0 \leq t \leq 1$) and even asymmetric (at the planning stage of new transactions, now no longer contract $w = (t_i v_i)_\mathcal{I}$, $0 \leq t_i \leq 1\ \forall i$). In (*Marakulin*, 2011, 2013, 2014) it is shown that the introducing of a partial break of contracts can be considered as a form of perfect competition conditions which are set for the model: contractual webs that are stable relative to the (simultaneous) conclusion of a new contract and the partial breaking of existing ones, implement equilibrium allocations. One can see that the actual method of perfect competition modeling is much simpler than those known in the literature. Now the equilibrium allocations and core allocations become objects of the same type: they are implemented by a web of contracts that is stable in a specific sense, and the stability of equilibrium is certainly stronger.

Under what conditions, in addition to the actual *construction* of economic model, the coincidence of the core and equilibria is realized—that is the implementation of the perfect competition conditions? In essence, these conditions are almost the same under which in modern literature the existence of equilibrium is proved: the convexity of the sets $\mathcal{P}_i(x_i)$ of all preferred consumption bundles provides a description of the core elements in the form of (nontrivial) quasiequilibrium—this is a surrogate of equilibrium such that $\exists p \neq 0 : py_i \geq px_i = p\mathbf{e}_i\ \forall y_i \in \mathcal{P}_i(x_i)$, $\forall i$. Now we have the result if the economy obeys conditions which ensure the transformation of quasiequilibrium into equilibrium. Usually this is the so-called survival assumption, which for (nontrivial) quasiequilibrium prices ensures the fulfillment of the Slater condition in the consumer problem: $\inf_{X_i} y_i p < \mathbf{e}_i p$, where X_i is the consumer set of the agent i (one also needs the openness of $\mathcal{P}_i(x_i)$). This provides most notably in the literature assuming of the irreducibility of the economy (see Section 5, Definition 8).

Thus, the coincidence of the core and equilibria under perfect competition and the fact of the existence of equilibrium are proved within the framework of the survival condition. However, for the existence of a core this condition does not play a role—its meaning is to ensure the existence of a standard equilibrium. At the same time, in its absence, there exist equilibria with non-standard prices introduced in (*Marakulin*, 1988, 2012). However, do these equilibria coincide in conditions of perfect competition with allocations from the core? This is a question to which we address in this paper. The following answer was obtained: in the economy without the survival condition equilibria with non-standard prices and only they can be described as fuzzy contractual allocations (allocations implemented by a net of contracts stable relative to the conclusion of a new mutually beneficial contract with simultaneous asymmetric break of current ones). Thus, it is established that the contractual model is the most accurate method for modeling of perfect competition.

The paper is organized as follows: in Section 1 I present a very short introduction to non-standard analysis; in Sec. 2 there are formulated the basic concepts of the general equilibrium theory which are applied in this paper: the model of the pure exchange economy, the notion of competitive equilibrium, the core, Edgeworth equilibria and their analogues and the concept of fuzzy contractual allocation. Section 3 is devoted to the analysis of the concept of equilibrium with non-standard prices; in Sec. 4 I introduce fuzzy contractual allocations and present their value characterization. Sec. 5 contains the main results about the equivalence of equilibria with nonstandard prices and fuzzy contractual allocations, as well as the analysis of Edgeworth equilibria. The conclusion ends the paper.

1 Methods of non-standard analysis

A non-standard analysis, sometimes also called as infinitesimal analysis, is a mathematical technique, rather than an independent theory. Non-standard analysis operates with ideal elements that can be both infinitely close to the object of interest or are infinitely far from it. In this section I will try to explain the meaning of this theory, although, of course, for a completely unfamiliar reader, it will not be easy to understand proofs. One can see details in (*Davis*, 1980; *Anderson*, 1992; *Loeb*, 2000).

Most of the nonstandard analysis applications are based on the idea of a hyperfinite set: this is a set, which can be enumerated by non-standard natural numbers, that are not exceeding some fixed non-standard natural one. Using this concept, one can approximate infinite (and even infinite-dimensional) objects by sets to which standard conclusions about the finite objects are applicable. In particular, in the economic-mathematical literature there is a variety of studies using the idea of a hyperfinite set[2]. Settings of this kind lead to the models with a hyperfinite set of economic agents, equipped with standard limited possibilities to influence the current situation (in the exchange model initial endowments are near-standard) (*Brown, Robinson*, 1974; *Rashid*, 1987; *Anderson*, 1992). There are also many other applications of nonstandard analysis methods to mathematical economics and game theory. Among them, the studies of models with overlapping generations, with infinite time horizon, economies with public goods, coalition games with "large" number of players, etc. Using the methods of nonstandard analysis, the problem of the existence of economic equilibrium is also solved if there is no Slater's condition (very unrealistic) in the customer's problem or any of its analogs (survival assumption) (*Marakulin*, 1988, 2012).

[2]Mainly these are models of "big" economy: with an infinite number of agents or products; this is a way to model the conditions of perfect competition—these are economies that contain "many" individuals, each of them has a negligible impact on the economy as a whole.

The theory of nonstandard analysis works with two structures: *standard universe* U and *nonstandard universe* ${}^*U$ that connects a one-to-one mapping (injection) $* : U \to {}^*U$. In addition, there is a formal language that is used to formulate different kinds of statements in each of these structures. It is shown that the set ${}^*U$ contains a huge number of new ideal objects, including infinitesimal and infinitely large numbers. The elements from ${}^*U$ of the form ${}^*x = *(x)$ for some $x \in U$ are called standard; the other elements of ${}^*U$ are called nonstandard ones. The set $S \subset {}^*U$ of elements of a nonstandard universe is called *internal* if S is itself an element of a nonstandard universe; otherwise S is called *external* one. External sets exist; already the set of natural numbers is an external one. The set ${}^*\mathbb{R}$ is a non-standard extension of real numbers and includes infinitesimal and infinitely large numbers having those properties of the numerical line that are expressed in a certain formal language. The value $\xi \in {}^*\mathbb{R}$ is infinitesimal if $|\xi| < 1/n$ for every natural $n \in \mathbb{N}$. The set of all such numbers is denoted $\mu(0)$ and is called the monad (of zero); $\mu(0) \ni \xi \iff \xi \approx 0$. The monad of $x \in {}^*\mathbb{R}$ is $\mu(x) \ni y \iff |x - y| \approx 0$. The standard part $\mathrm{st}(x) \in \mathbb{R}$ of a number $x \in {}^*\mathbb{R}$ is a real value such that $|\mathrm{st}(x) - x| \approx 0$. The numbers from ${}^*\mathbb{R}$ that have the standard part are called near-standard (for $\xi \approx 0$, $\xi \neq 0$ the value $1/\xi \approx \infty$ exists but has not a standard part). The notions of the monad, the standard part and so on are transferred to any topological spaces; the map $*(\cdot)$ preserves the vector and algebraic properties of the objects of standard mathematics.

There are three main tools (a kind of mathematical technique) of non-standard analysis. The first is *transfer principle*, which states that any statement, true in the standard universe, will be true in the nonstandard one, and vice versa. The second one is so called *concurrence theorem*. The theorem guarantees that the extended structure contains quite a lot of ideal elements and, in particular, it includes all possible completions, compactification, and so on. The third tool is the principle of proof from the contrary, assuming that a set is *internal* one. Appendix contains a summary of some of the results of the nonstandard analysis that are applied in our paper.

2 The equilibrium market model: non-standard prices or stable systems of contracts?

We study a standard model of the exchange economy (no production). In this model $L = \mathbb{R}^l$ denotes (finite-dimensional) *commodity space* and there is a (finite) set of agents (traders or consumers) $\mathcal{I} = \{1, \ldots, n\}$. The consumer $i \in \mathcal{I}$ is standardly characterized his/her consumption set $X_i \subset L$, a bundle of initial endowments $\mathbf{e}_i \in L$ and a preference relation described as a point-set mapping $\mathcal{P}_i : X_i \rightrightarrows X_i$, where the set $\mathcal{P}_i(x_i)$ presents the set of all consumption bundles strictly preferred by the agent i to the bundle x_i. We also use the notation $y_i \succ_i x_i$, which by definition is equivalent to $y_i \in \mathcal{P}_i(x_i)$. Thus, the exchange economy under study can be represented as a triple

$$\mathcal{E}^m = \langle \mathcal{I}, L, (X_i, \mathcal{P}_i, \mathbf{e}_i)_{i \in \mathcal{I}} \rangle.$$

We denote by $\mathbf{e} = (\mathbf{e}_i)_{i \in \mathcal{I}}$ the initial endowments vector of all traders of the model $\mathcal{E}^m$, put $\mathbb{X} = \prod_{i \in \mathcal{I}} X_i$ and define the set of all *feasible allocations* of the model $\mathcal{E}^m$:

$$\mathcal{A}(\mathbb{X}) = \{\, x \in \mathbb{X} \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i \,\}.$$

In the sequel we always assume that $\mathcal{E}^m$ satisfies the following assumption.

Assumption A. For each $i \in \mathcal{I}$: the set X_i is convex, closed, $\mathcal{P}_i(x_i) \cup \{x_i\}$ and $\mathcal{P}_i(x_i)$ are convex and $x_i \notin \mathcal{P}_i(x_i)$ for each $x_i \in X_i$.

By virtue of the assumption **A**, preferences can be satiated, i.e. $\mathcal{P}_i(x_i) = \emptyset$ is possible for some i and $x_i \in X_i$. However, if $\mathcal{P}_i(x_i) \neq \emptyset$, then the preferences *are locally non-satiated* at the point x_i. The most significant results of this paper are obtained for an economy model in which consumer sets are polyhedral (that is, they are an intersection of a finite number of half-spaces). Therefore, to simplify the exposition, we give the following definition.

We call the model $\mathcal{E}^m$ *polyhedral* if all sets X_i are polyhedral. In addition, we will usually assume that $\mathbf{e}_i \in X_i$ and $\mathcal{P}_i(x_i)$ are open in X_i $\forall i \in \mathcal{I}$. This model is a special case of the Arrow–Debreu economy; now let us recall the concepts of competitive and cooperative equilibria known in the literature, as well as the presentation of perfect competition.

2.1 Competitive equilibrium

Definition 1 *A couple (x,p), where $x \in \mathbb{X} = \prod_{\mathcal{I}} X_i$ and $p \neq 0$ is a linear functional on L, is said to be **quasiequilibrium** of economy $\mathcal{E}^m$, if x is feasible, i.e. $\sum_{\mathcal{I}} x_i = \sum_{\mathcal{I}} \mathbf{e}_i$, and*

$$\langle p, \mathcal{P}_i(x_i)\rangle \geq px_i = p\mathbf{e}_i, \quad \forall i \in \mathcal{I}. \tag{1}$$

A quasiequilibrium such that $x'_i \in \mathcal{P}_i(x_i)$ actually implies $px'_i > px_i$[3] *is called **Walrasian or competitive** equilibrium.*

Clearly that if one specifies for consumer $i \in \mathcal{I}$ its *budget set*:

$$B_i(p) = \{y \in X_i \mid py \leq p\mathbf{e}_i\},$$

then (in strict form of inequality!) condition (1) can be rewritten in a form $\mathcal{P}_i(x_i) \cap B_i(p) = \emptyset$. With all its merits, this classical concept of equilibrium has one essential shortcoming: it may not exist even if all the necessary (and "natural") assumptions are provided, ensuring continuity, convexity and compactness of all necessary model parameters. The reason for this is possible violation of Slater's condition in the consumer's problem: in order to provide it, additional strong requirements must be imposed. The problem of Slater's condition appears (albeit in a slightly different form) and in a number of other more general models, for example, for an economy with public goods. For this reason, in general equilibrium theory a number of generalized concepts are considered; one of the most productive approaches is the concept of equilibrium with nonstandard prices (*Marakulin*, 1988, 2012), a preliminary analysis of which is presented in this section.

