

Valeriy M. MARAKULIN

**Equilibria and core as an essence of economic
contractual interactions: mathematical foundations**

monograph

Incomplete draft. Please do not circulate without author's permission.

Novosibirsk
2024

To my young friend and son Anton

Marakulin Valeriy M. Equilibria and core as an essence of economic contractual interactions: mathematical foundations, ~330 p.

Original “theory of contracts” is consistently developed in the monography: this is a new economic theory formalizing contractual interactions of economic agents. There are barter (exchange) and production contracts. Stable systems (webs) of contracts are studied: this is stability relative to the possibility to enter into new contracts and to break old ones, in joint or separate modes. The possible break of contracts is a key feature of the approach and the breaking can be complete, partial, asymmetric, and even from the specific system of equivalent virtual contracts. This is a new theoretical language, and the contractual paradigm is very fruitful: it represents a new model of perfect competition and successfully fills many gaps of modern economic theories.

Designed for professionals in the field of mathematical economics and general equilibrium theory, for graduated and non-graduated students specializing in this area. This volume is approximately 330 of A4-pages, there are many illustrations. Bibliography of 111 titles.

Keywords and Phrases: Arrow–Debreu economy, incomplete market, contract, contractual allocation, competitive equilibrium, core, contractual process (trajectory), perfect competition, differentiated information, contractual approach, WEE, REE.

JEL Classification: C6, C7, D5

Contents

Survey of notations	6
Preface	8
Introduction	11
I Statics	15
1 Contract-based approach in Arrow–Debreu–McKenzie model	17
1.1 The model of a contractual exchange economy	18
1.2 Contracts in a standard exchange economy	32
1.2.1 Preliminary results and an example	34
1.2.2 Properly and perfectly contractual allocations	41
1.2.3 Fuzzy core and fuzzy contractual allocations	47
1.2.4 Contracts with constrained coalition structure	52
1.2.5 Existence of fuzzy contractual allocations	56
1.3 Production Arrow–Debreu economies	62
1.3.1 Production economies	63
1.3.2 Contractual approach in convex Arrow–Debreu model	66
1.3.3 Contractual economies with non-convex smooth production	75
1.3.4 Contractual economies with non-convex and nonsmooth production	79
Conclusion to Chapter 1	91
2 Contracts and the production of public goods	93
2.1 Public goods and main classical notions	95
2.2 Contractual economy with public goods	100
2.2.1 Fuzzy core vs. Foley core—what is the best?	100
2.2.2 Partial breaking and properly allocations with public goods	105
2.2.3 Fuzzy contractual allocations	112
Conclusion to Chapter 2	117
3 Contract-based incomplete markets	119
3.1 Incomplete market model	120
3.1.1 The contractual approach in incomplete markets	125
3.2 Contractual analysis and results	130

3.2.1	Preliminary analysis and technical results	130
3.2.2	Contractual characterization of <i>GEI</i> -equilibria	138
3.2.3	<i>GEI</i> -equilibrium as a fuzzy contractual allocation	140
3.2.4	Core for incomplete markets	149
3.2.5	Fuzzy contractual core in incomplete markets	156
3.2.6	Replicas and asymptotic properties of core	159
3.3	Contractual incomplete core in examples	164
3.3.1	Incomplete market with one asset	164
3.3.2	Hart's example	168
	Conclusion to Chapter 3	173
II Dynamics		177
4	Processes driving economy to equilibrium	181
4.1	Survey of the literature	182
4.1.1	Walrasian tâtonnement process of equilibrium prices	183
4.1.2	Processes of the prices change, using Jacobi matrix of excess demand function	185
4.1.3	Disequilibrium models of trade processes	187
4.1.4	Edgeworth's processes	189
4.1.5	Strategic approach	190
	Conclusion to Chapter 4	192
5	Dynamical contractual process as cooperative tâtonnement	193
5.1	On the definition of contractual trajectory	193
5.1.1	Contractual trajectory: preliminary analysis	195
5.1.2	Contracting and recontracting processes	198
5.2	Proper-contractual <i>UB</i> -processes: uniform breaking of contracts	200
5.3	<i>CUB</i> -processes: coalitional breaking of contracts	205
5.3.1	Breaking of contracts in active coalitions	206
5.3.2	Proper contractual bilateral <i>CUB</i> -process	208
5.4	Contractual processes: final specifications and associated price process	211
5.4.1	Trade rule for a contractual trajectory	212
5.4.2	Price process, associated with a contractual trajectory	214
	Conclusion to Chapter 5	217
6	Convergence of contractual processes	219
6.1	Contractual process in 2×2 economy	219
6.1.1	Two examples	219
6.1.2	Analysis of contractual processes in 2×2 economies	223
6.2	Local stability in contractual processes	231
6.3	Convergence of benevolent <i>UB</i> -processes	236
6.4	Examples of contractual processes: convergence and cycling	245
6.4.1	The convergence of <i>UB</i> -processes in a 2×3 economy	246

6.4.2	The absence of convergence for a 4×2 economy under assumptions CUB, IBA	247
6.4.3	Absence of convergence for 3×2 economy and UB -process with a piecewise-continuous trade rule	251
	Conclusion to Chapter 6	255
III Information		257
7	Information as a proper part of mathematical model	261
7.1	Differential information economic model	264
7.1.1	Agents and their information	264
7.1.2	Information sharing rule. Limit information	267
7.1.3	Preferences	272
7.2	Allocations, core and equilibria	276
7.2.1	Allocations	277
7.2.2	Core	277
7.2.3	Equilibria	278
7.3	Incentive compatibility or profitable lie	280
7.3.1	Definitions and examples	281
7.3.2	Maximin expected utilities	288
	Conclusion to Chapter 7	291
8	Private core and equilibrium in a contractual economy	293
8.1	Private core. Lemma on beneficial contract	293
8.1.1	Scarf's theorem with fractional coalitions	301
8.1.2	Existence of core in economy with differentiated information	304
8.1.3	Private core and equilibria under perfect competition	307
	Conclusion to Chapter 8	314
9	Contractual interim core and equilibrium	315
9.1	REE -equilibrium: preliminary analysis	315
9.2	Contract-based approach and differentiated agents	319
9.2.1	Differentiated agents	320
9.2.2	Ambiguity and maximin expected preferences	329
9.3	Interim core and REE -equilibria	334
9.3.1	Universal equilibrium and core notions in DI -economies	346
9.4	Computer simulation of contractual processes in DI -economies	349
	Conclusion to Chapter 9	355
References		357

Survey of notations

In this monograph, together with standard notations, we applied the following ones. Everywhere below L denotes a usually finite dimensional vector space over \mathbb{R} and L^* is its dual (conjugated) space.

$\text{co } A$ is a convex hull of $A \subseteq L$,

$A + x = \{a + x \mid a \in A\}$ for all $A \subseteq L, x \in L$,

$A + B = \{a + b \mid a \in A, b \in B\}$ for all $A \subseteq L, B \subseteq L$,

$A \overset{*}{-} B = \{a \in L \mid a + B \subseteq A\}$, for all $A \subseteq L, B \subseteq L$, it is a (geometrical) difference of Minkowski,

$\langle p, x \rangle = p(x) = px$ denotes inner product of vectors $p \in L^*, x \in L$,

$\langle p, A \rangle = \{p(x) \mid x \in A\}$, where $p \in L^*, A \subseteq L$,

$\langle A, B \rangle = \{\langle x, y \rangle \mid x \in A, y \in B\}$, where $B \subseteq L^*, A \subseteq L$,

$A \geq B \iff a \geq b, \forall a \in A, \forall b \in B$ for all $A \subseteq \mathbb{R}, B \subseteq \mathbb{R}$,

$A > B \iff a > b, \forall a \in A, \forall b \in B$ for all $A \subseteq \mathbb{R}, B \subseteq \mathbb{R}$,

$A \setminus B = \{x \in A \mid x \notin B\}$ is set-theoretical difference.

Possible kinds of *linear* intervals in L with endpoints $a, b \in L$ are denoted as follows:

$[a, b] = \text{co}\{a, b\} = \{\lambda a + (1 - \lambda)b \mid 0 \leq \lambda \leq 1\}$,

$(a, b] = [a, b] \setminus \{a\} = \{\lambda a + (1 - \lambda)b \mid 0 < \lambda \leq 1\}$,

$[a, b) = [a, b] \setminus \{b\} = \{\lambda a + (1 - \lambda)b \mid 0 \leq \lambda < 1\}$

$(a, b) = (a, b] \cap [a, b)$.

If L is endowed with a topology then for $A \subseteq L$:

$\overline{A} = \text{cl}A$ is closure of A , and

$\text{int}A$ is its interior.

Ω is a set of all elementary events of nature,

$P = P(\Omega)$ is a partition of set Ω ,

$(\Omega, P(\Omega))$ is a measurable space of nature events,

Ω^* is a partition consisting on singleton elements (complete information),

\mathfrak{P} is a set of all partitions Ω ,

$P(\mathbf{w}) : \mathbf{w} \in P(\mathbf{w}) \in P(\Omega), \mathbf{w} \in \Omega$,

$P \succeq P' \iff P(\mathbf{w}) \subseteq P'(\mathbf{w}), \forall \mathbf{w} \in \Omega$,

$P \vee P' \ \& \ P \wedge P'$ denote supremum and infimum of partitions $P, P' \in \mathfrak{P}$,

$\mathbb{P} = (P_i)_{\mathcal{I}} \ \& \ \mathbb{P}' = (P'_i)_{\mathcal{I}}$ denote informational structures of an economy,

$\mathbb{P} \succeq \mathbb{P}' \iff P_i \succeq P'_i, \forall i \in \mathcal{I}$,

$k_S : (\mathfrak{P})^S \rightarrow (\mathfrak{P})^S$ is a coalition $S \subseteq \mathcal{I}$ information rule,

$k = (k_S)_{S \in C}$ is an information rule of economy,

$k = (k_S)_{S \in C}$ is a rule of informational sharing $\iff k_S((P_i)_{i \in S}) \succeq (P_i)_{i \in S}, \forall S, \mathbb{P}$,

\mathfrak{K} is a set of all informational sharing rules,

$k \succeq \kappa \iff k_S^i((P_j)_{j \in S}) \succeq \kappa_S^i((P_j)_{j \in S})$ for each coalition $S \subseteq \mathcal{I}, \forall \mathbb{P} \in \mathfrak{P}^{\mathcal{I}}$,

$\mathfrak{M}(\Omega, \mathbb{R}^l) = (\mathbb{R}^l)^\Omega = L$ is a space of contingent commodities,

$\mathfrak{M}(\Omega, \mathbb{R}^l) \times \mathfrak{P}$ is a generalized commodity space,

$L_E = \{y \in \mathfrak{M}(\Omega, \mathbb{R}^l) \mid y(\mathbf{w}) = 0, \forall \mathbf{w} \notin E\}$ is a subspace induced by $E \subseteq \Omega$,

$\mathcal{L}_E = \{y \cdot \chi^E(\cdot) \in L \mid y \in \mathbb{R}^l\}$ is a subspace of constant on E functions,

$\mathcal{L}_P = \mathfrak{M}_P(\Omega, \mathbb{R}^l) := \{f : \Omega \rightarrow \mathbb{R}^l \mid f|_{P(\mathbf{w})} = \text{const}\}$ is a subspace of all P -measurable mappings, $P \in \mathfrak{P}$,

$\mathcal{L}_i = \mathcal{L}_{P_i}$ is a subspace corresponded to information $P_i \in \mathfrak{P}$ of agent $i \in \mathcal{I}$,

$\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^l$.

Preface

I would like to say some words on the history of the contractual approach appearance and the presented monograph: In the 1990s years of the last century, I was engaged to develop the theory of incomplete (financial) markets—argued the existence of equilibrium, and so on. However, during that time I discovered that the theory is imperfect and contains a significant gap—there is no concept of the core. At first it surprised me, but later I realized that it is indeed a problem: in the literature I could find only a few clumsy attempts to introduce something similar to the core, all clearly unsatisfactory.

When I started to think about this problem, it became clear that one has somehow separate the flows of goods on-the-spot markets of the future (states of nature) between themselves and separate them with the deliveries across the states of nature, that in the model is realized through trade of assets. And then I was reminded of an interesting and long forgotten idea of 70–80 years by my former scientific supervisor Valeriy Makarov. It was the idea about a contract as a specific form of agreement among agents in delivering of commodities that has to be an elementary and basic form of economic interaction. As I understood much later, in the early 70s similar ideas have appeared in the works of Victor Polterovich. In those years, Makarov's idea had not obtained a profound development, the possible outcomes were unclear and seemed somewhat unsolvable. However, rudiments of the new terminology were presented and, in the works of another scholar from Novosibirsk, Anatoly Kozyrev, an idea of a partial breaking of the contract appeared. Partial break means that agents involved in the contract carry out the obligations in short measure, but this measure is the same for all individuals (for example, all execute 60%) and it is also (renewed) contract. Kozyrev showed that the allocation implemented via a web of contracts, stable with respect to the partial break and signing a new contract (under assumptions!) is equilibrium. This fact was very important, and for me it meant that having properly developed contractual approach can be applied to determine the core in incomplete markets.

So, at the beginning of the 21st century, I turned to develop a contractual approach in an attempt correctly introduce incomplete market core. A bit later, when the outcome started to be observable, I applied for a grant from Economic Education Research Consortium (EERC): a foundation that has played an important role in the further promotion of contractual approach. Being uncertain about success, I still believed in this barter idea and hoped that EERC experts will be able not only to appreciate it but also to provide important directions for its development. Then, being invited to EERC workshop, I really saw the friendly and intellectually challenging working atmosphere. The EERC experts, including Richard Ericson, Victor Polterovich, Shlomo Weber and others, turned out such nice resource persons, that the great amount of helpful important comments became a real surprise for me. I obtained new opinions and new literature on the topic. I would have satisfied even if the project wasn't supported. But it was. So after the seminar I was satisfied and simultaneously filled responsibility for the project and successfully completed it at summer EERC workshop in Kyiv, 2002. Soon the Expert Committee of EERC

awarded this project “Contracts and domination in incomplete markets” with Zvi Griliches prize as the best study of the year and I felt very proud.

Having this lucky experience and understanding now how valuable EERC workshops are, I was surely eager to apply again, but did not hurry to satisfy the high quality standards of EERC. The idea was to further elaborate the contractual approach. So, I presented my second proposal in winter 2004 workshop, and suggested studying the contractual version of disequilibrium dynamic of GE economy. It was one of the new ideas emerging at previous EERC workshops. In spite the fact my proposal had a lacuna (I even tried to cancel or postpone the application), I was invited to the next workshop and even won a grant, because the experts were thinking deeply enough to appreciate the main idea, the difficulties and possible resolutions of the problem. And really, after 1.5 years of work, in 2006, I did successfully complete this investigation. The elaborated theory is presented in the Part II of this monography.

The third EERC grant I won at winter 2008 workshop and successfully presented the final report in June 2010. This study further elaborates the contractual approach, now applied to an economy with asymmetrically informed agents. Again, I am developing the idea that was suggested in general form within the experts’ comments on my first EERC grant. Generally, my experience of cooperation with EERC is quite positive and I am very grateful this Foundation. The results of the last EERC project investigations compiled Part III of the monography. Also, these results were partially carried out when the author was visiting Fellow of European University Institute, Florence, Italy in 2009 (three months) and he is specially grateful to the Institute and its Economic department for very kind hospitality and creative atmosphere that helped me in the work.

Also, I would like to express gratitude to Central European University, Budapest, Hungary, where I was visiting research Fellow via Special and Extension program. I carried out studies on contractual views on public goods economics. The obtained results formed Chapter 3 and partially Chapter 1 (in the part of production models) of the monography. I very grateful to the University and its Economic department for the support of my studies.

Introduction

During the initial stages of developing the economic theory, many important basic notions and principles appeared and different prominent methods of their mathematical analysis were elaborated. However, it was also obvious that there are a lot of real economy features which are not taken into account by the classical theory. This is why up to the end of the 1980s a crisis of models was generated—classical constructions being well studied do not suggest fully satisfactory answers. This situation, starting mainly from the early 80s, motivated constructing and investigating *non-perfect* market models.

In modern economic theory, one can see a number of *non-perfect* market models including incomplete (financial) markets (the trade with specific financial tools, so-called assets, is incorporated into the model), markets with informational asymmetry (about future events, *etc.*), sequential markets (time factor and trust), and so on. In our opinion, the diversity of models and the difficulties in their analysis are caused, on one hand, by the complexity of the object (economy) and, on the other hand, by the absence of sufficiently universal tools for the model investigation. The latter resulted in a variety of solution concepts primarily related to the notion of domination (via coalition) and therefore to the concept of the core. The reason for this is that, following the classical tradition, the main attention is paid to the analysis of the final resource allocation. The commonly missed fact is that in the real economy this allocation is a result of many exchange dealings among economic agents (coalitions). It is important that not every exchange is permissible in a real economy—there are many reasons for this: institutional, physical, informational, ethical, behavioral, *etc.* I believe that the focus of the theory should be shifted in order to be concentrated directly on the exchange bargains of commodities, contracts, which should be included in the model as primitives and form (together with the other model elements) the basis for theoretical constructions, instead of allocations. Thus I propose a *contract-based approach* as a tool for economic modeling. I hope that this approach is able to help to clarify many problems of economic theory arising in the analysis of non-perfect markets. It may also help in an understanding of perfect competition conditions delivering another view on this subject, and we suggest this approach to be a cornerstone of domination and core for the economic models of different types. Moreover, this approach may also be fruitful for the analysis of market processes even in classical frameworks, for example, to understand better the tâtonnement process, to avoid the idea of an auctioneer who rules prices while they are not equilibrium ones (see, *e.g.*, [Arrow, Hahn \(1991\)](#) and also survey from [Section 4.1](#) of this monograph).

The first attempts to introduce the formalized notion of contract in exchange

economies were made by *Polterovich* (1970), *Makarov* (1980, 1982), and *Kozyrev* (1982a,b). In *Polterovich* (1970) some kinds of contractual processes were studied (a break of contracts was not allowed), in *Makarov* (1980, 1982) general ideas of contractual approach as a kind of new language were suggested. *Kozyrev* (1982a,b) first suggested partially broken contracts that implied the study of appropriate forms of stability, and some preliminary positive results were obtained (see Section 1.2). This study is a result of the author's investigations of the last two decades presented in the papers of 2002–2022 years. Already terminology was essentially transformed and expanded: now it includes a wider class of contractual notions and interactions relative to mentioned papers (a term “contract” already has been changed). Our theory of contractual barter and production interactions was not elaborated in western literature in its specific form. About relationships, it will explain later.

In the framework of an ordinary pure exchange model, every (barter) contract is simply an elementary, possible, and permissible exchange of commodities among consumers. Contracts may be added to one another, and with every (finite) set of contracts, an allocation of resources can be associated—as a result of the summation of contracts and the allocation of the initial endowments. It is presumed that every feasible set of (permissible) contracts—let us call it “*a web of contracts*”—may be changed during economic life. Each consumer can *break contracts* in which he/she participates, and each coalition of consumers can also *sign a new contract(s)*. In a classical model with *private* commodities, a production contract can be associated with a collection of production plans and with an ongoing parallel barter exchange. Contract related with public goods has a really cooperative production specification since joint consumption commodities are produced and each individual has to have his own input into a deal that specifies contractual functioning.

Moreover, a consumer can be able to partially break contracts (assigned in the past) if it is beneficial for him. The partial breaking of a contract means its replacement by a smaller volume contract with the same exchange proportions. This leads to the concept of “*properly contractual allocations*” and approaches contractual processes to market processes under perfect competition conditions. An allocation is called properly contractual if it can be realized by a web of contracts that is stable relative to the procedure of both parties *partially breaking* existing contracts and signing new contracts. At the same time, the core allocations are described in terms of “*contractual allocations*.” These are the allocations that can be realized by a web of contracts and which are stable relative to the procedure of (fully) breaking contracts and signing new contracts. Thus the only difference between these two notions of contractual allocation is that in the first case, the partial breaking of contracts is allowed, while in the second case, only the complete breaking is possible. Thereby, an equilibrium can be described in purely game-theoretical terms and does not address any kind of value parameters. The mathematical nature of this phenomenon is quite similar to the coincidence of equilibrium allocations and the fuzzy core elements (or Edgeworth's equilibria, see *Aliprantis et al.* (1989)), which is one possible way to model the conditions of perfect competition.

What is the major theoretical meaning of the contractual approach? The crucial point is that being based on extended views on coalitional stability of allocation, this

allows us to redefine equilibria in micro-terms: indifference with tâtonnement that does not model local interactions. It gives us an opportunity to describe economic interactions under equilibrium prices. The partial breaking of contracts plays a key role in the analysis, and it can be even asymmetrical (for the newly planned contractual stages). By introducing of partial break, we enter into an economic model perfect competition conditions in a way that is essentially simpler than known in the literature. Really in a classical equilibrium theory, perfect competition conditions are described (see *e.g.* [Vind \(1995\)](#)) as a presence of continuum agents, each of them being infinitesimally small relative to the economy as a whole, (it is said there is a non-atomic measurable space of economic agents, [Aumann \(1964\)](#)). Under these assumptions (non-atomic space etc.) a nontrivial theorem on the coincidence of core and equilibria is proven, it applies involved techniques on the integration of point-to-set mappings, *etc.* (see also [Hildenbrand \(1974\)](#)). There are other delicate models of perfect competition conditions applying among others nonstandard analysis: economies with hyperfinite numbers (infinite natural number) of agents are studied here. However, do we really need to apply so complicated and prominent techniques to substantiate perfect competition conditions? Not at all and contract-based approach suggests the simplest way of perfect competition conditions presentation: all that we need is to allow the agents to *partially* break contracts. Moreover, now equilibrium and core allocations are the objects of the same rank: they are implemented by the webs of contracts stable in a specific sense where a fortiori equilibrium stability is stronger one.

In general, partial breaking can be interpreted in different ways, one of which is indirectly paying attention to the dynamical component of the contractual process (it is out of a static model frame). Suppose an economy lives during the long-duration interval of time. The contract assumes mutual deliveries of goods among agents and, after its execution, an opportunity of renewal, *i.e.*, the same contract can be signed again, but now it is realized during *another* time period. The agents can agree with the contract's renewal (prolongation) or disagree, first studying an opportunity to prolong the contract in smaller volumes. Thus, instead of *breaking the contract*, even if partial, for an economy in dynamics living a long time period one *can speak about renewal and non-renewal* of the contracts. So agents learn do not to renew non-profitable and iteratively realize profitable contracts (in optimal volume for a current time moment and saving exchange proportions) at the next stage of contracting. If nobody wants to re-contract again, it means the system as a whole enters a stable functioning. One can note that admitting partial breaking contracts we tolerate an individual to competitive myself to myself, and this is a competition between individual interests at different time moments of realizations, and similarly for other agents (specific realization of R. H. Coase conjecture...)

The monograph consists of three more or less independent parts: Statics, Dynamics, and Information. The first part consists of three chapters devoted to the analysis and presentation of the results for the contractual approach applied to pure exchange models, Arrow–Debreu, and incomplete (financial) markets. The second part presents a theory of contractual processes that drive the economy to equilibrium. Here there are also three chapters, with the first presenting a detailed review of the literature

on the processes that transit economy in equilibrium. The last chapter presents a series of model examples that demonstrate the convergence and possible looping of contractual processes. The third and last part of the monograph concerns the theory of contracts being applied to models with asymmetrically informed agents. In particular, the application of the contractual approach reveals the relationship between classical concepts such as Walrasian expectations equilibrium (*WEE*) and in the rational expectations equilibrium (by Radner, *REE*). A lot of new concepts of specific contractual equilibrium are also introduced: equilibria and core with differentiated agents, interim (intermediate) core and equilibrium, and so on. Each chapter of the monograph ends with a conclusion, which highlights the main results of this chapter.

Part I

Statics

Chapter 1

Contract-based approach in Arrow–Debreu–McKenzie model

The chapter develops the *theory of contractual interactions* in the framework of an abstract economy with private goods. It applies this theory to classical markets and production economies of Arrow–Debreu kind with convex and non-convex technological sectors. We consider and study the formal rules of operating with the sets of contracts. The difference in these rules differentiates the types of a web’s stability and therefore in the stability of allocations realized by webs. The types of these “stabilities,” together with the property of contracts to be permissible, reflect different behavioral, physical, and institutional principles formally given in a game-theoretical form, which one can find in real life and in neoclassical economic theory. So, different types of web stabilities correspond to different types of contractual allocations, as well as their modifications, which can relax or strengthen the property for an allocation to be stable. We introduce a new formalism and several original contractual concepts: coherent, perfect, fuzzy, and complex contractual allocations; some relationships among them are revealed. Thereby, depending on the structure of permissible contracts, one can describe notions well known in economic theory such as core, competitive equilibria, Pareto boundary, and so on in terms of a stable web of contracts. The relationship between the elements of fuzzy core and fuzzy contractual allocations is important in its own right: it provides a natural interpretation for fuzzy core and can be applied even for infinity dimensional models, *e.g.* to state the existence of equilibria (see [Marakulin \(2006a\)](#) generalizing results from [Florenzano, Marakulin \(2001\)](#)).

The *inadmissibility* of some exchange contracts is a specific feature of many modern non-classical models, and it reflects the essence of our approach. However, as a first step we consider only the application of the contract-based approach to economies *without admissibility constraints* for contracts, this corresponds to the case of classical markets. In subsequent sections really non-perfect markets are appearing: public goods, financial and asymmetrically informed markets, and so on: not all versions are well studied. In non-perfect economies, the permissibility constraints for contracts are usually implemented as a requirement for a contract to be situated in a subspace of all possible barter exchanges or in a subspace of technological possibilities. It may be the subspace of measurable functions relative to informational structure (algebra)

for economies with asymmetrically informed agents or a specific subspace formed via real assets in incomplete markets (because the direct commodity exchange between different states of the world is impossible) and so on.

Notice that our theory of contractual interactions does not address bargaining theory (e.g. see [Thomson \(1994\)](#), [Muthoo \(1999\)](#)) and solutions elaborated in its context (Nash’s, etc.). These results may be partially incorporated into it in further investigations. We have not also studied any bargaining procedures for driving agents to optimal or fair contracts.¹

From the middle of the 1980s, studies aimed to clarify the basic hypotheses of competitive equilibrium theory in a context of a strategic game began to appear. An idea was to apply game theoretical methods to give the answers to such questions as whence the prices are undertaken and who defines them, why the agents should accept the prices as given and why they cannot change them (a consumer is said to be a “price-taker”), that is equilibrium and perfect competition? The answers to these and other important theoretical questions are given in the analysis of some games in extensive form. These games belong to a class of DMBG-games (dynamical matching and bargaining games), constructed by a model of an economy in a special way. In general, an ideology of this approach is quite close to contractual. This approach and its substantial sense are described in §4.1.5 from Chapter 4. Here we only note that certainly, the contract-based approach does not present such widespread variants of disequilibrium interactions as in the strategic case. However, it is not still a shortage at all because a cooperative component of implemented allocation (presented in an aggregated form via a stable web of contracts) plays a crucial role in subsequent analysis. Moreover, relative to the strategic contractual model is simpler and easier to be analyzed.

1.1 The model of a contractual exchange economy

We consider a typical exchange economy in which L denotes the (finite-dimensional) *space of commodities*. Let $\mathcal{I} = \{1, \dots, n\}$ be a set of agents (traders or consumers). A consumer $i \in \mathcal{I}$ is characterized by a consumption set $X_i \subset L$, an initial endowment $\mathbf{e}_i \in L$, and a preference relation described by a point-to-set mapping $\mathcal{P}_i : X_i \rightrightarrows X_i$ where $\mathcal{P}_i(x_i)$ denotes the set of all consumption bundles strictly preferred by the i -th agent to the bundle x_i . It is also applied the notation $y_i \succ_i x_i$ which is equivalent to $y_i \in \mathcal{P}_i(x_i)$. So, the pure exchange model may be represented as a triplet

$$\mathcal{E} = \langle \mathcal{I}, L, (X_i, \mathcal{P}_i, \mathbf{e}_i)_{i \in \mathcal{I}} \rangle.$$

¹In general there is a possibility to apply the bargaining theory correctly to define a specific contractual tâtonnement process that drives the economy to equilibrium, but it still is not properly elaborated.

Let us denote by $\mathbf{e} = (\mathbf{e}_i)_{i \in \mathcal{I}}$ the vector of initial endowments of all traders of the economy. Denote $X = \prod_{i \in \mathcal{I}} X_i$ and let

$$\mathcal{A}(X) = \{x \in X \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\}$$

be the set of all *feasible allocations*.

Everywhere below we assume that model \mathcal{E} satisfies the following assumption.

(A) For each $i \in \mathcal{I}$, X_i is a convex solid² closed set, $\mathbf{e}_i \in X_i$, and for every $x_i \in X_i$ there exists an open convex $G_i \subset L$ such that $\mathcal{P}_i(x_i) = G_i \cap X_i$ and if $\mathcal{P}_i(x_i) \neq \emptyset$ then $x_i \in \overline{\mathcal{P}_i(x_i)} \setminus \mathcal{P}_i(x_i)$.³

Notice that due to **(A)** preferences may be satiated, *i.e.*, $\mathcal{P}_i(x_i) = \emptyset$ is possible for some agent i and $x_i \in X_i$. However, if $\mathcal{P}_i(x_i) \neq \emptyset$, then preference is *locally non-satiated* at the point x_i .

Let $\mathfrak{L} = L^{\mathcal{I}}$ denote the space of allocations of the economy \mathcal{E} . In the framework of model \mathcal{E} , we are going to introduce and study a formal mechanism of contracting and recontracting. This mechanism reflects the idea that any group of agents can find and realize some (permissible) within-the-group exchanges of commodities referred to as contracts. The mechanism defines the rules of contracting.

By the formal definition, any reallocation of commodities $v = (v_i)_{i \in \mathcal{I}} \in \mathfrak{L}$, *i.e.* any vector $v \in \mathfrak{L}$ satisfying

$$\sum_{i \in \mathcal{I}} v_i = 0,$$

is called a *contract*.

Not every kind of possible reallocation may be realized in the economy; there are some institutional, physical, and behavioral restrictions in the economic models of different types. This is why we equip the abstract contractual economy model with a new element, the set of *permissible* contracts $\mathcal{W} \subset \mathfrak{L}$. Thus, the contractual (exchange) economy under study may be shortly represented by the 4-tuple

$$\mathcal{E}^c = \langle \mathcal{I}, L, \mathcal{W}, (X_i, \mathcal{P}_i, \mathbf{e}_i)_{i \in \mathcal{I}} \rangle.$$

In addition to **(A)**, we only assume everywhere below that for a contractual economy

(C) The set \mathcal{W} is star-shaped at zero in \mathfrak{L} , *i.e.*,

$$v \in \mathcal{W} \implies \lambda v \in \mathcal{W}, \quad \forall 0 \leq \lambda \leq 1.$$

The economy \mathcal{E}^c as well as the economy \mathcal{E} is called *smooth* if for every $i \in \mathcal{I}$,

$$\mathcal{P}_i(x_i) = \{y \in X_i \mid u_i(y) > u_i(x_i)\}, \quad \forall x_i \in X_i$$

for some *differentiable* quasi-concave function u_i defined on an open neighborhood of X_i .

²Here “solid” is equivalent to “having a nonempty interior.”

³The symbol \overline{A} denotes the closure of A and \setminus is set for the set-theoretical difference.

For a contractual economy, we study the sets of contracts that represent *feasible* allocations and introduce the operation of breaking a part of a given set of contracts. This motivates the following important definition.

A finite collection V of permissible contracts is called *a web of contracts relative to* $y \in \mathcal{A}(X)$ if

$$y + \sum_{v \in U} v \in X, \quad \forall U \subseteq V.$$

We denote by $x_y(U)$ the feasible allocation sustained by U relative to y , *i.e.*, we put

$$x_y(U) := y + \sum_{v \in U} v.$$

Similarly, $U_y(x)$ denotes the web which realizes x relative to y .

A web of contracts V relative to \mathbf{e} is called *a web of contracts* or simply *a web*. Note that $V = \emptyset$ is a web relative to every $y \in \mathcal{A}(X)$. Denoting

$$\Delta(V) = \sum_{v \in V} v,$$

where V is an arbitrary collection of contracts (by convention, we write $\Delta(\emptyset) = 0$), we can write

$$x_y(V) = y + \Delta(V), \quad x(V) = x_{\mathbf{e}}(V) = \mathbf{e} + \Delta(V)$$

so that V being a web simply means that

$$x_{\mathbf{e}}(U) \in X, \quad \forall U \subseteq V.$$

Now we are going to introduce the operations of breaking existing contracts and signing new ones. For any contract $v \in V$, let us set

$$S(v) = \text{supp}(v) = \{i \in \mathcal{I} \mid v_i \neq 0\},$$

the support of the contract v . It is assumed that any contract $v \in V$ may be *broken* by any trader in $S(v)$, since he/she simply may not keep his/her contractual obligations. Also, a nonempty group (coalition) of consumers can *sign* any number of new contracts. Being applied jointly, *i.e.*, as a simultaneous procedure, these operations allow coalition $T \subseteq \mathcal{I}$ to yield new webs of contracts. The set of all such webs is denoted by $F(V, T)$. Formally, we require that each element $U \in F(V, T)$ has to satisfy the following properties:

- (i) $v \in V \setminus U \Rightarrow S(v) \cap T \neq \emptyset$,
- (ii) $v \in U \setminus V \Rightarrow S(v) \subseteq T$,
- (iii) $\sum_{v \in U \setminus V} \lambda_v v \in W$ for all $0 \leq \lambda_v \leq 1$, $v \in U \setminus V$.

Condition (i) means that only members of T can break contracts in V , condition (ii) means that only members of T may sign new contracts, and (iii) is a kind of joint permissibility of new contracts, which is useful in applications of the contractual economy. Notice also that due to the definition of a web of contracts, a coalition can break any subset of contracts of a given web that satisfies (i).⁴

Further, for the webs of contracts, it is introduced the notion of domination via a coalition. This property, being written as $U \succ_T V$ (U dominates V via coalition T), means that

$$(i) \quad U \in F(V, T),$$

$$(ii) \quad x_i(U) \succ_i x_i(V) \quad \text{for all } i \in T.$$

Definition 1.1.1 *A web of contracts V is called stable if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \succ_T V$.*

An allocation x is called contractual if $x = x(V)$ for a stable web V .

The property that a web of contracts is stable may be relaxed as well as strengthened. The most important possibilities are described below.

Definition 1.1.2 *A web of contracts V is called lower stable if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \succ_T V$ and $U \subset V$.*

A web of contracts V is called upper stable if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \succ_T V$ and $V \subset U$.

An upper and lower stable web of contracts V is called weakly stable.

An allocation x is called lower, upper, or weakly contractual if $x = x(V)$ for some lower, upper, or weakly stable web V , respectively.

It has to be clear that all the above notions of stability and domination may be considered as “relative to some given feasible allocation,” simply use this allocation instead of the initial e . Also, it has to be clear that the notion of a weakly stable web (weakly contractual allocation) is really weaker than the corresponding notion of a stable web (contractual allocation). The difference is that in the first case the operations of breaking existing contracts and signing new contracts are applied separately, whereas in the second case they are applied simultaneously. In the framework of a market economy, we consider below the relationships among the sets of contractual, lower, upper, and weakly contractual allocations. They correspond to notions well known in economic theory.

How can the process of recontracting (breaking existing contracts and signing new ones) be expressed in economic terms? We can assume that this is something like a tâtonnement process (cooperative tâtonnement), which, for example, may be as follows. To simplify the argument, let us imagine that there is an ordered list of all coalitions. At the first stage (iteration), the coalitions, in the given order

⁴Otherwise, it would occur that an allocation realized via breaking contracts is not feasible.

of appearance, start to sign and/or break contracts (transiting to webs in $F(V_\xi, T_\xi)$, where ξ is the order number of coalition T_ξ). Here the first coalition “starts” from the given initial endowment allocation \mathbf{e} and since there were no contracts signed before from the web $V_1 = \emptyset$. The stage, the iterative loop, is finished when the last coalition has made its choice. Next, the second stage starts where the same process is going on, assuming that the first coalition in the list deals with the web of contracts realized at the end of the first stage. The fixed points of this iterative process correspond to the contractual allocations and to the stable webs of contracts. Clearly, the order of coalitions’ “appearance” during a stage is not essential. Moreover, a coalition can appear several times during one stage, and the order of coalitions’ appearance can vary from stage to stage. What is really important is that each coalition has a chance to appear in infinitely many iterations. In general, this scenario does not impose any time restrictions on the duration of an iteration, and the number of iterations is potentially unlimited (therefore the duration of a stage is infinitely small). Informally, it is just presumed that the process finishes in “a reasonable time” and the economy transits to a stationary state. It is these potentially possible stationary states that are the subject of my analysis. Notice also that if in the iterative process, starting from some stage, one forbids the signing of new contracts for coalitions, one can realize stationary states and webs of contracts which are lower stable (in one of the senses described above or below: it depends on which kind of contract breaking is permissible, *i.e.*, whether one can break a contract only in its entirety, partially, or even with a transition to equivalent contracts). Similarly, by forbidding breaking contracts or by breaking contracts in a mixed regime (at one stage forbidding signing new contracts, at another stage forbidding breaking them, and so on), one can realize upper and weakly stable webs and contractual allocations, respectively. A conventional presentation of the contracting and recontracting process, a kind of timing, is presented in Figure 1.1.1. Here coalition $T_\xi = \{1, 4\}$ breaks a part of contracts $W \subseteq V_{\xi-1}$ from the web

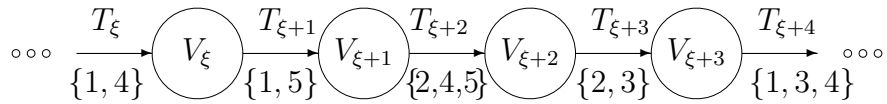


Figure 1.1.1: *Contracting and recontracting process*

$V_{\xi-1}$ and signs a new contract $w = (w_1, 0, 0, w_4, 0, \dots, 0) \in \mathbb{R}^{ln}$ (l is a number of commodities), forming a new web V_ξ such that

$$\begin{aligned} \mathbf{e}_1 + \sum_{v^{\xi-1} \in V_{\xi-1} \setminus W} v_1^{\xi-1} + w_1 \succ_1 \mathbf{e}_1 + \sum_{v^{\xi-1} \in V_{\xi-1}} v_1^{\xi-1}, \\ \mathbf{e}_4 + \sum_{v^{\xi-1} \in V_{\xi-1} \setminus W} v_4^{\xi-1} + w_4 \succ_4 \mathbf{e}_4 + \sum_{v^{\xi-1} \in V_{\xi-1}} v_4^{\xi-1}. \end{aligned}$$

On the left-hand side of these relations, the summation is taken over contracts from $V_{\xi-1}$, which are not broken by coalition $\{1, 4\}$. Conversely, on the right-hand side of these relations, the summation is taken over all contracts from the web $V_{\xi-1}$. Further, the web V_ξ is transformed by coalition $T_{\xi+1} = \{1, 5\}$ in a similar way, and so on.

Remark 1.1.1 Similar views on commodity exchange processes, sometimes called barter processes, can be found in *Madden (1975)*, *Graham et al. (1976)*, and other papers. However, all these papers studied other problems and did not elaborate contract-based approach properly. ■

Now we continue the list of stability concepts, strengthening the stability relative to the procedure of breaking contracts. It is clear that a web that is not lower stable cannot be long-lived in a market. This is why we restrict our attention below only to the lower stable webs. First, let us introduce an equivalence relation on the set of all such webs. This equivalence will allow us to partially divide some contracts. To this end, we can define a partial ordering on the set of all webs as follows:

$$U \geq V \iff \exists \text{ a map } \mathbf{onto} \ f : U \rightarrow V \text{ such that}$$

$$(i) \ \lambda f(u) = u \text{ for some } 0 \leq \lambda \leq 1 \text{ and for every } u \in U,$$

$$(ii) \ \sum_{u \in f^{-1}(v)} u = v \text{ for every } v \in V.$$

One can easily see that the set of contracts $f^{-1}(v)$ is a *partition* of contract v and so the web U consists of (finite) *partitions* of contracts from V . The minimal elements of the set of all webs may be called *root webs*. Note that for $U \geq V$ due to definition we have $\Delta(U) = \Delta(V)$. Now the equivalence relation may be defined as follows:

$$U \simeq V \iff \exists \text{ a web } W \text{ such that } V \geq W \ \& \ U \geq W.$$

Clearly, $U \simeq V$ simply means that these webs have a common root web.