In order to clarify the problem under study, we first consider one representative example borrowed from (*Marakulin* (1988, 2012), Example 1.2.1 page. 89), and then we will outline the formal concept of equilibrium with non-standard prices.

[3] Below it is often written as $\langle p, \mathcal{P}_i(x_i)\rangle > px_i$.

Example 1 Let us consider an economy $\mathcal{E}^m$ having two agents and two commodities (see Fig. 1), such that:

$$X_1 = \{(x,y) \mid (0,0) \le (x,y) \le (10,10)\}, \quad X_2 = X_1 \cap \{(x,y) \mid y \ge 4 - x\},$$

$$u_1(x,y) = 16 - (x-4)^2 - (y-4)^2, \quad u_2(x,y) = y, \quad \mathbf{e}_1 = (1,3), \quad \mathbf{e}_2 = (2,2).$$

Let us further consider the available possibilities for a finding of that could be called an equilibrium. At the prices $p = (p^x, p^y)$ such that $p^x \le 0$, the optimal reaction of agent 1 is such that his demand for the first good is ≥ 4, i.e. it is greater then total quantity of this good. This is why equilibrium or even semi-equilibrium[4] does not exist in this case. For $p^y \le 0$ agent 2 produces unreal demand for commodity y, it is: $y_2 = 10$. Therefore it has to be $p \gg 0$. If $p^y \gtrnsim p^x$ then optimal reaction of agent 2 coincides with $\mathbf{e}_2$, and for 1st agent it is so that $x_1 > 1$, i.e., again there is no balance even in the form of inequality. For $p^x \gtrnsim p^y$ optimal reaction of agent 2 is $(0, y_2)$, where $y_2 \ge 4$, but optimal reaction of agent 1 is so that summarized demand for second good is more than 5 and it is greater than supply. The latter follows from the fact that all bundles of (x_1, y_1), which are better than $\mathbf{e}_1$ and are so that $x_2 \le 1$ (to keep the balance), at these prices are outside the budget set of participant 1. At the prices $p = (1,1)$ as at non-standard prices $p = (1+\varepsilon, 1-\varepsilon)$, $\varepsilon \approx 0$, $\varepsilon > 0$, demand for good y exceeds supply. Prices $(1-\varepsilon, 1+\varepsilon)$ produce demand for x exceeding its supply. All cases where non-standard prices play a role, are exhausted by two considered above (due to the homogeneity of income). In other situations, the budget sets of participants at prices p and st$p$ coincide[5] (see *Marakulin* (2012), Proposition 1.1.5, p. 64). Thus, at any prices (including non-standard

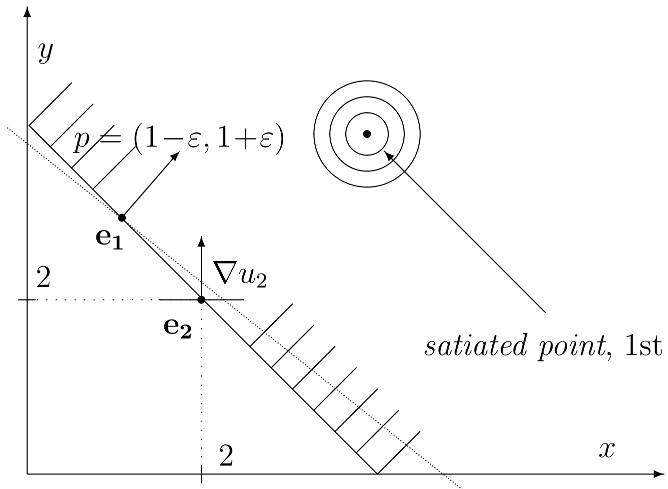

Figure 1: *Equilibrium with non-standard prices and transferable values.*

ones), the balance condition of the final allocation is violated. Nevertheless, there is a kind of non-standard equilibrium in this economy, it is an allocation $(x_1, y_1) = (2,2)$, $(x_2, y_2) = (1,3)$, which is implemented at the prices $p = (1-\varepsilon, 1+\varepsilon)$, $\varepsilon \approx 0$, $\varepsilon \ne 0$ and under an additional condition that one adds the value 2ε to the income of agent 2 (right hand side of budget constrain). In general, such construction is called an equilibrium

[4]Balance in the form of inequality: $\sum_{\mathcal{I}} x_i \le \sum_{\mathcal{I}} \mathbf{e}_i$.

[5]It is so because at prices $\bar{p} = \mathrm{st}p \ne 0$ *or* $(1,1)$ for consumer problem Slater's condition is fulfilled: $\inf_{X_i} \bar{p}x_i < \bar{p}\mathbf{e}_i$.

with non-standard prices and transferable values $\delta = (0, 2\varepsilon)$ (the equilibrium is also implemented for $\delta = (\varepsilon', 2\varepsilon)$ and any $\varepsilon \approx \varepsilon' \approx 0$, $\varepsilon > 0$, $\varepsilon' > 0$).

So, in order to determine a reasonable equilibrium allocation in an economy, we needed to resort to a very non-trivial value mechanism using non-standard prices and transferable values. However, the same allocation can be achieved if agents are allowed to exchange goods (and how can this be forbidden?). Indeed, in the (equilibrium) allocation $(x_1, y_1) = (2, 2)$, $(x_2, y_2) = (1, 3)$ the agent 1 only exchanged the unit of the first good for the unit of 2nd one from the agent 2, while both significantly increased their utility. This kind of operation is called the conclusion of a mutually beneficial barter contract, and now we are coming to the notion of a stable web (collection) of contracts that is introducing within the general theory of contracts (developed by the author). ■

In the modern theory of contracts (barter interaction), there are many types of stable contractual systems (webs), weak and strong, but the key point of this theory is the possibility of breaking contracts. In different settings, this break can be different: complete, partial, asymmetric partial and even the breaking contracts from an equivalent (in a sense) web of contracts. Previous studies show that for the modelling of equilibrium the most appropriate form of contractual interaction is the concept of fuzzy contractual allocation, the essence of which is that during contractual interaction at the time of conclusion of a new mutually beneficial contract an asymmetric partial breaking of contracts is allowed; allocations that are stable in this sense (i.e. this kind of manipulation can not be profitable) are called *fuzzy contractual* (see *Marakulin* (2011, 2013)). In the Section 2.3 this approach will be described formally. However, in which model framework does the equivalence between equilibrium and contractual allocations take place? Is there any difference between the allocations that correspond to "non-standard equilibria" and "fuzzy contractual" ones?

Now let us consider the formal concept of equilibrium with non-standard prices and transferable values in the simplest case of an exchange model with private ownership (*Marakulin*, 1988, 2012).

The following concepts and notations are regularly used in the paper: for *internal* subset $A \subset {}^*L$ there are specified the standard part and the standard interior, respectively,

$$\mathrm{st}A = \{y \in L \mid \mu(y) \cap A \neq \emptyset\}, \quad \mathrm{si}A = \{y \in L \mid \mu(y) \subset A \neq \emptyset\}.$$

Here $\mu(y)$ denotes the *monad* of point $y \in L$. It is known that st$A$ is closed and si$A$ is an open set (see *Marakulin* (1988), Proposition 2.14).

Given $p \in {}^*L$ and $\delta \in {}^*\mathbb{R}^{\mathcal{I}}$, $\delta \geq 0$ let us consider non-standard budget sets ${}^*B_i(p, \delta_i)$, specified via constrains $px' \leq p\mathbf{e}_i + \delta_i$, $x' \in {}^*X_i$, i.e. we define

$${}^*B_i(p, \delta_i) = \{x' \in {}^*X_i \mid px' \leq p\mathbf{e}_i + \delta_i\}, \quad i \in \mathcal{I}.$$

Notice that this is an internal subset of ${}^*X_i$. The vector $\delta = (\delta_1, \ldots, \delta_n) \geq 0$, specifying here the values added to the right hand sides of budget constrains, is called *scheme of redistribution for surplus values*.

Definition 2 *An allocation $x = (x_i)_{i \in \mathcal{I}} \in \mathbb{X}$ is said to be an **equilibrium with non-standard prices** $p \in {}^*L'$ under a **scheme of redistribution for surplus values** $\delta \in {}^*\mathbb{R}^{\mathcal{I}}$, $\delta \geq 0$, if the following conditions are fulfilled:*

(i) $x_i \in \mathrm{st}^*B_i(p, \delta_i) \ \ \forall i \in \mathcal{I}$;

(ii) $\mathcal{P}_i(x_i) \cap \mathrm{st}^*B_i(p, \delta_i) = \emptyset \ \ \forall i \in \mathcal{I}$;

(iii) $\sum_{\mathcal{I}} x_i = \sum_{\mathcal{I}} \mathbf{e}_i$.

For brevity, the triple (x, p, δ) is also called δ-equilibrium with nonstandard prices. An illustration of Definition 2 is given in presented above Example 1 of the exchange economy. The following theorem ensures the existence of nonstandard δ-equilibria.

Theorem 1. *Let $\mathcal{E}^m$ satisfies to assumption* $\mathbf{A}$, $\mathbf{e}_i \in X_i \ \forall i \in \mathcal{I}$ *and* $\mathcal{A}(\mathbb{X})$ *is bounded. Let in addition for each* $i \in \mathcal{I}$ *preference* $\mathcal{P}_i(\cdot)$ *has an open graph*[6] *in* $X_i \times X_i$. *Then for every* $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in {}^*\mathbb{R}^{\mathcal{I}}$ *such that* $\varepsilon \gg 0$, *there exists* δ*-equilibrium such that* $\delta = \tau \cdot \varepsilon$ *for a some non-standard* $\tau \geq 0$.

Example 1 and others show that for the existence of *standard* equilibria the assumptions given in Theorem 1 are not sufficient, since the assumption $\mathbf{e}_i \in X_i$ does not provide Slater's condition ($\inf_{X_i} px_i < p\mathbf{e}_i$) in the consumer problem for all potentially equilibrium (standard) prices $p \in L'$. However, non-standard ones do exist. Note that for a standard equilibrium with transferable values does exist, it is sufficient to postulate $\mathbf{e}_i \in \operatorname{int} X_i \ \forall i \in \mathcal{I}$. From a substantive point of view, this is of course an excessive requirement (actually means that each agent has non-zero stocks of each commodity). Details and proof of Theorem 1 can be found in (*Marakulin*, 2012).

Remark 1 The term "equilibrium with non-standard prices under a scheme of redistribution for surplus values" appeals to the fact that any equilibrium obeys the financial balance $\sum_{\mathcal{I}} px_i = \sum_{\mathcal{I}} p\mathbf{e}_i$ and, therefore, the agent i can spend more money than he has only if he/she receives money $px_i - p\mathbf{e}_i = \delta_i > 0$, transferred from agents who do not need them. In a standard setting they can only be satiated agents.

For the equilibrium with non-standard prices the situation is more delicate and the inequality $px_j < p\mathbf{e}_j$ may appear even if all individuals $j \in \mathcal{I}$ are non-satiated. This is possible in the case when the quantities $\delta_i > 0$ are infinitely small and formed from residual (excess!) values $\gamma_j = p\mathbf{e}_i - px_j > 0$ of individuals, for which any increase their utility is possible only for "infinitely larger" values (dividing it by γ_j one gets an infinitely large value). Thus, non-spent surplus values are somehow collected and transferred to those are interested in.

In Theorem 1 this redistribution can be performed among agents in any given proportions defined by the vector $\varepsilon \gg 0$. The fact that something is added to the budget of *each* agent is illusory: for satiated individuals this money is not spent, but somehow transferred to others, according to the "scheme" $\varepsilon \gg 0$. As a result, agents additionally get the values $\delta_i = \tau\varepsilon_i \geq 0$, where $\tau \geq 0$ can be interpreted as the price for excess financial resources. Theorem 1 says that such equilibrium does exist for any $\varepsilon \gg 0$ and, moreover, as it is proved in *Konovalov, Marakulin* (2006), for a fixed scheme of redistribution of surpluses one can guarantee (almost always) only a finite number of equilibria: otherwise infinitely many of them may exit. A corresponding example (stable with respect to variations of utility) is available in (*Konovalov, Marakulin*, 2006; *Marakulin*, 2012). ■

[6]For a mapping $\mathcal{P}_i : X_i \Rightarrow X_i$ the graph of it is $Gr(\mathcal{P}_i) = \{(x_i, y_i) \in X_i \times X_i \mid y_i \in \mathcal{P}_i(x_i)\}$.