Definition 1.1.3 *An allocation x is called properly contractual (resp. lower properly contractual, weakly properly contractual) if there exists a web V such that $x = x(V)$ and for every $U \simeq V$ the allocation $x = x(U)$ is contractual (resp. lower contractual, weakly contractual).*

So, for properly contractual allocations we allow the agents to partially break contracts as well as to sign new contracts (simultaneously or separately). This may be interpreted in two ways. First, speaking *in behavioral terms*, the agents are going to sign many small volume contracts instead of signing one contract of a large volume. This way they gain more economic freedom through the ability to break off some small contracts if a necessity arises. The second way is to treat a proper contract as a kind of *preliminary agreement*. In this agreement, only the rates of exchange are rigidly defined in contrast to the volume of the contract, which is flexible and will be defined rigidly at the end of the contracting procedure. It should be clear that due to the last definition, the property of an allocation to be stable (in any of the senses) is essentially strengthened when we add the word “proper” to the term “contractual” allocation. Below, we refine and define the term “proper” for a single contract and for a web.

Definition 1.1.4 Let V be a web. A contract $v \in V$ is coherent if every web U such that $U \simeq \{v\}$ is lower stable relative to $(x(V) - v)$, when it is taken as the initial endowments or equivalently for ordered preferences, if

$$x_i(V) \succeq_i x_i(V) - \lambda v_i, \quad \forall 0 \leq \lambda \leq 1, \quad \forall i \in \mathcal{I}.$$

A subweb $U \subseteq V$ consisting of coherent contracts is called coherent.

A subweb $U \subseteq V$ is called proper if for every web $W \simeq U$ the web $(V \setminus U) \cup W$ is lower stable.

An allocation x realized by a coherent web V , i.e., $x = x(V)$, is called (lower) coherent.

Notice that the only difference between coherent and proper webs is that in the first case the web is stable relative to the partial breaking of any *single* contract, whereas in the second case the agents may partially break *any number* of contracts in the web. In general, these notions are not equivalent (see Example 1.2.1). Moreover, even the notions of coherent and proper contracts are not equivalent; in the latter case, breaking more than one contract in the web is also allowed. The case of a proper web of contracts is geometrically presented in Figure 1.1.2 a) in the coordinate system of consumer $i \in \mathcal{I}$. Figure 1.1.2 b) presents the geometry of stable but non-coherent and non-proper webs.

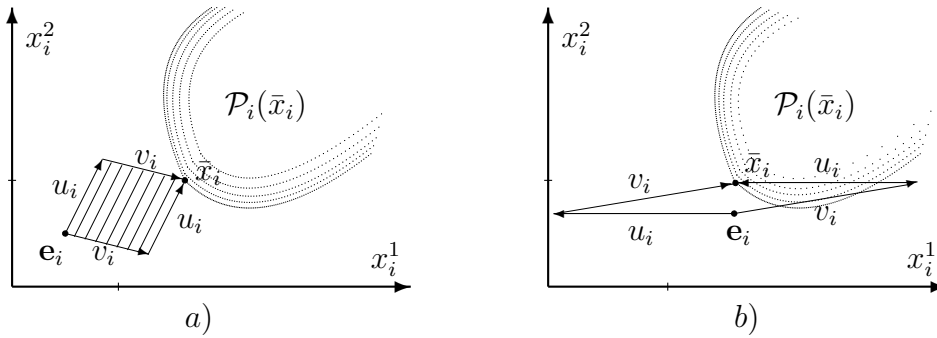


Figure 1.1.2: a) —the web $\{u, v\}$ is proper, b) —the web $\{u, v\}$ is not proper

The difference is that while in the first case the whole “parallelogram of contracts” does not intersect $\mathcal{P}_i(\bar{x}_i)$, in the second case it does intersect, but no vertex belongs to $\mathcal{P}_i(\bar{x}_i)$.

The next proposition fully characterizes coherent contracts for convex contractual economies.

Proposition 1.1.1 Let V be a web. Contract $v \in V$ is coherent iff there exist linear functionals $p_i^v \neq 0$ such that

$$\langle p_i^v, \mathcal{P}_i(x_i(V)) \rangle > \langle p_i^v, x_i(V) \rangle^5 \quad \& \quad \langle p_i^v, v_i \rangle \geq 0 \quad (1.1.1)$$

for every $i \in \mathcal{I}$. Moreover, if the utility functions are differentiable, (1.1.1) is fulfilled for $p_i^v = \nabla u_i(x_i(V)) \neq 0$, $i \in \mathcal{I}$.

⁵ $\langle A, p \rangle$ denotes the set $\{\langle a, p \rangle \mid a \in A\}$ and $A > b$ ($A \geq b$) means $a > b$ ($a \geq b$) for all $a \in A$. By convention we assume $\langle p, \emptyset \rangle > b$, $\forall b \in \mathbb{R}$.

Proof of Proposition 1.1.1. Let us show that (1.1.1) is sufficient. Assume that for every $i \in \text{supp}(v)$, inequalities (1.1.1) are true, but the contract v is not coherent. Then after partially breaking v , the broken part being $0 \leq \alpha_v \leq 1$, agents realize the new allocation

$$x_i^\alpha = x_i - \alpha_v v_i, \quad i \in \mathcal{I}$$

such that $x_i^\alpha \succ_i x_i$ for some $i \in \text{supp}(v)$ so that the first part of (1.1.1) gives $p_i^v x_i^\alpha > p_i^v x_i$. But then, using the second part of (1.1.1) for this i , we obtain $p_i^v x_i^\alpha \leq p_i^v x_i$, which contradicts the previous inequality.

To establish that (1.1.1) is necessary, assume that the contract $v \in V$ is coherent, *i.e.*, the web V is stable relative to the partial breaking of contract v . For each consumer $i \in \mathcal{I}$, let us consider the set

$$\mathcal{U}_i(x) = \{x_i - \alpha_v v_i \mid 0 \leq \alpha_v \leq 1\} \subset X_i.$$

By definition, the sets \mathcal{U}_i are nonempty, convex and closed. Now, since the contract $v \in V$ is coherent, it follows that

$$\mathcal{P}_i(x_i) \cap \mathcal{U}_i(x) = \emptyset, \quad \forall i \in \mathcal{I}.$$

By assumption, there exists an open convex set $G_i \subset L$, such that in non-satiated case $G_i \cap X_i = \mathcal{P}_i(x_i) \neq \emptyset$, $x_i \in \overline{G}_i$ for the given $x \in \mathcal{A}(\mathcal{E})$ and i . Now for each non-satiated $i \in \mathcal{I}$ at x_i the last relations imply

$$G_i \cap \mathcal{U}_i(x) = \emptyset \quad \& \quad x_i \in \overline{G}_i.$$

Therefore, by the separation theorem, for each non-satiated $i \in \mathcal{I}$ there exists $p_i^v \in L'$ such that $p_i^v \neq 0$ and

$$\langle p_i^v, G_i \rangle > \langle p_i^v, x_i \rangle \geq \langle p_i^v, \mathcal{U}_i(x) \rangle.$$

Clearly, for differentiable utility functions if $\nabla u_i(x_i(V)) \neq 0$ then one can put $p_i^v = \nabla u_i(x_i(V))$, $i \in \mathcal{I}$. Now the first inequality implies the first part of (1.1.1), and the second one gives

$$p_i^v x_i \geq p_i^v x_i - \alpha_v \langle v_i, p_i^v \rangle, \quad 0 \leq \alpha_v \leq 1.$$

Therefore, $\langle v_i, p_i^v \rangle \geq 0$. ■

The following corollaries characterize properly contractual allocations in terms of coherent webs.

Corollary 1.1.1 *Let \mathcal{E}^c be a smooth contractual economy. Then allocation x is lower properly contractual iff there exists a coherent web V such that $x = x(V)$. Moreover, relation (1.1.1) is fulfilled for $p_i = p_i^v = \nabla u_i(x_i(V)) \neq 0$ and for every $v \in V$ and $i \in \mathcal{I}$.*

Proof of Corollary 1.1.1. It is enough to check that every coherent web U , such that $x = x(U)$ is stable relative to the procedure of partially breaking *any number* of contracts, *i.e.*, U is a proper web.

In fact, due to Proposition 1.1.1 for differentiable utilities, for every $v \in V_e(x)$ the functional (vector) $p_i^v = \nabla u_i(x_i) = p_i \neq 0$ satisfies condition (1.1.1) for all $i \in \mathcal{I}$. This implies

$$G_i \cap \mathcal{M}_i(x) = \emptyset, \quad \forall i \in \mathcal{I},$$

where

$$\mathcal{M}_i(x) = \left\{ x_i - \sum_{v \in V} \alpha_v v_i \mid 0 \leq \alpha_v \leq 1, v \in V \right\} \subset X_i.$$

The proof is completed. \blacksquare

In particular, the last corollary states that for smooth economies every coherent web is proper, *i.e.*, stable relative to the partially breaking any number of contracts. Note that the assumption of differentiability of utilities cannot be dropped here; appropriate examples can be easily constructed (see Example 1.2.1, second part).

The next property of proper and coherent contracts is that under a condition of saving aggregated exchange parameters they can be replaced by another proper web, keeping the lower stable property for the new web. This fact also follows from Proposition 1.1.1. Recall that $V_y(x) = V$ denotes a web realizing the allocation x relative to the initial endowments y , *i.e.*, $x = y + \Delta(V)$.

Corollary 1.1.2 *Let \mathcal{E}^c be a smooth economy and $x \in \mathcal{A}(X)$. Then any coherent web $V_e(x)$ has the following inheritance property: for every (coherent) contract $v \in V_e(x)$ and every coherent web $W_{x-v}(x)$, the new web $U = (V_e(x) \setminus \{v\}) \cup W_{x-v}(x)$ is lower stable relative to partially breaking (any number of) contracts and therefore is proper.*

Proof of Corollary 1.1.2. Due to Proposition 1.1.1 (necessity), for all $i \in \mathcal{I}$ there are unambiguously up to normalization defined functionals p_i satisfying (1.1.1) for any (coherent) contract $v \in V_e(x)$, as well as for contracts in the coherent web $W_{x-v}(x)$, since this web is stable relative to the partial breaking of contracts and $\sum_{w \in W_{x-v}(x)} w + x - v = x$. In other words, for every non-satiated i at x_i there is a $p_i \neq 0$ such that

$$\langle p_i, \mathcal{P}_i(x_i) \rangle > \langle p_i, x_i \rangle, \quad \langle p_i, v'_i \rangle \geq 0, \quad \forall v' \in V_e(x)$$

and

$$\langle p_i, w_i \rangle \geq 0, \quad \forall w \in W_{x-v}(x).$$

Hence, joining the second relation with the first one for contracts in U and applying Proposition 1.1.1 in the part of sufficiency yield the result via Corollary 1.1.1. \blacksquare

The next important property of proper and coherent contracts is that they can be replaced by another proper web, keeping the lower stable property for the new web. This fact also follows from Proposition 1.1.1. Recall that $V_y(x) = V$ denotes a web realizing the allocation x relative to the initial endowments y , *i.e.*, $x = y + \Delta(V)$.

Corollary 1.1.3 *Let \mathcal{E}^c be a smooth economy and $x \in \mathcal{A}(X)$. Then any coherent web $V_e(x)$ has the following inheritance property: for every (coherent) contract $v \in V_e(x)$*

and every coherent web $W_{x-v}(x)$, the new web $U = (V_e(x) \setminus \{v\}) \cup W_{x-v}(x)$ is lower stable relative to partially breaking (any number of) contracts and therefore is proper.

Proof of Corollary 1.1.3. Due to Proposition 1.1.1 (necessity), the functionals $p_i = \nabla u_i(x_i)$ satisfy (1.1.1) for any (coherent) contract $v \in V_e(x)$, as well as for contracts in the coherent web $W_{x-v}(x)$, since this web is stable relative to the partial breaking of contracts and $\sum_{w \in W_{x-v}(x)} w + x - v = x$. In other words, for every i there is a $p_i \neq 0$ such that

$$\langle p_i, \mathcal{P}_i(x_i) \rangle > \langle p_i, x_i \rangle, \quad \langle p_i, v'_i \rangle \geq 0 \quad \forall v' \in V_e(x)$$

and

$$\langle p_i, w_i \rangle \geq 0 \quad \forall w \in W_{x-v}(x).$$

Hence, joining the second relation with the first one for contracts in U and applying Proposition 1.1.1 in the part of sufficiency yields the result via Corollary 1.1.1. ■

The following proposition gives the characterization of proper webs in dual terms for the non-smooth case. The proof of this proposition is similar to the proof of Corollary 1.1.1.

Proposition 1.1.2 *Let V be a web of contracts and $x = x(V)$. Then web V is proper, i.e., x is a lower properly contractual allocation, if and only if, there exist linear functionals $p_i \neq 0$, such that*

$$\langle p_i, \mathcal{P}_i(x_i(V)) \rangle > \langle p_i, x_i(V) \rangle \quad \& \quad \langle p_i, v_i \rangle \geq 0 \quad \forall v \in V \quad (1.1.2)$$

for each $i \in \mathcal{I}$.

Proof of Proposition 1.1.2. Under assumption (A), the fact that V is a proper web is equivalent to the condition

$$G_i \cap \mathcal{M}_i(x) = \emptyset \quad \forall i \in \mathcal{I},$$

where

$$\mathcal{M}_i(x) = \left\{ x_i - \sum_{v \in V} \alpha_v v_i \mid 0 \leq \alpha_v \leq 1, v \in U \right\} \subset X_i.$$

Now applying the separation theorem to the sets G_i and $\mathcal{M}_i(x)$, we establish the necessity of (1.1.2). Its sufficiency can be checked directly. ■

In applications of contractual economies one can also use contracts with a stronger stability, so-called *perfect* contracts. To introduce this notion, let us first consider another kind of equivalence relation defined on the set of all proper webs. This (weak) equivalence relation may be defined as follows: Let U and V be *proper* webs, then

$$U \sim V \iff \sum_{u \in U} u = \sum_{v \in V} v.$$

Clearly, $U \sim V$ simply means that these webs are proper and realize the same allocation. It also has to be clear that $U \simeq V$ implies $U \sim V$ for all proper webs U and V . Given a proper web V , a proper web U such that $U \sim V$ may be referred to as a *virtual* web (relative to V).

Definition 1.1.5 *An allocation x is called perfectly contractual if there exists a proper web V such that $x = x(V)$, and for every proper web U such that $U \sim V$ the allocation $x = x(U)$ is contractual.*

Perfectly contractual allocations correspond with agents' perfectly contractual behavior, which can be treated in the following way. When the agents are signing contracts, they should take care that not only these contracts would be short enough, as for the properly contractual behavior, but also would be *differently directed* (*i.e.*, they have sufficiently many different exchange proportions) to provide the opportunity to break the “unluckily directed” ones. So agents are allowed not only to break contracts partially but also to change (in a sense) the “directions” of contracts without loss of low stable property. One can see an analogy with the hedge policy, the main difference being that here we are speaking about the exchange-of-goods based contracts (barter), by the signing of which and moving into agreements via “tacks or traverses,” the agents reach a final resource reallocation.

This may also be treated in terms of an *optional agreement*. In fact, in a perfectly contractual allocation, the society is protected from the possibility that some coalition initiates a new recontracting process. A coalition may hope that via recontracting, it will ‘gather more profitable harvest,’ *i.e.*, find better consumption programs for its members. The following scenario may take place. A coalition, acting through its members, may suggest to the non-members of the coalition, who are involved in the coalition contracting, that the contracts be rewritten so that

- (i) the same allocation is realized,
- (ii) nobody has incentives to *break* new contracts partially, *i.e.*, the new web inherits the lower stable property of the initial web relative to the partial breaking of contracts.

In such a case, the non-members of the coalition may sign these new agreements as long as they have no revealed incentives to refrain from doing so (possibly the coalition members are good negotiators). Once these new agreements are signed, the coalition breaks a part of the contracts and signs a new contract that as a whole provides the coalition members with better consumption bundles. However, for a perfectly contractual allocation, this hypothetical behavior of every coalition cannot be profitable.

Certainly the property of an allocation to be perfectly contractual is the strongest kind of stability. The following definition extends the notion of being *perfect* to a single contract.

Definition 1.1.6 Let V be a web. A coherent contract $v \in V$ is perfect if every web U such that $U \sim \{v\}$ is stable relative to $(x(V) - v)$, which is taken as the initial endowments.

A web (subweb) which contains only perfect contracts is called perfect.

Remark 1.1.2 We would also like to mention an alternative definition of a web's perfect subweb of contracts.

Let V be a web. A subweb $U \subset V$ is called perfect if it is proper and, for every web W proper relative to $(x(V) - \sum_{u \in U} u)$ and such that $W \sim U$, the web $(V \setminus U) \cup W$ is stable. A contract $v \in V$ is called perfect if the subweb $\{v\}$ is perfect.

Note that according to this definition, every web containing at least one perfect contract is stable. Notice also the difference between the two following statements for $U \subset V$ which arises in this case: “ U is a perfect subweb” and “ U is a perfect web relative to $(x(V) - \sum_{u \in U} u)$ ”. The first one implies the second, but in general the reverse is not true (since in the first case it is allowed to break contracts in $V \setminus U$, but in the second one it is not).

If one assumes this definition, the contracts and their webs (subwebs), perfect in the sense of Definition 1.1.6, may be renamed as perfectly coherent. Note also that due to Proposition 1.1.1 and its corollaries, for *smooth economies* both *variants of the definitions of a perfect contract and of a perfect web (subweb) are equivalent*. ■

The described scheme of agents' interaction during the contract process is illustrated in the following Example 1.1.1. Moreover, this example presents a properly contractual allocation, which *is not perfectly* contractual.

Example 1.1.1 Consider an exchange economy with *two* commodities and with *three* agents having the following characteristics. Let $X_i = \mathbb{R}_+^2$, $i = 1, 2, 3$ be the agents' consumption sets, and let preferences be defined via utility functions $u_i : X_i \rightarrow \mathbb{R}$. Let endowments $\mathbf{e}_i \in X_i$ and utilities have the form

$$\begin{aligned} u_1(z) &= \min\{4z_1 + 4z_2, z_1 + 7z_2\}, & \mathbf{e}_1 &= (2, \frac{1}{2}), \\ u_2(z) &= \min\{4z_1 + 6z_2, 3z_1 + 7z_2\}, & \mathbf{e}_2 &= (\frac{5}{4}, \frac{9}{4}), \\ u_3(z) &= \min\{20z_1 + z_2, z_1 + 20z_2\}, & \mathbf{e}_3 &= (\frac{3}{4}, \frac{5}{4}). \end{aligned}$$

Let us consider an allocation $x = (x_1, x_2, x_3) \in \mathbb{R}_+^6$, where

$$x_1 = (1, 1), \quad x_2 = (2, 2), \quad x_3 = (1, 1).$$

This allocation can be realized by a web consisting of the only contract $w = x - \mathbf{e}$. Since $u_1(x_1) = 8 > 5\frac{1}{2} = u_1(\mathbf{e}_1)$, $u_2(x_2) = 20 > 18\frac{1}{2} = u_2(\mathbf{e}_2)$, $u_3(x_3) = 21 > 16\frac{1}{4} = u_3(\mathbf{e}_3)$, the allocation x is individually rational. As in our case for each agent, as soon as the sets $\mathcal{P}_i(x_i)$ of strictly better consumption bundles are represented as the intersection of some (open) cone with the vertex at the point x_i and positive orthant, then the individual rationality of x implies that the web $W = \{w\}$ is *proper* relative to \mathbf{e} . Moreover, in the next section a stronger property will be established—this allocation $x(W)$ is properly contractual.

Further, coalition $\{1, 2\}$ can propose to agent 3 to rewrite contract $w = x - \mathbf{e}$, dividing it due to an allocation $y \in X$ into two contracts: $u = x - y$ and $v = y - \mathbf{e}$, forming a *virtual* proper web $\{u, v\} \sim \{w\}$. Agent 3 can accept this proposal, since he/she realizes the same consumption program, and moreover, via the partial breaking of new contracts, the agent spreads his/her playing abilities to dominate current allocation. Let us consider an allocation $y = (y_1, y_2, y_3)$ satisfying

$$h = x_1 + x_2 - (y_1 + y_2) = (-3\varepsilon, \varepsilon), \quad \varepsilon > 0.$$

This condition compromises the requirement that the web $\{u, v\}$ be proper. For example, one can take

$$y_1 = \left(\frac{7}{4}, \frac{2}{3}\right), \quad \varepsilon = \frac{1}{24} \implies y_2 = \left(\frac{11}{8}, \frac{55}{24}\right), \quad y_3 = \left(\frac{7}{8}, \frac{25}{24}\right).$$

To check that the web $\{u, v\}$ is proper, apply Proposition 1.1.2. Now for agent 1 consider functional $p_1 = (1, 7)$ supporting $\mathcal{P}_1(x_1)$ at point x_1 , obtaining

$$p_1(x_1 - y_1) = 1, \quad p_1(y_1 - \mathbf{e}_1) = \frac{11}{12}.$$

For agent 2 consider supporting functional $p_2 = (4, 6)$, having

$$p_2(x_2 - y_2) = p_2(y_2 - \mathbf{e}_2) = \frac{3}{4}.$$

For agent 3 take supporting functional $p_3 = (20, 1)$, for which

$$p_3(x_3 - y_3) = 2\frac{11}{24}, \quad p_3(y_3 - \mathbf{e}_3) = 2\frac{7}{24}.$$

All described functionals satisfy the sufficient condition (1.1.2) of Proposition 1.1.2, which proves the web $\{u, v\}$ to be proper. This analysis is illustrated by Figure 1.1.3, where the left figure presents Edgeworth's box,⁶ constructed separately for coalition $\{1, 2\}$, the right one describes the case for agent 3 in his/her coordinate system. Here $\tilde{\mathbf{e}}_2 = x_1 + x_2 - \mathbf{e}_2$, $\tilde{\mathcal{P}}_2(x_2) = x_1 + x_2 - \mathcal{P}_2(x_2)$, $\tilde{y}_2 = x_1 + x_2 - y_2$ are the presentations of initial endowments, preferences and consumption vector y_2 in 1st agent's coordinate system, correspondingly. Notice that $\tilde{\mathbf{e}}_2 \neq \mathbf{e}_1$ and $\tilde{y}_2 \neq y_1$, since $\mathbf{e}_1 + \mathbf{e}_2 \neq x_1 + x_2 \neq y_1 + y_2$.

Finally, when agent 3 accepts the proposition of coalition $\{1, 2\}$, the members of this coalition can break contract $u = x - y$, realizing the allocation y as a kind of “new initial one,” and can sign a new contract $g = (g_1, g_2)$, where $g_1 = x_1 - y_1 + \delta(1, 1)$, $g_2 = x_2 - y_2 - h - \delta(1, 1)$, $\delta > 0$. Due to the definition of h , this is a contract in fact such that new consumption bundles are $z_1 = x_1 + \delta(1, 1)$, $z_2 = x_2 + (3\varepsilon - \delta, -\varepsilon - \delta)$. The utility of the first agent for the new bundle increases due to preference monotonicity. The utility of the second agent also increases if δ is small enough:

$$u_2(z_2) = u_2(x_2) + \min\{6\varepsilon - 10\delta, 2\varepsilon - 10\delta\} = u_2(x_2) + 2\varepsilon - 10\delta > u_2(x_2) \quad \text{for } \delta < \varepsilon/5.$$

So, acting in the described way, coalition $\{1, 2\}$ can reach a higher consumption level for its members (let all contracts be permissible), and studied allocation x is *not perfectly* contractual. ■

⁶On Edgeworth's box, see also Example 1.2.1 from the next section.

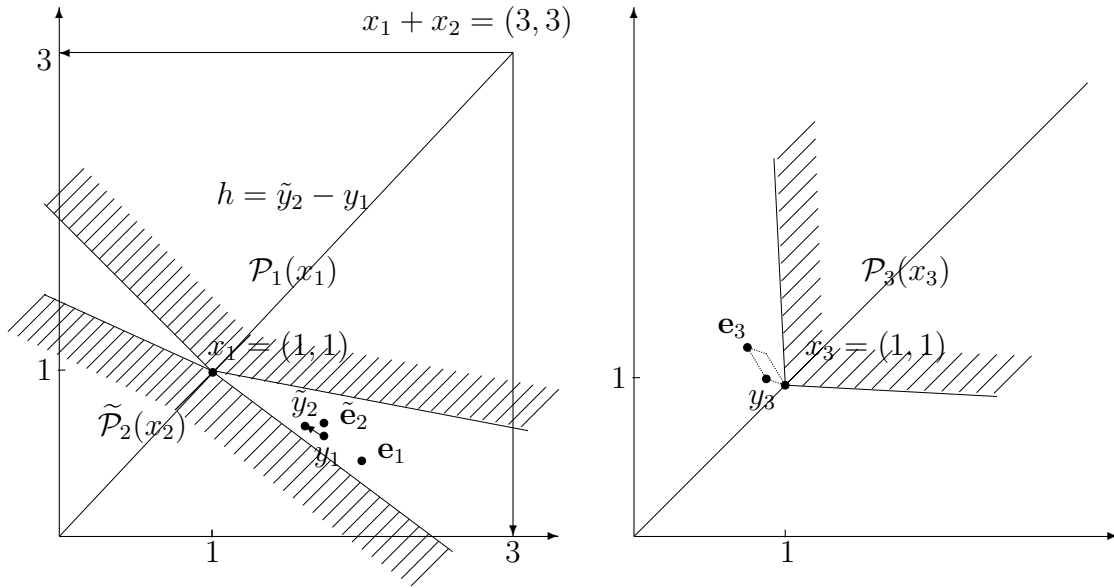


Figure 1.1.3: *Edgeworth's box for coalition $\{1, 2\}$ and agent 3's consumption separately*

Further, we consider the concept of *fuzzy contractual* allocation. The notion of properly contractual allocation presumes that agents are able to partially break contracts in such a way that every contract may be divided into several contracts with equal exchange proportions and some of these contracts may be broken, *i.e.*, instead of contract $v \in V$ the agents may deal with a finite family of contracts $\{u_\xi\}$, such that $\sum u_\xi = v$ and $u_\xi = \lambda_\xi v$ for some real $\lambda_\xi \geq 0$ for all ξ . Thus for partial breaking of contract v the members of coalition $S = \text{supp}(v)$ have to coordinate their actions. Relaxing this coordination requirement for fuzzy contractual allocation, we allow the agents to break contracts *asymmetrically* and together with $\sum u_\xi = v$ to require $(u_\xi)_i = \lambda_{\xi i} v_i$ for a real $\lambda_{\xi i} \geq 0$ for all ξ and i . Notice that now vector u_ξ may not be a contract at all, since $\sum_{i \in \mathcal{I}} u_{\xi i} = 0$ may not hold.

Let V be a web of contracts. For every $v \in V$ we consider and put into correspondence an n -dimension vector

$$t^v = (t_1^v, t_2^v, \dots, t_n^v), \quad 0 \leq t_i^v \leq 1, \quad \forall i \in \mathcal{I},$$

and let

$$v^t = (t_1^v v_1, t_2^v v_2, \dots, t_n^v v_n)$$

be the vector of commodity bundles formed from contract $v = (v_i)_{i \in \mathcal{I}}$ when all agents "break" individual bundles (fragments) of this contract in shares $(1 - t_i^v)_{i \in \mathcal{I}}$. Denote $T(V) = T = \{t^v \mid v \in V\}$ and introduce

$$V^T = \{v^t \mid v \in V, t^v \in T\}, \quad \Delta(V^T) = \sum_{v^t \in V^T} v^t. \quad (1.1.3)$$

Definition 1.1.7 An allocation $x \in \mathcal{A}(X)$ is called *fuzzy contractual* if there exists a proper web V such that $x = x(V)$ and for every $T(V)$ there is no barter contract $w = (w_1, \dots, w_n) \in \mathbb{R}^n$, $\sum_{i \in \mathcal{I}} w_i = 0$, such that for

$$\xi_i = \xi_i(T, V, w) = \mathbf{e}_i + \Delta_i(V^T) + w_i, \quad i \in \mathcal{I} \quad (1.1.4)$$

one has

$$\xi_i \succ_i x_i \quad \forall i : \xi_i \neq x_i. \quad (1.1.5)$$

So, for this kind of allocation the negation of domination means that the implementing web of contracts is *proper* and allocation is *stable* relative to asymmetric *partial breakings of contracts*. It is appropriate to compare this definition with Definition 1.3.5, which specifies fuzzy contractual allocations in an economy of Arrow–Debreu type (with a production sector): allowing only zero production, we have an equivalent result.

In economic terms, this notion can be explained in the following way. During recontracting agents may make mistakes, coordination among coalition members may work imperfectly, and so on. As a result, an agent i can (erroneously) think that after the partial breaking of current contracts he/she will have a commodity bundle $x_i^T = \mathbf{e}_i + \Delta_i(V^T)$ and that commodities from x_i^T may be mutually beneficial exchanged so that to dominate current allocation $x = (x_i)_{\mathcal{I}}$. If allocation $x(V)$ is not fuzzy contractual, then the last may (potentially) destroy agreements and allocation will be changed. Thus fuzzy contractual allocations are protected from this kind of agreement destructions.

Finishing the gallery of various kinds of allocation stability in a contractual economy, let us assume that the set of permissible contracts can be represented as a (finite) union of star-shaped sets, *i.e.*,

$$\mathcal{W} = \cup \mathcal{V}_\xi.$$

Note that \mathcal{W} is then a star-shaped set itself and \mathcal{V}_ξ may, in particular, be convex sets or, moreover, subspaces of \mathfrak{L} (as for incomplete markets or DI-economies).

Now, for a given web V , we can associate with each $v \in V$, $v \in \mathcal{W}$, certain sets in $\{\mathcal{V}_\xi\}$ and can require that if $v \in \mathcal{V}_\xi$ for a given ξ , then the contract v has to be coherent and either proper or perfect or neither proper nor perfect. When such an association is established, the allocation $x(V)$ is called *complex* (composition) contractual. In other words, a complex contractual allocation is stable relative to both the procedure of appropriately breaking contracts (depending on the set to which the contract belongs) and the procedure of signing new (permissible) contracts. Moreover, some additional requirements may be imposed on a web realizing an allocation. These requirements always take the form of joint stability of contracts from the web. Below we will see such kind of construction in the context of an incomplete market model.

1.2 Contracts in a standard exchange economy

In the classical setting, it is indirectly assumed that for a pure exchange economy all kinds of commodity exchanges are allowed, the only restriction being that the

realized consumption programs (bundles) have to belong to the agents' consumption sets, *i.e.*, the allocations have to be feasible. This is why when one complements this model with a contract-based mechanism it is logical to think that all contracts are permissible, *i.e.*, one may presume that $\mathcal{W} = \mathfrak{L}$, where \mathfrak{L} is the space of allocations. In fact, sometimes it suffices to require a bit less. So, speaking of a *standard* exchange economy, we always assume that the corresponding contractual economy is such that the set \mathcal{W} of all permissible contracts is *radial* (absorbing)⁷ at zero in \mathfrak{L} . In all other aspects the standard model coincides with model \mathcal{E} . Further let us recall some definitions.

A pair (x, p) is said to be a *quasi-equilibrium* of \mathcal{E} if $x \in \mathcal{A}(X)$ and there exists a linear functional $p \neq 0$ onto L such that

$$\langle p, \mathcal{P}_i(x_i) \rangle \geq px_i = p\mathbf{e}_i, \quad \forall i \in \mathcal{I}.$$

A quasi-equilibrium such that $x'_i \in \mathcal{P}_i(x_i)$ actually implies $px'_i > px_i$ is a *Walrasian or competitive equilibrium*.

On the other hand, $x \in \mathcal{A}(X)$ is said to be dominated (blocked) by a nonempty coalition $S \subseteq \mathcal{I}$ if there exists $y^S \in \prod_{i \in S} X_i$ such that $\sum_{i \in S} y_i^S = \sum_{i \in S} \mathbf{e}_i$ and $y_i^S \in \mathcal{P}_i(x_i) \forall i \in S$.

The *core* of \mathcal{E} , denoted by $\mathcal{C}(\mathcal{E})$, is the set of all $x \in \mathcal{A}(X)$ that are blocked by no (nonempty) coalition.

Weak Pareto boundary for \mathcal{E} , denoted by $\mathcal{PB}^w(\mathcal{E})$, is the set of all $x \in \mathcal{A}(X)$ that cannot be dominated by the coalition \mathcal{I} of all agents.

An allocation $x \in \mathcal{A}(X)$ is called *individually rational* if it cannot be dominated by singleton coalitions. $\mathcal{IR}(\mathcal{E})$ denotes the set of all these allocations.

The above definitions imply

$$\mathcal{C}(\mathcal{E}) \subset \mathcal{PB}^w(\mathcal{E}) \cap \mathcal{IR}(\mathcal{E}).$$

In general, the reverse inclusion is true only for a two-consumer economy.

Next we would like to make several remarks on the concept of Pareto optimality (Pareto boundary). First recall a stronger concept of optimality, sometimes called the *strong Pareto optimality*. For preordered preferences⁸ \succeq_i an allocation $x = (x_i)_{\mathcal{I}} \in \mathcal{A}(X)$ is a *strong Pareto optimum* if there is no $z = (z_i)_{\mathcal{I}} \in \mathcal{A}(X)$ such that

$$z_i \succeq_i x_i \quad \forall i \in \mathcal{I} \quad \& \quad \exists j \in \mathcal{I} : z_j \succ_j x_j.$$

Let us denote by $\mathcal{PB}^s(\mathcal{E})$ the *strong Pareto boundary*, the set of all strongly Pareto optimal allocations. The definitions imply $\mathcal{PB}^s(\mathcal{E}) \subset \mathcal{PB}^w(\mathcal{E})$.

There is one more possibility of defining the optimality concept in an economic model. It takes an intermediate position between the two notions considered above. We will see below that exactly this kind of optimality is realized by upper contractual allocations.

⁷A set $A \subset \mathfrak{L}$ is radial at a point $a \in A$ if, for every $b \in \mathfrak{L}$, $\lambda b \in (A - a)$ for all real $0 \leq \lambda \leq \lambda_b$ and some $\lambda_b > 0$. Notice that a *convex* radial set is star-shaped at its every point.

⁸This is a reflexive, complete and transitive *non-strict* binary relation.

Let us call an allocation $x = (x_i)_{\mathcal{I}} \in \mathcal{A}(X)$ *strictly Pareto optimal* if there is no coalition $S \subseteq \mathcal{I}$ for which there exists a $y^S \in \prod_{i \in S} X_i$ such that $\sum_{i \in S} y_i^S = \sum_{i \in S} x_i$ and $y_i^S \succ_i x_i$ for each $i \in S$. In other words, x is an allocation in the core of the other economy, which differs from the original one in only one aspect, namely, allocation x is taken as the initial endowments. In our opinion, the last concept of optimality presents the most precise form of Pareto optimality.

Denote by $\mathcal{PB}(\mathcal{E})$ the *strict Pareto boundary*. It is easily seen that

$$\mathcal{PB}^s(\mathcal{E}) \subset \mathcal{PB}(\mathcal{E}) \subset \mathcal{PB}^w(\mathcal{E}).$$

Therefore, if under some conditions one can show that an $x = (x_i)_{\mathcal{I}} \in \mathcal{PB}^w(\mathcal{E})$ is strongly Pareto optimal (this is the case if, for example, the preferences are locally non-satiated and $x \in \text{int}X$),⁹ then the allocation x is strictly Pareto optimal as well.

The next important notion, fruitfully working in the theory of economic equilibrium, is the concept of fuzzy core. Recall that any vector

$$t = (t_1, \dots, t_n) \neq 0, \quad 0 \leq t_i \leq 1, \quad \forall i \in \mathcal{I}$$

may be identified with a fuzzy coalition, where the real number t_i being interpreted as the measure of agent i in the coalition. A coalition t is said to dominate (block) an allocation $x \in \mathcal{A}(X)$ if there exists $y^t \in \prod_{\mathcal{I}} X_i$ such that

$$\sum_{i \in \mathcal{I}} t_i y_i^t = \sum_{i \in \mathcal{I}} t_i e_i \iff \sum_{i \in \mathcal{I}} t_i (y_i^t - e_i) = 0 \quad (1.2.1)$$

and

$$y_i^t \succ_i x_i, \quad \forall i \in \text{supp}(t) = \{i \in \mathcal{I} \mid t_i > 0\}. \quad (1.2.2)$$

The set of all feasible allocations which cannot be dominated by fuzzy coalitions is denoted by $\mathcal{C}^f(\mathcal{E})$ and is called the *fuzzy core* of the economy \mathcal{E} .

1.2.1 Preliminary results and an example

We start the analysis with the theorem, which establishes relationships between the core and contractual allocations.

Theorem 1.2.1 *Let \mathcal{E}^c be a contractual economy such that $\mathcal{W} = \mathfrak{L}$, and let x be a feasible allocation. Then:*

- (i) x is contractual $\iff x \in \mathcal{C}(\mathcal{E}) \cap \mathcal{PB}(\mathcal{E})$;
- (ii) x is upper contractual $\iff x \in \mathcal{PB}(\mathcal{E})$;
- (iii) x is lower contractual $\iff x \in \mathcal{IR}(\mathcal{E})$;
- (iv) x is weakly contractual $\iff x \in \mathcal{IR}(\mathcal{E}) \cap \mathcal{PB}(\mathcal{E})$.

⁹Moreover, it is well known that if the preferences are strictly monotonic and $X_i = \mathbb{R}_+^l$ (l is the number of commodities) for all i , then the concepts of strong and weak Pareto optimality are equivalent.

Proof of Theorem 1.2.1. The necessity of (i)–(iv) directly follows from the definitions. To check their sufficiency, let us consider the web $V_e(x)$ consisting of only one contract $v = x - e$. This is really a web since $x \in \mathcal{A}(X)$ and by assumption, $x - e \in \mathcal{W}$. A routine checking of Definitions 1.1.1, 1.1.2 completes the proof. ■

The following theorem characterizes the equilibrium allocations in terms of the properly contractual ones. So, the result of this theorem allows us to consider stability of an allocation relative to the partial breaking of contracts as a specific form of perfect competition conditions that delivers another contractual sight on this subject. This approach is developed via the idea of fuzzy contractual allocations and their relationships with equilibria (they coincide under weaker assumptions) that are revealed at the end of section, see Propositions 1.2.1, 1.2.2 and Lemma 1.2.2.

Theorem 1.2.2 *Let \mathcal{E}^c be a smooth contractual economy, the set $\mathcal{W} \subseteq \mathcal{L}$ be radial at zero and x be a feasible allocation such that $x \in \text{int}X$ and $\nabla u_i(x_i) \neq 0$ for some i and all agents are non-satiated. Then the following statements are equivalent:*

- (i) x is an equilibrium allocation;
- (ii) x is Pareto optimal and there exists a coherent web V realizing this allocation, i.e., $x = x(V)$ such that V is coherent and upper stable;
- (iii) x is a properly contractual allocation;
- (iv) x is a perfectly contractual allocation.

Moreover, if (x, p) is an equilibrium and V is a web realizing $x = x(V)$, then V is a coherent web if and only if $pv_i = 0, \forall v \in V, \forall i \in \mathcal{I}$.

Remark 1.2.1 The analysis of the theorem's proof demonstrates that the implication (i) \Rightarrow (iii) is true in the general case for the non-smooth, possibly satiated preferences and without the requirement $x \in \text{int}X$, i.e., in a standard exchange economy every equilibrium is a properly contractual allocation.

Recall also that due to Theorem 1.2.1 (ii), statement (ii) of Theorem 1.2.2, in which the Pareto optimality of the allocation x is claimed, is equivalent to the existence of a coherent and upper stable web realizing this allocation. ■

Proof of Theorem 1.2.2. To establish the equivalence of (i)–(iv) recall that under the theorem's conditions and via Second Welfare Theorem, $x \in \mathcal{A}(X)$ is Pareto optimal iff there exists an $i \in \mathcal{I}$ such that for $p = \nabla u_i(x_i)$

$$\langle \mathcal{P}_j(x_j), p \rangle > \langle p, x_j \rangle, \quad \forall j \in \mathcal{I}.^{10} \quad (1.2.3)$$

Let without loss of generality be $i = 1$.

¹⁰In particular, it implies that non-zero $\nabla u_i(x_i)$ and $\nabla u_j(x_j)$ coincide up to a normalization for all $i \neq j$. It is easy to analyze the omitted case when $\nabla u_i(x_i) = 0 \forall i \in \mathcal{I}$.

Further, let us notice that $(iv) \Rightarrow (iii) \Rightarrow (ii)$. To establish $(ii) \Rightarrow (i)$ define $p = \nabla u_1(x_1)$ and apply Proposition 1.1.1 to obtain $pv_i \geq 0$ for all $v \in V$ and $i \in \mathcal{I}$. Summing up over $v \in V$ for every $i \in \mathcal{I}$, we arrive at

$$\langle p, \Delta_i(V) \rangle \geq 0 \implies px_i \geq p\mathbf{e}_i.$$

This, due to the feasibility of x , implies $px_i = p\mathbf{e}_i$ for all i which by (1.2.3) yields the equilibrium properties of (x, p) .

Now let us prove $(i) \Rightarrow (iv)$. Let $v = v^r = (x - \mathbf{e})/r$, where the natural r is chosen based on the assumption that \mathcal{W} is absorbing, and define the set V consisting of the r identical copies of the contract v . Clearly, V is a web. Now the equilibrium properties of the pair (x, p) imply $pv_i = 0$, $\forall v \in V$, $\forall i \in \mathcal{I}$ and therefore (1.1.1) is true (one can take $p = \nabla u_i(x_i)$ as the equilibrium price vector for x). Applying the sufficiency part of Proposition 1.1.1, one concludes that V is a coherent and, moreover, proper web. Now let $U \sim V$ for a proper web U . Once again due to Proposition 1.1.1 (necessity), the properness of U implies $pu_i \geq 0$, $\forall u \in U$, $\forall i \in \mathcal{I}$. But then the contract specification ($\sum_{i \in \mathcal{I}} u_i = 0$) implies $pu_i = 0$, $\forall u \in U$, $\forall i \in \mathcal{I}$. Finally, if for a $T \subseteq \mathcal{I}$, $T \neq \emptyset$ and a web $W \in F(U, T)$ we have $W \succ_T U$, then it follows from the equilibrium definition that

$$\langle p, y_i(W) \rangle > \langle p, x_i(U) \rangle, \quad \forall i \in T.$$

Consequently, summing up these inequalities over $i \in T$, we arrive at the contradiction with the contract specification of $w \in W \setminus U$ (since $S(w) \subseteq T$, because of $W \in F(U, T)$ and (ii)). The final part of theorem is also clear. \blacksquare

Further, let us consider an example demonstrating the difference between the various notions of contractual allocation. Of course, for this difference to be realized when the partial breaking of contracts is allowed, the conditions of Theorem 1.2.2 have to be invalid, and either the utilities have to be non-smooth or the allocation has to belong to the boundary of X .

The following example, borrowed from [Kozyrev \(1982a\)](#), shows that for non-differentiable utility functions a properly contractual allocation may not be an equilibrium.

Example 1.2.1 Consider a two-commodities exchange economy with two consumers, where $X_i = \mathbb{R}_+^2$. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ denote the consumption bundles of 1st and 2nd agent, respectively. Let preferences be defined onto \mathbb{R}_+^2 by the strictly monotonic utility functions

$$u_1(x_1, x_2) = 2\sqrt{x_1x_2} + x_1 + x_2, \quad u_2(y_1, y_2) = 2\sqrt{y_1y_2} + y_1 + y_2 + \min\{y_1, y_2\}.$$

For the initial endowments, we take the vectors

$$\mathbf{e}_1 = (1, 0), \quad \mathbf{e}_2 = (0, 1), \quad \bar{\mathbf{e}} = \mathbf{e}_1 + \mathbf{e}_2 = (1, 1), \quad \mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2) = ((1, 0), (0, 1)).$$

In further considerations, we make use of the ‘‘Edgeworth’s box,’’ the well known subset of \mathbb{R}^2 :

$$EB(\bar{\mathbf{e}}) = \{x \in \mathbb{R}^2 \mid 0 \leq x \leq (1, 1) = \bar{\mathbf{e}}\}.$$

One can interpret $x \in EB(\bar{\mathbf{e}})$ as the consumption of the first consumer and $(\bar{\mathbf{e}} - x) = y$ as the consumption of the second one. The point x may also be associated with the allocation $(x, \bar{\mathbf{e}} - x)$.

A simple analysis shows that in this example Pareto boundary is the set

$$\mathcal{PB} = \text{co}\{(0, 0), (1, 1)\} = \{x \in EB(\bar{\mathbf{e}}) \mid x_1 = x_2 = \alpha, 0 \leq \alpha \leq 1\},$$

i.e., it is the diagonal of $EB(\bar{\mathbf{e}})$.

Since every equilibrium allocation is Pareto optimal and due to the fact that if it is an interior point of the box, the price vector has to coincide up to a normalization with $\nabla u_1(x)$, the vector $(1, 1)$ has to be an equilibrium price vector. Clearly, the points $(1, 1)$ and $(0, 0)$ are not equilibrium allocations. Consequently, $p = (1, 1)$ is *the only* (up to a normalization) *equilibrium price*. Using the budget constraints $px_i = pe_i$, one can easily find the unique equilibrium allocation which corresponds to the first agent's consumption bundle $(\frac{1}{2}, \frac{1}{2})$ in the Edgeworth's box.

The core in this economy with two consumers coincides with the set $\mathcal{PB} \cap \mathcal{IR}$ which, in turn, is the set of all contractual and weakly contractual allocations and can be easily calculated to be

$$\mathcal{PB} \cap \mathcal{IR} = \{x \in \mathcal{PB} \mid u_1(x) \geq u_1(\mathbf{e}_1), u_2(\bar{\mathbf{e}} - x) \geq u_2(\mathbf{e}_2)\} = \text{co}\{(\frac{1}{4}, \frac{1}{4}), (\frac{4}{5}, \frac{4}{5})\}.$$

Next let us find the set of all properly contractual allocations. Clearly, this is the set of all points $x = (\alpha, \alpha)$ in $\mathcal{PB} \subset EB$ for which the derivatives of the utility functions u_1 and u_2 are not positive in the directions of $h_1 = \mathbf{e}_1 - (\alpha, \alpha)$ and $h_2 = \mathbf{e}_2 - (1 - \alpha, 1 - \alpha)$, *i.e.*, we need to solve the system of equations

$$\partial_{h_1} u_1(\alpha, \alpha) \leq 0, \quad \partial_{h_2} u_2(1 - \alpha, 1 - \alpha) \leq 0.$$

A direct calculation gives

$$\nabla u_1(\alpha, \beta) = (\sqrt{\beta/\alpha} + 1, \sqrt{\alpha/\beta} + 1)$$

for all $\alpha > 0, \beta > 0$ and

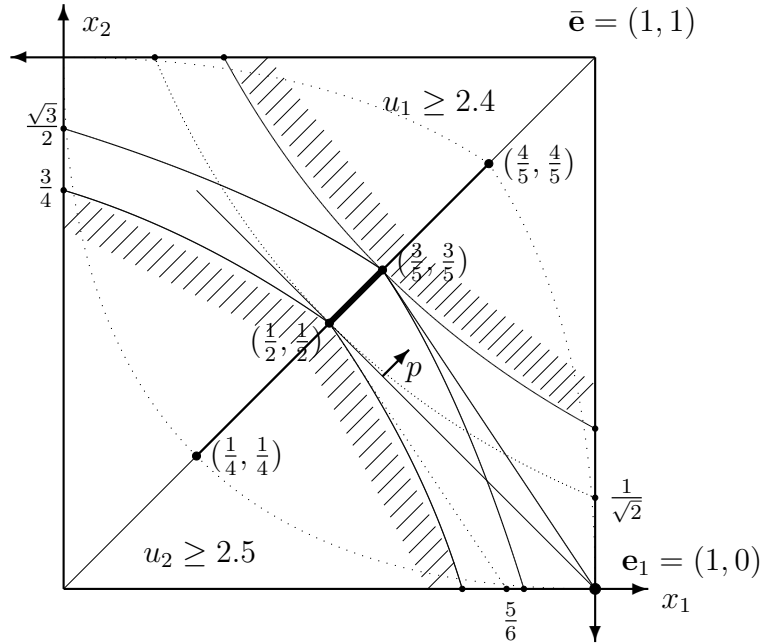
$$\nabla u_2(1 - \alpha, 1 - \beta) = (\sqrt{(1 - \beta)/(1 - \alpha)} + 2, \sqrt{(1 - \alpha)/(1 - \beta)} + 1)$$

for $\alpha > \beta > 0, 1 - \alpha > 0$. Calculating the inner products and substituting $\alpha = \beta$ (*i.e.*, passing to the limit for $\beta \rightarrow \alpha$) yields

$$1 - 2\alpha \leq 0, \quad 5\alpha - 3 \leq 0.$$

As a result, the set of properly contractual allocations is described as $\text{co}\{(\frac{1}{2}, \frac{1}{2}), (\frac{3}{5}, \frac{3}{5})\}$ and does *not coincide* with (but contains!) the set of *equilibrium* allocations. Figure 1.2.4 illustrates the analysis conducted.

In this example it seems to be interesting to clarify the structure of the set corresponding to another new theoretical concept, the set of all *lower* properly contractual allocations. With this in mind, let me describe allocations which cannot be dominated

Figure 1.2.4: *Non-smooth preferences*

by the coalition $\{1\}$ relative to the partial breaking of the contract $x - \mathbf{e}$ (i.e., the web consisting of only one contract). In the Edgeworth's box, they are the points (α, β) satisfying the condition

$$\partial_{h_1} u_1(\alpha, \beta) \leq 0, \quad h_1 = \mathbf{e}_1 - (\alpha, \beta).$$

Now calculating the directional derivative in the form of the inner product and substituting the value for \mathbf{e}_1 , we obtain for $(\alpha, \beta) \gg 0$

$$(1 - \alpha)(\sqrt{\beta/\alpha} + 1) - \beta(\sqrt{\alpha/\beta} + 1) \leq 0,$$

which can be rewritten as

$$\sqrt{\beta/\alpha} \leq (\sqrt{\alpha} + \sqrt{\beta})^2 - 1 \iff (\sqrt{\alpha} + \sqrt{\beta}) \leq (\sqrt{\alpha} + \sqrt{\beta})^2 \sqrt{\alpha}.$$

Dividing both sides by $\sqrt{\alpha} + \sqrt{\beta} > 0$ yields after transformations

$$\sqrt{\beta\alpha} \geq 1 - \alpha \iff \beta + 2 \geq \frac{1}{\alpha} + \alpha.$$

Therefore, the set we are interested in is the part of the *epigraph* of the curve $x_1^2 = \frac{1}{x_1} + x_1 - 2$ inside the rectangle $0 \leq (x_1^1, x_1^2) \leq (1, 1)$. This set is shown in Figure 1.2.5. It is harder to carry out a similar analysis for agent 2 by purely analytical means. This is why we also turn to geometrical considerations. Let us choose an individual-rational point (α, β) for agent 2 in the coordinate system of agent 1, which lies in the interior of the box. If this point *belongs to the diagonal*, i.e., $\alpha = \beta$, then as we have seen above, it satisfies the required property only if $\alpha \leq 3/5$. Now assume that the point (α, β) *does not belong to the diagonal*, i.e., $\alpha \neq \beta$ and consider the second

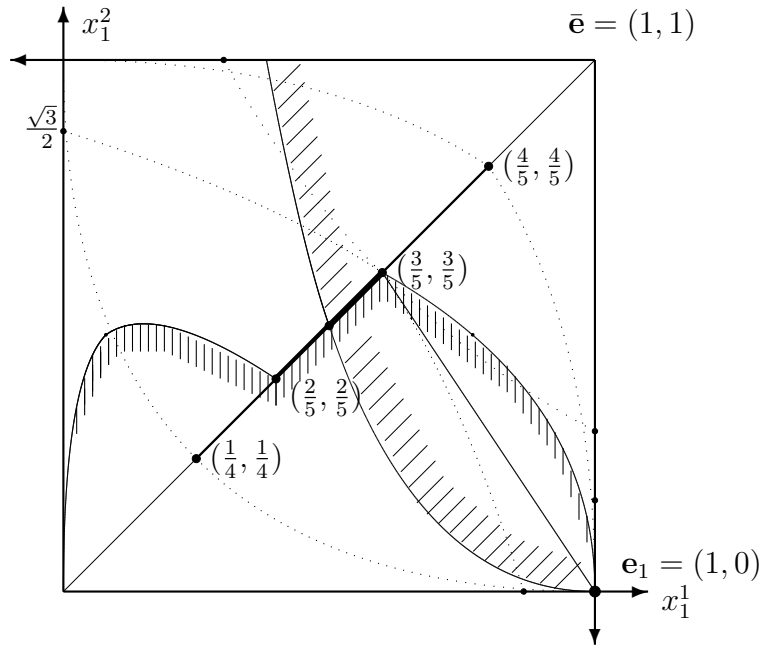


Figure 1.2.5: Lower proper contractual allocations

agent's indifference curve (in the coordinate system of the 1st agent) passing through this point. This curve corresponds to the graph of a concave function; therefore, if our point is desired, *i.e.*, $\partial_{h_2} u_2(1-\alpha, 1-\beta) \leq 0$ for $h_2 = \mathbf{e}_2 - (1-\alpha, 1-\beta)$, then *every* point of the curve to the *right* of (α, β) is also a desired one. Next, if $\alpha \leq \beta$, the right-hand side of the curve intersects the diagonal at a point (γ, γ) where, by the above argument $\gamma \leq 3/5$. Moreover, for $\alpha < \beta$, we have $u_2(1-\alpha, 1-\beta) = u_1(1-\alpha, 1-\beta) + 1-\beta$. Now calculate the gradient and pass to the limit for $(\alpha, \beta) \rightarrow (\gamma, \gamma)$, $\gamma \geq 0$. The result is that (γ, γ) must satisfy the condition

$$\langle (2, 3), (\gamma - 1, \gamma) \rangle \leq 0 \Rightarrow \gamma \leq 2/5.$$

Consequently, in the coordinate system of agent 1 the desired set can be viewed as the *ordinate set* of the curve consisting of three parts: for the interval $[0, \frac{2}{5}]$, this is the graph of a concave function which is equal to 0 at 0 and $\frac{2}{5}$ at $\frac{2}{5}$; for the interval $[\frac{2}{5}, \frac{3}{5}]$ the curve coincides with the diagonal; for the interval $[\frac{3}{5}, 1]$ this is again the graph of a concave function which is equal to $\frac{3}{5}$ at $\frac{3}{5}$ and 0 at 1. Thus we arrive at a *non-convex* set. The intersection of this second set with the first one represents the set of all *lower* properly contractual allocations in the Edgeworth's box. Being intersected with the Pareto boundary (the diagonal) in addition, this set represents the already known set of all properly contractual allocations; see Figure 1.2.5.

In the framework of this example, one can also demonstrate the difference between the notions of coherent and proper webs as well as between the corresponding stability properties. Let us consider an allocation, in which agent 1 consumes the bundle $(\frac{3}{5}, \frac{3}{5})$; see Figure 1.2.6. Set $\hat{h}_1 = (0, 1) - (\frac{3}{5}, \frac{3}{5}) = (-\frac{3}{5}, \frac{2}{5})$ and calculate the derivative of the

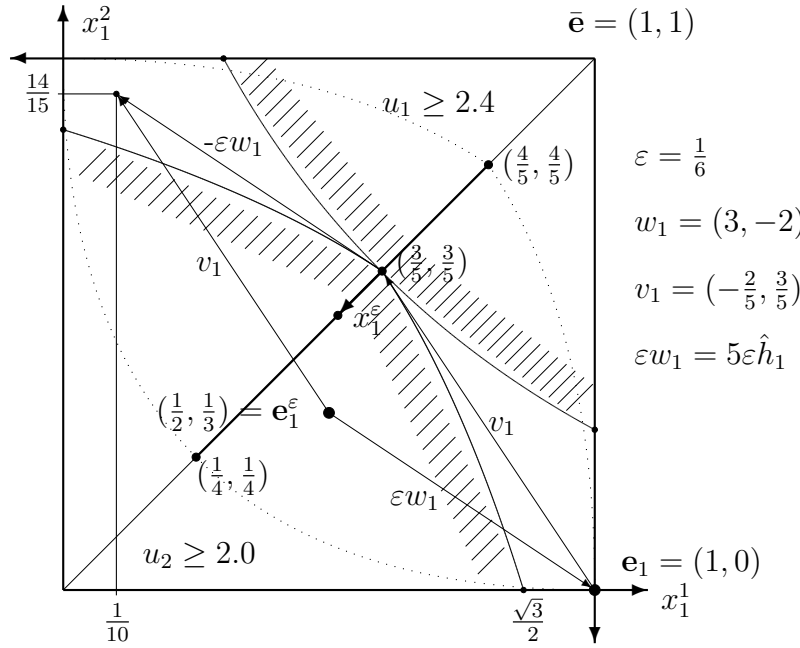


Figure 1.2.6: *Coherent non-proper contractual allocations* function u_1 at the point $(\frac{3}{5}, \frac{3}{5})$ in the direction of \hat{h}_1 :

$$\partial_{\hat{h}_1} u_1(\frac{3}{5}, \frac{3}{5}) = \langle (2, 2), \hat{h}_1 \rangle = -\frac{2}{5} < 0.$$

For the allocation considered, the second agent's consumption bundle is a point of non-differentiability of his/her utility function, but its derivative in the direction of $\hat{h}_2 = (1, 0) - (\frac{2}{5}, \frac{2}{5}) = (\frac{3}{5}, -\frac{2}{5})$ can be easily calculated as the limit for $(\alpha, \beta) \rightarrow (\frac{3}{5}, \frac{3}{5})$ of the inner product of the vector-gradient of u_2 calculated at the point $(1 - \alpha, 1 - \beta)$, $\beta > \alpha > 0$, and the vector \hat{h}_2 . Since for $\beta > \alpha > 0$, $1 - \beta > 0$,

$$\nabla u_2(1 - \alpha, 1 - \beta) = (\sqrt{(1 - \beta)/(1 - \alpha)} + 1, \sqrt{(1 - \alpha)/(1 - \beta)} + 2);$$

upon calculating the inner products and substituting $\alpha = \beta = \frac{3}{5}$, one obtains

$$\partial_{\hat{h}_2} u_2(\frac{2}{5}, \frac{2}{5}) = \langle (2, 3), \hat{h}_2 \rangle = 0.$$

Next, for a sufficiently small $\varepsilon > 0$, take allocation

$$\mathbf{e}^\varepsilon = \mathbf{e} - \varepsilon w$$

to be the initial endowments and consider web $V^\varepsilon = \{v, \varepsilon w\}$ where

$$v = (v_1, v_2), \quad v_1 = -v_2 = (\frac{3}{5}, \frac{3}{5}) - (1, 0) = (-\frac{2}{5}, \frac{3}{5}),$$

and

$$w = (w_1, w_2), \quad w_1 = -w_2 = (3, -2)$$

is the contract in which the first consumer exchanges 2 units of commodity 2 for 3 units of commodity 1. For example, one can take $\varepsilon = \frac{1}{6}$. Obviously

$$\mathbf{e}^\varepsilon + v + \varepsilon w = ((\frac{3}{5}, \frac{3}{5}), (\frac{2}{5}, \frac{2}{5})),$$

i.e., the allocation considered is realized by the web V^ε relative to the endowments \mathbf{e}^ε . The above calculations of the derivatives in the directions of \hat{h}_1, \hat{h}_2 together with the previous calculations of the derivatives in the directions of h_1 and h_2 (for $\alpha = \frac{3}{5}$), show that each contract in the web V^ε is *coherent* relative to \mathbf{e}^ε . However, for every $\varepsilon \in (0, \frac{1}{6}]$ the web V^ε is not proper since after the breaking of a half of εw and after the partial breaking of the contract v , the broken part being $\delta = \frac{5}{2}\varepsilon < 1$, it realizes the allocation for which the first agent's consumption program has the form

$$\frac{1}{2}\varepsilon(3, -2) + (1 - \frac{5}{2}\varepsilon)(-\frac{2}{5}, \frac{3}{5}) + (1, 0) - \varepsilon(3, -2) = (\frac{3}{5}, \frac{3}{5}) - \varepsilon(\frac{1}{2}, \frac{1}{2}) = x_1^\varepsilon.$$

Therefore, the proposed partial breaking of contracts is, in fact, profitable for the second agent since it results in an increase in her/his consumption by $\varepsilon(\frac{1}{2}, \frac{1}{2})$. This proves that the web V^ε is *not proper* relative to \mathbf{e}^ε . ■

The above example stimulates a more careful study of the mathematical properties of properly (and perfectly) contractual allocations in the situations where the conditions of Theorem 1.2.2 are invalid, that we are going to do below.

1.2.2 Properly and perfectly contractual allocations

Let us begin with the discussion of lower properly contractual allocations. Due to the definition, they are the allocations $x(V)$ which can be implemented by a web of contracts V , which is stable relative to the partially breaking contracts. As soon as for the standard market model every contract is permissible, web V can be replaced by web $\Delta V = \{u\}$ consisting of only one contract $u = \sum_{v \in V} v = x - \mathbf{e}$. It can be easily seen that the low stability of the original web implies the same type of stability for web ΔV . Therefore, one can restrict the analysis of properly (not only lower) contractual allocations to the webs having the form $\{x - \mathbf{e}\}$. Further, it will be clear that all the conclusions can be easily applied to the case of webs consisting of multiple contracts. However, in the case of webs consisting of a single contact, one can directly conclude from the definition that an allocation $x \in \mathcal{A}(X)$ is *lower properly contractual* if and only if

$$[x_i, \mathbf{e}_i] \cap \mathcal{P}_i(x_i) = \emptyset, \quad \forall i \in \mathcal{I}, \quad (1.2.4)$$

where

$$[x_i, \mathbf{e}_i] = \{\lambda x_i + (1 - \lambda)\mathbf{e}_i \mid 0 \leq \lambda \leq 1\}.$$

Now via Theorem 1.2.1 (ii) the definitions of stability clearly imply that *weakly properly contractual* allocations are exactly the *Pareto optimal* and *satisfying* (1.2.4) ones.

The properly contractual allocations can be characterized in similar terms as follows. Breaking the $(1 - \lambda)$ part of the contract $x - \mathbf{e}$ and signing a new contract v , the members of $S \subset \mathcal{I}$ realize the collection of consumption bundles $y^S = (y_i^S)_S$ such that $\sum_S y_i^S = \sum_S (\lambda x_i + (1 - \lambda)\mathbf{e}_i)$. Since the definition of properly contractual allocation forbids this type of domination, the allocation x must belong to the core of the economy with the initial endowments $\lambda x + (1 - \lambda)\mathbf{e} = \mathbf{e}_x^\lambda = \mathbf{e}_x \in [x, \mathbf{e}]$, which we denote by $\mathcal{C}(\mathcal{E}_x^\lambda)$. Thus an allocation x is *properly contractual* if and only if it belongs

to the core of each economy $\mathcal{C}(\mathcal{E}_x^\lambda)$, *i.e.*, when

$$x \in \bigcap_{\mathbf{e}_x \in [x, \mathbf{e}]} \mathcal{C}(\mathcal{E}_x^\lambda).$$

We can continue by similarly describing the perfectly contractual allocations. In order to do this, we actually only need to understand to which kinds of the “initial endowments allocations” one can transit upon breaking a “virtual web” for $\{x - \mathbf{e}\}$. First of all, note that in this case we may restrict the analysis to the webs consisting of two contracts. To see this, let V be a proper virtual web realizing the allocation x , *i.e.*, $V \sim \{x - \mathbf{e}\}$, and let $U \subseteq V$ be the set of all contracts broken by some coalition. Form the web $W = \{\Delta(U), \Delta(V \setminus U)\}$ in which all broken contracts are aggregated into one contract and all preserved contracts into another one. Clearly, web W is proper, $W \sim V$, and by breaking contract $\Delta(U)$ one realizes the allocation that coincides with the allocation obtained from V as a result of breaking contracts $U \subseteq V$, which completes the argument. So, let $W = \{u, v\} \sim \{x - \mathbf{e}\}$. Then by partially breaking contracts in this web, one can realize any point y in the convex hull of the set consisting of four points: x , \mathbf{e} , $\mathbf{e} + v$, $\mathbf{e} + u$. It is clear that the web $\{x - y, y - \mathbf{e}\}$ constructed for this y is also proper. The opposite is also true: if this web is proper, $y \in \mathcal{A}(X)$ can be realized via breaking a part of the contracts in a virtual web realizing x . Thus if for $x \in \mathcal{A}(X)$ one defines

$$PC = \{y \in \mathcal{A}(X) \mid \text{web } \{x - y, y - \mathbf{e}\} \text{ is proper}\},$$

then

$$x \text{ is perfectly contractual} \iff PC \neq \emptyset \ \& \ x \in \bigcap_{y \in PC} \mathcal{C}(\mathcal{E}_y), \quad (1.2.5)$$

where y is the vector of initial endowments in model \mathcal{E}_y . To better understand the meaning of this formula, the structure of the set PC needs to be clarified. Below, we will do it in dual terms.

Further, let us turn to the description of the objects under study in terms of dual cones. It is of mathematical interest in its own right and, as we will see below, can considerably contribute to the understanding of various concepts of contractual allocation in the situations we are interested in.

The cone

$$K^* = \{p \in L' \mid \langle p, K \rangle \geq 0\}$$

is said to be the dual cone of the set $K \subset L$. For every $i \in \mathcal{I}$, let us set

$$\Gamma(x_i) = \{p \in L' \mid \langle p, \mathcal{P}_i(x_i) - \{x_i\} \rangle \geq 0\}.$$

This is the dual cone of the $\mathcal{P}_i(x_i) - \{x_i\}$. It is well known (and easy to prove applying the separation theorem) that for every (strictly) Pareto optimal allocation, there corresponds a (non-zero) linear price functional $p \in L'$ such that

$$\langle p, \mathcal{P}_i(x_i) \rangle \geq \langle p, x_i \rangle, \quad \forall i \in \mathcal{I}.$$

The necessity part of this statement is always true (for convex, possibly satiated preferences), whereas its sufficiency part is true for the interior points of the consumption sets (if, in addition, the sets $\mathcal{P}_i(x_i)$ are open in X_i , that we have due to **(A)**). One can see that this description is very close to being a precise characterization of Pareto optimality. Accordingly, it is not inaccurate if we call the allocations which satisfy this property Pareto quasi-optimal. In the above terms, they can be described as the allocations satisfying

$$\Gamma(x) = \bigcap_{\mathcal{I}} \Gamma(x_i) \neq \{0\}. \quad (1.2.6)$$

For every $i \in \mathcal{I}$ set

$$G(x_i - \mathbf{e}_i) = \{p \in L' \mid \langle p, x_i - \mathbf{e}_i \rangle \geq 0\}, \quad G(x) = \bigcap_{\mathcal{I}} G(x_i - \mathbf{e}_i).$$

Now one can easily see that x is a *quasi-equilibrium* \iff

$$\bigcap_{\mathcal{I}} [G(x_i - \mathbf{e}_i) \cap \Gamma(x_i)] \neq \{0\} \iff G(x) \cap \Gamma(x) \neq \{0\}. \quad (1.2.7)$$

Assume that $x \in \text{int}X$.¹¹ Then similarly applying the separation theorem and (1.2.4), we see that an allocation x is *lower properly contractual* \iff

$$G(x_i - \mathbf{e}_i) \cap \Gamma(x_i) \neq \{0\}, \quad \forall i \in \mathcal{I} \iff$$

$$\forall i \in \mathcal{I} \exists p_i \in L', p_i \neq 0 : \langle p_i, \mathcal{P}_i(x_i) \rangle \geq \langle p_i, x_i \rangle \ \& \ p_i x_i \geq p_i \mathbf{e}_i. \quad (1.2.8)$$

Thus, in this case an allocation is *weak properly contractual* if (1.2.6) holds in addition.

Further, let us study the properties of the properly contractual allocations.

Lemma 1.2.1 *If an allocation x is properly contractual then for each coalition $S \subseteq \mathcal{I}$, $S \neq \emptyset$, containing only non-satiated agents, there exist $p_S \in L'$, $p_S \neq 0$ such that*

$$\langle p_S, \mathcal{P}_i(x_i) \rangle \geq \langle p_S, x_i \rangle, \quad \forall i \in S \ \& \ p_S \sum_S x_i \geq p_S \sum_S \mathbf{e}_i. \quad (1.2.9)$$

If in addition $x \in \text{int}X$, the opposite is true, i.e., if there exist linear $p_S \in L'$, $p_S \neq 0$ satisfying (1.2.9), $\forall S \subseteq \mathcal{I}$, then x is a properly contractual allocation.

In other words, the lemma states that for a properly contractual allocation, each coalition can find internal-coalition prices such that, first, they are “suitable” for every member of the coalition (the first inequality in (1.2.9), that may be treated as a form of the coalition efficiency) and, second, the contract $x - \mathbf{e}$ is coalition-profitable relative to these prices (the second inequality in (1.2.9)). Thus, the properly contractual allocations are precisely the allocations which satisfy the condition of *coalition-profitability* (1.2.9). The statement of the lemma is similar to the description of the weak properly contractual allocations given in (1.2.8), the only difference being

¹¹This condition together with **(A)** is essential for establishing sufficiency; as for necessity, it may be dropped (in view of **(A)**).

that the lemma claims the existence of internal-coalition prices, satisfying (1.2.9) for *every coalition*, whereas in (1.2.8) just for *singleton coalitions*. This is why the weak properly contractual allocations are just *individually profitable* and *Pareto optimal* (*i.e.*, the coalition of all agents is profitable as well). Notice also that the statement of Lemma 1.2.1 may be rewritten in the equivalent form

$$\bigcap_S \Gamma(x_i) \cap G\left(\sum_S x_i - \sum_S \mathbf{e}_i\right) \neq \{0\}, \quad \forall S \subseteq \mathcal{I}.$$

Of course, in the general case, this requirement is weaker than (1.2.7). This is why in order to establish that a properly contractual allocation is a (quasi)equilibrium, we need to make additional assumptions to guarantee that (1.2.7) is equivalent to the last relation (for example, that the utilities are differentiable and $x \in \text{int}X$, which is actually a strong assumption, that we made in Theorem 1.2.2).

Proof of Lemma 1.2.1. It follows from the above analysis that an allocation x is properly contractual iff it cannot be improved upon by any coalition $S \subseteq \mathcal{I}$ relative to the endowments $x^\lambda = \lambda x + (1 - \lambda)\mathbf{e}$ for all $\lambda \in [0, 1]$. Let $\mathcal{P}_S(y^S) = \prod_S \mathcal{P}_i(y_i)$ for $y^S = (y_i)_{i \in S} \in \prod_S X_i = X^S$ and let $x_S^\lambda = (x_i^\lambda)_{i \in S}$. Then for a fixed λ , the last property can be written in the form

$$\mathcal{P}_S(x^S) \cap (L_S + x_S^\lambda) = \emptyset, \quad L_S = \{y^S \in L^S \mid \sum_S y_i = 0\}. \quad (1.2.10)$$

Therefore,

$$\mathcal{P}_S(x^S) \cap (L_S + [x^S, \mathbf{e}^S]) = \emptyset, \quad x^S = (x_i)_{i \in S}, \quad \mathbf{e}^S = (\mathbf{e}_i)_{i \in S},$$

where $[x^S, \mathbf{e}^S]$ is the linear segment in L^S connecting the points x^S and \mathbf{e}^S (the convex hull of two points). Due to assumption **(A)**, since the set $L_S + [x^S, \mathbf{e}^S]$ is convex and each member of S is non-satiated (hence $\text{int} \mathcal{P}_S(x^S) \neq \emptyset$), one can apply the classical separation theorem, which gives the existence of a linear functional (vector) $p^S = (p_i)_{i \in S} \in (L')^S$, $p^S \neq 0$, such that

$$\langle p^S, \mathcal{P}_S(x^S) \rangle \geq \langle p^S, L_S + [x^S, \mathbf{e}^S] \rangle.$$

The right-hand side of this inequality is a subset of \mathbb{R} *bounded from above*. Hence the inequality may be true only if the set $\langle p^S, L_S \rangle$ is bounded. Since L_S is a subspace of L^S , it follows that $\langle p^S, L_S \rangle = \{0\}$. A standard argument then implies that $p_i = p_j = p \forall i, j \in S$ (because $p^S z^S = 0$ for all $z^S \in L^S$ such that $z_i^S = -z_j^S \in L$ and $z_t^S = 0$ for $t \neq i, j, t \in S$). Moreover, $p \neq 0$ since $p^S = (p, \dots, p) \neq 0$. Next, assumption **(A)** implies that $\mathcal{P}_S(x^S)$ is convex and $x^S \in \overline{\mathcal{P}_S(x^S)}$. Therefore, it follows from the last inequality that

$$\langle p^S, x^S \rangle \geq \langle p^S, [x^S, \mathbf{e}^S] \rangle \iff p \sum_S x_i \geq p \sum_S \mathbf{e}_i.$$

Moreover, arguing by contradiction, we obtain

$$\langle p^S, \mathcal{P}_S(x^S) \rangle \geq \langle p^S, x^S \rangle = \sup \langle p^S, [x^S, \mathbf{e}^S] \rangle \iff \langle p, \mathcal{P}_i(y_i) \rangle \geq \langle p, x_i \rangle, \quad \forall i \in S.$$

Now to complete the proof of the lemma's necessity, just substitute $p_S = p$.

The lemma's sufficiency follows from **(A)** and the condition $x \in \text{int}X$, since in this case the inequalities in the first part of (1.2.9) are actually strict, which together with the second part of (1.2.9) implies (1.2.10) for all $\lambda \in [0, 1]$. ■

Lemma 1.2.1 allows us to discover new interesting (and sometimes unexpected) properties specific to properly contractual allocations. For example, it follows from this lemma that every properly contractual allocation in a 2-replicated economy is an equilibrium if the economy contains just two agents or, alternatively, if there is one agent with differentiable preferences.¹²

First, we would like to recall the concept of a replicated economy. Given a natural $r \in \mathbb{N}$, the r -fold replica of \mathcal{E} is the model \mathcal{E}^r in which every consumer of the original model defines a *type of economic agent* represented by her/his r precise copies. For convenience, the agents in \mathcal{E}^r are numbered by double indexes (i, m) , $i \in \mathcal{I}$, $m = 1, \dots, r$. It is assumed that $X_{im} = X_i$, $\mathbf{e}_{im} = \mathbf{e}_i$, and the preferences, being defined on and taking values in X_{im} , are defined by $\mathcal{P}_{im} = \mathcal{P}_i$. Notice that for every allocation $x = (x_i)_{\mathcal{I}}$ in the initial model, there canonically corresponds an allocation in the replica according to the rule $x_{im} = x_i, \forall i, m$. The opposite is also true if the allocation is *symmetric*, *i.e.*, when identical agents consume equal bundles (equal treatment).

Replicas play an important role in the analysis of perfect competition, especially for proving the well known Edgeworth's conjecture, which states that under perfect competition conditions the core and equilibria coincide. It is the replica's symmetric allocations and the corresponding allocations in the original model that is the main subject of this analysis, with every coalition in the replica being allowed to dominate the original allocations by not necessarily symmetric inter-coalition allocations.

Theorem 1.2.3 *Assume that an economy has two agents or, alternatively, there exists an agent with a smooth preference whose consumption choice is an interior point of his/her consumption set. Then every allocation, which is properly contractual in the 2-fold replica economy, is a quasi-equilibrium if all agents are non-satiated.*

Proof of Theorem 1.2.3. Consider first the case of an economy with two agents. Let $\mathcal{I} = \{1, 2\}$ and let $x = (x_1, x_2)$ be a properly contractual allocation in the 2-fold replica economy. It suffices to establish (1.2.7) for x . To this end, apply Lemma 1.2.1 and relation (1.2.9) to the coalitions $S' = \{(1, 1), (1, 2), (2, 1)\}$ and $S'' = \{(1, 1), (2, 1), (2, 2)\}$. This results in the existence of *nonzero* vectors $p', p'' \in L'$ such that

$$p', p'' \in \Gamma(x_1) \cap \Gamma(x_2)$$

and

$$p'(2x_1 + x_2) \geq p'(2\mathbf{e}_1 + \mathbf{e}_2), \quad p''(x_1 + 2x_2) \geq p''(\mathbf{e}_1 + 2\mathbf{e}_2).$$

Since $x_1 + x_2 = \mathbf{e}_1 + \mathbf{e}_2$, the last inequalities are equivalent to

$$p'x_1 \geq p'\mathbf{e}_1 \quad \& \quad p''x_2 \geq p''\mathbf{e}_2.$$

¹²This result was first proved in [Kozyrev \(1982a\)](#), who applied the technique of subdifferential calculus (to *concave* utility functions, *etc.*), that restricts the generality. Lemma 1.2.1 and the proof of the next theorem are original.

If one of these inequalities is actually an equality, then due to the feasibility of x , either p' or p'' belongs to the intersection in (1.2.7). Suppose both inequalities are strict. Then, the first component of the 2-dimension vector $(p'(x_1 - \mathbf{e}_1), p'(x_2 - \mathbf{e}_2))$ is strictly more than zero, the second one is strictly less than zero, and their sum is equal to zero. The same is true for the vector $(p''(x_1 - \mathbf{e}_1), p''(x_2 - \mathbf{e}_2))$, in which the first component is strictly less than zero. Next find a real $0 < \alpha < 1$ such that $\alpha p'(x_1 - \mathbf{e}_1) + (1 - \alpha)p''(x_1 - \mathbf{e}_1) = 0$ (set $\alpha = -p''(x_1 - \mathbf{e}_1) / [p'(x_1 - \mathbf{e}_1) - p''(x_1 - \mathbf{e}_1)]$). Then it is clear that $[\alpha p' + (1 - \alpha)p''](x_2 - \mathbf{e}_2) = 0$. Now let us set $p = \alpha p' + (1 - \alpha)p'' \neq 0$. Hence, by construction, we obtain $p(x_i - \mathbf{e}_i) = 0 \Rightarrow p \in G(x_i - \mathbf{e}_i)$, $i = 1, 2$. Also $p \in \Gamma(x_i)$, $i = 1, 2$ due to the convexity of $\Gamma(x_i)$. This proves (1.2.7).

Now let us show (1.2.7), assuming that there exists an agent with a smooth preference. First note that due to this assumption, if there exists a nonzero vector $p \in \Gamma(x_i)$, $\forall i \in \mathcal{I}$, then it is *unique up to a normalization*. Now apply Lemma 1.2.1 and relation (1.2.9) to the coalitions $S^i = \{(i, 2)\} \cup \mathcal{I} \times \{1\}$, $i \in \mathcal{I}$, *i.e.*, all the coalitions in which the i -th type consumer is presented by two agents and all the other consumers just by one agent. It follows that there exists a nonzero vector $p \in \Gamma(x_i)$, $\forall i \in \mathcal{I}$, which is *common* for all coalitions S^i , such that for every $i \in \mathcal{I}$

$$p\left[\sum_{j \in \mathcal{I}} x_j + x_i\right] \geq p\left[\sum_{j \in \mathcal{I}} \mathbf{e}_j + \mathbf{e}_i\right] \implies px_i \geq p\mathbf{e}_i.$$

Since x is feasible, $px_i = p\mathbf{e}_i$ for all i and the proof is complete. \blacksquare

Arguing along the lines of the proof of Lemma 1.2.1, one can obtain a dual description of perfectly contractual allocations. Having this in mind, let us make use of formula (1.2.5) to describe the proper web $\{x - y, y - \mathbf{e}\}$ in dual terms. As in (1.2.4), we conclude that $\{x - y, y - \mathbf{e}\}$ is proper \iff

$$\text{co}\{x_i, y_i, \mathbf{e}_i, x_i - y_i + \mathbf{e}_i\} \cap \mathcal{P}_i(x_i) = \emptyset \quad \forall i \in \mathcal{I}.$$

Now applying the separation theorem, we see that this web is proper if and only if for each i there exists a nonzero $p_i \in L'$ such that

$$\langle p_i, \mathcal{P}_i(x_i) \rangle \geq \langle p_i, x_i \rangle \geq \langle p_i, \text{co}\{x_i, y_i, \mathbf{e}_i, x_i - y_i + \mathbf{e}_i\} \rangle.$$

In this chain of inequalities, the second one is equivalent to $p_i x_i \geq p_i y_i \geq p_i \mathbf{e}_i$. Thus we obtained a description of PC , the set of all allocations which can be realized via breaking a part of the contracts in a virtual web realizing x . We can apply this description to give a characterization of the perfectly contractual allocations.