2.2 Core, replicas and Edgeworth equilibrium

Let us recall that an allocation $x \in \mathcal{A}(\mathbb{X})$ *is dominated* (is blocked by) by a coalition (non-empty) $S \subset \mathcal{I}$, if there exists $y^S \in \prod_{i \in S} X_i$, such that $\sum_{i \in S} y_i^S = \sum_{i \in S} \mathbf{e}_i$ and $y_i^S \succ_i x_i$ for each $i \in S$.

Core of $\mathcal{E}^m$, denoted as $\mathcal{C}(\mathcal{E}^m)$, is the set of all $x \in \mathcal{A}(\mathbb{X})$ which can be dominated by no coalition.

The concept of the core of the economy formalizes the idea that there is no group of agents (coalition) that has incentives for the formation of an autonomous sub-economy: allocations from the core destroys such destructive cooperation and the economic system preserves integrity. It is well known in general equilibrium theory (it is easy to prove) that competitive equilibrium always belongs to the core and, consequently, possesses the same properties of cooperative stability.

However one needs to establish the reverse inclusion, i.e. it is required to identify the conditions under which for the economy *every element* of the core presents an equilibrium allocation, and this is the specification of perfect competition conditions.

The classical and historically first method of modeling perfect competition is based on the notion of a replicated model of the economy. In this way Gerard Debreu and Herbert Scarf firstly substantiated the Edgeworth conjecture, for which modern ideas of perfect competition are based on. In (*Debreu, Scarf*, 1963) it was proved that the core shrinks to equilibria if (roughly) the volume of the replica tends to infinity. Later, Debreu–Scarf approach was generalized and developed by many authors; moreover not only new methods for modeling perfect competition were developed, but also there were proved the most powerful theorems for the existence of equilibrium in economies with an infinite-dimensional commodity space, e.g. *Aliprantis et al.* (1989); *Marakulin* (2012). The basic design of Debreu–Scarf approach is described below.

Replica of the volume $r \in \mathbb{N}$ (r-fold replica) for the model $\mathcal{E}^m$ is a model $\mathcal{E}^m_r$, composed from r sub-economies, each of which coincides with the initial model $\mathcal{E}^m$. In other words, the parameters of the replicated model are obtained as r-times copying of parameters of the initial economy. More precisely, the model $\mathcal{E}^m_r$ includes consumers from $\mathcal{I}_r = \{(i,m)\}_{m=1, i \in \mathcal{I}}^{m=r}$ who have the property that their parameters (consumption sets, preferences, initial endowments, etc.) that have a double index (i,m), completely are determined by their "type" (first index) in the original model. Here consumers' preferences are also given by a point-to-set mapping $\mathcal{P}_{im}(\cdot) = \mathcal{P}_i(\cdot)$, but now for all $m = 1, 2, \ldots, r$ they are defined and taking values in X_{im} (potentially it can be $\mathbb{X}_m = \prod_{i \in \mathcal{I}} X_{im}$).

For an allocation $x = (x_i)_{\mathcal{I}}$ of the initial model $\mathcal{E}^m$ one can put into correspondence the allocation $x^r = (x^r_{im})_{\mathcal{I}_r}$ from the replica according to the rule $x_{im} = x_i \ \forall i, m$. Similarly, for an allocation $(x_{im})_{\mathcal{I}_r}$ from the replica, one can associate the allocation from initial model if $x_{im} = x_{im'}$ for all $i \in \mathcal{I}$, $m, m' = 1, \ldots, r$, i.e. when agents of the same type consume the same commodity bundles[7]. Following (*Aliprantis et al.*, 1989), now we define the main subject in the classical analysis of perfect competition, it is the concept of Edgeworth equilibrium.

Definition 3 *A feasible allocation $x \in \mathcal{A}(\mathbb{X})$ is called* ***Edgeworth equilibrium*** *if $x^r \in \mathcal{C}(\mathcal{E}^m_r)$ for every natural $r \geq 1$.*

The set of all Edgeworth equilibria is denoted by $\mathcal{C}^e(\mathcal{E}^m)$.

[7] In equilibrium theory this is known as the "equal treatment" property.

The concept of fuzzy core presents an effective tool for the analysis of perfect competition; the definition and main properties of it is given below. As we shall see later, this formally another concept implements allocations analogous to the Edgeworth equilibria, i.e. mathematically these concepts are equivalent. Recall that any vector

$$t = (t_1, \dots, t_n) \neq 0, \quad 0 \leq t_i \leq 1, \quad \forall i \in \mathcal{I}$$

is identified with the *fuzzy coalition*, where the real t_i is interpreted as a measure of participation of consumer i in this coalition.

Definition 4 *It is said that the fuzzy t coalition dominates (blocks) the allocation x in A if there is a $y^t \in \prod_{\mathcal{I}} X_i$, such that*

$$\sum_{i \in \mathcal{I}} t_i y_i^t = \sum_{i \in \mathcal{I}} t_i \mathbf{e}_i \iff \sum_{i \in \mathcal{I}} t_i (y_i^t - \mathbf{e}_i) = 0 \tag{2}$$

and wherein

$$y_i^t \succ_i x_i, \quad \forall i \in \operatorname{supp}(t) = \{i \in \mathcal{I} \mid t_i > 0\}. \tag{3}$$

*The set of all feasible allocations that are not blocked by fuzzy coalitions is denoted by $\mathcal{C}^f(\mathcal{E}^m)$ and is called the **fuzzy core** of economy $\mathcal{E}^m$.*

For non-satiated preferences, i.e. when $\mathcal{P}_i(x_i) \neq \emptyset$, $\forall i \in \mathcal{I}$, the fact of fuzzy domination can be rewritten in the equivalent form[8]:

$$0 \in \sum_{i \in \mathcal{I}} t_i (\mathcal{P}_i(x_i) - \mathbf{e}_i).$$

Thus, if $\mathcal{E}^m$ satisfies assumption **A** (convex preference), then the condition $x \in \mathcal{C}^f(\mathcal{E})$ is equivalent to[9]

$$0 \notin \operatorname{co}[\bigcup_{\mathcal{I}} (\mathcal{P}_i(x_i) - \mathbf{e}_i)], \tag{4}$$

which (applying separation theorem) implies that all elements of fuzzy core are quasiequilibria.

In *Marakulin* (2011) there was suggested another characterization of fuzzy core and the following lemma was proved.

Lemma 1 *Let Assumption* **A** *hold and $x = (x_i)_{\mathcal{I}} \in \mathcal{A}(\mathbb{X})$. Then $x \in \mathcal{C}^f(\mathcal{E}^m) \iff$*

$$\prod_{\mathcal{I}} \operatorname{co}(\mathcal{P}_i(x_i) \cup \{\mathbf{e}_i\}) \bigcap \{(z_1, \dots, z_n) \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\} = \{\mathbf{e}\}. \tag{5}$$

Note that the characterization (5) also takes place for satiated preferences. The result of the following lemma is well known in the theory of equilibrium, it reduces an analysis of Edgeworth's equilibria to the analysis of elements of the fuzzy core.

Lemma 2 *If Assumption* **A** *holds and $\mathcal{P}_i(x_i)$ are open in X_i $\forall i \in \mathcal{I}$, then*

$$\mathcal{C}^f(\mathcal{E}^m) = \mathcal{C}^e(\mathcal{E}^m).$$

[8] Allowing negligence, here and below in the formulas we identify the vector with the one-element set containing it.

[9] Since domination over arbitrary fuzzy coalitions is equivalent to dominating over normalized ones.

Proof of Lemma 2. First let us establish $\mathcal{C}^f(\mathcal{E}^{in}) \subseteq \mathcal{C}^e(\mathcal{E}^{in})$. Let us assume that for some r a coalition $S \subseteq \mathcal{I} \times \{1, \dots, r\}$ dominates an allocation x^r for $x \in \mathcal{C}^f(\mathcal{E}^m)$. Let $\mathcal{I}(S) \subseteq \mathcal{I}$ be a set of all types of agents that are nontrivially represented in the coalition S. By definition, domination means that for each $(i, m) \in S$ there is $y_{im} \in \mathcal{P}_i(x_i)$ such that

$$\sum_{(i,m) \in S} y_{im} = \sum_{(i,m) \in S} \mathbf{e}_{im}.$$

Now, if we "average" the dominating consumption bundles corresponding to each agent's type, i.e. if we put

$$y_i = \left(\sum_{m|(i,m) \in S} y_{im} \right) / s_i \quad \forall i \in \mathcal{I}(S),$$

where s_i is the number of elements (cardinality) of the set $S^i = \{m \mid (i, m) \in S\}$ (one has $i \in \mathcal{I}(S) \iff S^i \neq \emptyset$), then from the previous equations we obtain

$$\sum_{\mathcal{I}(S)} s_i y_i = \sum_{\mathcal{I}(S)} s_i \mathbf{e}_i.$$

Due to convexity of $\mathcal{P}_i(x_i)$, we also have $y_i \in \mathcal{P}_i(x_i)$ for all $i \in \mathcal{I}(S)$. Now define the vector $t = (t_1, \dots, t_n)$, specifying $t_i = s_i/r, \ i \in \mathcal{I}(S) \ \ \& \ \ t_i = 0, \ i \in \mathcal{I} \setminus \mathcal{I}(S)$. It is clear that in the preceding equality *natural numbers* s_i can be equivalently replaced by *rational* t_i. Thus, for Edgeworth's equilibria domination is impossible by fuzzy rational coalitions, but for fuzzy core elements for anyone. The inclusion is stated.

Next we state $\mathcal{C}^e(\mathcal{E}^m) \subseteq \mathcal{C}^f(\mathcal{E}^m)$. Assuming the opposite, find $x \in \mathcal{C}^e(\mathcal{E}^m)$, which is dominated by a fuzzy coalition $t \neq 0$. By definition there is $y^t \in \prod_{\mathcal{I}} X_i$ satisfying relations (2) and (3). We will show that the allocation x is dominated by a fuzzy coalition $q = (q_1, \dots, q_n)$ with *rational* components q_i, $i \in \mathcal{I}$. To this end, for $t_i > 0$ we set

$$x'_i = (t_i/q_i) y_i + (1 - t_i/q_i) \mathbf{e}_i \implies q_i(x'_i - \mathbf{e}_i) = t_i(y_i - \mathbf{e}_i),$$

where *rational* q_i satisfy the condition $t_i \leq q_i \leq 1$, and for $t_i = 0$ we define $q_i = 0$ and $x'_i = y^t_i$. Since $\mathbf{e}_i \in X_i$, by construction we have $x' = (x'_i)_{\mathcal{I}} \in \prod_{\mathcal{I}} X_i$ and

$$\sum_{i \in \mathcal{I}} q_i(x'_i - \mathbf{e}_i) = 0.$$

However, by virtue of the openness of $\mathcal{P}_i(x_i)$, the values q_i can be chosen so that $x'_i \in \mathcal{P}_i(x_i)$ be true for all i satisfying $q_i > 0$. Now we have a contradiction with the choice of $x \in \mathcal{C}^e(\mathcal{E}^m)$. Thus, the fuzzy core coincides with the set of Edgeworth equilibria. ■