In fact, being properly contractual, a perfectly contractual allocation has to satisfy (1.2.9) and, in addition, for every $y \in \mathcal{A}(X)$, the condition

$$\forall i \in \mathcal{I} \exists p_i \in L', p_i \neq 0 : \langle p_i, \mathcal{P}_i(x_i) \rangle \geq p_i x_i \quad \& \quad p_i x_i \geq p_i y_i \geq p_i \mathbf{e}_i \quad (1.2.11)$$

has to imply $x \in \mathcal{C}(\mathcal{E}_y)$. Moreover, since it has to be true for all $y' \in [x, y]$ (substitute y' for y in the last relations), *i.e.*, since x is properly contractual relative to y , one

can apply Lemma 1.2.1. Thus condition (1.2.11) has to imply that for each coalition $S \subseteq \mathcal{I}$, there exists a nonzero $p_S \in L^I$ such that

$$\langle p_S, \mathcal{P}_i(x_i) \rangle \geq p_S x_i, \quad \forall i \in S \quad \& \quad p_S \sum_S x_i \geq p_S \sum_S y_i.$$

Further, we turn to a comparative analysis of the contractual allocations and the fuzzy core allocations.

1.2.3 Fuzzy core and fuzzy contractual allocations

We begin with a study of the specific properties of the fuzzy core allocations. The elements of fuzzy core are defined via conditions (1.2.1), (1.2.2) which for non-satiated preferences, *i.e.*, when $\mathcal{P}_i(x_i) \neq \emptyset, \forall i \in \mathcal{I}$, domination may be equivalently rewritten in the form¹³

$$0 \notin \sum_{i \in \mathcal{I}} t_i (\mathcal{P}_i(x_i) - \mathbf{e}_i).$$

Thus in this case condition $x \in \mathcal{C}^f(\mathcal{E})$ is equivalent to¹⁴

$$0 \notin \text{co}[\cup_{\mathcal{I}} (\mathcal{P}_i(x_i) - \mathbf{e}_i)], \quad (1.2.12)$$

that after applying separation theorem allows to conclude that the elements of the fuzzy core are quasi-equilibria. Below we propose other useful in applications characterizations of fuzzy core points presented in “geometrical” terms. To this end, let us consider the sets

$$\Upsilon_i(x_i) = \text{co}(\mathcal{P}_i(x_i) \cup \{\mathbf{e}_i\}), \quad i \in \mathcal{I}.$$

Due to the convexity of $\mathcal{P}_i(x_i)$, for $\mathcal{P}_i(x_i) \neq \emptyset$, conclude

$$\text{co}(\mathcal{P}_i(x_i) \cup \{\mathbf{e}_i\}) = \bigcup_{0 \leq \lambda \leq 1} [\lambda \mathcal{P}_i(x_i) + (1 - \lambda) \mathbf{e}_i] = \bigcup_{0 \leq \lambda \leq 1} \lambda (\mathcal{P}_i(x_i) - \mathbf{e}_i) + \mathbf{e}_i, \quad i \in \mathcal{I}.$$

This implies that the condition $z + \mathbf{e} \in \prod_{\mathcal{I}} \Upsilon_i(x_i)$, where $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$, is equivalent to the existence of $0 \leq \lambda_i \leq 1$ and $[y_i \in \mathcal{P}_i(x_i) \neq \emptyset$ and $y_i = \mathbf{e}_i$, if $\mathcal{P}_i(x_i) = \emptyset]$, $i \in \mathcal{I}$ such that

$$z = (\lambda_1(y_1 - \mathbf{e}_1), \dots, \lambda_n(y_n - \mathbf{e}_n)).$$

Hence, due to (1.2.1), (1.2.2)

$$\begin{aligned} x \in \mathcal{C}^f(\mathcal{E}) &\iff \nexists z \in L^{\mathcal{I}}, z \neq 0: z + \mathbf{e} \in \prod_{\mathcal{I}} \Upsilon_i(x_i) \quad \& \quad \sum_{i \in \mathcal{I}} z_i = 0 \iff \\ &\prod_{\mathcal{I}} \Upsilon_i(x_i) \cap \{(z_1, \dots, z_n) \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\} = \{\mathbf{e}\}. \end{aligned} \quad (1.2.13)$$

Notice that characterization (1.2.13) is also valid for satiated preferences. In doing so, we have proven the following

¹³Admitting some inaccuracy in formulas here and below, we identify a vector with a one-element set containing it.

¹⁴Clearly, for a dominating fuzzy coalition t one may always think that $\sum_{i \in \mathcal{I}} t_i = 1$.

Proposition 1.2.1 *An allocation $x \in \mathcal{A}(X)$ is the element of fuzzy core if and only if relation (1.2.13) is true.*

In the case of a 2-agent economy, condition (1.2.13) may be rewritten in the form

$$\Upsilon_1(x_1) \cap (\bar{\mathbf{e}} - \Upsilon_2(\bar{\mathbf{e}} - x_1)) = \{\mathbf{e}_1\}, \quad \bar{\mathbf{e}} = \mathbf{e}_1 + \mathbf{e}_2.$$

Hence,

$$(x_1, x_2) \notin \mathcal{C}^f(\mathcal{E}) \iff \exists \text{ ray starting at the point } \mathbf{e}_1, \text{ which intersects} \\ \text{both sets, } \mathcal{P}_1(x_1) \text{ and } \bar{\mathbf{e}} - \mathcal{P}_2(\bar{\mathbf{e}} - x_1) = \tilde{\mathcal{P}}_2(x_2).$$

Figure 1.2.7 presents a graphic illustration of conducted analysis in the Edgeworth's box for a 2-goods economy. In this case, an allocation x lying in the fuzzy core is equivalent to the convex hulls of $\mathcal{P}_1(x_1) \cup \{\mathbf{e}_1\}$ and of $[\bar{\mathbf{e}} - \mathcal{P}_2(\bar{\mathbf{e}} - x_1)] \cup \{\mathbf{e}_1\}$ having only one point, \mathbf{e}_1 , in common.

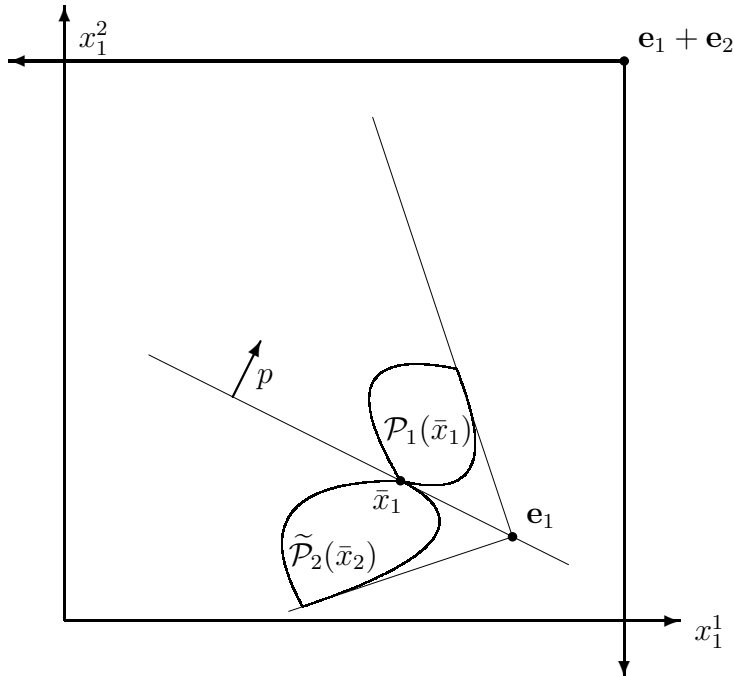


Figure 1.2.7: *Fuzzy core*

Further, we are going to reveal the specification of fuzzy core allocations in purely contractual terms. We start from a preliminary result describing mathematical properties of fuzzy contractual allocations (Definition 1.1.7, page 31), that is of interest in its own right.

Proposition 1.2.2 *A lower properly contractual allocation $x \in \mathcal{A}(X)$ is fuzzy contractual if and only if*

$$\prod_{\mathcal{I}} [(\mathcal{P}_i(x_i) + \text{co}\{0, \mathbf{e}_i - x_i\}) \cup \{\mathbf{e}_i\}] \cap \mathcal{A}(L^{\mathcal{I}}) = \{\mathbf{e}\}, \quad (1.2.14)$$

where $\mathcal{A}(L^{\mathcal{I}})$ is a subspace defined by the balance constraints of a pure exchange economy:

$$\mathcal{A}(L^{\mathcal{I}}) = \{(z_i)_{\mathcal{I}} \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\}.$$

Notice that in this proposition $\mathcal{P}_i(x_i) = \emptyset$ is possible for some $i \in \mathcal{I}$: by definition $\emptyset + A = \emptyset$ for any $A \subseteq L$. Also recall that being lower properly contractual an allocation $x \in \mathcal{A}(X)$ has to satisfy (1.2.4) that can be directly incorporated into the statement.

Figure 1.2.8 illustrates Proposition 1.2.2 result in the Edgeworth's box. Here $\tilde{\mathcal{P}}_2(x_2) = \bar{\mathbf{e}} - \mathcal{P}_2(\bar{\mathbf{e}} - x_1)$ and one can see that preferred bundles are extended along linear segment with endpoints x_1, \mathbf{e}_1 .

The statement of Proposition 1.2.2 can be reformulated in another form, that may be useful in applications.

Corollary 1.2.1 *A lower properly contractual allocation $x \in \mathcal{A}(X)$ is fuzzy contractual \iff*

$$(\mathcal{P}_i(x_i) + \text{co}\{0, \mathbf{e}_i - x_i\}) \times \prod_{j \neq i, j \in \mathcal{I}} [(\mathcal{P}_j(x_j) + \text{co}\{0, \mathbf{e}_j - x_j\}) \cup \{\mathbf{e}_j\}] \cap \mathcal{A}(L^{\mathcal{I}}) = \emptyset, \quad (1.2.15)$$

for each $i \in \mathcal{I} : \mathcal{P}_i(x_i) \neq \emptyset$.

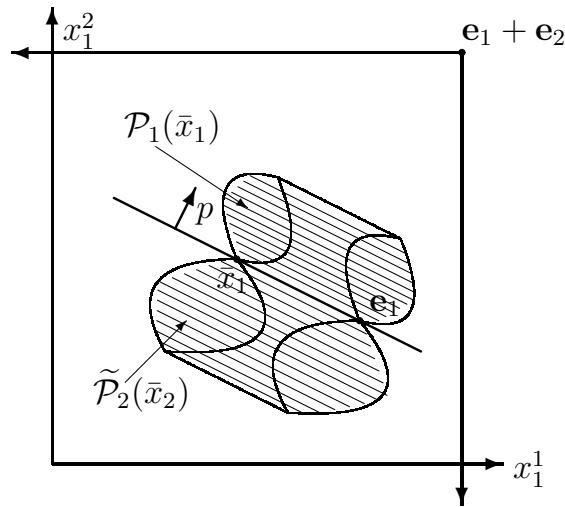


Figure 1.2.8: Fuzzy contractual allocations

Proof of Proposition 1.2.2. Let x be a fuzzy contractual allocation implemented by a proper web V , i.e., $x = x(V)$ for some web V , satisfying Definition 1.1.7. Suppose that (1.2.14) is false and therefore does exist $y = (y_i)_{\mathcal{I}} \neq \mathbf{e}$ which belongs to the left part of equality (1.2.14). Consider coalition $S = \{i \in \mathcal{I} \mid y_i \neq \mathbf{e}_i\}$. Notice $\mathcal{P}_i(x_i) \neq \emptyset$, $i \in S$ and find $z_i \in \mathcal{P}_i(x_i)$, $i \in S$ such that $y_i = z_i + \lambda_i(\mathbf{e}_i - x_i)$, for

some real $0 \leq \lambda_i \leq 1$, $i \in S$ and $y_i = \mathbf{e}_i$, $i \notin S$. Determine $w_i = y_i - \mathbf{e}_i$, $i \in \mathcal{I}$. Since $\sum_{i \in \mathcal{I}} y_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$ then $\sum_{i \in \mathcal{I}} w_i = 0$ and therefore $w = (w_i)_{i \in \mathcal{I}}$ is a contract with $\text{supp}(w) = S \neq \emptyset$. One can write

$$z_i = y_i - \mathbf{e}_i + \lambda_i(x_i - \mathbf{e}_i) + \mathbf{e}_i = w_i + \lambda_i \sum_{v \in V} v_i + \mathbf{e}_i, \quad i \in S.$$

Now for all $v \in V$ put $t_i = t_i^v = \lambda_i$, $i \in S$, and $t_i = 1$, $i \notin S$ and apply $T(V) = \{t^v\}_{v \in V}$ for allocation $x = x(V)$. We have $x^T = \mathbf{e} + \Delta(V^T)$, whereby construction $x_i^T = \mathbf{e}_i + t_i(x_i - \mathbf{e}_i)$, $\forall i \in \mathcal{I}$. Therefore, by construction

$$\xi_i = w_i + x_i^T = z_i \in \mathcal{P}_i(x_i), \quad \forall i \in S = \{j \in \mathcal{I} \mid x_j \neq \xi_j\},$$

that contradicts Definition 1.1.7.

Show that if a lower properly contractual allocation x satisfies (1.2.14) then it is fuzzy contractual relative to a web $V = \{x - \mathbf{e}\}$. Assume contrary and find $T = \{t\}$ and a contract $w = (w_i)_{i \in \mathcal{I}}$, $\text{supp}(w) = S \neq \emptyset$, such that

$$w_i + t_i(x_i - \mathbf{e}_i) + \mathbf{e}_i \in \mathcal{P}_i(x_i), \quad \forall i \in S \iff z_i = w_i + \mathbf{e}_i \in \mathcal{P}_i(x_i) + t_i(\mathbf{e}_i - x_i), \quad \forall i \in S.$$

Let us determine $z_i = \mathbf{e}_i$ for $i \notin S$. Now due to contract's definition conclude $\sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$ that implies the allocation $z \neq \mathbf{e}$ belongs to the left part of (1.2.14) and this is a contradiction. \blacksquare

Notice that as soon as for every feasible allocation $x = (x_i)_{i \in \mathcal{I}}$ we have

$$\mathbf{e}_i \in \Upsilon_i(x_i) \subset (\mathcal{P}_i(x_i) + \text{co}\{0, \mathbf{e}_i - x_i\}) \cup \{\mathbf{e}_i\}, \quad \forall i \in \mathcal{I},$$

then due to Propositions 1.2.1, 1.2.2 every fuzzy contractual allocation belongs to fuzzy core of the economy. However, in general, the property of an allocation to be fuzzy contractual is still a bit stronger than being an element of fuzzy core. The following result clarifies the relationships between two fuzzy notions.

Lemma 1.2.2 *Let $x \in \mathcal{A}(X)$ and $\mathcal{P}_i(x_i) \neq \emptyset$ for all $i \in \mathcal{I}$. Then $x \in \mathcal{C}^f(\mathcal{E})$ implies:*

$$\prod_{i \in \mathcal{I}} (\mathcal{P}_i(x_i) + \text{co}\{0, \mathbf{e}_i - x_i\}) \cap \{(z_1, \dots, z_n) \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\} = \emptyset. \quad (1.2.16)$$

The comparing of formulas (1.2.16) and (1.2.15) makes clearer the difference between fuzzy core allocation and fuzzy contractual one. One can see that this difference is not too big that allows us to interpret allocations from fuzzy core as fuzzy contractual ones.¹⁵ Moreover, the fact that every element of fuzzy core is a quasi-equilibrium (this is why fuzzy core is so popular in existence theory) can be also easily derived from formula (1.2.16). In fact, separating sets in (1.2.16) by a (non-zero) linear functional $\pi = (p_1, \dots, p_n) \in L^{\mathcal{I}}$ one can conclude:

¹⁵Earlier in literature allocations from fuzzy core were interpreted only as Edgeworth's equilibria and served as a technical tool more than an economic concept.

- (i) $p_i = p_j = p \neq 0$ for each $i, j \in \mathcal{I}$; this is so because π is bounded on $\mathcal{A}(L^{\mathcal{I}}) = \{(z_1, \dots, z_n) \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\}$. So, one can take p as a price vector.
- (ii) Due to construction and in view of preferences are locally non-satiated at the point $x \in \mathcal{A}(X)$ the points x_i and \mathbf{e}_i belong to the closure of $\mathcal{P}_i(x) + \text{co}\{0, \mathbf{e}_i - x_i\}$. Therefore, via separating property we have $\sum_{j \neq i} p\mathbf{e}_j + px_i \geq \sum_{\mathcal{I}} p\mathbf{e}_j \Rightarrow px_i \geq p\mathbf{e}_i \forall i \in \mathcal{I}$, that is possible only if $px_i = p\mathbf{e}_i \forall i \in \mathcal{I}$. So, we obtain budget constraints for consumption bundles.
- (iii) By separation property for each i we also have $\langle p, \mathcal{P}_i(x) + \text{co}\{0, \mathbf{e}_i - x_i\} \rangle \geq p\mathbf{e}_i$, that by (ii) implies $\langle p, \mathcal{P}_i(x) \rangle \geq px_i = p\mathbf{e}_i$. So we proved that p is (quasi)equilibrium prices for allocation $x = (x_i)_{i \in \mathcal{I}}$.

As a result, one can see that if an economic model is such that every quasi-equilibrium is equilibrium, then every fuzzy core allocation is fuzzy contractual one and therefore two fuzzy concepts are equivalent each other. Conditions delivering this fact are well known in literature; for example, it is the case when an economy is irreducible.

Proof of Lemma 1.2.2. The argument in the proving of lemma's result is based on Propositions 1.2.1 and relation (1.2.13), which characterize fuzzy core elements. We need to show that (1.2.13) implies (1.2.16).

Assume that x satisfies (1.2.13) and suppose that (1.2.16) is false. This implies that there is a vector $t = (t_1, \dots, t_n)$, $0 \leq t_i \leq 1$ and bundles $z_i \succ_i x_i$, $i \in \mathcal{I}$ such that

$$\sum_{\mathcal{I}} z_i + \sum_{\mathcal{I}} t_i(\mathbf{e}_i - x_i) = \sum_{\mathcal{I}} \mathbf{e}_i \quad (1.2.17)$$

holds. Now for a real $0 < \beta \leq \frac{1}{2}$ consider the vector $y = y(\beta) = (y_i)_{i \in \mathcal{I}}$, where

$$y_i(\beta) = \beta[z_i + t_i(\mathbf{e}_i - x_i)] + (1 - \beta)x_i, \quad i \in \mathcal{I}.$$

In view of (1.2.17), and $x \in \mathcal{A}(X)$ we have $\sum_{\mathcal{I}} y_i(\beta) = \sum_{\mathcal{I}} \mathbf{e}_i$ for every β . Now vectors $y_i(\beta)$ can be presented in the form

$$y_i(\beta) = (1 - \beta t_i)x_i + \beta t_i \mathbf{e}_i + (1 - \beta t_i) \frac{\beta}{1 - \beta t_i} (z_i - x_i), \quad i \in \mathcal{I},$$

whereby the choice of β we have $\mu_i = \frac{\beta}{1 - \beta t_i} \leq 1$. This due to **(A)** for $i \in \mathcal{I}$ implies

$$\mu_i(z_i - x_i) \in \mathcal{P}_i(x) - x_i \Rightarrow \exists \eta_i \in \mathcal{P}_i(x) : \mu_i(z_i - x_i) = \eta_i - x_i.$$

Therefore, the previous formula gives

$$y_i = (1 - \beta t_i)\eta_i + \beta t_i \mathbf{e}_i,$$

that implies $y_i \in \Upsilon_i(x_i)$, $i \in \mathcal{I}$. This allows us to apply relation (1.2.13), concluding $y = y(\beta) = \mathbf{e}$ for all real $0 < \beta \leq \frac{1}{2}$. Write this equality componentwise and due to $y_i(\beta)$ specification find

$$\beta[z_i + t_i(\mathbf{e}_i - x_i)] + (1 - \beta)x_i = \mathbf{e}_i \Rightarrow z_i + t_i(\mathbf{e}_i - x_i) = x_i + \frac{\mathbf{e}_i - x_i}{\beta}$$

that has to be true for all $i \in \mathcal{I}$ and all $0 < \beta \leq \frac{1}{2}$. However, these equalities (consider *different* β) can be true only if $x_i = \mathbf{e}_i = z_i$, $i \in \mathcal{I}$, that due to the choice of z_i implies $x_i \succ_i x_i$ and contradicts to **(A)**. Proof is completed. ■

1.2.4 Contracts with constrained coalition structure

A number of characterizations of the equilibrium were described above in the form of the contractual interaction, and the most significant among others—as a fuzzy contractual allocation. Despite the fact that this characterization was established under the weakened conditions (no differentiable utilities or interior point), the achieved result still can not be called quite satisfactory from the modeling point of view, because it assumes the existence of agreements in many unrealistic coalitions between agents living at great distances, etc.

In this section we consider a possible limit on the number of participants in the exchange transaction. We show that certain constraints of this type can be used without prejudice to its equilibrium properties of the final allocation. The ideas in this section go back to *Graham, Weintraub (1975)*; *Graham et al. (1976)*, where it was found that Pareto optimal allocation can be achieved via mutually beneficial exchanges carried out in coalitions limited by the dimension of the commodity space. In these works, the contractual approach itself was not developed and the possibilities of individuals to break contracts—partial or complete—were not considered, and obtained characterization does not appeal to Walrasian equilibria. Moreover, we also take into account the fact that an agent may not be interested in absolutely all existing products. The analysis can be reduced to an effective space of products of lower dimensionality—by eliminating products that are not of interest to the contracting parties. As a result, a coalition has a specific product space which dimension can be applied to restrict the size of coalitions. We will see that such restrictions on the size of coalitions does not prevent for fuzzy contractual allocations to be Walrasian equilibria. In further analysis, we need the following

Theorem 1.2.4 (CARATHÉODORY, 1907) *Let $A \subset L$ be a subset of a vector space L . If $\dim \text{aff}(A) = d < \infty$, then any element $x \in \text{co } A$ can be presented as a convex hull of not more than $d + 1$ elements of A .*

In a real economy, consumers may not be interested in all existing products, i.e., individuals may be indifferent to some products¹⁶. Excluding them from consideration, one can reduce the dimension of the actual product space for each agent. The exact definition is given below.

Definition 1.2.1 *A commodity j is **indifferent** for $i \in \mathcal{I}$ if $\forall x \in \mathcal{A}(\mathcal{E})$*

$$\forall y_i = ((y_i)_{-j}, y_i^j) \in \mathcal{P}_i(x_i) \iff ((y_i)_{-j}, \mathbf{e}_i^j) \in \mathcal{P}_i((x_i)_{-j}, \mathbf{e}_i^j) = \mathcal{P}_i(x_i).^{17}$$

¹⁶For example, an ordinary consumer on the market is not interested in all kinds of spare parts, parts and structural elements (bolts, nuts, gears, transistors ...).

¹⁷Here we indirectly assume that all bundles we need belong to consumption set, i.e., $((y_i)_{-j}, \mathbf{e}_i^j)$, $((x_i)_{-j}, \mathbf{e}_i^j) \in X_i$; it is a specific constraint for X_i , $i \in \mathcal{I}$.

Here $y_i = (y_i^j)_{j=1,\dots,l} \in \mathbb{R}^l$ and $(y_i)_{-j} = (y_i^k)_{k \neq j, k=1,\dots,l}$ is a vector consisting of all components of y_i excluding y_i^j .

Two properties are postulated in this definition: a product j is indifferent to a given individual i , if in any consumption bundle $y_i = ((y_i)_{-j}, y_i^j) \in X_i$ his/her consumption can be replaced by the initial one (to nullify?), i.e., one goes to a bundle $((y_i)_{-j}, \mathbf{e}_i^j)$ such that $((y_i)_{-j}, \mathbf{e}_i^j) \in X_i$ and this does not lead to the change of consumption properties of $y_i \in L$.

Let $L_i \subseteq L$ be the space of *non-indifferent* commodities (interesting) for individual i and let $L_S \subseteq L$ be a subspace of commodities that are interesting for the members of coalition $S \subseteq \mathcal{I}$:

$$L_S = \sum_{i \in S} L_i.$$

In this section, the notation z^S means the projection of the vector $z \in L$ onto the subspace $L_S \subseteq L$. Recall that for contracts $v \in \mathcal{W}$ there is defined $S(v) = \text{supp}(v)$, this is the support of the contract. Given the possible indifference to some products, as a product space for a coalition $S(v)$ (page 20) one can specify

$$L_{S(v)} = \sum_{i \in S(v)} L_i.$$

We begin the discussion from the concept of upper stable web and upper contractual (stable) allocation, see Definition 1.1.2. Theorem 1.2.1, which characterizes the contractual allocations for $\mathcal{W} = \mathfrak{L} = L^{\mathcal{I}}$, this is the set of all permissible contracts in the economy, can be clarified and one requires less:

$$v = (v_i)_{i \in \mathcal{I}} \in \mathcal{W} \iff v_i \in L_{S(v)}, i \in S(v), |S(v)| \leq \dim(L_{S(v)}) + 1. \quad (1.2.18)$$

It is said that a vector (consumption bundle) $\kappa \in L$ is *extremely desirable* if for each $x = (x_i)_{i \in \mathcal{I}}$ from a neighborhood of $\mathcal{A}(\mathcal{E})$ in X one has

$$x_i + \kappa \succ_i x_i, \quad i \in \mathcal{I}.$$

In the literature, it is standardly assumed that cumulative initial endowments $\sum_{i \in \mathcal{I}} \mathbf{e}_i = \bar{\mathbf{e}}$ presents an extremely desirable bundle.

Theorem 1.2.5 *If \mathcal{W} obeys (1.2.18) then every upper contractual allocation is strictly Pareto optimal. Moreover, if preferences of \mathcal{E} are transitive and there is an extremely desirable bundle $\kappa \in L$, then (1.2.18) can be weakened and one can require*

$$v \in \mathcal{W} \iff v_i \in L_{S(v)}, i \in S(v) \ \& \ |S(v)| \leq \dim(L_{S(v)}).$$

So, the Theorem states that the economic system can arrive at Pareto optimal allocation via a contractual process with coalitions size constrained by (1.2.18).

Proof of Theorem 1.2.5. Suppose that an upper contractual allocation $x \in \mathcal{A}(\mathcal{E})$ is not strictly Pareto optimal. Therefore, there exists a coalition $S \subset \mathcal{I}$ and contract $v = (v_i)_{i \in \mathcal{I}} \in \mathfrak{L} = L^{\mathcal{I}}$, $\text{supp}(v) = S$ such that

$$\forall i \in S \quad x_i + v_i \in \mathcal{P}_i(x). \quad (1.2.19)$$

Here by Definition 1.2.1 for each member of the coalition S , the components of v_i corresponding to indifferent products can be considered as zero, i.e. $v_i \in L_{S(v)} \forall i \in \mathcal{I}$. Since x is upper contractual one, then $v \notin \mathcal{W}$ (here $v_i = 0, i \in \mathcal{I} \setminus S$) and, therefore, $|S(v)| > \dim(L_{S(v)}) + 1$. Now we can assume that S is a coalition of minimal size among those having this property. We have $\frac{1}{|S|} \sum_{i \in S} v_i = 0, S = S(v)$. Using the Caratheodory theorem, one can find a coalition $T \subset S$ such that

$$\forall i \in T \quad \exists \alpha_i \in (0, 1] : \quad \sum_{i \in T} \alpha_i = 1, \quad \sum_{i \in T} \alpha_i v_i = 0 \quad \& \quad |T| \leq \dim(L_S) + 1.$$

Define $w_i = \alpha_i v_i \neq 0$, and think without loss of generality that $w_i \in L_T, i \in T$ (if necessary, one replaces some components with zeros). Now due to the main assumption **(A)** (page 19) one has $\lambda(\mathcal{P}_i(x_i) - x_i) \subseteq \mathcal{P}_i(x_i) - x_i \forall \lambda \in (0, 1]$, that implies $x_i + w_i \in \mathcal{P}_i(x), i \in T$. Since $\sum_{i \in T} w_i = 0$ and $|T| < |S|$, we come to a contradiction with the choice of S as a coalition of minimal size. Therefore, there are no such coalitions at all and $x \in \mathcal{PB}(\mathcal{E})$.

In the second part of the statement of the theorem, we again argue from the contrary and find a coalition $S \subset \mathcal{I}$ of minimal size and a contract $v \in \mathfrak{L}, \text{supp}(v) = S, v_i \in L_S, i \in \mathcal{I}$ satisfying (1.2.19) and such that $|S| > \dim L_S$. Let us specify

$$\Gamma = \text{co}\{v_i \in L_S \mid i \in S\}.$$

By construction one has $\frac{1}{|S|} \sum_{i \in S} v_i = 0 \in \Gamma$. Next, we take an extremely desirable $\kappa \in L$, consider its projection κ^S onto L_S and find a real $\lambda \geq 0$ such that $-\lambda \kappa^S$ belongs to the face of (bounded) polyhedron Γ . This can be done from the condition

$$\lambda = \max\{\lambda' \mid -\lambda' \kappa^S \in \Gamma\}.$$

Since the dimension of any proper face is at most $\dim L_S - 1$, there is a coalition $T \subset S$ such that $|T| \leq \dim L_S$ and

$$\forall i \in T \quad \exists \alpha_i \in (0, 1] : \quad \sum_{i \in T} \alpha_i = 1, \quad \sum_{i \in T} \alpha_i v_i = -\lambda \kappa^S.$$

Next one defines $w_i = \alpha_i(v_i + \lambda \kappa^S), i \in T$ and $w_i = 0, i \in \mathcal{I} \setminus T$. As a result one has:

$$x_i \prec_i x_i + \alpha_i v_i \prec_i x_i + \alpha_i v_i + \alpha_i \lambda \kappa^S = x_i + w_i, \quad i \in T,$$

$$\sum_{i \in T} w_i^S = \sum_{i \in T} \alpha_i v_i^S + \left(\sum_{i \in T} \alpha_i\right) \lambda \kappa^S = 0.$$

These relations indicate that $w = (w_i)_{i \in \mathcal{I}}$ is a mutually beneficial contract, the support of which is the coalition T , no larger than $\dim(L_S)$. Thus, we again have found the coalition that dominates the current allocation, and its size is strictly less than $|S|$, which is impossible. \blacksquare .

Let us turn now to the characterization of fuzzy contractual allocation, which represents the main result of the section.

Lemma 1.2.3 *Let x be a fuzzy contractual allocation and \mathcal{W} obeys (1.2.18). Then*

$$\prod_{i \in \mathcal{I}} [(\mathcal{P}_i(x_i) + \text{co}\{0, \mathbf{e}_i - x_i\}) \cup \{\mathbf{e}_i\}] \cap \mathcal{A}(L^{\mathcal{I}}) = \{\mathbf{e}\}.$$

Now by virtue of the characterization obtained in the Proposition 1.2.2 and the reasonings at the end of the previous section, we directly conclude

Corollary 1.2.2 *Let \mathcal{W} obey (1.2.18). Then every fuzzy contractual allocation is a quasi-equilibrium one.*

So, the Lemma and Corollary states that applying partial break and contracts specified in (1.2.18), a contractual process can arrive the economy to Walrasian equilibrium.

Proof of Lemma 1.2.3. Let x be a fuzzy contractual allocation, \mathcal{W} obeys (1.2.18) and conclusion of the theorem is false. Let us consider the left part of intersection (1.2.14). Now we first show that there is no $y = (y_i)_{\mathcal{I}} \neq \mathbf{e}$ such that the coalition

$$T(y) = \{i \in \mathcal{I} \mid y_i \neq \mathbf{e}_i\} \neq \emptyset \quad (1.2.20)$$

satisfies $|T(y)| \leq \dim(L_T) + 1$. Indeed, otherwise according to the construction one can find $z_i \in \mathcal{P}_i(x_i)$, $\alpha_i \in [0, 1]$, $i \in T$ such that

$$y_i = z_i + \alpha_i(\mathbf{e}_i - x_i) \neq \mathbf{e}_i, \quad i \in T, \quad \sum_T y_i = \sum_T \mathbf{e}_i.$$

Applying now Definition 1.2.1, we may think that $z_i, x_i \in X_i \cap (L_T + \mathbf{e}_i)$ (for i and the bundle x_i one has to change indifferent components with his/her initial endowments and do not change all other). Now, specifying $v_i = (y_i - \mathbf{e}_i) \in L_T \subseteq L$, $i \in \mathcal{I}$, via construction and Definition 1.2.1 we obtain

$$z_i = v_i + \alpha_i(x_i - \mathbf{e}_i) + \mathbf{e}_i \succ_i x_i, \quad i \in T, \quad \sum_{i \in \mathcal{I}} v_i = 0 \quad \& \quad \text{supp}(v) = T,$$

that contradicts Definition 1.1.7 and condition (1.2.18).

Thus, if the conclusion of the theorem is false, then $|T(y)| > \dim(L_T) + 1$ for each coalition specified by (1.2.20). But in the (finite) set of all such coalitions there is a coalition of minimal size, which we denote $S \subset \mathcal{I}$. Again, one can think $x_i \in X_i \cap (L_S + \mathbf{e}_i)$, $i \in S$. By construction there are $z_i \in \mathcal{P}_i(x_i) \cap (L_S + \mathbf{e}_i)$, $\alpha_i \in [0, 1]$, $i \in S$ such that

$$y_i = z_i + \alpha_i(\mathbf{e}_i - x_i) \neq \mathbf{e}_i, \quad i \in S, \quad \sum_S y_i = \sum_S \mathbf{e}_i.$$

We have $\frac{1}{|S|} \sum_S (y_i - \mathbf{e}_i) = 0$. Since by assumption $|S(y)| > \dim(L_S) + 1$, then using Caratheodory theorem, one concludes there exists $R \subset S$ and $\beta_i \in (0, 1]$, $i \in R$ such that $|R| < |S|$ and

$$\sum_{i \in R} \beta_i (y_i - \mathbf{e}_i) = 0, \quad \sum_{i \in R} \beta_i = 1 \quad \Rightarrow \quad \sum_{i \in R} (\beta_i z_i + \beta_i \alpha_i (\mathbf{e}_i - x_i) - \beta_i \mathbf{e}_i) = 0.$$

Since $\lambda(\mathcal{P}_i(x_i) - x_i) \subseteq \mathcal{P}_i(x_i) - x_i \forall \lambda \in (0, 1]$, the terms on the right-hand side of the latter equality can be rewritten in the form

$$\beta_i(z_i - x_i) + \beta_i\alpha_i(\mathbf{e}_i - x_i) - \beta_i(\mathbf{e}_i - x_i) = \xi_i - x_i - \beta_i(1 - \alpha_i)(\mathbf{e}_i - x_i) = v_i, \quad \xi_i \succ_i x_i, \quad i \in R.$$

By construction $\sum_{i \in R} v_i = 0$ and defining $y'_i = v_i + \mathbf{e}_i$, $i \in R$ and $y'_i = \mathbf{e}_i$ for $i \in \mathcal{I} \setminus R$ one obtains $\sum_{i \in \mathcal{I}} y'_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$ and

$$y'_i = \xi_i + (1 - \beta_i(1 - \alpha_i))(\mathbf{e}_i - x_i) \in \mathcal{P}_i(x_i) + \text{co}\{0, \mathbf{e}_i - x_i\}, \quad i \in R.$$

Thus, we found y' such that under condition (1.2.20) we have $T(y') = R$, where $|R| < |S|$, which contradicts the minimality of $S \subset \mathcal{I}$. This contradiction completes the proof. \blacksquare

1.2.5 Existence of fuzzy contractual allocations

In this section, we consider the problem of the existence of fuzzy contractual allocations in the context of the pure exchange economy. Our existence theorem is proved under the additional weak assumption of (lower semicontinuous) extendability of individual preferred sets to a neighborhood of the consumption sets, which, taking into account the supposition **(A)** used above, is very close to modern methods of existence theory. The proof is based on the application of Michael's theorem on the existence of a continuous selector for a multivalued lower semicontinuous mapping taking convex values, and Brouwer's theorem on the existence of a fixed point for a mapping from a convex compact set into itself.

For the existence of objects under study, we apply the following (weak) preference continuity assumption and which, without prejudice to the content, we consider defined on the set $X = \prod_{i \in \mathcal{I}} X_i$ of all possible allocations.

(SC) For each $i \in \mathcal{I}$ there is a point-to-set mapping $\hat{\mathcal{P}}_i : X \rightrightarrows L$ such that for every $x \in \mathcal{A}(X)$ the image $\hat{\mathcal{P}}_i(x)$ is convex, open in L , implements

$$\mathcal{P}_i(x) = \hat{\mathcal{P}}_i(x) \cap X_i$$

and for every $y_i \in \hat{\mathcal{P}}_i(x)$ the set

$$\hat{\mathcal{P}}_i^{-1}(y_i) = \{z \in X \mid y_i \in \hat{\mathcal{P}}_i(z)\}$$

is open one in X .

Thus, under the assumption **(SC)**, it is assumed that the “set of preferred” consumption plans $\mathcal{P}_i(x)$, $x \in X$ can be continuously extended to a neighborhood of the set X_i , $i \in \mathcal{I}$. Also, below due to **(A)** and without loss of generality to simplify notations I am assuming that X_i is convex and has *full dimension*, i.e., $\text{int}X_i \neq \emptyset \forall i \in \mathcal{I}$.

Theorem 1.2.6 *Let in an exchange economy the set $\mathcal{A}(X)$ be bounded and assumptions **(A)**, **(SC)** hold. Then fuzzy contractual allocations do exist.*

Corollary 1.2.3 *In Theorem 1.2.6 conditions fuzzy core is non-empty, i.e., $\mathcal{C}^f(\mathcal{E}) \neq \emptyset$.*

The existence of contractual core, fuzzy and fuzzy-contractual allocations can be established by applying Brouwer (or Kakutani) fixed point theorem and Michael's (*Michael, 1956*) continuous selector theorems. The proof of Theorem 1.2.6 is presented below; we use characterization described in Proposition 1.2.2.

For the further analysis we need auxiliary lemmas. Let $NS \subset \mathcal{A}(X)$ be an area of all lower unstable contractual allocations, i.e. $x \in \mathcal{A}(X)$ for which (1.2.4) is false; also let $\Omega \subset \mathcal{A}(X)$ be a subset consisting the points $x \in \mathcal{A}(X)$ for which (1.2.14) is false. Below I study some properties of these sets.

Lemma 1.2.4 *If economy \mathcal{E} obeys (A) and (SC), then NS and Ω are open in $\mathcal{A}(X)$.*

Proof. An area of all lower stable contractual allocations is specified as

$$LS = \{x = (x_i)_{i \in \mathcal{I}} \in \mathcal{A}(X) \mid \mathcal{P}_i(x) \cap [x_i, \mathbf{e}_i] = \emptyset \ \forall i \in \mathcal{I}\}$$

and now I consider its supplement $NS = \mathcal{A}(X) \setminus LS$, this is the set of all allocations for which there is an agent interested in a partial break of current contract $v = x - \mathbf{e}$. Suppose $\mathcal{P}_i(x) \cap [x_i, \mathbf{e}_i] \neq \emptyset$ for some $i \in \mathcal{I}$. It means there is $y_i \in \hat{\mathcal{P}}_i(x) \cap [x_i, \mathbf{e}_i]$, $y_i \neq x_i$. Since $\hat{\mathcal{P}}_i(x)$ is assumed to be an open one, there is a finite set $A \subset \hat{\mathcal{P}}_i(x)$ such that

$$y_i \in \text{int}(\text{co } A) \Rightarrow x \in \Theta = \bigcap_{a \in A} \hat{\mathcal{P}}_i^{-1}(a),$$

where $\Theta \subset X$ is open in X . Due to $\hat{\mathcal{P}}_i(z)$, $z \in \mathcal{A}(X)$ are also assumed to be convex ones we conclude $y_i \in \text{co } A \subset \hat{\mathcal{P}}_i(z) \ \forall z \in \Theta$. Now if $\varepsilon > 0$ is so that

$$z_i \in L, \|z_i - y_i\| < \varepsilon \Rightarrow z_i \in \text{co } A \ \& \ x' \in \mathcal{A}(X), \|x' - x\| < \varepsilon \Rightarrow x' \in \Theta,$$

then for these allocations one can conclude

$$[x'_i, \mathbf{e}_i] \cap \text{co } A \neq \emptyset \Rightarrow [x'_i, \mathbf{e}_i] \cap \mathcal{P}_i(x') \neq \emptyset \ \forall x' \in \mathcal{A}(X) : \|x' - x\| < \varepsilon.$$

As a result we conclude NS is the neighbourhood of every its point and, therefore, is an open subset of $\mathcal{A}(X)$.

Next I consider $\Omega \subset \mathcal{A}(X)$. The reasoning is similar to that presented above: for every $x \in \Omega$ one can find $t = (t_i)_{i \in \mathcal{I}} \in [0, 1]^{\mathcal{I}}$ and a contract $w = (w_i)_{i \in \mathcal{I}}$, $\sum_{i \in \mathcal{I}} w_i = 0$, such that

$$y_i = w_i + t_i(x_i - \mathbf{e}_i) + \mathbf{e}_i \in \mathcal{P}_i(x), \ \forall i : y_i \neq x_i.$$

For these $i \in \mathcal{I}$ there are finite $A_i \subset L$ such that

$$y_i \in \text{int}(\text{co } A_i) \subset \hat{\mathcal{P}}_i(z), \ \forall z \in \bigcap_{i: y_i \neq x_i} \hat{\mathcal{P}}_i^{-1}(A_i) \cap \mathcal{A}(X) \subset \Omega.$$

It implies there is $\varepsilon > 0$ such that

$$z = (z_i)_{i \in \mathcal{I}} \in \mathcal{A}(X), \|z - x\| < \varepsilon \Rightarrow y_i = w_i + t_i(z_i - \mathbf{e}_i) + \mathbf{e}_i \in \tilde{\mathcal{P}}_i(z), \ \forall i : y_i \neq x_i.$$

As a result $\Omega \subset \mathcal{A}(X)$ is a neighbourhood of every its point and therefore it is an open subset in $\mathcal{A}(X)$ —as we wanted to prove. \blacksquare

Let us study other properties of these allocations from Ω . Assuming $x \in \Omega$ we consider the set of contracts $\varphi(x)$ that fuzzily block this allocation:

$$\begin{aligned} \varphi(x) = \{ & (v_i, t_i)_{\mathcal{I}} \in (L \times [0, 1])^{\mathcal{I}} \mid \sum_{\mathcal{I}} v_i = 0, v \neq 0 : \forall i \notin \text{supp}(v), t_i = 1 \ \& \\ & \forall i \in \mathcal{I} \ v_i + t_i(x_i - \mathbf{e}_i) + \mathbf{e}_i = g_i(x), \ \forall i \in \text{supp}(v), \ g_i(x) \succ_i x_i \}. \end{aligned} \quad (1.2.21)$$

The following lemma presents crucial properties of the point-to-set mapping $\varphi(\cdot)$. First I recall the definition of lower hemicontinuous¹⁸ point-to-set mapping.

Definition 1.2.2 *Let Y, Z be topological spaces. A point-to-set mapping $\psi : Y \rightrightarrows Z$ is called lower hemicontinuous (l.h.c.) iff*

$$\psi^{-1}(V) = \{y \in Y \mid \psi(y) \cap V \neq \emptyset\}$$

is open for every open $V \subset Z$. For a metric spaces Y, Z a l.h.c. mapping can be equivalently characterized as follows:

For every $y \in Y, z \in \psi(y) \subset Z$ and every sequence $y_m \rightarrow y$ there is a subsequence $y_{m_k} \in Y$ and a sequence $z_k \in \psi(y_{m_k}), m, k \in \mathbb{N}$ such that $z_k \rightarrow z$ for $k \rightarrow \infty$.

Lemma 1.2.5 *If $x \in \Omega$ the set $\varphi(x)$ is convex and non-empty. Moreover, the mapping $\varphi : \Omega \rightrightarrows (L \times [0, 1])^{\mathcal{I}}$ is lower hemicontinuous one.*

Proof. I first state the convexity of $\varphi(x)$. Let $(w', t'), (w'', t'') \in \varphi(x)$ and $\alpha \in (0, 1)$. Then, from the convexity of preferences **(A)**, having in mind $t'_i = 1$ for $i \notin \text{supp}(w')$ and, similarly, $t''_i = 1$ for $i \notin \text{supp}(w'')$ we have:

$$\begin{aligned} & \forall i \in \text{supp}(w') \cup \text{supp}(w'') \\ & \alpha(w'_i + t'_i(x_i - \mathbf{e}_i) + \mathbf{e}_i) + (1 - \alpha)(w''_i + t''_i(x_i - \mathbf{e}_i) + \mathbf{e}_i) \succ_i x_i. \end{aligned}$$

Thus, with respect to $t = \alpha t' + (1 - \alpha)t''$, for any $\alpha \in [0, 1]$ the contract

$$w = \alpha w' + (1 - \alpha)w'' : (w, t) \in \varphi(x).$$

Next, we show that due to **(SC)** point-to-set mapping $\varphi(\cdot)$, defined in (1.2.21) is lower hemicontinuous.

Indeed, let $(v, t) \in \varphi(x)$ be fixed. Now according to (1.2.21), for $i \in \mathcal{I}$ such that $v_i \neq 0$ we have

$$g_i(x) = v_i + t_i(x_i - \mathbf{e}_i) + \mathbf{e}_i \in \mathcal{P}_i(x).$$

¹⁸According to the modern views, the term “semi-continuous” mapping is specifically applied for a function—point-to-point map—and “hemicontinuous” is used for a correspondence.

Clearly, without loss of generality it is enough to study the case $g_i(x) \in \text{int}\mathcal{P}_i(x)$. Now let $A_i \subset \text{int}\mathcal{P}_i(x)$ be a finite subset such that $g_i(x) \in \text{int}(\text{co } A_i)$, i.e. $\text{co } A_i$ is a neighborhood of $g_i(x)$. We specify

$$V_i = \bigcap_{a \in A_i} \mathcal{P}_i^{-1}(a).$$

Due to **(SC)** and **(A)** this is an open neighborhood of $x \in \mathcal{A}(X)$ such that $g_i(x) \in \text{co } A_i \subset \mathcal{P}_i(y)$ for every $y \in V_i$. So, if $x^m \in \mathcal{A}(X)$, $x^m \rightarrow x$ for the natural $m \rightarrow \infty$, then for some $k \in \mathbb{N}$ we have: $\forall m \geq k \forall i \in \text{supp}(v)$

$$g_i(x^m) = v_i + t_i(x_i^m - \mathbf{e}_i) + \mathbf{e}_i \in \text{co } A_i \subset \mathcal{P}_i(x^m) \quad \& \quad g_i(x^m) \rightarrow g_i(x).$$

As a result, via (1.2.21) one concludes $(v^m, t^m) = (v, t) \in \varphi(x^m)$ for all $m \in \mathbb{N}$ big enough. This proves, by definition, $\varphi(\cdot)$ is lower hemicontinuous in $x \in \Omega \subset \mathcal{A}(X)$. \blacksquare

In the proof of Lemma 1.2.6 below I apply the following Michael theorem (see *Michael (1956)* p. 368, Th 3.1''', (c)) on the existence of a continuous selector in its simplified finite-dimensional presentation.¹⁹

Theorem 1.2.7 (MICHAEL, 1956) *Let Y and Z be subsets of finite-dimensional linear spaces. Then every l.h.c. point-to-set mapping $\psi : Y \rightrightarrows Z$ having nonempty convex images $\psi(y) \subset Z \forall y \in Y$ has a continuous selector.*

Lemma 1.2.6 *There is a continuous function $h : \Omega \rightarrow \mathcal{A}(X)$ such that for some continuous $\xi_i : \Omega \rightarrow X_i$, $\gamma_i : \Omega \rightarrow [0, 1]$ such that $\xi_i(x) \in \mathcal{P}_i(x) \cup \{x_i\}$ one has*

$$h_i(x) = \xi_i(x) + \gamma_i(x)(\mathbf{e}_i - x_i), \quad \forall x \in \Omega, \quad i \in \mathcal{I}$$

and, moreover, for any $x \in \Omega$ there exists $i \in \mathcal{I}$ such that $h_i(x) = \xi_i(x) \succ_i x_i$, i.e. $\xi_i(x) \in \mathcal{P}_i(x)$ and $\gamma_i(x) = 0$.

Proof. According to assumptions and Lemma 1.2.5, the correspondence $\varphi(\cdot)$ specified in (1.2.21) obeys all requirements of Michael's theorem on the existence of continuous selector: a lower hemicontinuous correspondence having domain $\Omega \subset \mathcal{A}(X)$, and with convex non-empty images. Thus, there is a continuous mapping satisfying

$$(v, t)(\cdot) : \Omega \rightarrow (L \times [0, 1])^{\mathcal{I}} \quad \text{such that} \quad (v(x), t(x)) \in \varphi(x) \quad \forall x \in \Omega.$$

By definition, we have $\sum_{i \in \mathcal{I}} v_i(x) = 0$ and, $t_i(x) = 1$, $g_i(x) = x_i$ for $i \notin \text{supp}(v(x))$ and

$$\forall i \in \text{supp}(v), \quad v_i(x) + t_i(x)(x_i - \mathbf{e}_i) + \mathbf{e}_i = g_i(x) \in \mathcal{P}(x).$$

¹⁹Note that in original paper item (c) has a typo for the range of $\phi : X \rightarrow \mathcal{K}(Y)$. Author denoted $\mathcal{K}(Y)$ as a set of all convex subsets of Y , but speak and prove the result for a narrower class of sets $\mathcal{D}(Y) \subset \mathcal{K}(Y)$, see p. 372. Here I present a less general result, to avoid a cumbersome specification of $\mathcal{D}(Y)$.

Therefore, for $f_i(x) = v_i(x) + \mathbf{e}_i = g_i(x) + t_i(x)(\mathbf{e}_i - x_i)$, $i \in \mathcal{I}$ we obtain $\sum_{i \in \mathcal{I}} f_i(x) = \sum_{i \in \mathcal{I}} \mathbf{e}_i$. Thus, we find a continuous mapping $f : \Omega \rightarrow \mathcal{A}(L^{\mathcal{I}})$ such that

$$\forall x \in \Omega \quad \mathbf{e} \neq f(x) \in \prod_{i \in \mathcal{I}} [(\mathcal{P}_i(x_i) + \text{co}\{0, \mathbf{e}_i - x_i\}) \cup \{\mathbf{e}_i\}] \cap \mathcal{A}(L^{\mathcal{I}}).$$

Now we define $t^{\min}(x) = \min_{j \in \mathcal{I}}(t_j(x))$ and specify

$$h_i(x) = \frac{x_i + g_i(x) + (t_i(x) - t^{\min}(x))(\mathbf{e}_i - x_i)}{2}, \quad x \in \Omega, \quad i \in \mathcal{I}. \quad (1.2.22)$$

So, as

$$\sum_{i \in \mathcal{I}} (g_i(x) + t_i(x)(\mathbf{e}_i - x_i)) = \sum_{i \in \mathcal{I}} \mathbf{e}_i, \quad \sum_{i \in \mathcal{I}} (\mathbf{e}_i - x_i) = 0$$

we conclude

$$\sum_{i \in \mathcal{I}} h_i(x) = \sum_{i \in \mathcal{I}} \mathbf{e}_i.$$

Moreover, for $i \in \mathcal{I}$ such that $t_i(x) = t^{\min}(x)$ we have $h_i(x) = \frac{x_i + g_i(x)}{2}$ that due to $x_i \in \text{cl } \mathcal{P}_i(x)$ and $g_i(x) \in \mathcal{P}_i(x)$ gives $h_i(x) \in \mathcal{P}_i(x)$. Now we need to show only that $h_i(x) \in X_i \forall i \in \mathcal{I}$.

For $g_i(x) = x_i$ one can put $\frac{(t_i(x) - t^{\min}(x))}{2} = \alpha_i \in [0, 1]$ and by (1.2.22) conclude

$$h_i(x) = (1 - \alpha_i)x_i + \alpha_i \mathbf{e}_i \in X_i.$$

For $g_i(x) \in \mathcal{P}_i(x) \subset X_i$ via (1.2.22) for $\beta_i = t_i(x) - t^{\min}(x)$ we have $(1 - \beta_i)x_i + \beta_i \mathbf{e}_i \in X_i$ and therefore

$$h_i(x) = \frac{g_i(x)}{2} + \frac{(1 - \beta_i)x_i + \beta_i \mathbf{e}_i}{2} \in X_i.$$

This proves the map specified in (1.2.22) has range $\mathcal{A}(X)$. Now putting

$$\xi_i(x) = \frac{x_i + g_i(x)}{2}, \quad \gamma_i(x) = \frac{t_i(x) - t^{\min}(x)}{2}, \quad i \in \mathcal{I},$$

we can redefine the map $h : \Omega \rightarrow \mathcal{A}(X)$; it obeys all requirements of Lemma 1.2.6. ■

Proof of Theorem 1.2.6. Recall that

$$LS = \{x = (x_i)_{i \in \mathcal{I}} \in \mathcal{A}(X) \mid \mathcal{P}_i(x) \cap [x_i, \mathbf{e}_i] = \emptyset \forall i \in \mathcal{I}\}$$

is an area of all lower stable contractual allocations and we consider its supplement $NS = \mathcal{A}(X) \setminus LS$, this is the set of all allocations for which there is an agent interested in a partial break of current contract $v = x - \mathbf{e}$. For $x \in NS$ condition (1.2.4) is false. Also we specified $\Omega \subset \mathcal{A}(X)$ as a subset consisting the points $x \in \mathcal{A}(X)$ for which (1.2.14) is false. Now let us suppose that

$$\Omega \cup NS = \mathcal{A}(X)$$

and show that it is impossible.

Indeed, according to Lemma 1.2.4, NS and Ω are an open subsets of $\mathcal{A}(X)$. We specify $q : NS \rightarrow \mathcal{A}(X)$ by formula

$$q(x) = \frac{x + \mathbf{e}}{2}$$

and “glue” this mapping with $h(\cdot)$ defined in Lemma 1.2.6, setting

$$f(x) = \alpha(x)q(x) + \beta(x)h(x), \quad x \in \mathcal{A}(X),$$

where $\alpha : \mathcal{A}(X) \rightarrow [0, 1]$, $\beta : \mathcal{A}(X) \rightarrow [0, 1]$ are continuous functions, such that $\alpha(x) = 1$ for $x \in NS \setminus \Omega$, $\beta(x) = 1$ for $x \in \Omega \setminus NS$, and $\alpha(x) + \beta(x) = 1 \quad \forall x \in \mathcal{A}(X)$. For example they can be specified as

$$\alpha(x) = \frac{\rho(x, LS)}{\rho(x, LS) + \rho(x, \mathcal{A}(X) \setminus \Omega)}, \quad \beta(x) = \frac{\rho(x, \mathcal{A}(X) \setminus \Omega)}{\rho(x, LS) + \rho(x, \mathcal{A}(X) \setminus \Omega)}, \quad x \in \mathcal{A}(X),$$

where $\rho(x, S)$ is a distance from the point x to the set $S \subset \mathcal{A}(X)$.²⁰ Obviously, for $\Omega \cup NS = \mathcal{A}(X)$ the mapping $f : \mathcal{A}(X) \rightarrow \mathcal{A}(X)$ is continuous and, by Brouwer’s theorem, it must have a fixed point $\bar{x} = f(\bar{x})$. However, where is it?

Suppose $\bar{x} \in NS \setminus \Omega$. Then $f(\bar{x}) = q(\bar{x}) = \frac{\bar{x} + \mathbf{e}}{2} \neq \bar{x}$, since otherwise $\bar{x} = \mathbf{e} \notin NS$.

Suppose $\bar{x} \in \Omega \setminus NS$. Then $f(\bar{x}) = h(\bar{x}) = (h_j(\bar{x}))_{j \in \mathcal{I}}$ and by Lemma 1.2.6 there is $i \in \mathcal{I}$ such that $h_i(\bar{x}) \in \mathcal{P}_i(\bar{x})$ that is impossible by **(A)**.

Suppose $\bar{x} \in NS \cap \Omega$. Now $f(\bar{x}) = \alpha(\bar{x})q(\bar{x}) + \beta(\bar{x})h(\bar{x})$. Clearly $\alpha(\bar{x}) > 0$ and $\beta(\bar{x}) > 0$, since the contrary is impossible. Recall that we also have $h_i(\bar{x}) = \xi_i(\bar{x}) + \gamma_i(\bar{x})(\mathbf{e}_i - \bar{x}_i) \quad \forall i \in \mathcal{I}$. Now for $i_0 \in \mathcal{I}$, which is interested in a partial breaking of the contract $\bar{x} - \mathbf{e}$, at the fixed point we have

$$\begin{aligned} \bar{x}_{i_0} &= \alpha \left[\bar{x}_{i_0} + \frac{1}{2}(\mathbf{e}_{i_0} - \bar{x}_{i_0}) \right] + \beta[\xi_{i_0}(\bar{x}) + \gamma_{i_0}(\bar{x})(\mathbf{e}_{i_0} - \bar{x}_{i_0})] = \\ &= \bar{x}_{i_0} + \alpha \frac{1}{2}(\mathbf{e}_{i_0} - \bar{x}_{i_0}) + \beta[\xi_{i_0}(\bar{x}) - \bar{x}_{i_0} + \gamma_{i_0}(\bar{x})(\mathbf{e}_{i_0} - \bar{x}_{i_0})] \Rightarrow \\ &= \left[\frac{1}{2}\alpha + \beta\gamma_{i_0}(\bar{x}) \right] (\mathbf{e}_{i_0} - \bar{x}_{i_0}) + \beta(\xi_{i_0}(\bar{x}) - \bar{x}_{i_0}) = 0. \end{aligned} \quad (1.2.23)$$

Clearly $\xi_{i_0}(\bar{x}) = \bar{x}_{i_0}$ is impossible, otherwise (1.2.23) implies $\bar{x}_{i_0} = \mathbf{e}_{i_0}$. Therefore $\xi_{i_0}(\bar{x}) \in \mathcal{P}_{i_0}(\bar{x})$. Also at a fixed point $\bar{x} \in NS$ for some $\lambda \in (0, 1]$ we have

$$\lambda \left[\frac{1}{2}\alpha + \beta\gamma_{i_0}(\bar{x}) \right] = \mu > 0, \quad \mu(\mathbf{e}_{i_0} - \bar{x}_{i_0}) + \bar{x}_{i_0} \in \mathcal{P}_{i_0}(\bar{x}).$$

At the same time, due to $(\xi_{i_0}(\bar{x}) - \bar{x}_{i_0}) \in \mathcal{P}_{i_0}(\bar{x}) - \bar{x}_{i_0}$ and **(A)** we conclude

$$\lambda\beta(\xi_{i_0}(\bar{x}) - \bar{x}_{i_0}) \in \mathcal{P}_{i_0}(\bar{x}) - \bar{x}_{i_0} \Rightarrow \exists \eta_{i_0}(\bar{x}) \in \mathcal{P}_{i_0}(\bar{x}) : \lambda\beta(\xi_{i_0}(\bar{x}) - \bar{x}_{i_0}) = \eta_{i_0}(\bar{x}) - \bar{x}_{i_0}.$$

Now, due to (1.2.23) and **(A)** we have $\mu(\mathbf{e}_{i_0} - \bar{x}_{i_0}) + \eta_{i_0}(\bar{x}) - \bar{x}_{i_0} = 0 \Rightarrow$

$$\frac{\mu(\mathbf{e}_{i_0} - \bar{x}_{i_0}) + \bar{x}_{i_0}}{2} + \frac{\eta_{i_0}(\bar{x})}{2} = \bar{x}_{i_0} \Rightarrow \bar{x}_{i_0} \in \text{co } \mathcal{P}_{i_0}(\bar{x}) = \mathcal{P}_{i_0}(\bar{x}),$$

²⁰It is standardly defined as $\rho(x, S) = \inf_{y \in S} \rho(x, y)$.

which is impossible.

Thus, the assumption $\Omega \cup NS = \mathcal{A}(X)$ implies the existence of a continuous mapping $f : \mathcal{A}(X) \rightarrow \mathcal{A}(X)$ with no fixed point in $\mathcal{A}(X)$. This contradicts Brouwer's theorem. So, the assumption that there are no fuzzy contractual allocations lead us to a contradiction and it proves the theorem. ■

1.3 Production Arrow–Debreu economies

One of the main objectives of the economic theory and its basic part—general equilibrium theory—consists in the description of resource allocation implemented via the system of the markets. In the classical Arrow–Debreu model resulting allocation arrives as Walrasian (competitive) equilibrium that is the basic object of the theoretical analysis, see *Arrow, Debreu (1954)*, *McKenzie (1954)*, *Mas-Colell et al. (1995)*, *Aliprantis et al. (1989)*, etc. Arrow–Debreu model was developed and generalized in different directions, one of them being the study of models with non-convex technological sets and with public goods.

For the classical Arrow–Debreu model, the convexity of technological sets (and sets of the preferred consumption bundles) is a very important assumption, otherwise, equilibria may not exist. However, non-convexity in technologies is a characteristic for many industrial spheres, and it relays with increasing returns from scale (for example, for private municipal enterprises). Therefore, the case of non-convex technological sets is a very important theoretical problem. The non-convexity in technologies leads to the known concept of equilibrium with pricing by marginal costs, so-called \mathcal{MCP} -equilibrium. For the first time, the existence of \mathcal{MCP} -equilibrium for a monopolistic economy with one firm has been established in *Mantel (1979)*. Further, in the paper of *Beato, Mas-Colell (1985)* existence of an equilibrium with marginal costs pricing has been proved for several firms with non-convex technologies, most general results have been obtained in *Bonnisseau, Cornet (1988, 1990)*; a survey of literature one can find in *Brown (1991)*. So general equilibrium theory started to develop for non-convex technological sets. Notice in addition, that equilibrium with marginal costs pricing implements only necessary condition for Pareto optimality of current (equilibrium) production allocation (in general \mathcal{MCP} -equilibrium may not be Pareto optimal); however, it is also sufficient in a convex case and corresponds with profit maximization of producers.

So, we will consider models of the economy with production and consumption sectors. Structurally they are Arrow–Debreu models, in the first case it is a model with a convex production sector, while in the second one increasing returns from scale in production are possible. The analysis is based on a contractual approach developed in the previous sections and extended to the production model in an appropriate way. In the consumption sector contract is considered almost the same as in an exchange model: a barter contract delivering commodities for exchange. However, for production contract agents have taken material expenses related to the production of goods.

The collections of contracts can form webs that can be transformed via concluding

new mutually beneficial contracts and breaking existing ones similarly pure exchange economy. However, production plans can now also be changed. A partial break of contracts is also permissible. Stable webs of contracts are the subject of the study, these webs allow to characterize Walrasian equilibrium for the convex economy and with \mathcal{MCP} -pricing (marginal cost pricing) for non-convex: it is true if the partial breaking of contracts is possible. A specific property of our contractual approach is that all processes of production and exchange are going without any kind of value parameters. So, applying the contractual approach, we are interested in the stable web of contracts and this stability can be variable.

Further, we consider a basic Arrow–Debreu model and traditional notions applied in its context: core, Walrasian, and \mathcal{MCP} -equilibria. Mantel’s theorem as being the first known result on the existence of \mathcal{MCP} -equilibrium is also presented here. In the following section, we study convex case: model, main definitions presented here and equivalence theorems.

1.3.1 Production economies

The formal economy of Arrow–Debreu type in its shortest form is presented by the following bundle of parameters:

$$\mathcal{E}^{AD} = \langle \mathcal{I}, \mathcal{J}, \mathbb{R}^l, \{X_i, \mathcal{P}_i(\cdot), \mathbf{e}_i, \{\theta_i\}\}_{i \in \mathcal{I}}, \{Y_j\}_{j \in \mathcal{J}} \rangle. \quad (1.3.1)$$

Here $\mathcal{I} = \{1, \dots, n\}$ is the set of consumers, $\mathcal{J} = \{1, \dots, m\}$ is the set of producers (firms), l is a number of commodities and $\mathbb{R}^l = L$ is the commodity space. Consumption sets are denoted as $X_i \subset \mathbb{R}^l$ and $X = \prod_{i \in \mathcal{I}} X_i$; agents’ preferences are presented by point-to-set mappings $\mathcal{P}_i : X_i \rightrightarrows X_i$, $i \in \mathcal{I}$ where $\mathcal{P}_i(x_i) = \{y_i \in X_i \mid y_i \succ_i x_i\}$ is a set of all consumption bundles strictly preferred by the i -th agent to the bundle x_i . It is also applied notation $y_i \succ_i x_i$ which is equivalent to $y_i \in \mathcal{P}_i(x_i)$. Consumers have also initial endowments $\mathbf{e}_i \in X_i$, $i \in \mathcal{I}$. Determine $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ and let $\bar{\mathbf{e}} = \sum_{i=1}^n \mathbf{e}_i$.

A producer $j \in \mathcal{J}$ is described by a technological set $Y_j \subset L$, $Y = \prod_{j \in \mathcal{J}} Y_j$, defined in terms of material flows, *i.e.*, non-negative component of $y_j \in Y_j$ is an output but if it is negative, then it is input of commodity in the units of counting. Now for a vector of prices $p = (p_1, \dots, p_l) \in L'$ profit $\pi_j(p, y_j)$ for a plan $y_j \in Y_j$ can be calculated in the form of inner product $\pi_j(p, y_j) = \langle p, y_j \rangle = p \cdot y_j$. There are also nm scalar values $\theta_i^j \geq 0$ they being the components of vectors $\theta_i = (\theta_i^1, \dots, \theta_i^m)$ present the shares of i in the profits π_j of producers $j \in \mathcal{J}$; by the definition $\sum_{i=1}^n \theta_i = (1, \dots, 1)$. Further, let us recall the definition of **competitive (Walrasian) equilibrium**.

Definition 1.3.1 A triplet (x, y, p) , where $x = (x_i)_{i \in \mathcal{I}} \in X$ is a family of consumption plans, $y = (y_j)_{j \in \mathcal{J}} \in Y$ are production plans and $p = (p_1, \dots, p_l) \neq 0$, $p \in L'$ is a price vector, is said to be **quasi-equilibrium**, if:

$$p \cdot y_j \geq \langle p, Y_j \rangle, \quad \forall j \in \mathcal{J}, \quad ^{21} \quad (1.3.2)$$

²¹ $\langle A, B \rangle = \{ \langle a, b \rangle = a \cdot b \mid a \in A, b \in B \}$ for all $A, B \subset L$; $A \geq b \iff a \geq b \forall a \in A$.

$$\langle p, \mathcal{P}_i(x_i) \rangle \geq p \cdot \mathbf{e}_i + \sum_{j=1}^m \theta_i^j p \cdot y_j = p \cdot x_i, \quad \forall i \in \mathcal{I}, \quad (1.3.3)$$

$$\sum_{i=1}^n x_i = \sum_{i=1}^n \mathbf{e}_i + \sum_{j=1}^m y_j. \quad (1.3.4)$$

If all inequalities in (1.3.3) have a strict sign then the triplet (x, y, p) is called **competitive (Walrasian) equilibrium**.

Requirements (1.3.2)–(1.3.4) have a familiar economic sense. If inequality (1.3.3) has strict form it means that consumption plan x_i is an optimal choice (demand) for individual i under his/her budget constraint $p \cdot z_i \leq p \cdot \mathbf{e}_i + \sum_{j=1}^m \theta_i^j p \cdot y_j = r_i(p, y)$, $z_i \in X_i$, where the right-hand side presents total agent's income from all channels (the sale of commodities \mathbf{e}_i and the shares in firm's profits $\theta_i^j p y_j$) under prices $p = (p_1, \dots, p_l) \in L'$. Condition (1.3.2) says that producers maximize profit and (1.3.4) is a material balance condition that usually is presented as the equality of demand and supply.

Conditions guaranteeing existence of equilibria in Arrow–Debreu model are well known in literature, *e.g.* see [Mas-Colell et al. \(1995\)](#), [Aliprantis et al. \(1989\)](#), [Marakulin \(2012\)](#). In consumption sector they are the continuity (different versions are applied), open-convex values, irreflexivity and local non-satiation of agents' preferences $\mathcal{P}_i : X_i \Rightarrow X_i, \forall i \in \mathcal{I}$. Consumption sets have to be convex and closed; moreover, they have to provide a bounded (compact) set of all feasible allocations. These requirements are sufficient for quasi-equilibria or more refined notion of equilibria with non-standard prices exist in exchange economy (see [Marakulin \(2012\)](#), [Kononov, Marakulin \(2006\)](#)). For strict equilibria to exist one needs to require in addition survival assumptions, it may be resource relatedness, irreducibility or something like this one. Further, let us turn to production sector and consider it in more detail.

For production sector it is usually assumed that for all $j \in \mathcal{J}$ technological sets Y_j have the following properties:

- Y_j — *convex*, closed sets (*i.e.*, limit and mixed technological processes are permissible),
- $Y_j - \mathbb{R}_+^l \subset Y_j$ — free disposal condition,
- $Y_j \cap \mathbb{R}_+^l = \{0\}$ — no free lunch, where \mathbb{R}_+^l is positive orthant of commodity space.
- $Y \cap (-Y) = \{0\}$ — the irreversibility of production processes.

In spite of the latter three requirements have an own economic sense they are really needed to provide, together with consumption sets properties (boundedness from below), that the set of all feasible (balanced) allocations is bounded one. Nowadays one can often meet a direct requirement for the feasible allocation set to be bounded. The first assumption is for us now the most of interest and as a part of it the convexity of production sets. Without this requirement, equilibria may not exist.

In this section we deal with the cases of convex and nonconvex production sets. Nonconvexity in production may occur, for example, due to increasing returns to scale (firm revenues are increasing per unit costs). For example, the recording of the CD-ROM is costlier than its replication, overwriting occurs at low cost. However, this possibility (because of technical-mathematical reasons) has not been studied in the classical version of the existence theory. Of course, in order for an equilibrium to exist under non-convex technology, the concept should be appropriately modified. However, this modification should be such that the new concept is resulted in (or at least has chances) the Pareto optimal allocation, as it is in the convex case. Pricing on the basis of average costs does not satisfy this requirement. The key idea of \mathcal{MCP} -equilibrium is that profit maximization is replaced by the (necessary) first-order condition (expressed in terms of gradients of the functions that define the production sets), which in the convex case is also sufficient for a plan to be profit maximizer. Thus the concept directly generalizes the usual competitive equilibrium. As the subject of interpretation, one is usually talking about the social planner who has the ability to “evaluate” the price obtained according to the principle of marginal cost pricing (\mathcal{MCP}), and then force the manufacturers to adhere to the specified production plans. Let us consider the simplest version of the model with a nonconvex production sector, described by differentiable functions.

Assume that the production sets Y_j are described via *differentiable functions* φ_j by formula

$$Y_j = \{y \in L \mid \varphi_j(y) \leq 0\}, \quad j \in \mathcal{J}. \quad (1.3.5)$$

and, moreover, in this case the boundary of production sets can be defined as²²

$$\partial Y_j = \{y \in L \mid \varphi_j(y) = 0\} \neq Y_j, \quad j \in \mathcal{J}. \quad (1.3.6)$$

Define $X = \prod_{i \in \mathcal{I}} X_i$, $Y = \prod_{j \in \mathcal{J}} Y_j$. The following definition of \mathcal{MCP} -equilibrium can be found in [Mantel \(1979\)](#), [Brown \(1991\)](#).

Definition 1.3.2 *\mathcal{MCP} -equilibrium (marginal cost pricing) is a triplet (x, y, p) , where $x = (x_i)_{i \in \mathcal{I}} \in X$ is a family of consumption plans, $y = (y_j)_{j \in \mathcal{J}} \in Y$ are production plans and $p = (p_1, \dots, p_l)$ is a price vector, which satisfies the following conditions:*

$$y \in \prod_{j \in \mathcal{J}} \partial Y_j \quad \& \quad \exists \lambda_j > 0 : \quad p = \lambda_j \nabla \varphi_j(y_j), \quad \forall j \in \mathcal{J}, \quad (1.3.7)$$

$$\langle p, \mathcal{P}_i(x_i) \rangle > p \cdot \mathbf{e}_i + \sum_{j \in \mathcal{J}} \theta_i^j p \cdot y_j = p \cdot x_i, \quad \forall i \in \mathcal{I}, \quad (1.3.8)$$

$$\sum_{i \in \mathcal{I}} x_i = \sum_{j \in \mathcal{J}} y_j + \sum_{i \in \mathcal{I}} \mathbf{e}_i. \quad (1.3.9)$$

The requirement (1.3.7) is the above-mentioned first-order conditions which the equilibrium production plans must satisfy instead of the condition of profit maximization

²²Clearly, in the general case, the topological boundary of the set may be narrower than the set described below.

(1.3.2). Conditions (1.3.8), (1.3.9) present the optimum of consumer preferences under budget and other constraints and the balance of commodity markets.

In the latter definition, it is implicitly assumed that production can be unprofitable, but total taxes cover the losses of firms with nonconvex production sets. It is important to note that if all firms have convex technologies, the concept of equilibrium with marginal cost pricing is turned to be the Walrasian equilibrium in the classical Arrow–Debreu model. Further well-known in the literature and one of the simplest results (see *Brown (1991)*) on existence of \mathcal{MCP} -equilibria is stated.

Consider a model with one firm and let assumptions (1.3.5), (1.3.6) hold. In addition, let us assume $0 \in Y$, $Y - \mathbb{R}_+^l \subseteq Y$, the set $(Y + \sum_{\mathcal{I}} \mathbf{e}_i) \cap \mathbb{R}_+^l$ is bounded and if $y + \sum_{\mathcal{I}} \mathbf{e}_i \in \partial Y \cap \mathbb{R}_+^l$, then $\nabla \varphi(y) \gg 0$.

Let for all $i \in \mathcal{I}$ consumption sets $X_i = \mathbb{R}_+^l$, and preferences are determined via utility functions $u_i : X_i \rightarrow \mathbb{R}$, which are continuous, strictly concave and locally non-satiated ones.

Theorem 1.3.1 (MANTEL, 1979) *Under presented assumptions, an equilibrium with marginal cost pricing does exist.*

Essentially stronger results can be found in *Bonnisseau, Cornet (1988, 1990)*. Here many firms are considered and they may have non-smooth boundary, as well as a general rule of pricing is analyzed. In this context, the mapping $\psi : \prod_{\mathcal{J}} \partial Y_j \rightarrow \mathbb{R}_+^l$ is considered, which maps a vector of production plans to a set of production prices. Requirements on the map $\psi(\cdot)$ are very general and this approach can present the marginal cost pricing as soon as average cost pricing—*ACP*, and also other variants, see *Bonnisseau, Cornet (1988)*, *Brown (1991)*.

Further, we are passing to the main purpose of the section—the analysis of contractual approach in Arrow–Debreu model with convex technologies as well as non-convex ones. Further presentation is according to *Marakulin (2014, 2021)*. We start from the analysis of convex Arrow–Debreu model.

1.3.2 Contractual approach in convex Arrow–Debreu model

Consider the following model of an economy with production

$$\mathcal{E} = \langle \mathcal{I}, L, (X_i, \mathbf{e}_i, Y_i, \mathcal{P}_i)_{i \in \mathcal{I}} \rangle. \quad (1.3.10)$$

This model differs from the classical Arrow–Debreu model only in the part of production sector, which assumes the existence of individualized production sets $Y_i \subset L$, $i \in \mathcal{I}$. It is easy to see that the classical model can be reduced to a model of this kind: it is sufficient to specify the individual production sets Y_i as $Y_i = \sum_{j=1}^m \theta_i^j \cdot \bar{Y}_j$, where \bar{Y}_j are production sets in the model \mathcal{E}^{AD} . Therefore, the results obtained for model \mathcal{E} will be applied to model \mathcal{E}^{AD} and vice versa. In this case, however, theoretical constructions are considerably simplified. For this model, budget constraints are applied in the form

$$p \cdot x_i \leq p \cdot \mathbf{e}_i + p \cdot y_i, \quad i \in \mathcal{I},$$

i.e., consumer i income from production activities is presented by the value $p \cdot y_i$. Further, define $Z_i = X_i \times Y_i$, $Z = \prod_{i \in \mathcal{I}} Z_i$ and let $\mathfrak{L} = L^{\mathcal{I}} \times L^{\mathcal{I}}$ be a space of allocations.

Define

$$\mathcal{A}(\mathcal{E}) = \{z = (x, y) \in X \times Y \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i + \sum_{i \in \mathcal{I}} y_i\} -$$

the set of all feasible allocations in model \mathcal{E} . In the context of \mathcal{E} equilibrium is defined as follows:

• A triplet (x, y, p) , where $p \neq 0$, $p \in L'$, $(x, y) \in \mathcal{A}(\mathcal{E})$ is said to be **quasi-equilibrium**, if for each $i \in \mathcal{I}$:

$$\langle p, \mathcal{P}_i(x_i) \rangle \geq p \cdot \mathbf{e}_i + p \cdot y_i = p \cdot x_i, \quad (1.3.11)$$

$$p \cdot y_i \geq \langle p, Y_i \rangle. \quad (1.3.12)$$

holds. If (1.3.11) has a strict sign for each i , then triple (x, y, p) is called **competitive equilibrium**.

Everywhere below we assume model \mathcal{E} satisfies the following assumption.

(A) For each $i \in \mathcal{I}$, X_i is a convex solid closed set (nonempty interior), $\mathbf{e}_i \in X_i$ and for every $x_i \in X_i$ there exists an open convex $G_i \subset L$ such that²³

$$\mathcal{P}_i(x_i) = G_i \cap X_i \quad \& \quad x_i \in \overline{\mathcal{P}_i(x_i)} \setminus \mathcal{P}_i(x_i).$$

• Economy \mathcal{E} is called **convex** if it obeys (A) and has a **convex production sector**, *i.e.* sets Y_i are convex for all $i \in \mathcal{I}$.

For the convenience of further exposition, we introduce a specific notion of a smooth economy.

• Economy \mathcal{E} is called **smooth** if for each i :

$$\mathcal{P}_i(x_i) = \{x'_i \in X_i \mid u_i(x'_i) > u_i(x_i)\}, \quad \forall x_i \in X_i$$

holds for a differentiable concave function $u_i(\cdot)$ that is defined on an open neighborhood of X_i and the set Y_i has a boundary presented as a smooth manifold.

Economy \mathcal{E} has a **smooth consumption sector** if only the first requirement is true.

Recall further the conceptual apparatus of the theory of barter contracts, see Section 1.1 (Marakulin, 2003, 2011), while adapting it to the model with the production sector.

Any vector $v = (v_i)_{i \in \mathcal{I}} \in L^{\mathcal{I}}$ satisfying $\sum_{i \in \mathcal{I}} v_i = 0$ is called a barter (exchange) contract. Such barter contracts are used in pure exchange economies, as well as in the consumption sector in the economy with production. In what follows, we assume that any barter agreement is valid. With every finite collection V of (permissible) contracts, it can be associated allocation $x(V) = \mathbf{e} + \sum_{v \in V} v$, where $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_n) \in X$ is

²³Notice that we strengthened the above similar assumption, adding non-satiation in preferences.

an initial resource allocation. If $\mathbf{e} + \sum_{v \in U} v \in X \ \forall U \subseteq V$, *i.e.*, if for any part of contracts is broken one can get anyway a feasible allocation, then we call V as a *web* of contracts.