2.3 Cooperative equilibrium as a fuzzy contractual allocation

In (*Marakulin*, 2011) another "fuzzy" concept was proposed and studied, this is the notion of fuzzy contractual allocation. Here we present it in a somewhat simplified form (sufficient for our purposes), avoiding detailed explanations and terminology of the contractual approach. As before, let $t = (t_1, \dots, t_n)$, $0 \leq t_i \leq 1 \ \forall i \in \mathcal{I}$ be a vector similar to the fuzzy coalition (allowing $t = 0$) and $x \in \mathcal{A}(\mathbb{X})$ is a feasible allocation to which the gross contract $x-\mathbf{e}=v=(v_i)_{i\in\mathcal{I}}$ (net trade) corresponds. It is assumed that the agents of the economy can (fuzzy and asymmetrically) break contract $v=(v_i)_{i\in\mathcal{I}}$, decreasing the individual consumption (fragment) from this contract in shares $(1-t_i)_{i\in\mathcal{I}}$, forming a tuple[10]

$$v^t=(t_1v_1,t_2v_2,\ldots,t_nv_n)$$

of commodity bundles, which can be used in subsequent exchange transactions together with the initial endowments. After the conclusion of a new contract $w^S=(w_i)_{\mathcal{I}}\in L^{\mathcal{I}}$, $\sum_{\mathcal{I}}w_i=0$ by a coalition $S\subseteq\mathcal{I}$ ($i\notin S\Rightarrow w_i=0$) they yield (possibly unreachable!) "allocation"

$$\xi(t,v,w)=w+v^t+\mathbf{e}=(w_1+t_1v_1^t+\mathbf{e}_1,\ldots,w_n+t_nv_n^t+\mathbf{e}_n).$$

Definition 5 *An allocation $x\in\mathcal{A}(\mathbb{X})$ is called **fuzzy contractual** if for every $t=(t_i)_{i\in\mathcal{I}}$, $0\le t_i\le 1$, $\forall i\in\mathcal{I}$ there is no barter contract $w=(w_1,\ldots,w_n)\in\mathbb{R}^{ln}$, $\sum_{\mathcal{I}}w_i=0$, such that for $x-\mathbf{e}=v$ and*

$$\xi_i=\xi_i(t,v,w)=w_i+t_iv_i+\mathbf{e}_i,\quad i\in\mathcal{I} \tag{6}$$

$$\xi_i\succ_i x_i\quad\forall i:\ \xi_i\neq x_i. \tag{7}$$

hold. The set of all fuzzy contractual allocations is denoted $\mathcal{FC}(\mathcal{E}^m)$.

Note that by virtue of (7) $w=0$ is permissible, i.e. only partial breaking of contracts is possible. Denying the possibility of such domination means that the contract web is *proper* and the allocation is *stable* with respect to asymmetric *partial break* of contracts. The following characteristic lemma can be directly produced from Definition 5, see *Marakulin* (2011).

Lemma 3 *An allocation $x\in\mathcal{A}(\mathbb{X})$ is fuzzy contractual if and only if*[11]

$$\mathcal{P}_i(x_i)\cap[x_i,\mathbf{e}_i]=\emptyset\ \ \forall i\in\mathcal{I} \tag{8}$$

and

$$\prod_{\mathcal{I}}[(\mathcal{P}_i(x_i)+[0,\mathbf{e}_i-x_i])\cup\{\mathbf{e}_i\}]\bigcap\{(z_i)_{\mathcal{I}}\in L^{\mathcal{I}}\mid\sum_{i\in\mathcal{I}}z_i=\sum_{i\in\mathcal{I}}\mathbf{e}_i\}=\{\mathbf{e}\}. \tag{9}$$

Condition (8) indicates that a partial break of contracts without signing of a new one cannot be beneficial, i.e. $\{x-\mathbf{e}\}$ is a proper web. The requirement (9) denies the existence of a dominating coalition after the partial asymmetric break of the contract $v=(x-\mathbf{e})$. The statement of Lemma 3 can be reformulated in a different form, convenient for further analysis.

Corollary 1 *An allocation $x\in\mathcal{A}(\mathbb{X})$ is fuzzy contractual if and only if (8) is true and*

$$\prod_{S}(\mathcal{P}_i(x_i)+[0,\mathbf{e}_i-x_i])\bigcap\{(z_i^S)_S\in L^S\mid\sum_S z_i^S=\sum_S\mathbf{e}_i\}=\emptyset\ \ \forall S\subseteq\mathcal{I}. \tag{10}$$

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[10]This is not a contract, because its key property $\sum_{\mathcal{I}}t_iv_i=0$ is violated.

[11]*A linear* segment with ends $a,b\in L$ is the set $[a,b]=\mathrm{co}\{a,b\}=\{\lambda a+(1-\lambda)b\mid 0\le\lambda\le 1\}$.

We note that for every $x = (x_i)_\mathcal{I} \in \mathbb{X}$ via Assumption **A** we have

$$\mathbf{e}_i \in \mathrm{co}(\mathcal{P}_i(x_i) \cup \{\mathbf{e}_i\}) \subset (\mathcal{P}_i(x_i) + [0, \mathbf{e}_i - x_i]) \cup \{\mathbf{e}_i\}, \quad \forall i \in \mathcal{I},$$

whence, by the Lemma 1 and (5), we conclude directly that every fuzzy contractual allocation belongs to fuzzy core. The following statement partly justifies the inverse relationship between two "fuzzy" concepts.

Lemma 4 *Let Assumption* **A** *hold,* $x \in \mathcal{A}(\mathbb{X})$ *and* $\mathcal{P}_i(x_i) \neq \emptyset$ *be true for all* $i \in \mathcal{I}$*. Then* $x \in \mathcal{C}^f(\mathcal{E})$ *implies:*

$$\prod_\mathcal{I}(\mathcal{P}_i(x_i) + [0, \mathbf{e}_i - x_i]) \bigcap \{(z_1, \ldots, z_n) \in L^\mathcal{I} \mid \sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}} \mathbf{e}_i\} = \emptyset. \tag{11}$$

Proof of Lemma 4. One needs to show that (5) implies (11). Suppose that x satisfies (5), but (11) is false. Then there are a vector $t = (t_i)_{i\in\mathcal{I}}$, $0 \leq t_i \leq 1$ and bundles $z_i \succ_i x_i$, $i \in \mathcal{I}$ such that

$$\sum_\mathcal{I} z_i + \sum_\mathcal{I} t_i(\mathbf{e}_i - x_i) = \sum_\mathcal{I} \mathbf{e}_i. \tag{12}$$

Next consider the vector $y = y(\beta) = (y_i)_{i\in\mathcal{I}}$, defined for a real $0 < \beta \leq \frac{1}{2}$ as

$$y_i(\beta) = \beta[z_i + t_i(\mathbf{e}_i - x_i)] + (1-\beta)x_i, \quad i \in \mathcal{I}.$$

Due to (12) and $x \in \mathcal{A}(\mathbb{X})$, for every β we have $\sum_\mathcal{I} y_i(\beta) = \sum_\mathcal{I} \mathbf{e}_i$. Further, the vectors $y_i(\beta)$ can be represented as

$$y_i(\beta) = (1-\beta t_i)x_i + \beta t_i \mathbf{e}_i + (1-\beta t_i)\frac{\beta}{1-\beta t_i}(z_i - x_i), \quad i \in \mathcal{I},$$

where by choice of β we have $\mu_i = \frac{\beta}{1-\beta t_i} \leq 1$. By virtue of Assumption **A** for $i \in \mathcal{I}$ the last implies

$$\mu_i(z_i - x_i) \in \mathcal{P}_i(x_i) - x_i \;\Rightarrow\; \exists \eta_i \in \mathcal{P}_i(x_i) : \; \mu_i(z_i - x_i) = \eta_i - x_i.$$

Consequently, from the preceding formula we conclude

$$y_i = (1-\beta t_i)\eta_i + \beta t_i \mathbf{e}_i,$$

which implies $y_i \in \mathrm{co}(\mathcal{P}_i(x_i) \cup \{\mathbf{e}_i\})$, $i \in \mathcal{I}$. Now we can apply (5), concluding $y = y(\beta) = \mathbf{e}$ for *all* real $0 < \beta \leq \frac{1}{2}$. We now write this equality componentwise and by definition of $y_i(\beta)$ obtain

$$\beta[z_i + t_i(\mathbf{e}_i - x_i)] + (1-\beta)x_i = \mathbf{e}_i \;\Rightarrow\; z_i + t_i(\mathbf{e}_i - x_i) = x_i + \frac{\mathbf{e}_i - x_i}{\beta},$$

which must be fulfilled for all $i \in \mathcal{I}$ and *all* $0 < \beta \leq \frac{1}{2}$. However, these equalities are true for *different* β, which is possible only if $x_i = \mathbf{e}_i = z_i$, $i \in \mathcal{I}$, that by the choice of z_i implies $x_i \succ_i x_i$, but this contradicts to Assumption **A** (preferences are irreflexive). ■

3 Equilibrium with non-standard prices: characterization and properties

Now we consider a useful specification of non-standard equilibrium, which essentially simplifies the theoretical construction and is used in the subsequent analysis. In the case of *standard* market equilibrium with transferable values, consumer sets of individuals satisfy the budget inequality $\langle p, z\rangle \leq \langle p, \mathbf{e}_i\rangle + \delta_i$. Moreover, for locally non-satiated preferences for every equilibrium plan x_i, budget equality $px_i = p\mathbf{e}_i + \delta_i$ holds, which allows us to rewrite the restriction in the form

$$\langle p, z\rangle \leq \max\{\langle p, \mathbf{e}_i\rangle, \langle p, x_i\rangle\}, \quad i \in \mathcal{I}. \tag{13}$$

Now constraints of the form (13) can be applied to define the concept of generalized equilibrium. Obviously, the converse is also true: if we define the values as $\delta_i = \max\{\langle p, \mathbf{e}_i\rangle, \langle p, x_i\rangle\} - \langle p, \mathbf{e}_i\rangle \geq 0$, then the equilibrium on the basis of constraints (13) is transformed into a standard equilibrium with transferable values.

In the case of *non-nonstandard* equilibrium, the situation is not so trivial. Next, we study this construction in detail.

Definition 6 *A permissible allocation $x \in \mathbb{X}$ of economy $\mathcal{E}^m$ is called an* ***equilibrium with nonstandard prices*** $p \in {}^*L'$, *if it obeys:*

(i) $\mathcal{P}_i(x_i) \cap \mathrm{st}\{z \in {}^*X_i \mid \langle p, z\rangle \leq \max\{\langle p, \mathbf{e}_i\rangle, \langle p, x_i\rangle\} = \emptyset \quad \forall i \in \mathcal{I}$,

(ii) $\sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} \mathbf{e}_i$.

In this definition, the budget constraint has the form (13), but already with non-standard prices and consumption plans, and it corresponds to the following budget sets:

$$B_i^x(p) = \mathrm{st}\{z \in {}^*X_i \mid \langle p, z\rangle \leq \max\{\langle p, \mathbf{e}_i\rangle, \langle p, x_i\rangle\}\}, \quad p \in {}^*L', \quad i \in \mathcal{I}.$$

As before, the value on the right hand side of the budget constraint includes the value $\delta_i = \max\{\langle p, \mathbf{e}_i\rangle, \langle p, x_i\rangle\} - \langle p, \mathbf{e}_i\rangle \geq 0$, which non-satiated individuals received from the satiated ones. However, now this can be not only usual satiation, but also non-standard: satiation can occur in markets of "high value level", but with negligible residual value, which is transferred to the lower level of "poverty" (there may be several ones).

To establish the equivalence of two non-standard concepts: given via $B_i^x(p)$ and, alternatively, through $\mathrm{st}^*B_i(p, \delta_i)$, it is enough to show that $B_i^x(p) \subseteq \mathrm{st}^*B_i(p, \delta_i)$ for any given δ_i and $x_i \in \mathrm{st}^*B_i(p, \delta_i)$. However, the problem is that, unlike the standard case, for non-standard budget sets and for convex and closed X_i

$$z \in \mathrm{st}^*B_i(p, \delta_i) \quad \& \quad pz > p\mathbf{e}_i + \delta_i$$

is potentially possible. In *Marakulin* (2012), Example 1.2.2, p. 97, this kind of situation is analyzed and described formally. Hence, for $z \in \mathrm{st}^*B_i(p, \delta_i)$ the inclusion

$$\{y \in {}^*X_i \mid \langle p, y\rangle \leq \langle p, z\rangle\} \subseteq \{y \in {}^*X_i \mid \langle p, y\rangle \leq \langle p, \mathbf{e}_i\rangle + \delta_i\}$$

in general is *false*. However it will take place under additional assumptions, but only *after standardization* of right and left hand parts of inclusion. This is the subject of immediate consideration.