In this section, contractual concepts are modified and adapted to an economy with production. In the latter case contract is a pair $(v, y) \in L^{\mathcal{I}} \times L^{\mathcal{I}}$, where v is an ordinary barter contract but $y = (y_1, \dots, y_n)$ is a vector which corresponds to production programs y_i for individuals $i \in \mathcal{I}$. If $(v, y) \in L^{\mathcal{I}} \times Y$, *i.e.*, if each production program is feasible, $y_i \in Y_i$, $i \in \mathcal{I}$, then contract (v, y) is permissible. For a finite collection V of contracts in the model (1.3.10) one can put into correspondence (consumption) allocation

$$x(V) = \mathbf{e} + \sum_{(v,y) \in V} y + \sum_{(v,y) \in V} v.$$

On the other hand, clearly that any contract (v, y) can be decomposed into the sum of pure exchange contract $(v, 0)$ and production contract (program) $(0, y)$. In this case, referring to the web of contracts, there is no need to specify a different set of production programs, it suffices to take a gross contract, obtained by summing them. Moreover, the actual production component of the contract may be without prejudice to endure beyond the concept of a web of contracts. So, one arrives at the following definition:

A finite collection V of permissible contracts is called a web of contracts relative to pair $(\zeta, y) \in X \times Y$, if

$$\zeta + y + \sum_{v \in U} v \in X, \ \forall U \subseteq V \iff \zeta_i + y_i + \sum_{v \in U} v_i \in X_i, \ \forall i \in \mathcal{I}, \ \forall U \subseteq V,$$

i.e., under the current consumption $\zeta \in X$ and production plans $y \in Y$, individuals enter into contractual relationships so that they can break any contracts. Here, the vector of current consumption plans $\zeta = (\zeta_1, \dots, \zeta_n)$ actually plays a role (variable) of initial endowments. In the case where $\zeta = \mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_n) \in X$, *i.e.*, when there is a web of contracts relative to a pair $(\mathbf{e}, y) \in X \times Y$, one calls this web a *y-web*. Specify:

$$x(V, y) = \mathbf{e} + y + \sum_{v \in V} v \in L^{\mathcal{I}}, \quad z(V, y) = (x(V, y), y) \in L^{\mathcal{I}} \times L^{\mathcal{I}}.$$

Let $y_T = (y_i)_{i \in T}$ be a collection of production plans of individuals entering in coalition T . Define $\mathcal{I} \setminus T = -T$, then y_{-T} will present the vector consisting of the production plans of all individuals who were not included in the coalition T . Now the collection of all production plans $y = (y_i)_{i \in \mathcal{I}}$ can be written in the form $y = (y_T, y_{-T})$.

Being applied jointly, *i.e.*, as a simultaneous procedure, these operations allow coalition $T \subseteq \mathcal{I}$ to yield new webs of contracts. The set of all such webs is denoted by $F(V, T)$. Formally, we require that each element $U \in F(V, T)$ has to satisfy the following properties:

Let us say that y' -web U , $y' = (y'_T, y'_{-T}) \in Y$, dominates y -web V , $y = (y_T, y_{-T}) \in Y$ via coalition T (notation $U \succ_T V$), if:

$$(i) \ U \in F(V, T),$$

(ii) $x_i(U, y') \succ_i x_i(V, y)$ for all $i \in T$.

Notice that now production plans of supplementary coalition y_{-T} are unchangeable but y_T can be substituted for new plans $y'_T \in \prod_{i \in T} Y_i$. In other words, members of the coalition T can change not only the exchange contracts (breaking of the old (any involving)) and conclude new within the coalition contracts), but they are also able to modify their production plans.

Definition 1.3.3 *A y -web of contracts V is called stable if there is no \tilde{y} -web U and no coalition $T \subset \mathcal{I}$ such that $U \succ_T V$. Allocation $z = (x, y) \in \mathcal{A}(\mathcal{E})$ is called contractual if $x = x(V, y)$ for some stable y -web V .*

In order to introduce the operation of partial breaking of contracts, similarly to the pure exchange economy case consider the following partial order on the set of y -webs. This ordering is defined by the rule:

$$U \geq V \iff \exists \text{ a map } \mathbf{onto} \ f : U \rightarrow V \text{ such that}$$

(i) $\lambda f(u) = u$ for some $0 \leq \lambda \leq 1$ and for every $u \in U$,

(ii) $\sum_{u \in f^{-1}(v)} u = v$ for every $v \in V$.

By this definition, a web U consists of a finite partition of contracts from V (decomposition into the sum, see (ii)) subject to the exchange ratios are not changed (due to (i)). Minimal elements of the ordering among all y -webs are called root webs.

The ordering induces the following equivalence relation on the set of all y -webs:

$$U \simeq V \iff \exists \text{ } y\text{-web } W \text{ such that } V \geq W \text{ and } U \geq W. \quad (1.3.13)$$

Notice that if $U \simeq V$, then $z(U, y) = z(V, y)$, *i.e.*, equivalent webs yield identical allocations.

In terms of equivalent webs an allocation $z \in \mathcal{A}(\mathcal{E})$ is called *properly contractual*, if there is y -web V such that $z = z(V, y)$ and for every $U \simeq V$ allocation $z = z(U, y)$ is a contractual one.

This concept of a properly contractual allocation can be reformulated in the following simplified form. For real α define $\alpha V = \{\alpha \cdot v \mid v \in V\}$, *i.e.*, αV is a web, yielded from V by multiplying contracts on α . For $0 \leq \alpha \leq 1$ consider web $U = \alpha V \cup (1 - \alpha)V$, which obviously implements the same allocation $z(U, y) = z(V, y)$. The web $U = \alpha V \cup (1 - \alpha)V$ is called α -partition of the web V . An allocation $z = z(V, y)$ is properly contractual if α -partition of V is stable for every $\alpha \in [0, 1]$. Below, I present a narrative definition in substantial terms.

Definition 1.3.4 *A pair $(x, y) \in X \times Y$ is called properly contractual allocation if there is a web V such that the following conditions are satisfied:*

(i) $x = x(V, y) = \sum_V v + \mathbf{e} + y$.

(ii) *There is no coalition S , for which it is profitable:*

- (α) to partially break barter contracts;
- (β) to transit from the programs $y = (y_S, y_{-S})$ to new production programs $y' = (y'_S, y_{-S})$, where $y'_S \in \prod_{i \in S} Y_i$;
- (γ) to sign a new contract.

In Section 1.2, it has been proven that for the smooth pure exchange economies each interior properly contractual allocation is an equilibrium (Theorem 1.2.2, see also *Kozyrev* (1982a), *Marakulin* (2003, 2011)). Below this result is extended to the convex Arrow–Debreu model.

The following lemma gives a characterization of Pareto optimal allocations of the convex economy in value terms and, in fact, this is an analog of the Second Welfare Theorem. Recall that:

• A feasible allocation (x, y) is said to be (weakly) Pareto optimal if there is no a family $(x'_i, y'_i)_{i \in \mathcal{I}} \in \mathcal{A}(\mathcal{E})$ such that $x'_i \succ_i x_i$ for all $i \in \mathcal{I}$.

Lemma 1.3.1 Let \mathcal{E} be a convex economy and $\bar{z} = (\bar{x}_i, \bar{y}_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} X_i \times Y_i$ be a Pareto optimal allocation. Then there is a vector $p \neq 0$ such that:

$$\langle p, \mathcal{P}_i(\bar{x}_i) \rangle \geq p \cdot \bar{x}_i, \quad (1.3.14)$$

$$p \cdot \bar{y}_i \geq \langle p, Y_i \rangle. \quad (1.3.15)$$

holds for each $i \in \mathcal{I}$.

Remark 1.3.1 If \mathcal{E} is an economy with a smooth consumption sector and if $\bar{x}_i \in \text{int}X_i$ then there is a real $\lambda_i > 0$ such that $p = \lambda_i \cdot \nabla u_i(\bar{x}_i)$. ■

Proof of Lemma 1.3.1. First, note that assumption **(A)** implies that $\bar{x}_i \notin \mathcal{P}_i(\bar{x}_i)$ and $\bar{x}_i \in \overline{\mathcal{P}_i(\bar{x}_i)}$ is true for all $i \in \mathcal{I}$. Next, write the property that allocation $(\bar{x}_i, \bar{y}_i)_{i \in \mathcal{I}}$ is Pareto optimal in an equivalent form:

$$\prod_{i \in \mathcal{I}} (\mathcal{P}_i(\bar{x}_i) \times Y_i) \cap \mathcal{A}(\mathcal{L}) = \emptyset,$$

$$\mathcal{A}(\mathcal{L}) = \{(x, y) \in \mathcal{L} \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i + \sum_{i \in \mathcal{I}} y_i\}.$$

Since intersected sets are convex and nonempty (because of **(A)**), then by separation theorem there exists a linear functional $f \neq 0$ separating these sets, *i.e.*

$$f((x_i, y_i)_{i \in \mathcal{I}}) \geq f((\check{x}_i, \check{y}_i)_{i \in \mathcal{I}}), \quad \forall (x_i, y_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} (\mathcal{P}_i(\bar{x}_i) \times Y_i), \quad \forall (\check{x}_i, \check{y}_i)_{i \in \mathcal{I}} \in \mathcal{A}(\mathcal{L}).$$

Since our commodity space is finite-dimensional, it follows that the linear functional can be represented by the vector (through the inner product), *i.e.*, we can assume that $f = (f_1, \dots, f_n) \in \mathcal{L}$. Now the latter relation can be written in the form:

$$\sum_{i \in \mathcal{I}} \langle f_i, (x_i, y_i) \rangle \geq \sum_{i \in \mathcal{I}} \langle f_i, (\check{x}_i, \check{y}_i) \rangle. \quad (1.3.16)$$

Fix $(x_i, y_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} (\mathcal{P}_i(\bar{x}_i) \times Y_i)$. From (1.3.16) the value $\sum_{i \in \mathcal{I}} \langle f_i, (\check{x}_i, \check{y}_i) \rangle$ is bounded on $\mathcal{A}(\mathfrak{L})$, *i.e.*, f is bounded from above on $\mathcal{A}(\mathfrak{L})$. Show that in fact the functional is constant on $\mathcal{A}(\mathfrak{L})$.

Assume contrary. The set $\mathcal{A}(\mathfrak{L})$ can be written in the form:

$$\mathcal{A}(\mathfrak{L}) = (\mathbf{e}_1, \dots, \mathbf{e}_n, 0, \dots, 0) + H = (\mathbf{e}, 0) + H,$$

where $H = \{(x, y) \in \mathfrak{L} \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} y_i\}$. Notice that H is a linear space such that the summation $f(\mathbf{e}, 0) + f(h)$ is bounded for all $h \in H$. Therefore f is bounded from above on H that is possible only if $f(h) \equiv 0$ on H (hence functional f is constant on $\mathcal{A}(\mathfrak{L})$).

Further, one can note that the vectors of the form

$$((0, 0)_1, \dots, (0, 0)_{i-1}, (x, x)_i, (0, 0)_{i+1}, \dots, (0, 0)_n) \in H, \quad \forall x \in L.$$

Therefore, $f_i(x, x) = 0$ for all $x \in L$. Write $f_i = (f_i^x, f_i^y)$ and obtain $f_i^x \cdot x + f_i^y \cdot x = 0$. Now, since x is an arbitrary chosen, one concludes

$$f_i^x = -f_i^y = p_i. \quad (1.3.17)$$

Further show that $p_i = p_j$ for all $i, j \in \mathcal{I}$.

Due to (1.3.17) one has $f_i = (p_i, -p_i)$. Further note that for every pair $(x, y) \in \mathfrak{L}$ the vector of the form

$$((0, 0)_1, \dots, (0, 0)_{i-1}, (x, y)_i, (0, 0)_{i+1}, \dots, (0, 0)_{j-1}, -(x, y)_j, (0, 0)_{j+1}, \dots, (0, 0)_n)$$

belongs to H . Therefore, one has $f_i(x, y) - f_j(x, y) = 0$. The latter equality holds for every pair $(x, y) \in L \times L$, that implies

$$f_i = f_j = (p, -p) \quad \forall i, j \in \mathcal{I}.$$

Finally, the fact $f \neq 0$ implies $p \neq 0$. Now (1.3.16) can be rewritten in the form

$$\sum_{i \in \mathcal{I}} \langle (p, -p), (x_i, y_i) \rangle \geq \sum_{i \in \mathcal{I}} \langle (p, -p), (\check{x}_i, \check{y}_i) \rangle. \quad (1.3.18)$$

To prove inequality (1.3.14) let us consider inequality (1.3.18). Consider $y_i = \check{y}_i = \bar{y}_i$ and let $\check{x}_i = \bar{x}_i$ for all $i \in \mathcal{I}$. In view of (1.3.16) for all $x_i \in \mathcal{P}_i(\bar{x}_i)$, $i \in \mathcal{I}$ one has:

$$\sum_{i \in \mathcal{I}} \langle (p, -p), (x_i - \bar{x}_i, 0) \rangle \geq 0 \quad \iff \quad \langle p, x_i \rangle + \sum_{j \neq i} \langle p, (x_j - \bar{x}_j) \rangle \geq \langle p, \bar{x}_i \rangle \quad \forall i \in \mathcal{I}.$$

Fix i and consider $j \neq i$. From **(A)** it follows that $\bar{x}_j \in \overline{\mathcal{P}_j(\bar{x}_j)}$, which allows in the addends with numbers $j \neq i$ in the left hand side of the last inequality pass to the limits by $x_j \rightarrow \bar{x}_j$ and as a result one gets $\langle p, x_i \rangle \geq \langle p, \bar{x}_i \rangle$ for all $x_i \in \mathcal{P}_i(\bar{x}_i)$ and all $i \in \mathcal{I}$, this proves (1.3.14).

To prove relation (1.3.15) consider $y_j = \check{y}_j = \bar{y}_j$ and $\check{x}_j = \bar{x}_j$ for all $j \neq i$. Let $\check{y}_i = \bar{y}_i$ and $\check{x}_i = \bar{x}_i$. In view of (1.3.18), one obtains

$$\sum_{j \in \mathcal{I}} p(x_j - \bar{x}_j) - p(y_i - \bar{y}_i) \geq 0.$$

Similarly to the previous one, passing x_j to \bar{x}_j for all $j \in \mathcal{I}$ one concludes

$$-p(y_i - \bar{y}_i) \geq 0 \quad \forall y_i \in Y_i \quad \iff \quad \langle p, \bar{y}_i \rangle \geq \langle p, Y_i \rangle,$$

as we wanted to prove. ■

The following theorem establishes an equivalence between equilibrium and properly contractual allocation in the economy with the convex production sector.

Theorem 1.3.2 *Let \mathcal{E} be a convex contractual economy with smooth consumption sectors. Let $\bar{z} = (\bar{x}, \bar{y}) \in \mathcal{A}(\mathcal{E})$ and $\bar{x}_i \in \text{int}X_i$ for all $i \in \mathcal{I}$. Then pair (\bar{z}, p) is an equilibrium if and only if \bar{z} is a properly contractual allocation.*

Proof of Theorem 1.3.2. Let us start by checking the sufficiency. Suppose that $\bar{z} = (\bar{x}(V, \bar{y}), \bar{y})$ is a properly contractual allocation. This allocation is Pareto optimal since coalition \mathcal{I} is unable to conclude a new contract beneficial for all members of \mathcal{I} . From Lemma 1.3.1 it follows the existence of $p \neq 0$, which satisfies

$$\langle p, \mathcal{P}_i(x_i) \rangle \geq p \cdot x_i, \quad \forall i \in \mathcal{I}.$$

Now, to prove that (\bar{x}, \bar{y}, p) is a quasi-equilibrium, it suffices to show the following

$$p \cdot \bar{x}_i = p \cdot \bar{y}_i + p \cdot \mathbf{e}_i, \quad \forall i \in \mathcal{I}. \quad (1.3.19)$$

The proof is by contradiction. Assume that not all equalities (1.3.19) hold. Then, since the allocation \bar{z} is balanced, one concludes:

$$\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} \bar{y}_i + \sum_{i \in \mathcal{I}} \mathbf{e}_i \Rightarrow \sum_{i \in \mathcal{I}} p\bar{x}_i = \sum_{i \in \mathcal{I}} p\bar{y}_i + \sum_{i \in \mathcal{I}} p\mathbf{e}_i.$$

Therefore there is $i \in \mathcal{I}$, for which

$$p \cdot \bar{x}_i - p \cdot \bar{y}_i < p \cdot \mathbf{e}_i.$$

This inequality can be rewritten in the form

$$p \cdot \bar{x}_i < p \cdot (\mathbf{e}_i + \bar{y}_i) = p \cdot x_i,$$

where

$$\bar{x}_i = \mathbf{e}_i + \bar{y}_i + \sum_{v \in V} v_i, \quad x_i = \mathbf{e}_i + \bar{y}_i + 0.$$

Further, from $p \cdot \bar{x}_i < p \cdot x_i$ one concludes $p \cdot (x_i - \bar{x}_i) > 0$, that due to Remark 1.3.1 to Lemma 1.3.1 ($p = \lambda_i \cdot \nabla u_i(\bar{x}_i)$ for some $\lambda_i > 0$) one has

$$\nabla u_i(\bar{x}_i) \cdot (x_i - \bar{x}_i) > 0.$$

However, this inequality means that the function u_i from the point \bar{x}_i increases in the direction $x_i - \bar{x}_i = -\sum_{v \in V} v_i$, so there is $\mu \in (0, 1)$ such that

$$u_i(\bar{x}_i - \mu \cdot \sum_{v \in V} v_i) > u_i(\bar{x}_i).$$

Thus it is profitable for agent i partially break all barter contracts in the volume μV . Now one arrives at a contradiction with the definition of properly contractual allocation. It remains to note that under the theorem conditions ($\bar{x}_i \in \text{int} X_i \forall i$) and in view of assumption **(A)** every quasi-equilibrium is an equilibrium in fact.

To prove the necessity, define $v = \bar{x} - \bar{y} - \mathbf{e}$ and put $V = \{v\}$. In view of equilibrium properties of pair (\bar{z}, p) , one has $p \cdot v_i = 0$ for all $i \in \mathcal{I}$. Suppose that \bar{z} is not properly contractual allocation. Then there is a coalition $T \subseteq \mathcal{I}$, real $0 \leq \lambda_v \leq 1$, contract $w = (w_i)_{i \in T}$ and production plans $y_T \in \prod_T Y_i$ such that $\mathbf{e}_i + y_i + \lambda_v \cdot v_i + w_i \succ_i \bar{x}_i$, $i \in T$. However, from equilibrium properties (1.3.3) one concludes

$$p \cdot (\mathbf{e}_i + y_i + \lambda_v \cdot v_i + w_i) > p \cdot \bar{x}_i = p \cdot (\mathbf{e}_i + \bar{y}_i + v_i), \quad i \in T.$$

Therefore $p \cdot w_i > p \cdot (\bar{y}_i - y_i)$ for all $i \in T$, that via (1.3.2) implies $p \cdot w_i > 0$, hence $p \cdot \sum_{i \in T} w_i \neq 0$. One obtains a contradiction with the definition of contract because it has to be $\sum_{i \in T} w_i = 0$. Theorem 1.3.2 is proved. \blacksquare

A similar result avoided from so restrictive differentiable utilities assumption can be stated if one likes exchange economies to transfer the study for fuzzy contractual allocations. In Section 2.2.3, this notion is introduced and elaborated rather carefully for the model with public goods. This is why we present only the main definition and announce the major result.

Definition 1.3.5 An allocation $(\bar{x}, \bar{y}) \in \mathcal{A}(\mathcal{E})$ implemented by a web of barter contracts $V = \{v\}$, $v = \bar{x} - \bar{y} - \mathbf{e}$ is called *fuzzy contractual* if for every $t = (t_i)_{i \in \mathcal{I}}$, $0 \leq t_i \leq 1$, $\forall i \in \mathcal{I}$, there are no other production programs $y_i \in Y_i$, $i \in \mathcal{I}$ and a barter contract $w = (w_1, \dots, w_n) \in \mathbb{R}^{ln}$, $\sum w_k = 0$ such that for

$$\xi_i = \xi_i(t, v, w) = \mathbf{e}_i + t_i v_i + w_i + y_i, \quad i \in \mathcal{I} \quad (1.3.20)$$

$$\xi_i \succ_i \bar{x}_i \quad \forall i : \xi_i \neq \bar{x}_i \quad (1.3.21)$$

take place.

Notice that due to (1.3.21) possibility $w = 0$ is also possible and thereby only a partial break can be implemented. If there is no such possibility to dominate, an allocation is called *stable relative to asymmetrical partial break of barter contracts* and the admissible *change* of the (individualized) *production program*²⁴ (lower stable allocation).

The following lemma complete characterizes fuzzy contractual allocations in “geometrical” categories.

²⁴The ability to select a new production program is the only significant “new property” of this concept in the production model.

Lemma 1.3.2 *Let $(\bar{x}, \bar{y}) \in \mathcal{A}(\mathcal{E})$ be a lower stable relative to partial break allocation. Then it is fuzzy contractual if and only if*

$$\mathcal{A}(L^{\mathcal{I}}) \cap \prod_{i \in \mathcal{I}} \left((\mathcal{P}_i(\bar{x}_i) - Y_i + \text{co}\{(\mathbf{e}_i + \bar{y}_i - \bar{x}_i), 0\}) \cup \{\mathbf{e}_i\} \right) = \{\mathbf{e}\}. \quad (1.3.22)$$

Here $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ and $\mathcal{A}(L^{\mathcal{I}})$ is a subspace that corresponds to the material balance constraints:

$$\mathcal{A}(L^{\mathcal{I}}) = \{(z_i)_{\mathcal{I}} \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\}.$$

This characterization works very efficiently in applications. In particular, it is possible to state results on fuzzy contractual allocations similar to Theorem 1.3.2 but for the preferences of general form. To do it, one has to separate sets from the left-hand part of (1.3.22) by linear functional, and then one analysis the result (for example it is so during the proof of Theorem 2.2.4 below).

Proof of Lemma 1.3.2. Necessity. Let (1.3.22) be false. Therefore, in the left-hand side of the intersection (1.3.22) there is $z = (z_i)_{\mathcal{I}} \neq \mathbf{e}$. Define

$$S = \{i \in \mathcal{I} \mid z_i \neq \mathbf{e}_i\} \neq \emptyset.$$

Further, find a contract with this support and appropriate amounts of contracts breaking. Define $w_i = z_i - \mathbf{e}_i$ that gives $\sum w_i = 0$ since $z \in \mathcal{A}(L^{\mathcal{I}})$. For $i \notin S$ one obviously has $w_i = 0$, i.e., $\text{supp}(w) = S$. Also, for $i \in S$ one has $z_i \in (\mathcal{P}_i(\bar{x}_i) - Y_i + \text{co}\{0, (\mathbf{e}_i + \bar{y}_i - \bar{x}_i)\})$, that allows to conclude $\exists 0 \leq t_i \leq 1$ and $\xi_i \succ_i \bar{x}_i$ such that:

$$\mathbf{e}_i + w_i = z_i = \xi_i - y_i + t_i(\mathbf{e}_i + \bar{y}_i - \bar{x}_i) \Rightarrow \xi_i = \mathbf{e}_i + t_i v_i + y_i + w_i,$$

for $v_i = \bar{x}_i - \mathbf{e}_i - \bar{y}_i$, $i \in S$ ($\xi_i = \bar{x}_i$ for $i \notin S$). This contradicts the definition of fuzzy contractual allocation.

Sufficiency. Let (1.3.22) be true for $(\bar{x}, \bar{y}) \in \mathcal{A}(\mathcal{E})$, and in addition, let it be lower stable relative to the partial breaking. Assume this is not fuzzy contractual. Then there are real $t = (t_i)_{\mathcal{I}}$, plans $y_i \in Y_i$, $i \in \mathcal{I}$ and a barter contract $w = (w_1, \dots, w_n) \in \mathbb{R}^{ln}$, $\sum w_k = 0$, satisfying all Definition 1.3.5 requirements. This for $v_i = \bar{x}_i - \bar{y}_i - \mathbf{e}_i$ due to (1.3.20) for the members of a nonempty coalition yields

$$\exists \xi_i \in \mathcal{P}_i(\bar{x}_i) : x_i = \xi_i + t_i(\mathbf{e}_i + \bar{y}_i - \bar{x}_i) = \mathbf{e}_i + w_i + y_i. \quad (1.3.23)$$

Summing over i by the definition of contract one concludes $\sum_{\mathcal{I}} x_i = \sum_{\mathcal{I}} \mathbf{e}_i + \sum_{\mathcal{I}} y_i$, i.e., for $z_i = x_i - y_i$, $i \in \mathcal{I}$ allocation $z = (z_i)_{\mathcal{I}}$ belongs to the intersection in the left hand part of (1.3.22). If one supposes $z = \mathbf{e}$, then $x_i = \mathbf{e}_i + y_i \forall i \in \mathcal{I}$, that being substituted to the right-hand part of (1.3.23) yields $w_i = 0$, $\forall i \in \mathcal{I} \Rightarrow \text{supp}(w) = \emptyset$. Hence, domination is carried out without the exchange and only via a partial break of the gross contract $v = \bar{x} - \bar{y} - \mathbf{e}$. However, this contradicts the lower stability relative to a partial break. Therefore, one finds an allocation $z \neq \mathbf{e}$

which belongs to the intersection of the left-hand side (1.3.22); it is a contradiction. ■

Finishing the section note that similarly to the case of exchange model for Arrow–Debreu kind model, one can introduce the notion of fuzzy core closely related with fuzzy contractual allocations. Recall that allocation $(\bar{x}, \bar{y}) \in \mathcal{A}(\mathcal{E})$ in the model (1.3.10) is dominated by a fuzzy coalition $t = (t_1, \dots, t_n) \neq 0$, $0 \leq t_i \leq 1$, $i \in \mathcal{I}$ if there are $x_i \in \mathcal{P}_i(\bar{x}_i)$, $y_i \in Y_i$, $i \in \mathcal{I}$, such that

$$\sum_{i \in \mathcal{I}} t_i(x_i - \mathbf{e}_i) = \sum_{i \in \mathcal{I}} t_i y_i.$$

Fuzzy core $\mathcal{C}^f(\mathcal{E})$ is formed as a set of all allocations which are non-dominated by fuzzy coalitions. If the preferences of all agents are non-satiated, then for the convex economy an allocation belongs to the fuzzy core if and only if

$$0 \notin \text{co}\{\cup_{\mathcal{I}}[\mathcal{P}_i(x_i) - Y_i - \mathbf{e}_i]\}.$$

The main theoretical value of the fuzzy core is that it can be effectively applied to develop a theory of the existence of competitive equilibria and, in view of results presented here, fuzzy contractual allocations. The general methodology of this approach (with regard to the existence of equilibrium) one can find in *Aliprantis et al.* (1989), and the most advanced mathematical results in *Florenzano* (1989, 1990) and *Marakulin* (2012), *etc.*

In conclusion, we only present another geometrical interpretation for fuzzy core elements which supports all ongoing analysis. Let economy be convex. Define

$$\Upsilon_i(x_i, y_i) = \text{co}[(\mathcal{P}_i(x_i) - Y_i) \cup \{\mathbf{e}_i\}], \quad i \in \mathcal{I}.$$

Then allocation $(x, y) \in \mathcal{C}^f(\mathcal{E}) \iff$

$$\prod_{i \in \mathcal{I}} \Upsilon_i(x_i, y_i) \cap \mathcal{A}(L^{\mathcal{I}}) = \{\mathbf{e}\}.$$

This result is appropriate to compare with that was obtained in Chapter 1 for the exchange model, see (1.2.13) and Proposition 1.2.1. An analogous analysis is provided also in Section 2.2.1 in the public goods economies context (next chapter).

1.3.3 Contractual economies with non-convex smooth production

Consider the model of the economy (1.3.10) for which one assumes the presence of non-convex production sets Y_i , $i \in \mathcal{I}$ and the other standard assumptions: closeness, free disposal and so on. For this model a specified concept of properly contractual allocation is applied and this concept uses the notion of a star-shaped set. Recall that a set $A \subseteq L$ is called a star-shaped with respect to $x \in A$ if for all $y \in A$ linear segment $[x, y] \subset A$; here by definition $[x, y] = \{z \in L \mid \exists \lambda \in [0, 1] : z = \lambda x + (1 - \lambda)y\}$.

Let $y = (y_i)_{i \in \mathcal{I}} \in Y$ be any given family of production plans. For each $i \in \mathcal{I}$ specify a set M_i^y as a star-shaped relative to $y_i \in Y_i$ subset of Y_i such that it is the maximal by inclusion among other star-shaped sets of this kind.

Notice that for a *convex* case, the sets M_i^y and Y_i are *matched* but in the non-convex variant it can be both ways, see Figure 1.3.9 a). In the case of nonconvex production sets the concept of properly contractual allocation is modified, but it coincides with the above in the convex context.

Definition 1.3.6 *A pair $(x, y) \in X \times Y$ is called **marginally contractual** allocation if there is a web V such that the following conditions are satisfied:*

$$(i) \quad x = x(V, y) = \sum_V v + \mathbf{e} + y.$$

(ii) *There is no coalition S , for which it is profitable (in a separate or simultaneous regime):*

(α) *to partially break barter contracts;*

(β) *to transit from the given production programs $y = (y_S, y_{-S})$ to new production programs $y' = (y'_S, y_{-S})$, where $y'_S \in \prod_{i \in S} M_i^y$;*

(γ) *to sign a new contract.*

Here, in contrast to the convex case, the freedom of the individual in choosing a production plan is limited. It is assumed that the mutual cooperation, or some authority establishes joint production plans, and the individuals still have the right to decide: do it or not (a specific form of non-binding agreement). However, the individual deviations from a given production plan is only possible within the sets $M_i^y \subset Y_i$. Again, in the case of convex production M_i^y and Y_i coincide.

Suppose further that the production sets Y_i are described by *differentiable functions* φ_i as follows:

$$Y_i = \{y \in L \mid \varphi_i(y) \leq 0\}, \quad i \in \mathcal{I}.$$

Without loss of generality assume that the boundary of production sets is defined as²⁵

$$\partial Y_i = \{y \in L \mid \varphi_i(y) = 0\} \neq Y_i, \quad i \in \mathcal{I}.$$

Further, we need the following

Lemma 1.3.3 *Let M_i^y be the maximal star-shaped subset of Y_i relative to a point $y_i \in \partial Y_i$. Then*

$$M_i^y \subset \{z \in L \mid \varphi_i(z) \leq 0 \ \& \ \nabla \varphi_i(y_i) \cdot z \leq \nabla \varphi_i(y_i) \cdot y_i\}.$$

Proof of Lemma 1.3.3. Take $z \in M_i^y$. By M_i^y definition, one has

$$z(\lambda) = (1 - \lambda)y_i + \lambda z \in M_i^y \subseteq Y_i, \quad \forall \lambda \in [0, 1].$$

²⁵Note that this implies $\text{int}Y_i \neq \emptyset$.

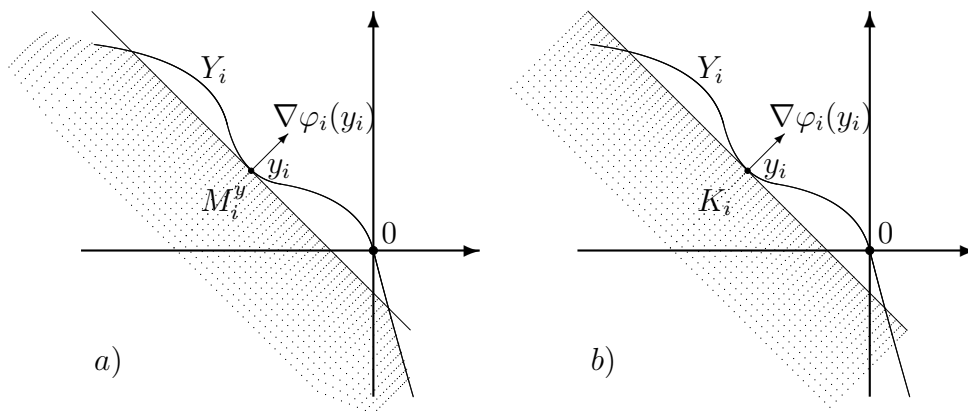


Figure 1.3.9: a)—Maximal star-shaped subset M_i^y , b)—semi-spaces K_i

Therefore, $\varphi_i(z(\lambda)) \leq 0$. As soon as $\varphi_i(y_i) = 0$, then

$$\lim_{\lambda \rightarrow +0} \frac{\varphi_i(y_i + \lambda(z - y_i)) - 0}{\lambda} = \langle \nabla \varphi_i(y_i), z - y_i \rangle \leq 0,$$

that completes the proof of lemma. ■

Relationship between marginal contractual allocation and \mathcal{MCP} -equilibrium is established in the following

Theorem 1.3.3 *Let \mathcal{E} be smooth economy and let $x = (x_i)_{i \in \mathcal{I}} \in \text{int}X$ and $y = (y_i)_{i \in \mathcal{I}} \in Y$ be the families of consumption and production plans. Then the triplet (x, y, p) is the \mathcal{MCP} -equilibrium if and only if (x, y) is marginal contractual allocation.*

Proof of Theorem 1.3.3. Necessity. Let $(\bar{x}, \bar{y}, \bar{p})$ be a \mathcal{MCP} -equilibrium and let $V = \{v\}$ be a web where $v = \bar{x} - \bar{y} - \mathbf{e}$. Show that $(x(V, \bar{y}), \bar{y}) = (\bar{x}, \bar{y})$ is a marginally contractual allocation. Assume contrary. Then there is a coalition S , a real $\lambda_v \in [0, 1]$, a new contract $u = (u_i)_{i \in \mathcal{I}}$ and new production programs $y'_S = (y'_i)_{i \in S} \in \prod_{i \in S} M_i^{\bar{y}}$ such that

$$\lambda_v \cdot v_i + \mathbf{e}_i + y'_i + u_i = x'_i \succ_i \bar{x}_i, \quad i \in S$$

holds. By equilibrium definition $\bar{p}x'_i > \bar{p}\mathbf{e}_i + \bar{p}y_i$. Since the deviation from the production plan is possible only within $M_i^{\bar{y}}$ then by Lemma 1.3.3 one obtains

$$\nabla \varphi_i(\bar{y}_i) \cdot y'_i \leq \nabla \varphi_i(\bar{y}_i) \cdot \bar{y}_i.$$

As soon as $\bar{p} = \lambda_i \cdot \nabla \varphi_i(\bar{y}_i)$ for some $\lambda_i > 0$ then $\bar{p} \cdot y'_i \leq \bar{p} \cdot \bar{y}_i$, that for each $i \in S$ yields

$$\bar{p}x'_i > \bar{p}\mathbf{e}_i + \bar{p}y_i \geq \bar{p}\mathbf{e}_i + \bar{p}y'_i \Rightarrow \bar{p}x'_i - \bar{p}\mathbf{e}_i - \bar{p}y'_i > 0.$$

Since $\lambda_v \bar{p}v_i + \bar{p}\mathbf{e}_i + \bar{p}y'_i + \bar{p}u_i = \bar{p}x'_i$, one has

$$\bar{p} \sum_S u_i = \bar{p} \sum_S x'_i - \bar{p} \sum_S \mathbf{e}_i - \bar{p} \sum_S y'_i - \lambda_v \bar{p} \sum_S v_i. \quad (1.3.24)$$

However, one already has $\bar{p} \sum_S x'_i - \bar{p} \sum_S \mathbf{e}_i - \bar{p} \sum_S y'_i > 0$. Now note that the properties of \mathcal{MCP} -equilibrium (1.3.8) imply $\bar{p} v_i = 0$ for all $i \in \mathcal{I}$ that due to (1.3.24) allows to deduce $\bar{p} \sum_S u_i > 0$, that contradicts to the contract definition.

Now establish the sufficiency. Let $(x(V, y), y)$ be marginally–contractual allocation. Show that there exists a vector $p \neq 0$ such that (x, y, p) is a \mathcal{MCP} -equilibrium. Consider the following convex sets (see Figure 1.3.9 b):

$$K_i = \{y'_i \in L \mid \langle \nabla \varphi_i(y_i), y'_i \rangle < \langle \nabla \varphi_i(y_i), y_i \rangle\} \cup \{y_i\}, \quad i \in \mathcal{I}.$$

Note that

$$\bar{K}_i \supset \{y'_i \in Y_i \mid \langle \nabla \varphi_i(y_i), y'_i \rangle \leq \langle \nabla \varphi_i(y_i), y_i \rangle\} \supset M_i^y, \quad i \in \mathcal{I}.$$

This is a tangent cone to the Y_i going from a point y_i .

Put $K = \prod_{i \in \mathcal{I}} K_i$. Next, consider an economy \mathcal{E}_K similar to one under study in which the sets Y_i are replaced by K_i . Show that the pair (x, y) is Pareto optimal in \mathcal{E}_K . Assuming contrary, one finds an attainable allocation $\hat{z} = (\hat{x}, \hat{y}) \in \mathcal{A}(\mathcal{E}_K)$, where $\hat{y} \in \prod_{i \in \mathcal{I}} K_i$ and $\hat{x}_i \succ_i x_i$ for all $i \in \mathcal{I}$. Let

$$y_i(\alpha) = y_i + \alpha(\hat{y}_i - y_i), \quad \alpha \in (0, 1], \quad i \in \mathcal{I}.$$

Since $\hat{y}_i \in K_i$, then for $\hat{y}_i \neq y_i$ one has

$$\langle \nabla \varphi_i(y_i), y_i(\alpha) - y_i \rangle = \alpha \langle \nabla \varphi_i(y_i), \hat{y}_i - y_i \rangle < 0, \quad i \in \mathcal{I}.$$

Recall that $\varphi_i(y_i) = 0$. But then for all sufficiently small $\alpha > 0$ one obtains

$$\varphi_i(y_i(\alpha)) = \varphi_i(y_i) + \langle \nabla \varphi_i(y_i), y_i(\alpha) - y_i \rangle + o(|y_i(\alpha) - y_i|) < 0, \quad i \in \mathcal{I}.$$

Consequently, for some $\varepsilon > 0$ and all $\alpha \in [0, \varepsilon]$, for each $i \in \mathcal{I}$ one has $y_i(\alpha) \in M_i^y$.

Further, fix $\alpha \in (0, \varepsilon]$ and put $\tilde{y}_i = y_i(\alpha)$, $\tilde{x}_i = \alpha \hat{x}_i + (1 - \alpha)x_i$. Let $w_i = \hat{x}_i - \hat{y}_i - \mathbf{e}_i$, $i \in \mathcal{I}$. As soon as $\mathcal{P}_i(x_i)$ is convex, one has $\tilde{x}_i \succ_i x_i$. On the other hand,

$$\tilde{x}_i = \alpha(\hat{y}_i + w_i + \mathbf{e}_i) + (1 - \alpha)(y_i + \sum_{v \in V} v_i + \mathbf{e}_i) = \mathbf{e}_i + (1 - \alpha) \sum_{v \in V} v_i + \alpha w_i + \tilde{y}_i.$$

Define $W = (1 - \alpha)V \cup \{\alpha w\}$, $w = (w_1, \dots, w_n)$. By construction, αw is a (new) contract and $\tilde{y}_i \in M_i^y$ is a new production plan constructed via the initial \hat{y}_i . Here new consumption plans $\tilde{x}_i = \tilde{x}_i(W, \tilde{y}_i)$ are strictly preferred to the plans x_i for all $i \in \mathcal{I}$. Now one comes to the contradiction with the definition of marginally contractual allocation since the allocation (\tilde{x}, \tilde{y}) is received from the current one through a partial break of initial contracts in the amount of $\alpha > 0$, the new choice of admissible production plans and the signing of a new barter contract αw .

Thus, now we are in the conditions of Lemma 1.3.1 according to which there is a vector $p \neq 0$ such that for all $i \in \mathcal{I}$

$$\langle p, \mathcal{P}_i(x_i) \rangle \geq p \cdot x_i \quad \& \quad p \cdot y_i \geq \langle p, K_i \rangle. \quad (1.3.25)$$

We prove further that p is a price vector for \mathcal{MCP} -equilibrium. To this end, we first show that

$$px_i = py_i + pe_i, \quad i \in \mathcal{I}.$$

However, this fact is proved in the same way as it was done in the convex case (see Theorem 1.3.2). As soon as assumptions **(A)** and $x_i \in \text{int}X_i$ for $p \neq 0$ in view of (1.3.25) together imply

$$\langle p, \mathcal{P}_i(x_i) \rangle > p \cdot x_i, \quad i \in \mathcal{I},$$

then condition (1.3.8) of Definition 1.3.2 is proven. Condition (1.3.7) of Definition 1.3.2 follows from (1.3.25) and specification of K_i . Finally, (1.3.9) holds by the definition of contractual allocation. Theorem 1.3.3 is proved. \blacksquare

1.3.4 Contractual economies with non-convex and nonsmooth production

In this section, we study the general case of the model (1.3.1) with non-convex technological sets and non-smooth boundaries.

Pareto frontier and tangent cones

In our analysis we shall apply several notions of tangent cones, see [Clarke \(1983\)](#); [Demianov, Rubinov \(1990\)](#); [Polovinkin \(1985\)](#). Below we recall some Minkowski algebraic operations applied to sets from a vector space. The product by the number, the sum of $A, B \subset L$ are determined in a standard way (see Notation on page 6) but in addition we apply the (geometrical) difference of Minkowski, specified as

$$A \overset{*}{-} B = \{a \in L \mid a + B \subseteq A\},$$

and

$$\varrho(b, A) = \inf\{\|b - a\| \mid a \in A\} - \text{a distance from a point } b \in L \text{ to the set } A.$$

Let $\Omega \subset \mathbb{R}^l$ be a set and $a \in \text{cl}(\Omega)$. A vector g is called an *admissible* direction of the set Ω at a point a if there is a number $\alpha_g > 0$ such that

$$a + \alpha g \in \Omega, \quad \alpha \in (0, \alpha_g).$$

The set $T_A(\Omega, a)$ of all *admissible* directions is called *adjacent* tangent cone to $\Omega \subset \mathbb{R}^l$ at $a \in \text{cl}(\Omega)$. It can be shortly presented as

$$T_A(\Omega, a) = \lim_{\alpha \rightarrow +0} \frac{\Omega - a}{\alpha} = \bigcup_{\alpha_0 > 0} \bigcap_{\alpha \in (0, \alpha_0)} \frac{\Omega - a}{\alpha}.$$

The vector g is called the *tangent* direction at the point a to the set Ω , if for $\alpha \rightarrow +0$ there is $o(\alpha) : (0, \alpha_0) \rightarrow \mathbb{R}^l$ such that

$$a + \alpha g + o(\alpha) \in \Omega,$$

where $o(\alpha)$ is so that $\frac{o(\alpha)}{\alpha} \rightarrow 0$ for $\alpha \rightarrow +0$. The set $T_L(\Omega, a)$ of all tangent directions is called (lower) *tangent cone* to $\Omega \subset \mathbb{R}^l$ at $a \in \text{cl}(\Omega)$ and can also be presented in the form

$$T_L(\Omega, a) = \liminf_{\alpha \downarrow 0} \frac{\Omega - a}{\alpha} = \{g \in \mathbb{R}^l \mid \lim_{\alpha \downarrow 0} \varrho(g, \alpha^{-1}(\Omega - a)) = 0\}.$$

An element $g \in \mathbb{R}^l$ is called the *possible* tangent direction to Ω at a , if there is a sequence $g_k \in \mathbb{R}^l$, $k \in \mathbb{N}$ and a sequence of positive real numbers $\alpha_k > 0$, $k \in \mathbb{N}$ so that

$$g_k \rightarrow g, \quad \alpha_k \downarrow 0, \quad a + \alpha_k g_k \in \Omega.$$

The set $T_B(\Omega, a)$ of all tangent vectors is called the *contingent cone* (or the *Bouligand tangent cone*) to $\Omega \subset \mathbb{R}^l$ at $a \in \text{cl}(\Omega)$. In a short form, it is

$$T_B(\Omega, a) = \limsup_{\alpha \downarrow 0} \frac{\Omega - a}{\alpha} = \{g \in \mathbb{R}^l \mid \liminf_{\alpha \downarrow 0} \varrho(g, \alpha^{-1}(\Omega - a)) = 0\}.$$

An element $g \in \mathbb{R}^l$ belongs to *Clarke cone* if for any sequences $x_k \in \Omega$ and positive real numbers $\alpha_k > 0$, $k \in \mathbb{N}$ such that $x_k \rightarrow a$, $\alpha_k \downarrow 0$ there exists a sequence $g_k \in \mathbb{R}^l$ for which $x_k + \alpha_k g_k \in \Omega$.²⁶ So, the Clarke tangent cone to a set Ω at a point a can be described as

$$T_C(\Omega, a) = \liminf_{\alpha \downarrow 0, x \rightarrow a} \frac{\Omega - x}{\alpha} = \{g \in \mathbb{R}^l \mid \lim_{\alpha \downarrow 0, x \rightarrow a} \varrho(g, \alpha^{-1}(\Omega - a)) = 0\}.$$

where the convergence $x \rightarrow a$ occurs over the set Ω .

The inclusions are obvious

$$T_C(\Omega, a) \subseteq T_L(\Omega, a) \subseteq T_B(\Omega, a).$$

If the set Ω is convex (or locally convex), then all the three indicated cones are equal. Clearly also that $T_A(\Omega, a) \subseteq T_L(\Omega, a)$.

We start our analysis from the characterization of Pareto optimality. Everywhere below we suppose (\mathbf{A}) , see page 67. As above we specify the set of all feasible allocations in model \mathcal{E}^{AD}

$$\mathcal{A}(\mathcal{E}^{AD}) = \{z = (x, y) \in X \times Y \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i + \sum_{j \in \mathcal{J}} y_j\}.$$

Now I recall a standard definition of Pareto optimality.

• A feasible allocation $(x, y) \in \mathcal{A}(\mathcal{E}^{AD})$ is said to be (weakly) Pareto optimal if there is no a family $((x'_i)_{i \in \mathcal{I}}, (y'_j)_{j \in \mathcal{J}}) \in \mathcal{A}(\mathcal{E}^{AD})$ such that $x'_i \succ_i x_i$ for all $i \in \mathcal{I}$.

²⁶This is not Clarke's original definition from [Clarke \(1983\)](#), but it is equivalent to it.

Lemma 1.3.4 *Let an economy \mathcal{E}^{AD} obey (A) and be non-satiated at a point $((\bar{x}_i)_{i \in \mathcal{I}}, (\bar{y}_j)_{j \in \mathcal{J}}) \in \mathcal{A}(\mathcal{E}^{AD})$. Assume that for each $j \in \mathcal{J}$ there is a **convex** cone $K_j(\bar{y}_j) \subset \mathbb{R}^l$ such that $K_j(\bar{y}_j) \subset T_B(Y_j, \bar{y}_j)$. Now if the allocation $(\bar{x}, \bar{y}) \in \mathcal{A}(\mathcal{E}^{AD})$ is Pareto optimal then $\bar{y}_j \in \partial Y_j \forall j \in \mathcal{J}$ and there is $p \in \mathbb{R}^l, p \neq 0$ such that*

$$\langle \mathcal{P}_i(\bar{x}_i), p \rangle \geq \langle \bar{x}_i, p \rangle, \quad i \in \mathcal{I}, \quad (1.3.26)$$

$$\langle K_j(\bar{y}_j), p \rangle \leq 0, \quad j \in \mathcal{J}. \quad (1.3.27)$$

Proof of Lemma 1.3.4. The Pareto optimality of an allocation can be rewritten in the following form: $((\bar{x}_i)_{i \in \mathcal{I}}, (\bar{y}_j)_{j \in \mathcal{J}}) \in \mathcal{A}(\mathcal{E}^{AD})$,

$$\left[\sum_{i \in \mathcal{I}} \mathcal{P}_i(\bar{x}_i) - \sum_{i \in \mathcal{I}} \mathbf{e}_i \right] \cap \sum_{j \in \mathcal{J}} Y_j = \emptyset. \quad (1.3.28)$$

Assume $\bar{y}_{j'} \in \text{int} Y_{j'}$ for some $j' \in \mathcal{J}$. Now via (A) one can find $x'_i \in \mathcal{P}_i(\bar{x}_i)$ enough close to $\bar{x}_i, i \in \mathcal{I}$ and such that

$$\sum_{i \in \mathcal{I}} (x'_i - \bar{x}_i) + \bar{y}_{j'} = y'_{j'} \in \text{int} Y_{j'} \Rightarrow \sum_{i \in \mathcal{I}} x'_i = \sum_{j \neq j', j \in \mathcal{J}} \bar{y}_j + y'_{j'} + \sum_{i \in \mathcal{I}} \mathbf{e}_i.$$

So we find an allocation $(x', (y'_j)_{j \in \mathcal{J}})$, where $y'_j = \bar{y}_j$ for $j \neq j'$, which dominates (\bar{x}, \bar{y}) , that is impossible for Pareto optimal allocation.

Further, due to $\sum_{i \in \mathcal{I}} \bar{x}_i - \sum_{i \in \mathcal{I}} \mathbf{e}_i = \sum_{j \in \mathcal{J}} \bar{y}_j$, (1.3.28) and Bouligand tangent cone definition one concludes

$$\left[\sum_{i \in \mathcal{I}} \text{int} \mathcal{P}_i(\bar{x}_i) - \sum_{i \in \mathcal{I}} \mathbf{e}_i \right] \cap \left(\sum_{j \in \mathcal{J}} \bar{y}_j + T_B\left(\sum_{j \in \mathcal{J}} Y_j, \sum_{j \in \mathcal{J}} \bar{y}_j\right) \right) = \emptyset.$$

Now since $T_B(\sum_{j \in \mathcal{J}} Y_j, \sum_{j \in \mathcal{J}} \bar{y}_j) \supseteq \sum_{j \in \mathcal{J}} T_B(Y_j, \bar{y}_j)$ we find

$$\left[\sum_{i \in \mathcal{I}} \text{int} \mathcal{P}_i(\bar{x}_i) - \sum_{i \in \mathcal{I}} \mathbf{e}_i \right] \cap \sum_{j \in \mathcal{J}} [\bar{y}_j + T_B(Y_j, \bar{y}_j)] = \emptyset \Rightarrow$$

$$\left[\sum_{i \in \mathcal{I}} \text{int} \mathcal{P}_i(\bar{x}_i) - \sum_{i \in \mathcal{I}} \mathbf{e}_i \right] \cap \sum_{j \in \mathcal{J}} [\bar{y}_j + K_j(\bar{y}_j)] = \emptyset.$$

Here two convex sets have empty intersection, one of which has a nonempty interior. We can apply separation theorem and find $p \in \mathbb{R}^l, p \neq 0$ such that

$$\langle p, \sum_{i \in \mathcal{I}} \text{int} \mathcal{P}_i(\bar{x}_i) - \sum_{i \in \mathcal{I}} \mathbf{e}_i \rangle \geq \langle p, \sum_{j \in \mathcal{J}} [\bar{y}_j + K_j(\bar{y}_j)] \rangle.$$

The set of values on the right-hand side of this inequality is upper-bounded, therefore because every $K_j(\bar{y}_j)$ is a conic set with a vertex at zero, we conclude

$$\langle p, K_j(\bar{y}_j) \rangle \leq 0 \quad \forall j \in \mathcal{J} \quad \& \quad \langle p, \sum_{i \in \mathcal{I}} \text{int} \mathcal{P}_i(\bar{x}_i) - \sum_{i \in \mathcal{I}} \mathbf{e}_i \rangle \geq \langle p, \sum_{j \in \mathcal{J}} \bar{y}_j \rangle.$$

The left inequality states (1.3.27). Now applying $\bar{x}_i \in \text{cl } \mathcal{P}_i(\bar{x}_i)$, $i \in \mathcal{I}$, taking into account $\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i + \sum_{j \in \mathcal{J}} \bar{y}_j$, passing to the limits²⁷ and transforming the right inequality we yield: for each $i' \in \mathcal{I}$

$$\sum_{i \neq i', i \in \mathcal{I}} \langle p, \bar{x}_i \rangle + \langle p, \mathcal{P}_{i'}(\bar{x}_{i'}) \rangle \geq \langle p, \sum_{i \in \mathcal{I}} \mathbf{e}_i \rangle + \langle p, \sum_{j \in \mathcal{J}} \bar{y}_j \rangle = \sum_{i \neq i', i \in \mathcal{I}} \langle p, \bar{x}_i \rangle + \langle p, \bar{x}_{i'} \rangle.$$

Omitting identical terms this proves (1.3.26). \blacksquare

So, Lemma 1.3.4 presents specific necessary conditions for an allocation to be Pareto optimal and this is an analog of the Second Welfare Theorem. Here the family of cones $K_j(\bar{y}_j) \subset T_B(Y_j, \bar{y}_j)$, $j \in \mathcal{J}$ accumulates the form of efficiency in this specification: wider cones provide a better description, and the best among them is a collection of half-spaces. Of course one can also try to apply the asymptotic cones described above, but among them only the Clarke cone is convex. The way to overcome this obstacle is to transform these cones using the method suggested in [Polovinkin \(1985\)](#); [Polovinkin, Balashov \(2004\)](#) for specifying the derivatives of the point-to-set mapping: I describe this construction below. It is based on the crucial property of Minkowski geometrical difference presented in the following lemma (see Lemma 1.1.4 in [Polovinkin, Balashov \(2004\)](#)).

Lemma 1.3.5 *For any cone K , the set $K \overset{*}{-} K$ is its **convex** subcone. For the convex cone K , the equality $K = K \overset{*}{-} K$ holds. If K is closed then $K \overset{*}{-} K$ is closed and*

$$T_C(K, 0) = K \overset{*}{-} K.$$

Proof. For $x \in (K \overset{*}{-} K)$ and real $\lambda > 0$ one has $\lambda(x + K) \subset \lambda K = K \Rightarrow \lambda x + K \subset K$. So, $K \overset{*}{-} K$ is a cone. Moreover, since for $x, y \in (K \overset{*}{-} K)$ one has $x + (y + K) \subseteq x + K \subseteq K$ one concludes $x + y \in (K \overset{*}{-} K)$, *i.e.*, $K \overset{*}{-} K$ —convex cone. The closeness of $K \overset{*}{-} K$ for closed K is obvious. The proof of relation $T_C(K, 0) = K \overset{*}{-} K$ can be found in [Polovinkin, Balashov \(2004\)](#), Lemma 1.4.5, p. 45. \blacksquare

There is another way to associate a cone with the given set.

Definition 1.3.7 *Let $\Omega \subset L$. The set*

$$O^+\Omega = \{x \in L \mid \forall a \in \Omega, \forall \lambda \in \mathbb{R}, \lambda \geq 0, a + \lambda x \in \Omega\}. \quad (1.3.29)$$

*is called the **asymptotic** cone of Ω .*

It is easy to see that $O^+\Omega$ is convex and in general, it can be non-closed. The asymptotic cone is closely related to the cones specified via Minkowski geometrical difference, it can be alternatively defined as

$$O^+\Omega = \Omega \overset{*}{-} \Omega.$$

²⁷Notice that $\forall p \in \mathbb{R}^l \langle p, \text{cl } \mathcal{P}_i(\bar{x}_i) \rangle \geq b \iff \langle p, \mathcal{P}_i(\bar{x}_i) \rangle \geq b \iff \langle p, \text{int } \mathcal{P}_i(\bar{x}_i) \rangle \geq b, \forall b \in \mathbb{R}$.

Now applying the geometrical difference to the cones introduced above, we are going to the specification of convex approximating cones. They are the following: if $a \in \text{cl } \Omega$, $\Omega \subset L$; then **asymptotic lower tangent** cone is

$$T_{AL}(\Omega, a) = T_L(\Omega, a) \overset{*}{-} T_L(\Omega, a),$$

and the **asymptotic upper tangent** cone is

$$T_{AB}(\Omega, a) = \text{cl} \left[(T_B(\Omega, a) \overset{*}{-} T_B(\Omega, a)) + T_{AL}(\Omega, a) \right].$$

The most important properties of these cones are summarized in the following theorem (see Theorem 1.4.1 in [Polovinkin, Balashov \(2004\)](#)).

Theorem 1.3.4 (POLOVINKIN) *The cones $T_{AL}(\Omega, a)$ and $T_{AB}(\Omega, a)$ are convex and closed. Moreover, the equalities and inclusions are valid:*

$$T_{AL}(\Omega, a) = T_C(T_L(\Omega, a); 0), \quad (1.3.30)$$

$$T_{AB}(\Omega, a) = \text{cl} [T_C(T_L(\Omega, a); 0) + T_C(T_B(\Omega, a); 0)] \quad (1.3.31)$$

$$T_C(\Omega, a) \subseteq T_{AL}(\Omega, a) \subseteq T_{AB}(\Omega, a) \subseteq T_B(\Omega, a). \quad (1.3.32)$$

Moreover, all inclusions in (1.3.32) can be strict.

Proof of the theorem can be found in [Polovinkin, Balashov \(2004\)](#). A useful characterization of the Clarke cone is presented in the following theorem. This specification is based on Theorem 2.1 from [Demianov, Rubinov \(1990\)](#), p. 84, here I omit a detailed proof.

Theorem 1.3.5 *Let $Y \subset \mathbb{R}^l$ be a technological set, satisfying standard assumptions and $\bar{y} \in \partial Y$. Then*

- *For almost all boundary points $\Omega \subset \partial Y$, every point has a tangent hyperplane, i.e. it is the differentiability point of a function f which graph presents*

$$Y = \{y \in \mathbb{R}^l \mid f(y) \leq 0\}, \quad \partial Y = \{y \in \mathbb{R}^l \mid f(y) = 0\}.$$

- *Clarke tangent cone at $\bar{y} \in Y$ is a normal cone to*

$$\Delta(\bar{y}) = \left\{ h \in \mathbb{R}^l \mid \exists y^t \in \Omega, t \in \mathbb{N} : y^t \xrightarrow[t \rightarrow \infty]{} \bar{y} \ \& \ \frac{\nabla f(y^t)}{\|\nabla f(y^t)\|} \xrightarrow[t \rightarrow \infty]{} h \right\}, \quad \text{i.e.}$$

$$T_C(Y, \bar{y}) = \{\kappa \in \mathbb{R}^l \mid \langle h, \kappa \rangle \leq 0 \ \forall h \in \Delta(\bar{y})\}.$$

Proof of Theorem 1.3.5. According to the assumptions for every $z \in \mathbb{R}^l$ one can find the only real $\lambda = \lambda(z)$ such that $[z - \lambda(1, 1, \dots, 1)] \in \partial Y$. Now we have

$$Y = \{z \in \mathbb{R}^l \mid \lambda(z) \leq 0\}.$$

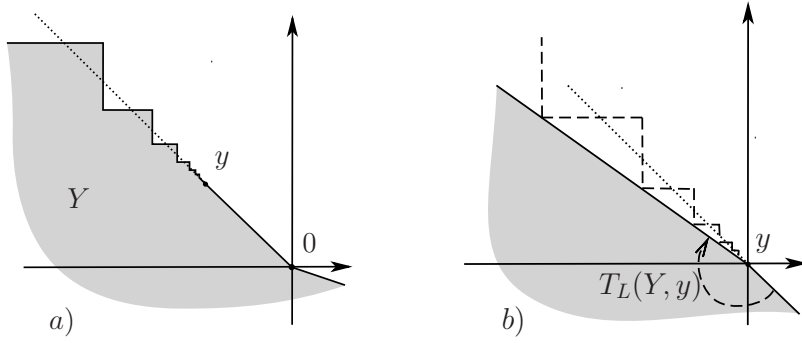


Figure 1.3.10: a) A non-convex set Y ; b) Lower tangent cone $T_L(Y, y)$

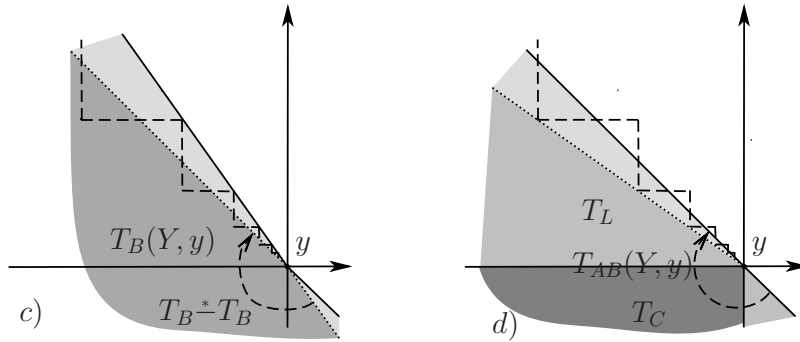


Figure 1.3.11: c) Bouligand tangent cone $T_B(Y, y)$; d) Tangent cones to Y at $y \in Y$

Define $f(z) = \lambda(z)$, $z \in \mathbb{R}^l$. Clearly, because of assumptions $f(\cdot)$ satisfies the Lipschitz condition and, therefore, is an absolutely continuous function. It is known that functions of this type are differentiable almost everywhere. The second statement of the theorem follows from the characterization of Clarke subdifferential for Lipschitz functions presented in Theorem 1.2 from [Demianov, Rubinov \(1990\)](#). ■

So, as an application one can describe the necessary conditions for an allocation to be Pareto optimal, the best option is to choose an **asymptotic upper tangent** cone $T_{AB}(Y_j, \bar{y}_j)$. In contrast, the worst choice is the Clarke cone $T_C(Y_j, \bar{y}_j)$. Let us call a convex closed cone $K \subseteq L$ **marginal** (at a point $\bar{y}_j \in Y_j$) iff

$$T_C(Y_j, \bar{y}_j) \subseteq K \subseteq T_{AB}(Y_j, \bar{y}_j). \tag{1.3.33}$$

A specification of the related equilibrium concepts and related contractual notions is presented in the next section. Below, I give an example showing that the cones presented above can be really different ones.

Example 1.3.1 On the two-dimensional plane, Figure 1.3.10 and 1.3.11 present possible non-convex technological set Y and mentioned above cones associated with the point $y \in Y$. In Figure 1.3.10 b) dark area presents a lower tangent cone at y (it is convex now), in Figure 1.3.11 c) non-convex Bouligand tangent cone is denoted as a

light gray area with a solid boundary line; from this, an asymptotic cone $T_B^* = T_B$ is derived, it is shown as a dark gray area. Figure 1.3.11 d) presents all conic types: negative orthant coincides with $T_C(Y, y)$ since $(1, 0), (0, 1) \in \Delta(y)$ (see Theorem 1.3.5), wider middle gray area corresponds to lower tangent cone $T_L(Y, y)$ and the widest one is asymptotic upper cone $T_{AB}(Y, y)$, now it is a half-space, bounded by the diagonal solid line. ■

\mathcal{K} – MCP-equilibria and contracts in non-convex economies

In Section 1.3.2 we studied the contractual approach in the convex Arrow–Debreu economy, which can be efficiently reduced to the model (1.3.10) with individualized production sets. However, this method is working only for convex production economies; for a non-convex economy, the presented reduction works incorrectly because it leads to an arbitrary change of the model since now relation $\sum_{i \in \mathcal{I}} [\sum_{j=1}^m \theta_i^j Y_j] = \sum_{j \in \mathcal{J}} Y_j$ is violated and is true only for convex $Y_j, j \in \mathcal{J}$. How this obstacle can be overcome?

The crucial idea for the comparing of production programs is that the difference between a new production plan $y'_j \in Y_j$ and an old one $y_j \in Y_j$ has to belong to a (convex) marginal tangent cone specified at the point y_j for the set Y_j . In other words one needs

$$y'_j - y_j \in K_j(y_j), \quad j \in \mathcal{J}$$

to be able to compare a bundle $(y'_j)_{j \in \mathcal{J}} = y' \in Y$ of production plans with the initial bundle $(y_j)_{j \in \mathcal{J}} = y \in Y$. Cones $K_j(y_j)$ are associated with the contractual interaction and present the first-order conditions specified in subsection above in Lemma 1.3.4. In particular these cones can be chosen as $T_C(Y_j, y_j)$ either $T_{AL}(Y_j, y_j)$, or $T_{AB}(Y_j, y_j)$. Now we introduce the concept of the web of contracts and their domination in production economies.

Again we specify contract as is a pair $(v, y) \in L^{\mathcal{I}} \times L^{\mathcal{J}}$, where v is an ordinary barter contract $v = (v_1, v_2, \dots, v_n)$ and $y = (y_1, \dots, y_m)$ is a bundle of production programs $y_j \in Y_j, j \in \mathcal{J}$. Now a reallocation $z = (z_1, \dots, z_n)$ of production plans can be added to individual consumptions, it can be specified as follows

$$z_i = \sum_{j \in \mathcal{J}} \theta_i^j y_j, \quad i \in \mathcal{I}.$$

So, if we have $(v, y) \in L^{\mathcal{I}} \times Y$ and $\sum_{i \in \mathcal{I}} v_i = 0$ then contract (v, y) is permissible and it implies consumptions $x_i = \mathbf{e}_i + v_i + z_i \in L, i \in \mathcal{I}$. Now we can define a web of contracts (V, y) associated with the family of production plans $y \in Y$: here V is a finite set of barter contracts such that

$$\sum_{v \in U} v + z + \mathbf{e} \in X, \quad \forall U \subseteq V \iff \mathbf{e}_i + z_i + \sum_{v \in U} v_i \in X_i, \quad \forall i \in \mathcal{I}, \forall U \subseteq V,$$

i.e., under the current endowments $\mathbf{e} \in X$ and production plans $y \in Y$, individuals enter into contractual relationships so that they can break any contracts. Specify:

$$x(V, y) = \mathbf{e} + z(y) + \sum_{v \in V} v \in L^{\mathcal{I}}.$$

I pay your attention that we apply the only family of production plans $y = (y_j)_{j \in \mathcal{J}}$. Now let us pass to the definition of webs domination. This domination assumes that some contracts from a web can be broken (as a whole), some new contracts concluded and new production plans generated.

A web of contracts (W, y') dominates a web (V, y) by coalition $S \subseteq \mathcal{I}$, written as $(W, y') \succ_S (V, y)$, iff

- $\forall v \in V \setminus W, S \cap \{i \in \mathcal{I} \mid v_i \neq 0\} \neq \emptyset$ — consumers from S can break contracts from $V \setminus W$;
- $\forall v \in W \setminus V, S \supseteq \{i \in \mathcal{I} \mid v_i \neq 0\}$ — only consumers from S can sign new contracts;
- $(y'_j - y_j) \in K_j(y_j), j \in \mathcal{J}$;
- $x_i(W, y') \succ_i x_i(V, y) \quad \forall i \in S$.

An allocation that cannot be dominated is called K -contractual.

The described contractual interaction also allows us to introduce a concept of contractual core, specified for non-convex production. This weak concept is related with a bundle of marginal conic sets $K_j(y_j) \subseteq L, j \in \mathcal{J}$ and is so that \mathcal{MCP} -equilibrium with prices from the normal cone, *i.e.*, prices $p \in L', p \neq 0$ satisfying

$$\langle p, K_j \rangle \leq 0, \quad j \in \mathcal{J},$$

belongs to the core.

Definition 1.3.8 *Coalition $S \subseteq \mathcal{I}$ dominates an allocation $(x, y) \in \mathcal{A}(\mathcal{E})$ via marginal cones $K = (K_j)_{j \in \mathcal{J}} \subset L^{\mathcal{J}}$ if there is $(x^S, z^S) \in \prod_{i \in S} X_i \times L^S$ such that for some $y'_j \in Y_j$ satisfying $(y'_j - y_j) \in K_j, j \in \mathcal{J}$ one has $z_i^S = \sum_{j \in \mathcal{J}} \theta_i^j y'_j$ and $\sum_S x_i^S = \sum_S \mathbf{e}_i + \sum_S z_i^S$ so that $x_i^S \succ_i x_i \forall i \in S$.*

*The set $\mathcal{C}^K(\mathcal{E}) \subset \mathcal{A}(\mathcal{E})$ of all allocations that can be dominated by no coalition is called **weak K -core**.*

This core is really weaker than the standard one as soon as the possibilities of a coalition to dominate are reduced and the core may contain allocations that are not Pareto optimal. An advantage of the concept is that generalized \mathcal{MCP} -equilibrium allocation (presented below) always is an element of the core and it is nonempty. It can be easily proved by arguing by contradiction. Now we are passing to a discussion of partially broken contracts. It leads us to the notions of proper and fuzzy contractual allocations.

Definition 1.3.9 *Let $K = (K_j)_{j \in \mathcal{J}} \subset L^{\mathcal{J}}$ be a family of marginal cones. An allocation $(\bar{x}, \bar{y}) \in \mathcal{A}(\mathcal{E})$ implemented by a web of barter contracts $V = \{v\}$, where*

$$v = \bar{x} - \bar{z} - \mathbf{e}, \quad \bar{z} = (\bar{z}_i)_{i \in \mathcal{I}}, \quad \bar{z}_i = \sum_{j \in \mathcal{J}} \theta_i^j \bar{y}_j$$

is called ***K*-marginal fuzzy contractual** if for every $t = (t_i)_{i \in \mathcal{I}}$, $0 \leq t_i \leq 1$, $\forall i \in \mathcal{I}$, there are no other production programs $y_j^i \in L$, $y_j^i - \bar{y}_j \in K_j$, $j \in \mathcal{J}$ and a barter contract $w = (w_1, \dots, w_n) \in L^{\mathcal{I}}$, $\sum_{i \in \mathcal{I}} w_i = 0$ such that for

$$\xi_i = \xi_i(t, v, w, y) = \mathbf{e}_i + t_i v_i + w_i + z_i, \quad z_i = \sum_{j \in \mathcal{J}} \theta_i^j y_j^i, \quad i \in \mathcal{I} \quad (1.3.34)$$

one takes place

$$\xi_i \succ_i \bar{x}_i \quad \forall i : \xi_i \neq \bar{x}_i. \quad (1.3.35)$$

This definition describes the following contractual interaction. First, the individuals are going to partially break barter contracts, each in the measure $(1 - t_i)$ and it is private information: nobody else knows about it. Second, they change production plans for $y_j^i \in L$ in a part θ_i^j that they can control and under the condition $y_j^i \in \bar{y}_j + K_j$, $j \in \mathcal{J}$: this is a form of social production efficiency (notice that since K_j is a convex set one has $\sum_{i \in \mathcal{I}} \theta_i^j y_j^i = y_j \in \bar{y}_j + K_j$). Finally, agents conclude a new barter contract $w = (w_1, \dots, w_n)$. As a result of this interaction, each involved individual has to benefit. Notice that due to (1.3.35) contract $w = 0$ is also possible and thereby only partial break can be realized. So, if there is no domination of this kind, then an allocation is called *K*-marginal fuzzy contractual.

The following lemma characterizes fuzzy contractual allocations in “geometrical” categories. Recall that an allocation $(\bar{x}, \bar{y}) \in \mathcal{A}(\mathcal{E}^{AD})$ is called *stable relative to asymmetrical partial break (lower stable)* iff $\forall i \in \mathcal{I} \quad \forall z_i \in Z_i = \sum_{j \in \mathcal{J}} \theta_i^j (\bar{y}_j + K_j)$

$$z_i + (1 - \lambda)\bar{x}_i + \lambda(\mathbf{e}_i + \bar{z}_i) \not\prec_i \bar{x}_i \quad \forall 0 \leq \lambda \leq 1 \iff \forall i \in \mathcal{I} \quad (Z_i + [\bar{x}_i, \mathbf{e}_i + \bar{z}_i]) \cap \mathcal{P}_i(\bar{x}_i) = \emptyset. \quad (1.3.36)$$

The specificity of condition (1.3.36) is that if it is not true, then there is an individual who acting separately certainly changes the web as soon as it is beneficial for him. Here, there is no contractual interaction.

Lemma 1.3.6 *Let $(\bar{x}, \bar{y}) \in \mathcal{A}(\mathcal{E}^{AD})$ be an allocation lower stable relative to a partial break and $K = (K_j)_{j \in \mathcal{J}} \subset L^{\mathcal{J}}$ be a family of marginal cones specified for $(\bar{y}_j)_{j \in \mathcal{J}} = \bar{y}$. Then (\bar{x}, \bar{y}) is *K*-fuzzy contractual if and only if*

$$\mathcal{A}(L^{\mathcal{I}}) \cap \prod_{i \in \mathcal{I}} \left((\mathcal{P}_i(\bar{x}_i) - Z_i + \text{co}\{\mathbf{e}_i + \bar{z}_i - \bar{x}_i, 0\}) \cup \{\mathbf{e}_i\} \right) = \{\mathbf{e}\}. \quad (1.3.37)$$

Here $Z_i = \sum_{j \in \mathcal{J}} \theta_i^j (\bar{y}_j + K_j)$, $\bar{z}_i = \sum_{j \in \mathcal{J}} \theta_i^j \bar{y}_j$, $i \in \mathcal{I}$, $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ and $\mathcal{A}(L^{\mathcal{I}})$ is a subspace that corresponds to the material balance constraints:

$$\mathcal{A}(L^{\mathcal{I}}) = \{(\zeta_i)_{i \in \mathcal{I}} \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} \zeta_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\}.$$

²⁸It means that nobody wants to partially break a barter contract without concluding a new one.

This characterization works very efficiently in applications, but one has to apply (1.3.37) and (1.3.36) simultaneously. In particular, applying it below we state the equivalence between fuzzy contractual allocations and \mathcal{MCP} -equilibria. To do it one has to separate sets from the left-hand part of (1.3.37) by linear functional and analyze the result.

Proof of Lemma 1.3.6. Necessity. Let (1.3.37) be false. Therefore, in the left-hand side of the intersection (1.3.37), there is $\zeta = (\zeta_i)_{i \in \mathcal{I}} \neq \mathbf{e}$. Define

$$S = \{i \in \mathcal{I} \mid \zeta_i \neq \mathbf{e}_i\} \neq \emptyset.$$

Further, find a new contract with this support and the appropriate amounts of contracts breaking. Define $w_i = \zeta_i - \mathbf{e}_i$ that gives $\sum_{i \in \mathcal{I}} w_i = 0$ since $\zeta \in \mathcal{A}(L^{\mathcal{I}})$. For $i \notin S$ one obviously has $w_i = 0$, i.e., $\text{supp}(w) = S$. Also for $i \in S$ one has

$$\zeta_i \in (\mathcal{P}_i(\bar{x}_i) - Z_i + \text{co}\{0, (\mathbf{e}_i + \bar{z}_i - \bar{x}_i)\}),$$

that allows to conclude $\exists 0 \leq t_i \leq 1$ and $\xi_i \succ_i \bar{x}_i$ such that:

$$\mathbf{e}_i + w_i = \zeta_i = \xi_i - z_i + t_i(\mathbf{e}_i + \bar{z}_i - \bar{x}_i) \Rightarrow \xi_i = \mathbf{e}_i + t_i v_i + z_i + w_i,$$

for $v_i = \bar{x}_i - \mathbf{e}_i - \bar{z}_i$, $i \in S$ (for $i \notin S$ one puts $t_i = 1$ & $z_i = \bar{z}_i \Rightarrow \xi_i = \bar{x}_i$). Here due to $\sum_{\mathcal{I}} \bar{x}_i = \sum_{\mathcal{I}} \mathbf{e}_i + \sum_{\mathcal{I}} \bar{z}_i$ we have $\sum_{\mathcal{I}} v_i = 0$. This contradicts the definition of K -fuzzy contractual allocation.

Sufficiency. Let (1.3.37) be true for $(\bar{x}, \bar{y}) \in \mathcal{A}(\mathcal{E}^{AD})$, and in addition, let the allocation be lower stable relative to the partial breaking. Assume this is not K -fuzzy contractual. Then there are $t = (t_i)_{\mathcal{I}}$, plans $z_i \in Z_i$, $i \in \mathcal{I}$, and a barter contract $w = (w_1, \dots, w_n) \in \mathbb{R}^{ln}$, $\sum_{\mathcal{I}} w_i = 0$, satisfying all Definition 1.3.9 requirements. This for $v_i = \bar{x}_i - \bar{z}_i - \mathbf{e}_i$ due to (1.3.34) for the members of a nonempty coalition yields

$$\exists \xi_i \in \mathcal{P}_i(\bar{x}_i) : x_i = \xi_i + t_i(\mathbf{e}_i + \bar{z}_i - \bar{x}_i) = \mathbf{e}_i + w_i + z_i. \quad (1.3.38)$$

Summing over i by the definition of contract one concludes $\sum_{\mathcal{I}} x_i = \sum_{\mathcal{I}} \mathbf{e}_i + \sum_{\mathcal{I}} z_i$, i.e., for $\zeta_i = x_i - z_i$, $i \in \mathcal{I}$ allocation $\zeta = (\zeta_i)_{\mathcal{I}}$ belongs to the intersection in the left-hand part of (1.3.37). If one supposes $\zeta = \mathbf{e}$, then $x_i = \mathbf{e}_i + z_i \forall i \in \mathcal{I}$, that being substituted to the right-hand part of (1.3.38) yields $w_i = 0$, $\forall i \in \mathcal{I} \Rightarrow \text{supp}(w) = \emptyset$. Hence, domination is carried out without the exchange and only via a partial break of the gross contract $v = \bar{x} - \bar{z} - \mathbf{e}$. However, this contradicts the lower stability relative to a partial break. Therefore, one finds an allocation $\zeta \neq \mathbf{e}$ which belongs to the intersection of the left-hand side (1.3.37); it is a contradiction. ■

Now we formalize generalized \mathcal{MCP} -equilibrium and then state equivalence of it and K -marginal fuzzy contractual allocation.

Definition 1.3.10 *Let (x, y, p) be a triplet such as $x = (x_i)_{i \in \mathcal{I}} \in X$ is a family of consumption plans, $y = (y_j)_{j \in \mathcal{J}} \in Y$ are production plans and $p = (p_1, \dots, p_l) \neq 0$ is a price vector. Given a family $K = (K_j(y_j))_{j \in \mathcal{J}}$ of marginal cones at points $(y_j)_{j \in \mathcal{J}}$,*

triplet (x, y, p) is called \mathcal{K} – \mathcal{MCP} -**quasi-equilibrium** if it obeys the following conditions:

$$y_j \in \partial Y_j \quad \& \quad \langle p, K_j \rangle \leq 0 \quad \forall j \in \mathcal{J}, \quad (1.3.39)$$

$$\langle p, \mathcal{P}_i(x_i) \rangle \geq p \cdot \mathbf{e}_i + \sum_{j \in \mathcal{J}} \theta_i^j p \cdot y_j = p \cdot x_i \quad \forall i \in \mathcal{I}, \quad (1.3.40)$$

$$\sum_{i \in \mathcal{I}} x_i = \sum_{j \in \mathcal{J}} y_j + \sum_{i \in \mathcal{I}} \mathbf{e}_i. \quad (1.3.41)$$

If all inequalities in (1.3.40) have **strict** sign then the triplet (x, y, p) is called \mathcal{K} – \mathcal{MCP} -**equilibrium**.

Here, in contrast to the convex case, the freedom of the individual in choosing a production plan is limited. It is assumed that mutual cooperation or some authority establishes joint production plans, and the individuals still have the right to decide: do it or not (a specific form of non-binding agreement). However, the individual deviations from a given production plan are only possible within the sets $K_j \cap Y_j$ in a part that an individual $i \in \mathcal{I}$ can control, *i.e.* $\theta_i^j K_j$. Again, in the case of convex production, this requirement and standard one coincide.

Finally, we note that similarly to the convex Arrow–Debreu economy for non-convex case, one can introduce the notion of a fuzzy core that is closely related to fuzzy contractual allocations. The main theoretical value of the fuzzy core is that it can be effectively applied to develop a theory of the existence of competitive equilibria and, in view of results presented here, fuzzy contractual allocations. I omit this presentation, the general methodology of this approach is well known, for details see Section 1.2.3, *Aliprantis et al.* (1989), and the most advanced mathematical results in *Florenzano* (1989, 1990) and *Marakulin* (2012), *etc.* Now we pass to study an equivalence between equilibrium and contractual approaches.