Now I recall some important facts from the general theory of equilibrium with non-standard prices. The first is a specific decomposition of the vector of non-standard prices via a standard basis, see *Marakulin* (2012), Lemma 1.1.2.

Lemma 5 *For every $p \in {}^*L'$, $p \neq 0$ there is the only system of orthonormal standard vectors $\{e_1, \ldots, e_k\}$ from $L' = \mathbb{R}^l$, such that*

$$p = \lambda_1 e_1 + \cdots + \lambda_k e_k, \quad \lambda_j \in {}^*\mathbb{R}, \quad j = 1, \ldots, k. \tag{14}$$

Here coefficients $\lambda_j > 0$ and obey the relations

$$\lambda_{j+1}/\lambda_j \approx 0, \quad j = 1, \ldots, k-1.$$

The system of vectors indicated in the lemma is unique, but it may not be a basis although, if necessary, it can be supplemented to a basis. The structure of budget sets was also studied in *Marakulin* (1988, 2012): based on the representation of the non-standard functional described in the Lemma 5 in the form (14), there was obtained a complete description of budget sets with non-standard prices and transferable values.

Using Lemma 5 and representation (14), for a given $p \in {}^*L' = {}^*\mathbb{R}^l$, we assign to a non-standard value $\gamma > 0$ natural number $j = 1, \ldots, k+1$ such that

$$\gamma/\lambda_{j-1} \approx 0 \;\; \& \;\; \gamma/\lambda_j \not\approx 0. \tag{15}$$

For definiteness, in (15) we set $\lambda_0 = +\infty$ and $\lambda_{k+1} = 0$. According to the statement of Lemma 5 such number $j = j(p, \gamma)$ is well defined. We set

$$\mu = \mathrm{st}(\gamma/\lambda_j) \;\; \text{for} \;\; \gamma/\lambda_j \not\approx \infty.$$

Theorem 2 (Marakulin 2012, Theorem 1.1.9) *Let $X \subset L$ be a convex polyhedral set, $\mathrm{w} \in X$, $p \in {}^*L'$, and a non-standard $\gamma > 0$ and natural $j = j(p, \gamma)$ are related by (15). Let*

$$B^{\mathrm{w}}(p, \gamma) = \mathrm{st}\{z \in {}^*X \mid \langle p, z \rangle \leq \langle p, \mathrm{w} \rangle + \gamma\}.$$

Then one of the following alternatives is true:

(*i*) $B^{\mathrm{w}}(p, \gamma) = X$ *for* $\gamma/\lambda_1 \approx +\infty$*;*

(*ii*) *there is $m < j$ such that*
$B^{\mathrm{w}}(p, \gamma) = \{x \in X \mid e_r y = e_r \mathrm{w},\ r < m,\ e_m y \leq e_m \mathrm{w}\} = B(m, p, \mathrm{w})$ *and there is $y \in X$ such that $e_m y < e_m \mathrm{w}$;*

(*iii*) $B^{\mathrm{w}}(p, \gamma) = \{x \in X \mid e_r y = e_r \mathrm{w},\ r < j\}$ *for* $\gamma/\lambda_j \approx +\infty$*;*

(*iv*) $B^{\mathrm{w}}(p, \gamma) = \{x \in X \mid e_r y = e_r \mathrm{w},\ r < j,\ e_j y \leq e_j \mathrm{w} + \mu\}$ *for* $\gamma/\lambda_j \not\approx +\infty$.

Moreover, in cases (*ii*) $-$ (*iv*)*:*

$$\nexists y \in X: \;\; e_r y = e_r \mathrm{w},\ r < s,\ e_s y < e_s \mathrm{w} \quad \forall 1 \leq s < m.$$

This theorem was established by Konovalov, A. V., it generalizes a similar result from *Marakulin* (1988) (proved for $\gamma = 0$, see *Marakulin* (2012), Theorem 1.1.8). Note that case $\gamma = 0$ is also described in the conclusion of Theorem 2, it corresponds to alternative (ii) and, for $j = k+1$, to (iii). In general, Theorem 2 represents a nontrivial result, in the proof of this the assumption for X to be polyhedral is essential (a counterexample is available in *Marakulin* (2012)). Now the inclusion we need can be obtained as a corollary of this theorem.

Remark 2 Theorem 2 reveals the "internal arrangement" of the budget set with a nonstandard prices $p \in {}^*\mathbb{R}^l$ and a transferable value $\gamma \geq 0$. In substantional terms: the first group of constraints $e_r y = e_r \mathrm{w}$, $r < m \leq j$, given by the functionals e_r, represented in decomposition (14) of the non-standard vector $p \in {}^*\mathbb{R}^l$, specifies a face of the polyhedron X. Within this face the "last" budget constraint $e_m y \leq e_m \mathrm{w} + \mu$, for which Slater's condition (for the found face) is satisfied (this is the first such situation). All non-essential elements are discarded in this construction: in the decomposition of p these are terms of the form $\lambda_r e_r$ for $\lambda_r/\gamma \approx 0$. In addition, extreme cases are also described: the case when $\gamma > 0$ is "too large"—(i), and when the "Slater" execution does not occur at all—(iii). ■

Corollary 2 *Under the conditions of Theorem 2, for any* $z \in B^{\mathrm{w}}(p,\gamma)$

$$\mathrm{st}\{y \in {}^*X \mid \langle p, y\rangle \leq \langle p, z\rangle\} \subseteq B^{\mathrm{w}}(p,\gamma).$$

Proof of Corollary 2 . Applying Theorem 2 for $\gamma = 0$ in the case of (ii), we have:

$$\exists q < j : \quad \mathrm{st}\{y \in {}^*X \mid \langle p, y\rangle \leq \langle p, z\rangle\} = \{x \in X \mid e_r y = e_r z,\ r < q,\ e_q y \leq e_q z\},$$

and moreover there is $y \in X$ such that $e_q y < e_q z$. Since $z \in B^{\mathrm{w}}(p,\gamma)$, then by Theorem 2 we have $q \geq m$ and

$$e_r z = e_r \mathrm{w},\ r < m, \quad \& \quad e_m z \leq e_m \mathrm{w} \quad \textbf{\textit{or}} \quad e_j z \leq e_j \mathrm{w} + \mu \ \textit{for}\ \gamma/\lambda_j \not\approx +\infty.$$

Now the previous formula in each alterative $(i) - (iv)$ of Theorem 2 yields the desired result. ■

Corollary 3 *Under the conditions of Theorem 2 if* $x \in B^{\mathrm{w}}(p,\gamma)$, $\mathcal{P}(x) \cap B^{\mathrm{w}}(p,\gamma) = \emptyset$ *and if* $\mathcal{P}(x) \neq \emptyset$ *is open in* X*, then*

$$\langle p, \mathcal{P}(x)\rangle > \langle p, x\rangle.$$

Proof. Let us apply Corollary 2 and rewrite $\mathcal{P}(x) \cap \mathrm{st}\{y \in {}^*X \mid \langle p, y\rangle \leq \langle p, x\rangle\} = \emptyset$ in the form

$$\mu(z) \cap \{\xi \in {}^*X \mid \langle p, \xi\rangle \leq \langle p, x\rangle\} = \emptyset \quad \forall z \in \mathcal{P}(x).$$

Since $z \in \mu(z)$ then $pz \leq px$ is impossible, that implies $pz > px$ $\forall z \in \mathcal{P}(x)$. ■

Now applying Corollary 2 we directly establish the equivalence of two definitions of nonstandard equilibrium for *polyhedral* economies.

Theorem 3 *Let* ***polyhedral*** *economy* $\mathcal{E}^m$ *satisfy* ($\mathbf{A}$). *Then an allocation* $x \in \mathcal{A}(\mathbb{X})$ *is equilibrium with non-standard prices* $p \in {}^*L'$ *and (some) transferable values* $\delta \in {}^*\mathbb{R}^{\mathcal{I}}_+$ *if and only if*

$$\mathcal{P}_i(x_i) \cap \mathrm{st}\{y \in {}^*X_i \mid py \leq \max\{\langle p, \mathbf{e}_i\rangle, \langle p, x_i\rangle\}\} = \emptyset \quad \forall i \in \mathcal{I}.$$

Remark 3 The fact that Theorem 3 deal with the polyhedral economies is important in the presented proof and it can not be replaced by a weaker requirement of convexity. The overall solution has not been found yet, but there are a number of particular options, which allows us to confidently hope that in the future this restriction can be removed. ■

Proof of Theorem 3. It was already noted earlier that for

$$\delta_i = \max\{\langle p, \mathbf{e}_i\rangle, \langle p, x_i\rangle\} - \langle p, \mathbf{e}_i\rangle \geq 0$$

the equilibrium according to Definition 6 becomes an equilibrium by Definition 2. The converse follows from

$$\mathrm{st}\{y \in {}^*X_i \mid \langle p, y\rangle \leq \langle p, x_i\rangle\} \subseteq B_i^{\mathbf{e}_i}(p, \delta_i) = \mathrm{st}^*B_i(p, \delta_i) -$$

by Corollary 2 and the trivial

$$\mathrm{st}\{y \in {}^*X_i \mid \langle p, y\rangle \leq \langle p, \mathbf{e}_i\rangle\} \subseteq \mathrm{st}^*B_i(p, \delta_i).$$

Consequently (since the union of the left-hand sides is contained in the right-hand side),

$$\mathrm{st}\{y \in {}^*X_i \mid \langle p, y\rangle \leq \max\{\langle p, x_i, \langle p, \mathbf{e}_i\rangle\}\} \subseteq \mathrm{st}^*B_i(p, \delta_i).$$

As a result, by (*ii*) of Definition 2, we have the desired result. ■

4 Fuzzy contractual allocation and its value characterization

In this section, we examine the characterization of fuzzy contractual allocation in the form of an equilibrium with non-standard prices. In the subsequent analysis, the sets of the following construction will play an important role. Let $x = (x_i)_{\mathcal{I}} \in \mathcal{A}(\mathbb{X})$ be an allocation of $\mathcal{E}^m$ and let $A_i \subset \mathcal{P}_i(x_i)$ be any *finite* subset, $i \in \mathcal{I}$. Define

$$D_i = \mathrm{co}[(\mathrm{co}A_i + [0, \mathbf{e}_i - x_i]) \cup \{\mathbf{e}_i\}]^{12}.$$

Due to assumption $\mathbf{A}$ we have $\mathrm{co}A_i \subset \mathcal{P}_i(x_i)$, that implies

$$D_i \subset (\mathcal{P}_i(x_i) + [0, \mathbf{e}_i - x_i]) \cup \{\mathbf{e}_i\}, \quad i \in \mathcal{I}.$$

Therefore by Lemma 3 and requirement (9), for fuzzy contractual x we have:

$$\prod_{\mathcal{I}} D_i \bigcap \{(z_i)_{\mathcal{I}} \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\} = \{\mathbf{e}\}. \tag{16}$$

[12] If $\mathcal{P}_i(x_i) = \emptyset$ then $A_i = \emptyset$ and, therefore, $D_i = \{\mathbf{e}_i\}$.

Lemma 6 *Let Assumption* **A** *hold and* $x \in \mathcal{A}(\mathbb{X})$ *be a fuzzy contractual allocation. Then* $\mathbf{e}_i$ *is the vertex in* $D_i \subset \mathbb{R}^l$ $\forall i \in \mathcal{I}$ *and* $\mathbf{e}$ *is the vertex in* $\prod_{\mathcal{I}} D_i$.

Proof. First we establish $\mathbf{e}_i \notin (\text{co}A_i + [0, \mathbf{e}_i - x_i])$. Assuming contrary one finds $y_i \in \text{co}A_i$ and $\lambda \in [0, 1]$ such that

$$\mathbf{e}_i = y_i + \lambda(\mathbf{e}_i - x_i) \quad \Rightarrow \quad y_i = \lambda x_i + (1 - \lambda)\mathbf{e}_i = \lambda(x_i - \mathbf{e}_i) + \mathbf{e}_i$$

and therefore $y_i \in [x_i, \mathbf{e}_i] \cap \mathcal{P}_i(x_i)$, i.e. this consumption can be achieved by breaking of existing contracts in volume $(1 - \lambda)$, that contradicts (8).

It is proved that $C_i = \text{co}A_i + [0, \mathbf{e}_i - x_i]$ is a bounded polyhedron such that $\mathbf{e}_i \notin C_i$. By virtue of the (second) classical separation theorem, there is a linear functional h (vector) such that for some real γ

$$\langle h, \mathbf{e}_i \rangle < \gamma < \langle h, C_i \rangle.$$

It follows that the hyperplane defined by the equation $hz = h\mathbf{e}_i$, $z \in \mathbb{R}^l$ is supporting for D_i and intersects it at a single point $\mathbf{e}_i$. This characterizes $\mathbf{e}_i \in D_i$ as a vertex (face of zero dimension). ■

Lemma 7 *Let Assumption* **A** *hold and* $x \in \mathcal{A}(\mathbb{X})$ *be a fuzzy contractual allocation. Then there exists* $p \in \mathbb{R}^l$ *such that*

$$\langle p, \text{co}A_i \rangle > \max\{\langle p, \mathbf{e}_i \rangle, \langle p, x_i \rangle\} \quad \forall i \in \mathcal{I} : \quad \mathcal{P}_i(x_i) \neq \emptyset.$$

In proving this lemma, we need a specific theorem on the separability of polyhedra, see *Marakulin* (2012), Proposition 1.1.1, p. 26.

Proposition 1 (Marakulin, 2001) *Let* $\{A_j\}_{j=1}^{j=k}$ *be a collection of convex polytops*[13] *in a liner space* E*, which satisfies*

$$\bigcap_{j=1}^{k} A_j = \{x\}.$$

Let F_{A_j} *be a face of* A_j *such that* $x \in \text{ri}F_{A_j}$*,*[14] $j = 1, \dots, k$. *Then there is a set of linear standard functionals* $\{f_j\}_{j=1}^{j=k}$ *not each of them is equal to zero, which obey*

$$\sum_{j=1}^{k} f_j = 0$$

and are such that $f_j(A_j) \geq f_j(x)$ *and* $F_{A_j} = \{y \in A_j \mid f_j(y) = f_j(x)\}$ *for all* $j = 1, \dots, k$.

[13] This set represented as convex hull of a finite number of points.

[14] This means that F_{A_j} is a face of minimal dimension which includes the point x.

Proof of Lemma 7. Let us apply Proposition 1 to relation (16), where two sets are intersected. Now we conclude the existence of a nonzero vector $f \in (\mathbb{R}^l)^{\mathcal{I}}$ such that for $\mathcal{A}(L^{\mathcal{I}}) = \{(z_i)_{\mathcal{I}} \in L^{\mathcal{I}} \mid \sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}} \mathbf{e}_i\}$

$$\langle f, \prod_{\mathcal{I}} D_i\rangle \geq \langle f, \mathbf{e}\rangle \quad \& \quad \langle -f, \mathcal{A}(L^{\mathcal{I}})\rangle \geq \langle -f, \mathbf{e}\rangle$$

holds and equality in inequalities is realized only for points from the minimal face containing element $\mathbf{e}$. However $\mathbf{e}$ is a vertex in $\prod_{\mathcal{I}} D_i$, and $\mathcal{A}(L^{\mathcal{I}})$ is affine space. Thus due to Proposition 1 we conclude: $\forall i \in \mathcal{I} \mid \mathcal{P}_i(x_i) \neq \emptyset$

$$\langle f, \prod_{\mathcal{I}} D_i \setminus \{\mathbf{e}\}\rangle > \langle f, \mathbf{e}\rangle \quad \& \quad \langle f, \mathcal{A}(L^{\mathcal{I}})\rangle = \langle f, \mathbf{e}\rangle.$$

Moreover by definition of $\mathcal{A}(L^{\mathcal{I}})$ the last equality entails that vector $f = (f_1, \ldots, f_n) \in (\mathbb{R}^l)^{\mathcal{I}}$ representing separation functional obeys $f_i = f_j = p \neq 0$ $\forall i, j \in \mathcal{I}$. Due to this fact the first inequality gives: $\forall i \in \mathcal{I} \mid \mathcal{P}_i(x_i) \neq \emptyset$,

$$\langle p, D_i\setminus\{\mathbf{e}_i\}\rangle + \sum_{j\in\mathcal{I}, j\neq i} \langle p, D_j\rangle > \sum_{j\in\mathcal{I}} \langle p, \mathbf{e}_j\rangle \;\Rightarrow\; \langle p, D_i\setminus\{\mathbf{e}_i\}\rangle + \sum_{j\in\mathcal{I}, j\neq i} \langle p, \mathbf{e}_j\rangle > \sum_{\mathcal{I}} \langle p, \mathbf{e}_j\rangle \;\Rightarrow$$

$$\langle p, \mathrm{co}A_i + [0, \mathbf{e}_i - x_i]\rangle > \langle p, \mathbf{e}_i\rangle \quad\Rightarrow\quad \langle p, \mathrm{co}A_i\rangle > \max\{\langle p, \mathbf{e}_i\rangle, \langle p, x_i\rangle\},$$

as we wanted to prove. ■

A key theorem of this paper will be proved with the help of so called concurrence theorem from the theory of non-standard analysis. Let U be an universe of standard mathematics and ${}^*U$ be a non-standard one. Now we recall the concept of *concurrent* relation.

Definition 7 *A binary relation $r \in U$ is called **concurrent in** U (or just **concurrent**), if whenever $a_1, \ldots, a_k \in$ $\mathrm{dom}(r)$ there is an element b, such that $(a_i, b) \in r$ for all $i = 1, \ldots, k$.*

Theorem 4 (CONCURRENCE THEOREM) *Let r be a concurrent relation in U. Then there is an element $b \in {}^*U$ such that $({}^*a, b) \in {}^*r$ for all $a \in \mathrm{dom}(r)$.*

So, for each finite subset A of $\mathrm{dom}(r)$, the definition of concurrence yields an element $q_A \in U$ such that $(a, q_A) \in r$ for all $a \in A$. Intuitively, concurrence theorem yields an (non-standard) element b, that is a sort of "limit" of these q_A as A "approaches" $\mathrm{dom}(r)$.

Theorem 5 *Let $\mathcal{E}^m$ be a polyhedral economy, $x = (x_i)_{\mathcal{I}} \in \mathcal{A}(\mathbb{X})$, Assumption* **A** *be true and $\mathcal{P}_i(x_i)$ be open in X_i, $\forall i \in \mathcal{I}$. If x is fuzzy contractual then there is a non-standard $p \in {}^*\mathbb{R}^l$ such that*

$$\mathcal{P}_i(x_i) \cap \mathrm{st}\{y \in {}^*X_i \mid py \leq \max\{\langle p, \mathbf{e}_i\rangle, \langle p, x_i\rangle\}\} = \emptyset \quad \forall i \in \mathcal{I}. \tag{17}$$

Proof of Theorem 5. Let us define a *concurrence relation* $\mathcal{U}$. Let $\mathbb{A} = \{A_1, \dots, A_n\}$ be any collection of *finite* (possibly empty) subsets $A_i \subset \mathcal{P}_i(x_i)$. Due to Lemma 7 the following statement is true:

$$\forall i \in \mathcal{I}\ \forall A_i \subset \mathcal{P}_i(x_i), A_i \neq \emptyset, |A_i| < +\infty \ \ \exists p \in \mathbb{R}^l \mid \langle p, A_i \rangle > \max\{\langle p, \mathbf{e}_i \rangle, \langle p, x_i \rangle\}.$$

Let $p(\mathbb{A})$ denote a liner functional that correspond to collection $\mathbb{A} = \{A_i\}_{\mathcal{I}}$ and satisfy Lemma 7 conclusion. Now define relation $\mathcal{U}$ as a set of all pairs $(\mathbb{A}, p(\mathbb{A}))$ of mentioned form, i.e. we specify

$$\mathcal{U} = \{(\mathbb{A}, p) \mid \mathbb{A} = \{A_i\}_{i=1}^{i=n},\ |A_i| < \infty,\ A_i \subset \mathcal{P}_i(x_i),\ p = p(\mathbb{A})\}.$$

The *concurrence* property of the relation $\mathcal{U}$ is easily verified. Indeed, for a finite family $\{\mathbb{A}^t\}$ from $\operatorname{dom}\mathcal{U}$ one can put $\widehat{A}_i = \cup_t A_i^t$ and applying Lemma 7, find a vector $\widehat{p}$, that corresponds to the bundle $\widehat{\mathbb{A}} = \{\widehat{A}_i\}_{\mathcal{I}}$. It is easy to see that $(\widehat{\mathbb{A}}, \widehat{p}) \in \mathcal{U}$. So, $\mathcal{U}$ is concurrence relation and by concurrence theorem there is a non-standard vector ${}^*p$ such that $(\mathbb{A}, {}^*p) \in {}^*\mathcal{U}$ for every $\mathbb{A} \in \operatorname{dom}\mathcal{U}$.

Notice that by specification of $\mathcal{U}$ vector ${}^*p$ satisfies to Lemma 7 for every $\mathbb{A} \in \operatorname{dom}\mathcal{U}$ and as soon as one can add to A_i any point from $\mathcal{P}_i(x_i)$ and this does not lead us outside of $\operatorname{dom}\mathcal{U}$, then

$$\langle {}^*p, y \rangle > \max\{\langle {}^*p, \mathbf{e}_i \rangle, \langle {}^*p, x_i \rangle\} \ \ \forall y \in \mathcal{P}_i(x_i) \neq \emptyset.$$

This relation can be rewritten in the form

$$\mathcal{P}_i(x_i) \cap \{y \in {}^*X_i \mid \langle {}^*p, y \rangle \leq \max\{\langle {}^*p, \mathbf{e}_i \rangle, \langle {}^*p, x_i \rangle\}\} = \emptyset \ \ \forall i \in \mathcal{I}.$$

Pay your attention that in this intersection left set $\mathcal{P}_i(x_i)$ is external and the right is internal one. This does not allow us to immediately obtain the result applying operation si(.) and st(.).[15]

Consider an arbitrary $y \in \mathcal{P}_i(x_i)$. It was assumed that $\mathcal{P}_i(x_i)$ is open in $X_i \subset \mathbb{R}^l$ and X_i are polyhedral sets. Therefore there is polyhedral neighbourhood of y in $\mathcal{P}_i(x_i)$, i.e. there is a finite set $A_i \subset \mathcal{P}_i(x_i)$ such that $\operatorname{co}A_i$ is a neighborhood y in X_i. But then from the construction

$${}^*\operatorname{co}A_i \cap \{y \in {}^*X_i \mid \langle {}^*p, y \rangle \leq \max\{\langle {}^*p, \mathbf{e}_i \rangle, \langle {}^*p, x_i \rangle\}\} = \emptyset,$$

that in view of $\mu(y) \subset {}^*\operatorname{co}A_i$, yields

$$\mu(y) \cap \{z \in {}^*X_i \mid \langle {}^*p, z \rangle \leq \max\{\langle {}^*p, \mathbf{e}_i \rangle, \langle {}^*p, x_i \rangle\}\} = \emptyset \ \ \Rightarrow$$
$$y \notin \operatorname{st}\{z \in {}^*X_i \mid \langle {}^*p, z \rangle \leq \max\{\langle {}^*p, \mathbf{e}_i \rangle, \langle {}^*p, x_i \rangle\}\},$$

as we wanted to prove. ■

5 Perfect competition: the equivalence of non-standard and contractual approach

In the previous sections, theorems characterizing perfect competition in the economy without a survival assumption were obtained. Recall that $x = (x_i)_{\mathcal{I}} \in \mathcal{A}(\mathbb{X})$ is called an *equilibrium with non-standard prices* if there are non-standard $p \in {}^*\mathbb{R}^l$ such that

$$\mathcal{P}_i(x_i) \cap \operatorname{st}\{y \in {}^*X_i \mid py \leq \max\{\langle p, \mathbf{e}_i \rangle, \langle p, x_i \rangle\}\} = \emptyset \ \ \forall i \in \mathcal{I}.$$

[15] It easy to see that for any internal sets $A \cap B = \emptyset \ \Rightarrow \ \operatorname{si}(A) \cap \operatorname{st}(B) = \emptyset$.

Theorem 6 *Let $\mathcal{E}^m$ be a polyhedral economy, $x = (x_i)_{\mathcal{I}} \in \mathcal{A}(\mathbb{X})$, Assumption* **A** *be true and $\mathcal{P}_i(x_i)$ be open in X_i, $\forall i \in \mathcal{I}$. Then x is fuzzy contractual allocation if and only if when this is the allocation for an equilibrium with non-standard prices.*

Combining this result with Theorem 1 (the existence of non-standard equilibria), we directly conclude

Corollary 4 *If under the assumptions of Theorem 6 additionally $\mathcal{P}_i(\cdot)$ has an open graph in $X_i \times X_i$ and $\mathbf{e}_i \in X_i$ $\forall i \in \mathcal{I}$, then $\mathcal{FC}(\mathcal{E}^m) \neq \emptyset$—fuzzy contractual allocations do exist.*

Proof of Theorem 6. In Theorem 5 it has been established that each fuzzy contractual allocation is an equilibrium with non-standard prices. Let us establish the opposite conclusion. Let $x \in \mathcal{A}(\mathbb{X})$ be a non-standard equilibrium. One needs to check properties (8), (9), characterising fuzzy contractual allocations. As $[x_i, \mathbf{e}_i] \subset \mathrm{st}\{y \in {}^*X_i \mid py \leq \max\{\langle p, \mathbf{e}_i\rangle, \langle p, x_i\rangle\}\}$, then the truth of (8) follows directly from the definition of equilibrium. Condition (9) requires verification, which can be carried out by reasoning from the contrary.

Suppose (9) is false, so there is $y = (y_i)_{i\in\mathcal{I}} \neq (\mathbf{e}_i)_{i\in\mathcal{I}} = \mathbf{e}$ from the left side of the intersection (9). Form a coalition

$$S = \{i \in \mathcal{I} \mid y_i \neq \mathbf{e}_i\} \neq \emptyset.$$

By construction (9) yields

$$y_i \in (\mathcal{P}_i(x_i) + [0, \mathbf{e}_i - x_i]) \;\; \forall i \in S \;\; \& \;\; \sum_S y_i = \sum_S \mathbf{e}_i.$$

This implies

$$\forall i \in S \; \exists \lambda_i \in [0,1] : \quad z_i = y_i + \lambda_i(x_i - \mathbf{e}_i) \in \mathcal{P}_i(x_i).$$

Definition of equilibrium yields $pz_i > \max\{p\mathbf{e}_i, px_i\}$ $\forall i \in S$ (because the other is impossible). On the other hand

$$pz_i = py_i + \lambda_i(x_i - \mathbf{e}_i)p \;\; \forall i \in S \quad \Rightarrow$$

$$\sum_S pz_i = \sum_S py_i + \sum_S \lambda_i p(x_i - \mathbf{e}_i) = \sum_S p\mathbf{e}_i + \sum_S \lambda_i p(x_i - \mathbf{e}_i) \leq \sum_S \max\{p\mathbf{e}_i, px_i\},$$

that contradicts to the previous conclusion. ■

In the general case of the economy $\mathcal{E}^m$ without the Slater condition, the fuzzy core, and hence the Edgeworth equilibria are characterized by the following

Theorem 7 *Let $\mathcal{E}^m$ be a polyhedral economy, $x = (x_i)_{\mathcal{I}} \in \mathcal{A}(\mathbb{X})$, Assumption* **A** *be true and $\mathcal{P}_i(x_i)$ be open in X_i, $\forall i \in \mathcal{I}$. Then $x \in \mathcal{C}^f(\mathcal{E}^m) = \mathcal{C}^e(\mathcal{E}^m)$ if and only if there is non-standard prices $p \in {}^*\mathbb{R}^l$ such that*

$$\langle p, \mathcal{P}_i(x_i)\rangle > \langle p, \mathbf{e}_i\rangle \quad \Longleftrightarrow \quad \mathcal{P}_i(x_i) \cap \mathrm{st}\{y \in {}^*X_i \mid py \leq \langle p, \mathbf{e}_i\rangle\} = \emptyset \;\; i \in \mathcal{I}. \tag{18}$$

Note that the equivalence in (18) also requires proof.

Proof of Theorem 7. The general logic of the proof of necessity (18) follows the proof of Theorem 5, but instead of D_i one applies sets

$$\Upsilon_i = \mathrm{co}(A_i \cup \{\mathbf{e}_i\}), \quad A_i \subset \mathcal{P}_i(x_i), \ |A_i| < +\infty, \quad i \in \mathcal{I},$$

which due to Lemma 1 and (5) and since $\{\mathbf{e}_i\}$ is a vertex in ${\Upsilon_i}^{16}$, allow us to reduce the problem to the application of the separation theorem from the Proposition 1. Further, similarly to Lemma 7, we conclude the existence of a vector $f = (f_1, \dots, f_n) \in (\mathbb{R}^l)^{\mathcal{I}} \neq 0$ such that

$$\langle f, \prod_{\mathcal{I}} \Upsilon_i \setminus \{\mathbf{e}\}\rangle > \langle f, \mathbf{e}\rangle \quad \& \quad \langle f, \mathcal{A}(L^{\mathcal{I}})\rangle = \langle f, \mathbf{e}\rangle.$$

Her the last equality implies $f_i = f_j = p \neq 0 \ \forall i, j \in \mathcal{I}$. With this in mind, the first inequality gives: $\forall i \in S = \{i \in \mathcal{I} \mid \mathcal{P}_i(x_i) \neq \emptyset\}$

$$\langle p, \Upsilon_i \setminus \{\mathbf{e}_i\}\rangle + \sum_{j \in \mathcal{I}, j \neq i} \langle p, \Upsilon_j\rangle > \sum_{j \in \mathcal{I}} \langle p, \mathbf{e}_j\rangle \ \Rightarrow \ \langle p, \Upsilon_i \setminus \{\mathbf{e}_i\}\rangle + \sum_{j \in \mathcal{I}, j \neq i} \langle p, \mathbf{e}_j\rangle > \sum_{\mathcal{I}} \langle p, \mathbf{e}_j\rangle \ \Rightarrow$$

$$\langle p, \lambda\, \mathrm{co} A_i + (1-\lambda)\mathbf{e}_i\rangle > \langle p, \mathbf{e}_i\rangle, \quad \forall \lambda \in (0,1] \quad \Longleftrightarrow \quad \langle p, \mathrm{co} A_i\rangle > \langle p, \mathbf{e}_i\rangle \ \ \forall i \in S.$$

Now applying the concurrence theorem in exactly the same way as in Theorem 5, we conclude that there exists non-standard $p' \in {}^*L'$ such that

$$\langle p', \mathcal{P}_i(x_i)\rangle > \langle p', \mathbf{e}_i\rangle \ \ \forall i \in S.$$

Since for $\mathcal{P}(x_i) = \emptyset$ this relation holds automatically, we can think that it is established for all $i \in \mathcal{I}$. The proof of the equivalence in (18) exactly repeats the reasoning of Theorem 5.

The sufficiency of (18) is stated immediately. In fact if $t = (t_i)_{\mathcal{I}}$ is a dominating fuzzy coalition then there exist $y_i \in X_i$ such that

$$y_i \succ_i x_i \ \ \forall i \in \mathcal{I}: \ t_i > 0 \ \ \& \ \ \sum_{\mathcal{I}} t_i(y_i - \mathbf{e}_i) = 0.$$

Now scalar multiplying the left-hand side of the equation by the price vector $p \in {}^*\mathbb{R}^l$, satisfying (18) via $py_i > p\mathbf{e}_i$ we find

$$\langle p, \sum_{\mathcal{I}} t_i(y_i - \mathbf{e}_i)\rangle > 0,$$

that is impossible. ■

Theorems reffc=nonst and 7 reveal the difference between the classical model of perfect competition and contractual one. Indeed, every fuzzy contractual allocation is Edgeworth's equilibrium, but when is the contractual approach equivalent to the classical one? It will be so if the economy is irreducible.

[16]Allocations from the core can not be dominated by singleton coalitions, so $\{\mathbf{e}_i\} \notin \mathrm{co} A_i \subset \mathcal{P}_i(x_i)$.

Definition 8 *An economy* $\mathcal{E}^m$ *is called* ***irreducible****, if for every feasible* $x = (x_i)_{i\in\mathcal{I}} \in \mathcal{A}(\mathbb{X})$ *and a nontrivial partition* $\mathcal{I} = S \cup T$, $S \cap T = \emptyset$ *there is a reallocation of resources of the complementary coalition* $T \neq \emptyset$, $z = (z_i)_{\mathcal{I}} \in L^{\mathcal{I}}$, *such that*

$$\sum_S z_i = \sum_T z_j \;\; \& \;\; \forall j \in T: \; \mathbf{e}_j - z_j \in X_j \;\; \&$$

$$\forall i \in S: \; x_i + z_i \in \operatorname{cl} \mathcal{P}_i(x_i) \;\; \& \;\; \exists i \in S: \; x_i + z_i \in \mathcal{P}_i(x_i).^{17}$$

The last formula says that the group T is able not only to provide resources in which are interested members of the group S, but somehow survive: the supply of commodities is possible to realize. Note also that the concept of an irreducible model includes *local non-satiation* for each economic agent.

Theorem 8 *Let* $\mathcal{E}^m$ *be a polyhedral economy,* $x = (x_i)_{\mathcal{I}} \in \mathbb{X}$, *Assumption* **A** *hold and* $\mathcal{P}_i(x_i)$ *be open in* X_i, $\forall i \in \mathcal{I}$. *Let also* $\sum_{\mathcal{I}} \mathbf{e}_i \in \operatorname{int} \sum_{\mathcal{I}} X_i$ *and* $\mathcal{E}^m$ *be irreducible economy. Then alternatives* $(i) - (iv)$ *are equivalent:*

(i) *x is an equilibrium with non-standard prices,*

(ii) *x is a fuzzy contractual allocation,*

(iii) *x is an element of fuzzy core,*

(iv) *x is an Edgeworth equilibrium.*

Remark 4 In the part of the coincidence of the classical competitive equilibria and Edgeworth equilibria, this is well known result in the theory of general equilibrium, which goes back to the pioneer work (*Debreu, Scarf*, 1963)[18]. We presented it here exclusively for the completeness. The standard logic is the following one: one firstly establishes the quasi-equilibrium properties for the elements of fuzzy core and then, using irreducibility and an internal point, shows that each equilibrium turns into a normal competitive equilibrium. Moreover, now the assumption of polyhedrality can be removed and instead of Theorem 7 we can use the relation (4) or (5) so that, using the separation theorem immediately to find quasi-equilibrium prices. The assumption $\sum_{\mathcal{I}} \mathbf{e}_i \in \operatorname{int} \sum_{\mathcal{I}} X_i$ can be interpreted as a condition that economy has no fictitious products. ■

Proof of Theorem 8. If we have non-standard prices $p \in {}^*\mathbb{R}^l$, e.g. specified in Theorems 6 and 7, we can define $\bar{p} = \operatorname{st}(p/||p||)^{19}$ and now for contractual model we shall have:

$$\forall i \in \mathcal{I} \;\; \langle \bar{p}, \mathcal{P}_i(x_i) \rangle \geq \max\{\langle \bar{p}, x_i \rangle, \langle \bar{p}, \mathbf{e}_i \rangle\},$$

and for the classical one

$$\langle \bar{p}, \mathcal{P}_i(x_i) \rangle \geq \langle \bar{p}, \mathbf{e}_i \rangle.$$

[17] It is often postulated $z_j = 0 \; \forall j \neq i, \; j \in S$.

[18] Here it is established under substantially weaker assumptions.

[19] The case $p = 0$ considered separatelly (trivial) but st(p/——p——) exists because of the compactness of the sphere of unit radius and by the nonstandard criterion of compactness.

If $\mathcal{P}_i(x_i) \neq \emptyset$, then due to $x_i \in \operatorname{cl}\mathcal{P}_i(x_i)$ (by Assumption **A**) the last inequality gives $\langle \bar{p}, x_i \rangle \geq \langle \bar{p}, \mathbf{e}_i \rangle$. Now from $\sum_{\mathcal{I}} x_i = \sum_{\mathcal{I}} \mathbf{e}_i$ we conclude $\langle \bar{p}, x_i \rangle = \langle \bar{p}, \mathbf{e}_i \rangle \ \forall i \in \mathcal{I}$. Thus, in the non-satiated economy with standardized prices, budget constraints are fulfilled and the allocation x implements *quasi-equilibrium*. In turn, by virtue of Theorem 7 quasi-equilibria become normal market equilibriums if for $\bar{p} = \operatorname{st}(p) \neq 0$

$$\operatorname{st}\{z_i \in {}^*X_i \mid pz_i \leq p\mathbf{e}_i\} = \{\bar{z}_i \in X_i \mid \bar{p}\bar{z}_i \leq \bar{p}\mathbf{e}_i\} \ \ \forall i \in \mathcal{I},$$

that is ensured by the Slater's condition: $\inf_{\bar{z}_i \in X_i} \bar{p}\bar{z}_i < \bar{p}\mathbf{e}_i \ i \in \mathcal{I}$. To see this it suffices to prove $\supseteq$ (the reverse inclusion follows from the standardization of the inequality $pz_i \leq p\mathbf{e}_i$); let us do it.

Let $\bar{z}_i \in X_i$ satisfy the budget equality $\bar{p}\bar{z}_i = \bar{p}\mathbf{e}_i$ (for strict inequality there is nothing to prove). Now we take $\hat{z}_i \in X_i$ from the condition $\bar{p}\hat{z}_i < \bar{p}\mathbf{e}_i$ and find $\varepsilon \approx 0$, $\varepsilon > 0$ such that $p(\varepsilon\hat{z}_i + (1-\varepsilon)\bar{z}_i) \leq p\mathbf{e}_i$. With this in mind we specify $\Delta p = p - \bar{p} \approx 0$ and discover the required inequality: after transformations and eliminations we find

$$(\bar{p} + \Delta p)(\varepsilon\hat{z}_i + (1-\varepsilon)\bar{z}_i) \leq (\bar{p} + \Delta p)\mathbf{e}_i \ \ \Rightarrow \ \ \varepsilon(\bar{p}\hat{z}_i - \bar{p}\mathbf{e}_i) \leq \Delta p\mathbf{e}_i - \Delta p(\varepsilon\hat{z}_i + (1-\varepsilon)\bar{z}_i).$$

Here the value on the right can be estimated as

$$0 \approx -||\Delta p||(||\mathbf{e}_i|| + ||\hat{z}_i|| + ||\bar{z}_i||) \leq \Delta p\mathbf{e}_i - \Delta p(\varepsilon\hat{z}_i + (1-\varepsilon)\bar{z}_i)$$

and as soon as $\bar{p}\hat{z}_i - \bar{p}\mathbf{e}_i < 0$ is a standard value then there is $\varepsilon \approx 0$, $\varepsilon > 0$ such that

$$\varepsilon(\bar{p}\hat{z}_i - \bar{p}\mathbf{e}_i) < -||\Delta p||(||\mathbf{e}_i|| + ||\hat{z}_i|| + ||\bar{z}_i||).$$

Now for this ε the vector $z_i = \varepsilon\hat{z}_i + (1-\varepsilon)\bar{z}_i \in {}^*X_i$ is such that $\operatorname{st}(z_i) = \bar{z}_i$ and z_i satisfies the non-standard budget constraint.

Next we show that for $\bar{p} = \operatorname{st}(p) \neq 0$ in an irreducible economy, the budget constraint of each individual satisfies Slater's condition. Assuming the contrary, one forms a (nonempty) group $S \subset \mathcal{I}$ of all individuals satisfying $\langle \bar{p}, \mathbf{e}_i \rangle > \inf_{X_i}\langle \bar{p}, X_i \rangle$ and defines its complement $T \neq \emptyset$. Further we consider reallocation (contract!) $z = (z_i)_{\mathcal{I}} \in L^{\mathcal{I}}$ of coalition S resources, defined by the definition of an irreducible model, and find

$$\forall i \in S \ \langle \bar{p}, x_i + z_i \rangle \geq \langle \bar{p}, x_i \rangle \ \ \& \ \ \exists i \in S : \ \langle \bar{p}, x_i + z_i \rangle > \langle \bar{p}, x_i \rangle \ \ \Rightarrow$$

$$\langle \bar{p}, \sum_S z_i \rangle > 0 \ \ \Rightarrow \ \ \langle \bar{p}, \sum_S z_i \rangle = \langle \bar{p}, \sum_T z_j \rangle > 0.$$

However by construction $\langle \bar{p}, \mathbf{e}_j \rangle = \inf_{X_j}\langle \bar{p}, X_j \rangle$ for $j \in T$ and via irreducibility

$$\langle \bar{p}, \mathbf{e}_j - z_j \rangle \geq \inf_{X_j}\langle \bar{p}, X_j \rangle \ \ \Longrightarrow \ \ \langle \bar{p}, z_j \rangle \leq 0 \ \ \forall j \in T.$$

Summing over $j \in T$, we obtain $\langle \bar{p}, \sum_T z_j \rangle \leq 0$, which contradicts to the previous conclusion. Consequently, $T = \emptyset$ and quasi-equilibrium is an equilibrium. ∎

Conclusion

The paper studies the modeling of perfect competition conditions in economic models, where the survival condition of economic agents (Slater's condition) is violated. In such models, the concept of equilibrium with non-standard prices and transferable values presents correct notion of equilibrium. In accordance with classical approaches, perfect competition is a condition in which the core coincides with equilibria, but how can this be done for non-standard equilibria in the case when they differ from the usual ones (there is no Slater)?

Our analysis is based on a contractual approach: we argue that the correct model of perfect competition is the notion of fuzzy contractual allocation: these are allocations such that there is no a coalition for members of which it would be beneficial to conclude a new mutually beneficial contract together with a simultaneous partial and potentially asymmetric break of current contracts.

Under the weakest assumptions, there was proved the theorem on the coincidence of the set of all "nonstandard" equilibria and the set of fuzzy contractual allocations: it is Theorem 6 in the text of the paper. As a consequence of this theorem, a (new) fact of the existence of fuzzy contractual allocations was established: these are assumptions that ensure the existence of non-standard equilibria.

There was also investigated the classical method of perfect competition modeling based on the idea of Edgeworth equilibrium and developing the approach of Debreu–Scarf. Its description in terms of non-standard prices is derived from which it follows that this model is less qualified than the contractual one, this is stated in Theorem 7. In addition, there are presented conditions (known earlier) justifying that Edgeworth equilibria coincide with competitive ones. Under these conditions, all studied concepts are equivalent ones, see Theorem 8.

Appendix: summary of some non-standard analysis results

In applications of non-standard analysis, sometimes it is very important to be able show that a particular set is internal.

Theorem 9 *Let A be an internal set, and $B \subseteq {}^*U$ be a definable one. Then $A \cap B$ is an internal set.*

Hyperfinite sets

Suppose that $A \in U$ is a set. Let $F(A)$ denote the set of finite subsets of A. A set $B \in {}^*U$ is said to be *hyperfinite*, if $B \in {}^*F(A)$ for some $A \in U$, in which case, of course, $B \subseteq {}^*A$.

Theorem 10 *If $A \in U$ and $n \in {}^*\mathbb{N} \setminus \mathbb{N}$ is an infinite natural number, then there exists a hyperfinite set D with $|D| < n$ such that $x \in A \Rightarrow {}^*x \in D$.*

Another important fact to know about hyperfinite sets is that every internat subset of a hyperfinite set is hyperfinite one.

Hyperreal numbers and standard parts

Because $\mathbb{R}$ is an ordered field, it follows that (by transfer principle) ${}^*\mathbb{R}$ is also an ordered field under the operations $+, \cdot$, and the relation $<$. A non-standard real number $r \in {}^*\mathbb{R}$ is said to be *finite*, if $|r| < n$ for some $n \in \mathbb{N}$. If $r \in {}^*\mathbb{R}$ is finite then there exists a unique real number which is infinite close to r. This number is called the *standard part* of r and is denoted by $^\circ r$ (or $\mathrm{st}(r) = \mathrm{st}\, r$). Conversely, if the standard part of r exists, then r is finite.

Monads and topology

Suppose that $(X, \mathcal{T})$ is a topological space, where $\mathcal{T}$ denotes the set of all open sets. If $x \in X$, the *monad* of x is the set

$$\mu(x) = \bigcap_{x \in T \in \mathcal{T}} {}^*T.$$

For an *internal* subset $A \subset {}^*X$ there are defined standard part and standard interior, respectively

$$\mathrm{st}A = \{y \in L \mid \mu(y) \cap A \neq \emptyset\}, \quad \mathrm{si}A = \{y \in L \mid \mu(y) \subset A \neq \emptyset\}.$$

It is known that $\mathrm{st}A$ is closed set and $\mathrm{si}A$ is open one.

Let (X, d) be a metric space. Then for $x \in {}^*X$, *metric* monad of x is the set

$$\mu_m(x) = \{y \in {}^*X | {}^*d(x, y) \approx 0\}.$$

Theorem 11 *Suppose (X, d) is a metric space, and $x \in X$. Then the monad of x is equal of metric monad of x.*

If $x, y \in {}^*X$ and y is an element of the (metric) monad of x, we write $x \approx y$ (read "y is infinitely close to x"). This definition is consistent with the previous notation introduced for elements of ${}^*\mathbb{R}$. The notation $x \approx y$ is also used for an arbitrary topological spaces, but only if $x \in X$; in this case $x \approx y \iff y \in \mu(x)$. An example of $\mathbb{R}$ shows that, generally speaking, monads need not be internal sets. However, each monad contains an internal subset, similar to an open neighborhood of a point. More accurately, the following is true

Theorem 12 *For every $x \in X$ there is an internal set $D \in {}^*\mathcal{T}$ such that $D \subseteq \mu(x)$.*

Theorem 13 *A set $A \subset X$ is open if and only if $\mu(x) \subset {}^*A$ for every $x \in A$.*

Theorem 14 *A set $A \subset X$ is closed if and only if for every $x \in X$ the condition $\mu(x) \cap {}^*A \neq \emptyset$ implies $x \in A$.*

Theorem 15 *A set $K \subset X$ is compact if and only if for every $y \in {}^*K$ there exists $x \in K$ such that $y \in \mu(x)$.*

For a topological space X, a point $y \in {}^*X$ is called *near-standard*, if $y \approx x$ for some $x \in X$; otherwise y is called *remote*.

Theorem 16 *The space X is compact if and only if every $y \in {}^*X$ is near-standard.*

Theorem 17 *Let f be a map from the topological space X into the topological space Y and assume that $x \in X$. Then f is continuous at the point x if and only if*

$$x' \approx x \;\Rightarrow\; {}^*f(x') \approx f(x).$$

The last condition can be written in the equivalent form as ${}^*f(\mu(x)) \subseteq \mu(f(x))$.

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