Remark 1.3.2 The existence of \mathcal{K} – \mathcal{MCP} -equilibria and the non-emptiness of weak K-core is not studied here, I left this problem for further research. In the literature, it is well known that these problems are closely related and the existence of one can be derived from another one. For Clarke cones $K = K_C(Y_j, y_j)$ we deal with \mathcal{MCP} -equilibrium by Bonnisseau–Cornet, which existence was stated in *Bonnisseau, Cornet* (1990) that clearly implies the non-emptiness of our weak core. Notice that under standard assumptions for production set $Y \subset \mathbb{R}^l$ a point-to-set mapping $y \Rightarrow T_C(Y, y)$ has good mathematical properties, it is lower semicontinuous correspondence with non-empty, convex, and closed values. This simplifies the existence problem, for other chosen cones (e.g. $K = T_{AB}(Y, y)$) it is not true and it can be a difficult obstacle.

The relationship between K -marginal fuzzy contractual allocation and \mathcal{K} – \mathcal{MCP} -equilibrium is established in the following

Theorem 1.3.6 *Let $(\bar{x}, \bar{y}) \in \mathcal{A}(\mathcal{E}^{AD})$ be an allocation in Arrow–Debreu economy and $K = (K_j(\bar{y}_j))_{j \in \mathcal{J}}$ be a family of closed convex cones such that*

$$T_C(Y_j, \bar{y}_j) \subseteq K_j(\bar{y}_j) \subseteq T_{AB}(Y_j, \bar{y}_j), \quad j \in \mathcal{J}.$$

If (\bar{x}, \bar{y}) is K -marginal fuzzy contractual then it is K -MCP-quasi-equilibrium. On the contrary, assume that there are prices $\bar{p} \in L'$ such that $(\bar{x}, \bar{y}, \bar{p})$ is K -MCP-equilibrium. Then (\bar{x}, \bar{y}) is K -marginal fuzzy contractual.

In the theory of the existence of Walrasian equilibrium, the conditions under which each quasi-equilibrium turns into a strict equilibrium are well known: it is necessary to require additionally some survival assumptions, the representative of which can be irreducibility. Here we omit the description of these methods and refer the interrogated reader to [Marakulin \(2012\)](#).

Proof of Theorem 1.3.6. Necessity. Our analysis is based on Lemma 1.3.2 and the application of relation (1.3.37). Indeed, (\bar{x}, \bar{y}) is K -fuzzy contractual iff (1.3.37) and (1.3.36) hold. Excluding \mathbf{e}_i from the left-hand side of (1.3.37), one concludes

$$\mathcal{A}(L^{\mathcal{I}}) \cap \prod_{i \in \mathcal{I}} (\mathcal{P}_i(\bar{x}_i) - Z_i + \text{co}\{\mathbf{e}_i + \bar{z}_i - \bar{x}_i, 0\}) = \emptyset.$$

Here affine subspace $\mathcal{A}(L^{\mathcal{I}})$ is intersected with a convex solid set. Applying classical separation theorem one finds a nonzero linear functional (vector) $f = (f_1, \dots, f_n) \in L^{\mathcal{I}}$ separating these sets, that is

$$\langle f, \mathcal{A}(L^{\mathcal{I}}) \rangle \leq \langle f, \prod_{i \in \mathcal{I}} (\mathcal{P}_i(\bar{x}_i) - Z_i + \text{co}\{\mathbf{e}_i + \bar{z}_i - \bar{x}_i, 0\}) \rangle.$$

The left-hand side of the inequality is upper bounded and, since $\mathcal{A}(L^{\mathcal{I}})$ is a subspace, one concludes

$$\begin{aligned} \langle (f_1, \dots, f_n), \{(\zeta_1, \dots, \zeta_n) \in L^n \mid \sum_{i=1}^n \zeta_i = 0\} \rangle = 0 &\Rightarrow \\ f_i = f_k = p \neq 0 \quad \forall i, k \in \mathcal{I}. \end{aligned}$$

Now, substituting this to the latter inequality and transforming it we obtain

$$\sum_{i \in \mathcal{I}} \langle p, \mathcal{P}_i(\bar{x}_i) - Z_i + \text{co}\{\mathbf{e}_i + \bar{z}_i - \bar{x}_i, 0\} \rangle \geq \sum_{i \in \mathcal{I}} p \mathbf{e}_i. \quad (1.3.42)$$

Now we recall that $Z_i = \bar{z}_i + \sum_{j \in \mathcal{J}} \theta_i^j K_j$ where every K_j is a convex conic set, that due to $\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \theta_i^j K_j = \sum_{j \in \mathcal{J}} K_j$ and the last inequality allows us to conclude

$$\langle p, -\sum_{j \in \mathcal{J}} K_j \rangle \geq 0 \quad \Rightarrow \quad \langle p, K_j \rangle \leq 0 \quad \forall j \in \mathcal{J},$$

that proves (1.3.39). Further, notice that $\bar{x}_i \in \text{cl} \mathcal{P}_i(\bar{x}_i) \quad \forall i \in \mathcal{I}$ implies

$$\bar{x}_i - \bar{z}_i + \mathbf{e}_i + \bar{z}_i - \bar{x}_i = \mathbf{e}_i \in \text{cl}[\mathcal{P}_i(\bar{x}_i) - Z_i + \text{co}\{\mathbf{e}_i + \bar{z}_i - \bar{x}_i, 0\}],$$

that substituting it into left-hand side of (1.3.42) for $j \neq i$ and then omitting identical terms yields

$$\forall i \in \mathcal{I} \quad \langle p, \mathcal{P}_i(\bar{x}_i) - Z_i + \text{co}\{\mathbf{e}_i + \bar{z}_i - \bar{x}_i, 0\} \rangle \geq p \mathbf{e}_i \quad \Rightarrow \quad \langle p, \mathcal{P}_i(\bar{x}_i) - \bar{z}_i \rangle \geq p \mathbf{e}_i.$$

Again, taking into account $\bar{x}_i \in \text{cl } \mathcal{P}_i(\bar{x}_i)$ and specification of \bar{z}_i the last one implies

$$\langle p, \mathcal{P}_i(\bar{x}_i) \rangle \geq p\mathbf{e}_i + \sum_{j \in \mathcal{J}} \theta_i^j p\bar{y}_j \quad \& \quad p\bar{x}_i \geq p\mathbf{e}_i + \sum_{j \in \mathcal{J}} \theta_i^j p\bar{y}_j \quad i \in \mathcal{I}.$$

Because $(\bar{x}, \bar{y}) \in \mathcal{A}(\mathcal{E}^{AD})$, *i.e.* it is feasible allocation, one concludes: for all $i \in \mathcal{I}$ $p\bar{x}_i = p\mathbf{e}_i + \sum_{j \in \mathcal{J}} \theta_i^j p\bar{y}_j$ that finishes the proving (\bar{x}, \bar{y}, p) is in fact a quasi-equilibrium by Definition 1.3.10.

Sufficiency. Let $(\bar{x}, \bar{y}, \bar{p})$ be a K -MCP-equilibrium. It easy to see that now condition (1.3.40) in its strict form and (1.3.39) for each (fixed) $i \in \mathcal{I}$ imply that

$$\langle \bar{p}, (\mathcal{P}_i(\bar{x}_i) - Z_i + \text{co}\{(\mathbf{e}_i + \bar{z}_i - \bar{x}_i), 0\}) \rangle > \bar{p}\mathbf{e}_i.$$

We always have

$$\sum_{i' \in \mathcal{I}, i' \neq i} \langle \bar{p}, (\mathcal{P}_{i'}(\bar{x}_{i'}) - Z_{i'} + \text{co}\{(\mathbf{e}_{i'} + \bar{z}_{i'} - \bar{x}_{i'}), 0\}) \cup \{\mathbf{e}_{i'}\} \rangle \geq \sum_{i' \in \mathcal{I}, i' \neq i} \bar{p}\mathbf{e}_{i'}.$$

Summing this and latter one we get a strict inequality that is true for every $i \in \mathcal{I}$. This can be true only if (1.3.22) holds. Condition (1.3.36) is also true, it follows from (1.3.39), (1.3.40) and via Z_i , $i \in \mathcal{I}$ specification. Theorem 1.3.3 is proved. ■

Conclusion to Chapter 1

In this chapter, the basic elements of the theory of barter contractual interactions were presented and first developed for exchange economies; then they were expanded to the models with the production of private goods. A series of stability concepts of contractual allocations and the webs of contracts were considered, their relationships with notions known in classical theory were revealed. Now equilibrium, Pareto boundary, core, fuzzy core obtain a clear description in contractual categories. It was shown that contract-based approach is rather convenient to model perfect competition conditions, efficiently supplements classical methods and presents alternative forms of it.

The graphical presentation of logical relationships among various stability concepts of the webs of contacts and related contractual allocations is given in Figure 1.3.12. Here the arrows show a stronger property from the nearest ones.

Main results for standard market are the following:

- Theorem 1.2.2 on the contractual characterization of equilibria.
- The results of Section 1.2.3, 1.2.4, characterizing properly, perfectly, and fuzzy contractual allocations; moreover, the number of participants in the contracts can be limited to the number of interesting for contracting parties commodities plus one.

In the second part of the chapter, the contractual approach was analyzed in the context of Arrow–Debreu model with the production of private goods. Economies with convex and nonconvex production sets were studied here and contractual description of

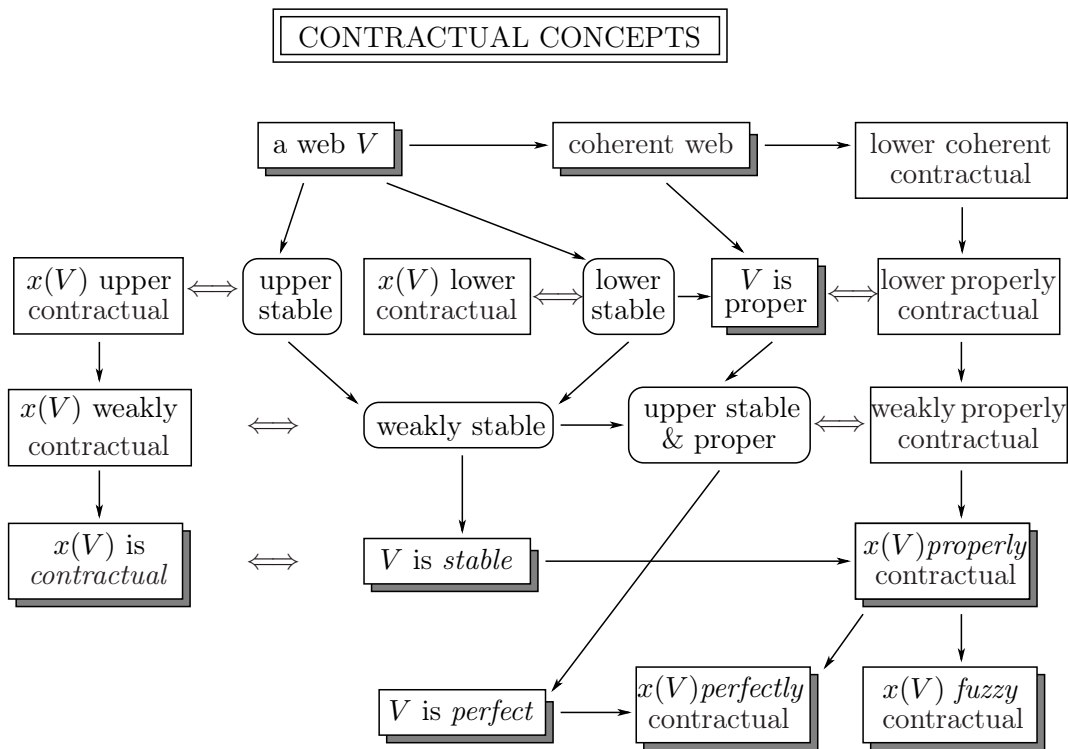


Figure 1.3.12: *The relationships and logical connections among the array of contractual concepts*

a variate of known theoretical equilibrium concepts was obtained. They are Walrasian equilibrium in a convex model and equilibrium with marginal cost pricing in a non-convex setting (\mathcal{MCP} -equilibrium and $\mathcal{K} - \mathcal{MCP}$ -equilibrium). At the same time, a number of new contractual concepts were proposed: specific variants of properly contractual, and also marginally contractual allocations. These concepts characterize equilibria in cooperative terms and without value categories, and this is an advantage of a contractual approach. The main results are the following:

- Theorem 1.3.2 and Lemma 1.3.2 on the equivalence of properly and fuzzy contractual allocations and competitive equilibrium in a convex Arrow–Debreu economy.
- Theorem 1.3.3 on the equivalence of \mathcal{MCP} -equilibrium and marginally contractual allocation for smooth economy.
- Lemma 1.3.4 and Theorem 1.3.6 on the equivalence of $\mathcal{K} - \mathcal{MCP}$ -equilibrium and \mathcal{K} -marginally contractual allocation for nonsmooth economy.

Chapter 2

Contracts and the production of public goods

In this chapter, an appropriate notion of production contract is introduced and the contractual approach is expanded to the model with public goods. The main results of the chapter are the theorems on the equivalence between Lindahl equilibria and properly and fuzzy contractual allocations. The relationship of equilibria with a specifically introduced fuzzy core for an economy with public and semi-public goods is also revealed.

An economy with public goods is specified by the presence of special commodities, which by their physical characteristics are the goods of public consumption. Examples of public goods are public television and radio, street lighting, roads, production of “security” (police, national defense, *etc.*). The list of examples can be continued, but clearly, a public good is a commodity that is simultaneously consumed by many agents and there is a need to reproduce it (one needs to repair roads, produce and broadcast television programs) that has to be financed somehow. It is clear that the funding of production of a collective consumption commodity should be carried out by all its consumers. In the neoclassical theory of a decentralized economy, the concept of individual valuations is considered as a basis for public goods financing; these valuations are calculated as the product of individual prices and a (total) consumption bundle. Of course, ordinary products may exist in the economy, and processes of their allocation and production are carried out under usual market rules. In theory, an appropriate concept of equilibrium (by Lindahl) is defined and studied in such a way that the related allocation is a Pareto optimal one. The correct determination of individual prices is a difficult theoretical problem in practice. In the case of private commodities, this issue is resolved via the market mechanism, based on a large number of exchange transactions, by the method of price “*tatônnement*.” This method does not work for public goods, because individuals are incapable of exchanging the parts of public goods.¹ From a theoretical point of view, individual prices should be proportional to the marginal rates of substitution (exchange), but in terms of utility functions, they should be proportional to a fragment of the gradient, corresponding to

¹Therefore, it is recommended in practice to transform the “public goods” into private ones whenever possible, through various specific techniques in order to enable the market mechanism. An example of this is the transition to the counters in water provision.

public goods. Thus, in order to “evaluate” the individual prices, one must have purely private information about the preferences of individuals, which is not practicable in real life. Below, I suggest a theoretical way to resolve this problem.

Modern views on the theory of financing of public goods stem from Samuelson’s papers and the results of some other authors; see the survey by *Milleron (1972)* and *Ruys (1974)*. A public good is a product of joint consumption of all economic agents. The Pareto efficient mechanism of value regulation of public goods is based on the individual valuations calculated as a product of the individual price and total consumption.² Clearly, there can be ordinary commodities in the economy: their exchange and production are governed by usual market rules. An appropriate theoretical concept of Pareto efficient equilibrium defined in the literature is known as the Lindahl equilibrium. However, practical implementation of the individual price apparatus being applied to public goods raises a problem of calculation of these prices (and taxes). This is in fact a difficult theoretical question that still does not have a clear answer in classical theory. One way to solve this problem could be based on the cooperative description of equilibrium, in such a way as is done in models with purely private goods, via a theorem on the coincidence of the core and equilibria under perfect competition conditions. However, examples show that under ordinary replication³ of an economy the Foley core does not shrink to equilibrium (see *Muench (1972)*, *Milleron (1972)*, and *Buchholz, Peters (2007)*). If one does not take into account such extreme results as *Conley (1994)*, the problem finds its resolution in the modern literature only through the transformation of the concept of coalitional domination. Instead of the familiar public goods so-called semi-public goods are introduced, as was done in *Vasil’ev et al. (1995)* (see also *Weber, Wiesmeth (1991)*, *Vasil’ev (1996)* and *Florenzano, Mercato (2006)*). The difference is that now the utility of consumed goods depends on the total number of customers.⁴ One can say that consumers are interested in the average level of consumption of public goods. In theory, there appeared such concepts as “returns to group size”, see *Roberts (1974)*, congestion of goods, *etc.* *Vasil’ev et al. (1995)* proved subtle theorems on the coincidence of the core and equilibria under the assumption of constant returns to the size of the coalition. The condition proves to be necessary for the coincidence between the core and equilibria, which is a remarkable result. In this chapter, the theorems on the coincidence of the core and the Lindahl equilibrium are stated within the contractual approach and under assumptions that are similar to those made in the papers mentioned above. However, here we are not talking about coalition sizes or a measure of congestion in public goods: in the model, individuals are engaged in ordinary economic activity; they sign and break (partially and asymmetrically) production and barter contracts, thereby producing a stable regime of functioning, which corresponds to a Lindahl equilibrium allocation.

²There are also other, non-Pareto efficient, equilibrium notions, such as based on a private provision of public goods (*e.g.* see *Florenzano (2009)*).

³This is one of the most popular ways to model perfect competition conditions, which goes back to Debreu and Scarf. There are other methods, *e.g.* Aumann’s approach, applying a non-atomic measure space of economic agents.

⁴Perception of the good is inversely proportional to the number of users of the same type; not only the total amount, but mainly the density of vehicles and traffic congestion affect the satisfaction of such a good as “transport infrastructure.”

The presentation below is based on [Marakulin \(2013\)](#).

2.1 Public goods and main classical notions

The structure of an economy with public goods is similar to the Arrow–Debreu model: the main difference is in the commodity space, which also represents public goods. The mechanism of public goods decentralization is also different and specific. The model has n consumers forming a set $\mathcal{I} = \{1, \dots, n\}$ and m producers (firms) $\mathcal{J} = \{1, \dots, m\}$. There are l types of private good, their nomenclature being $\{1, \dots, l\}$, and s kinds of public good, numbered by the indices $\{l + 1, \dots, l + s\}$. Thus, the total number of products is $l + s$. Consumers are equipped with individualized *private goods consumption sets* $X_i^p \subset \mathbb{R}^l$ and a *common* for all consumers set of *permissible for consumption public goods* $X^c \subset \mathbb{R}^s$. So, here $\mathbb{R}^{l+s} = L$ is a commodity space. In addition, consumers have initial endowments of private goods $\mathbf{e}_i \in X_i^p$, $i \in \mathcal{I}$, and the economy as a whole has endowments of public goods $\mathbf{e}^c \in X^c$. In general, firms can produce and spend private as well as public goods; their production capacities are presented by technological sets $Y_j \subset \mathbb{R}^{l+s}$, $0 \in Y_j$, $j \in \mathcal{J}$. Production plans $y_j \in Y_j$ are written in the form $y_j = (y_j^p, y_j^c)$, where $y_j^p \in \mathbb{R}^l$ is associated with the private goods and $y_j^c \in \mathbb{R}^s$ with the public ones. The set

$$\mathbb{Z} = \prod_{\mathcal{I}} X_i^p \times X^c \times \prod_{\mathcal{J}} Y_j$$

is identified with the set of all *admissible states* and the space $\mathfrak{L} = \mathbb{R}^{ln+s+m(l+s)} \supset \mathbb{Z}$ is a *space of allocations*. Consumers' preferences are defined and take values in $X_i^p \times X^c$, *i.e.*, $\mathcal{P}_i : X_i^p \times X^c \Rightarrow X_i^p \times X^c$. The sets $X_i^p \times X^c$ are associated with consumption sets of individuals; they are assumed to be convex. One can see that this model has externalities, concentrated in the area of public goods. In addition, as in the Arrow–Debreu model, $\theta_i^j \geq 0$ (a component of vector $\theta_i = (\theta_i^1, \dots, \theta_i^m)$) is the *share* of consumer i in the profit of producer j . These quantities satisfy $\sum_{i \in \mathcal{I}} \theta_i^j = 1$ for all $j \in \mathcal{J}$ (*i.e.*, profit is completely distributed among all shareholders).

Assume that the processes of exchange and production of goods are regulated by the individual prices for public goods $q_i \in \mathbb{R}^s$ and the market prices $p \in \mathbb{R}^l$ for the private commodities. In this case, the budget constraint for consumption plans $(x_i, x^c) \in X_i^p \times X^c$ of individual $i \in \mathcal{I}$ are specified by

$$\langle x_i, p \rangle + \langle x^c, q_i \rangle \leq \langle \mathbf{e}_i, p \rangle + \langle \mathbf{e}^c, q_i \rangle + \sum_{j \in \mathcal{J}} \theta_i^j (p y_j^p + \bar{q} y_j^c),$$

where $y \in \prod_{\mathcal{J}} Y_j$, $q = (q_1, \dots, q_n) \in [\mathbb{R}^s]^{\mathcal{I}}$, $p \in \mathbb{R}^l$ and $\bar{q} = \sum_{\mathcal{I}} q_i$. So, one can see that agents' incomes are formed from three sources: the sale of private endowments \mathbf{e}_i for market prices p , the individualized value of public goods $\langle q_i, \mathbf{e}^c \rangle$, and as the “sum of dividends” from the profits of producers. It is also worth repeating that the producers' profits are determined by “production” prices (p, \bar{q}) . Now in its shortest form the model under study can be written as

$$\mathcal{E}^{pg} = \langle \mathcal{I}, \mathcal{J}, \mathbb{R}^l, \mathbb{R}^s, \{X_i^p, \mathcal{P}_i(\cdot), \theta_i, \mathbf{e}_i\}_{i \in \mathcal{I}}, \{Y_j\}_{j \in \mathcal{J}}, X^c, \mathbf{e}^c \rangle.$$

In the neoclassical setting the Lindahl equilibrium is considered as the fundamental solution concept.

Definition 2.1.1 An allocation $z = (x, x^c, y) \in \mathbb{Z}$ is said to be a **Lindahl equilibrium** with a price bundle $(p, q_1, \dots, q_n) \in \mathbb{R}^{l+ns}$ if for $\bar{q} = \sum_{\mathcal{I}} q_i$ it satisfies

$$py_j^p + \bar{q}y_j^c \geq \langle (p, \bar{q}), Y_j \rangle, \quad j \in \mathcal{J}, \quad (2.1.1)$$

$$\langle (p, q_i), \mathcal{P}_i(x_i, x^c) \rangle > \mathbf{e}_i p + \mathbf{e}^c q_i + \sum_{j \in \mathcal{J}} \theta_i^j (py_j^p + \bar{q}y_j^c) = x_i p + x^c q_i, \quad i \in \mathcal{I}, \quad (2.1.2)$$

$$\sum_{\mathcal{I}} x_i = \sum_{\mathcal{I}} \mathbf{e}_i + \sum_{\mathcal{J}} y_j^p, \quad (2.1.3)$$

$$x^c = \mathbf{e}^c + \sum_{\mathcal{J}} y_j^c. \quad (2.1.4)$$

With non-strict inequalities in (2.1.2) it is said to be a **quasi-equilibrium**.

Requirements (2.1.1)–(2.1.4) have their usual substantial sense. Condition (2.1.1) implements the principle of producers' profit maximization, (2.1.2) states that (x_i, x^c) is an optimal budget acceptable plan, condition (2.1.3) presents the balance for private consumption commodities and (2.1.4) is the balance for public goods.

Commenting on the equilibrium definition, one notes that individuals evaluate the consumption of public goods according to individual prices q_i , while in production specific industrial prices are applied: they are equal to the sum of individual prices, $\bar{q} = \sum_{\mathcal{I}} q_i$. Also note the specific requirement of the public goods balance: this is so because all individuals consume the same quantity of good, if it is a public one. Further, any allocation from \mathbb{Z} satisfying (2.1.3) and (2.1.4) is called a *feasible* (valid) one and the set of all such allocations is denoted $\mathcal{A}(\mathcal{E}^{pg})$.

Without going into detail, we note that for a convex economy the Lindahl equilibrium is always Pareto optimal and does exist under almost the same assumptions as the Walrasian equilibrium; see, for example, *Ruys (1974)*, *Florenzano, Mercato (2006)*, *Florenzano (2009)*, and *Marakulin (2012)* §1.2.3. Similarly to the usual Arrow–Debreu model in an economy with public goods there is an analog of the 2nd Welfare theorem: every Pareto optimal allocation can be presented in the form of a Lindahl equilibrium for a specific redistribution of initial endowments. In other words, for a Pareto optimal allocation one can provide a dual characterization in value categories (a similar result can be found *e.g.* in *Florenzano, 2009*). Insofar as this is involved in the subsequent analysis, I present a formalization of an appropriate mathematical result and give its proof. Recall that:

• A feasible allocation $(\bar{x}, \bar{x}^c, \bar{y}) = ((\bar{x}_i)_{i \in \mathcal{I}}, \bar{x}^c, (\bar{y}_j)_{j \in \mathcal{J}}) \in \mathbb{Z}$ is said to be (weakly) **Pareto optimal** if there is no allocation $((x_i)_{i \in \mathcal{I}}, x^c, (y_j)_{j \in \mathcal{J}}) \in \mathcal{A}(\mathcal{E}^{pg})$ such that $(x_i, x^c) \succ_i (\bar{x}_i, \bar{x}^c)$ for each $i \in \mathcal{I}$.

In the following lemma and the subsequent analysis we shall use the assumption⁵ (A) and apply a specific notion of locally *non-satiated* preferences of each individual in

⁵Here, one has to consider $X_i^p \times X^c$ as a consumption set for individual i .

the groups of *private* and (separately) *public* goods. The latter means that changing the consumption bundle (x_i^p, x_i^c) only in part of the private or (separately) public goods while the consumption from another group of commodities is the same, it is possible to obtain a strictly preferred consumption bundle:

$$\mathcal{P}_i(x_i^p, x_i^c) \cap \mathbb{R}^l \times \{x_i^c\} \neq \emptyset \quad \& \quad \mathcal{P}_i(x_i^p, x_i^c) \cap \{x_i^p\} \times \mathbb{R}^s \neq \emptyset, \quad \forall i \in \mathcal{I}. \quad (2.1.5)$$

Lemma 2.1.1 *Let the allocation $\bar{z} = ((\bar{x}_i)_{i \in \mathcal{I}}, \bar{x}^c, (\bar{y}_j)_{j \in \mathcal{J}}) \in \mathcal{A}(\mathcal{E}^{pg})$ be Pareto optimal. Then there is a vector of prices for private goods $p \in \mathbb{R}^l$ and individualized price vectors for public goods $q_i \in \mathbb{R}^s$, $i \in \mathcal{I}$, which are **not all** equal to **zero** and are such that*

$$\langle (p, q_i), \mathcal{P}_i(\bar{x}_i, \bar{x}^c) \rangle \geq \langle (p, q_i), (\bar{x}_i, \bar{x}^c) \rangle, \quad i \in \mathcal{I}, \quad (2.1.6)$$

$$\langle (p, \sum_{i \in \mathcal{I}} q_i), \bar{y}_j \rangle \geq \langle p, Y_j \rangle, \quad j \in \mathcal{J}. \quad (2.1.7)$$

If for \bar{z} in addition, the consumption of each individual is **non-satiated** by private and (separately) public goods, and if $(\bar{x}_i, \bar{x}^c) \in \text{int}(X_i^p \times X^c) \forall i \in \mathcal{I}$, then all these price vectors **are non-zero**, i.e. $p \neq 0$, $q_i \neq 0 \forall i \in \mathcal{I}$.

Remark 2.1.1 If \mathcal{E}^{pg} is an economy with a smooth consumption sector and all the lemma's assumptions are satisfied then for interior allocation and for every i there is a real $\lambda_i > 0$: $(p, q_i) = \lambda_i \cdot \nabla u_i(\bar{x}_i, \bar{x}^c)$. ■

Proof of Lemma 2.1.1. To establish the lemma, one expresses the Pareto optimality of the allocation \bar{z} in a form suitable for the application of the separation theorem. For this purpose, let us define the following affine space:

$$\mathcal{L} = \{(x_1^p, x_1^c, \dots, x_n^p, x_n^c, y_1^p, y_1^c, \dots, y_m^p, y_m^c) \in \mathbb{R}^{(l+s)(n+m)} : \}$$

$$x_1^p + \dots + x_n^p - (y_1^p + \dots + y_m^p) = \sum_{\mathcal{I}} \mathbf{e}_i; \quad (2.1.8)$$

$$x_2^c - x_1^c = 0, \quad x_3^c - x_1^c = 0, \dots, \quad x_n^c - x_1^c = 0; \quad (2.1.9)$$

$$x_1^c - (y_1^c + \dots + y_m^c) = \mathbf{e}^c. \quad (2.1.10)$$

Now, Pareto optimality can be written as

$$\prod_{\mathcal{I}} (\mathcal{P}_i(\bar{x}_i, \bar{x}^c) \times \prod_{\mathcal{J}} Y_j) \cap \mathcal{L} = \emptyset. \quad (2.1.11)$$

Since the intersected sets are convex and nonempty (by **(A)**), then, by the separation theorem, there exists a linear functional $f \neq 0$, which separates these sets, i.e. if one presents this functional in the form of inner product, then there exists a vector $f = (f_1, \dots, f_n, g_1, \dots, g_m)$ such that

$$\langle f, \prod_{\mathcal{I}} (\mathcal{P}_i(\bar{x}_i, \bar{x}^c) \times \prod_{\mathcal{J}} Y_j) \rangle \geq \langle f, \mathcal{L} \rangle.$$

Further one reveals the structure of the functional (vector) f . Since f is bounded from above on \mathcal{L} , then it must be constant on \mathcal{L} and, hence, its representing vector must be located in the subspace orthogonal to \mathcal{L} . However, \mathcal{L}^\perp is represented as a linear hull of the normal vectors to the hyperplanes defined by relations (2.1.8)–(2.1.10) (all equations in vector form). The analysis of these relations gives the following representation: $\exists p \in \mathbb{R}^l, \exists \bar{q}, q_2, \dots, q_n \in \mathbb{R}^s$ such that for $q_1 = \bar{q} - \sum_{i=2}^n q_i$ one has $f_i = (p, q_i), g_j = (-p, -\bar{q}) \forall i, j, i.e.$

$$f = (p, q_1, p, q_2, \dots, p, q_n, (-p, -\bar{q}), \dots, (-p, -\bar{q})).$$

Further, to establish (2.1.6), consider the value of the functional f on the vector

$$(\bar{x}_1^p, \bar{x}^c, \dots, \bar{x}_n^p, \bar{x}^c, \bar{y}_1^p, \bar{y}_1^c, \dots, \bar{y}_m^p, \bar{y}_m^c) \in \mathcal{L} \quad (2.1.12)$$

and compare it with the value on a similar vector where at the place of i 's consumption a vector $(x_i^p, x_i^c) \in \mathcal{P}_i(\bar{x}_i^p, \bar{x}_i^c)$ is written. By (A) one has $(\bar{x}_k, \bar{x}^c) \in \overline{\mathcal{P}_k(\bar{x}_k, \bar{x}^c)}, \forall k \in \mathcal{I}$ and, therefore, constructed the vector belongs to the closure of the set, recorded in the left side of the intersection (2.1.11). Consequently, the value of the functional on the constructed vector must be not less than its value on the vector (2.1.12), hence, reducing the common terms in the right-hand and the left-hand sides one arrives at (2.1.6).

Inequalities (2.1.7) are proved in a similar way: the value of the functional on the vector (2.1.12) should be compared with the value on a similar vector where instead of production plan $(\bar{y}_j^p, \bar{y}_j^c)$ (arbitrary chosen) plan $(y_j^p, y_j^c) \in Y_j$ is written. In so doing one arrives at

$$\begin{aligned} \langle (-p, -\bar{q}), (y_j^p, y_j^c) \rangle &\geq \langle (-p, -\bar{q}), (\bar{y}_j^p, \bar{y}_j^c) \rangle, \forall (y_j^p, y_j^c) \in Y_j \iff \\ &\langle (p, \bar{q}), \bar{y}_j \rangle \geq \langle (p, \bar{q}), Y_j \rangle, \end{aligned}$$

which proves the first part of the lemma. We now prove the second part.

Consider an inequality of (2.1.6) such that $(p, q_i) \neq 0$. Suppose, for example, that $p = 0$. Now substitute consumption bundle \bar{x}_i by x_i so that $(x_i, \bar{x}^c) \succ_i (\bar{x}_i, \bar{x}^c)$ is true. Next find x^c such that $q_i x^c < q_i \bar{x}^c$ but still $(x_i, x^c) \succ_i (\bar{x}_i, \bar{x}^c)$. The assumptions of the lemma (interior point and non-satiation separately for private and public goods) allows us to do it. However, now one obtains $\langle (p, q_i), (x_i, x^c) \rangle < \langle (p, q_i), (\bar{x}_i, \bar{x}^c) \rangle$, which contradicts (2.1.6). Therefore, $q_i \neq 0$ implies that $p \neq 0$. It can be proven similarly that $p \neq 0 \Rightarrow q_i \neq 0$ for each i . Lemma 2.1.1 is proved. \blacksquare

In the model \mathcal{E}^{pg} with public goods the concept of the Foley core is usually considered in literature, and it has a familiar substantial sense: the set of all production allocations, which can be dominated by no coalition, *i.e.*, no group of individuals would benefit to live as a separate economy.

• An allocation $z = (x, x^c, y) \in \mathcal{A}(\mathcal{E}^{pg})$ is said to be dominated (blocked) by coalition $\emptyset \neq S \subseteq \mathcal{I}$ if there exist production $y^S = (y^{Sp}, y^{Sc}) \in Y_S = \sum_{i \in S} \sum_{j \in \mathcal{J}} \theta_i^j Y_j$ and consumption $((x_i^S)_S, x^{Sc}) \in \prod_S X_i \times X^c$ programs such that

$$\sum_{i \in S} x_i^S = \sum_{i \in S} \mathbf{e}_i + y^{Sp}, \quad x^{Sc} = \mathbf{e}^c + y^{Sc} \quad \& \quad (x_i^S, x^{Sc}) \succ_i (x_i, x^c) \quad \text{for each } i \in S.$$

The set of all allocations that are **dominated by no coalition** is denoted as $\mathcal{C}(\mathcal{E}^{pg})$ and is called **Foley core**.

Foley introduced this concept (Foley, 1970) and proved under certain assumptions that the Lindahl equilibrium belongs to the core: below this result follows directly from the analogous fact for the fuzzy core; see Proposition 2.2.1. However, does the Foley core shrink to equilibria under infinite replication of the model? It is well-known from the literature that an infinite replication of the model *does not* imply that *Foley core shrinks* to (equal treatment) Lindahl equilibria, unlike economies with only private goods. The following example, borrowed from Buchholz, Peters (2007), illustrates the fact. So, how is perfect competition to be presented in public goods economy? As we will see below there are at least two approaches: the fuzzy core approach and the contractual approach.

Example 2.1.1 Consider an economy with two individuals $i = 1, 2$ and two goods: one private and another public one. Let $(x_i, x^c) \in \mathbb{R}_+^2$ be permissible consumption plans, which agents $i = 1, 2$ can consume in any non-negative quantities. Let consumers have identical Cobb–Douglas utilities specified in logarithmic form as $u_i(x_i, x^c) = \ln(x_i) + \ln(x^c)$ and let the individuals' endowments be $\mathbf{e}_i = (1, 0)$, $i = 1, 2$ (public goods endowments be zero). Finally, suppose there is a technological set specified as a cone

$$Y = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \leq 0, \quad y_2 \leq -y_1\},$$

i.e., technology is linear and one unit of private good produces a unit of public one. Now let us calculate a Lindahl equilibrium according to Definition 2.1.1.

Due to (2.1.1) and the fact that public good in the equilibrium is not in zero quantity, for the (normalized) equilibrium prices, one has $(p, \bar{q}) = (1, 1)$ and the zero profit for producer. From the first-order conditions in the consumer's problem, one concludes $\nabla u_i(x_i, x^c) = (\frac{1}{x_i}, \frac{1}{x^c}) = \lambda_i(1, q_i)$, $\lambda_i > 0 \Rightarrow q_i = \frac{x_i}{x^c}$, that from the budget equality allows to conclude $x_i + x^c \cdot \frac{x_i}{x^c} = 1 \Rightarrow x_i = \frac{1}{2} \Rightarrow x^c = 2 - x_1 - x_2 = 1$. Thus, in the equilibrium one has

$$(x_1, x^c) = (x_2, x^c) = \left(\frac{1}{2}, 1\right),$$

where the price for the private good is $p = 1$, and the individualized prices are $q_1 = q_2 = \frac{1}{2}$. Further let us find the core.

In an economy with two agents core is a part of the Pareto boundary, where the consumption bundles for each agent are individually rational (utility is not less than in an economy with one agent). Pareto frontier can be found using the above analysis and Lemma 2.1.1; now the individual rationality constraint is given by $x_i \cdot x^c \geq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, that allows to describe core structure.

$$\mathcal{C}(\mathcal{E}^{pg}) = \{(x_1, x_2, x^c) \in \mathbb{R}_+^3 \mid x^c = 1, \quad x_1 + x_2 = 1, \quad x_1 \geq \frac{1}{4}, \quad x_2 \geq \frac{1}{4}\}.$$

Using an analysis similar to the above, it is easy to understand that in the n -times replicated economy the Lindahl equilibrium has the following form.

$$(\bar{x}_{11}, \bar{x}_{21}, \dots, \bar{x}_{1n}, \bar{x}_{2n}, \bar{x}^c, (y, y^c)) = \left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2}, n, (-n, n)\right),$$

under prices $p = 1$, $q_{ik} = \frac{1}{2n}$, $i = 1, 2$, $k = 1, \dots, n$.

Let us consider now the following allocation.

$$z = (x_{11}, x_{21}, \dots, x_{1n}, x_{2n}, x^c, (y, y^c)) = \left(\frac{5}{8}, \frac{3}{8}, \dots, \frac{5}{8}, \frac{3}{8}, n, (-n, n) \right),$$

i.e., agents of the second type are spending for the public good $\frac{5}{8}$ of the private commodity, and first type agents only $\frac{3}{8}$. Now applying Lemma 2.1.1 one can easily show that this is a Pareto optimal allocation (here similarly to the above $q_{ik} = \frac{x_{ik}}{x^c}$). Moreover, it belongs to the core. To verify this, consider a coalition S , consisting of l 1st type agents and m of 2nd ones. The optimal production of public goods in this group (through inter-coalitional Pareto frontier) is $x_S^c = \frac{l+m}{2} \leq n$. Hence, in order to allow agents of 1st type to reach the level of utility no less than in the allocation z , they have to consume the private good *more* than $\frac{5}{8}$. But then (by balance) in the group of 2nd type individuals in S there is an individual whose consumption of private good x_{2k}^S is strictly less than $\frac{1}{m}(\frac{l+m}{2} - \frac{5}{8}l) = \frac{1}{2} - \frac{1}{8}\frac{l}{m}$, *i.e.*, $\exists (2, k) \in S: x_{2k}^S < \frac{1}{2} - \frac{1}{8}\frac{l}{m}$
 \Rightarrow

$$\begin{aligned} x_{2k}^S \cdot \frac{l+m}{2} &< \left(\frac{1}{2} - \frac{1}{8}\frac{l}{m} \right) \cdot \frac{l+m}{2} < \left(\frac{1}{2} - \frac{1}{8}\frac{l}{n} \right) \cdot \frac{l+n}{2} = \\ &= \frac{1}{16n}(4n-l)(n+l) < \frac{1}{16n}(6n^2 - 2(l-n)^2) < \frac{6n^2}{16n} = \frac{3}{8}n. \end{aligned}$$

Taking the logarithm of both sides of the inequality, we conclude that the utility of the individual $(2, k) \in S$ is lower than in the proposed allocation, which contradicts S is blocking. Since l and m were chosen arbitrarily, then the resulting contradiction proves that allocation z belongs to Foley core.

The presented analysis holds for arbitrarily large n , which proves that Foley core does not shrink to Lindahl equilibrium when the model of economy is replicating to infinity. \blacksquare

In the next section, we introduce and study the concept of the fuzzy core for a model with public goods. In Section 2.2.3, fuzzy contractual allocations will be introduced and studied. As we shall see, these concepts are closely related to each other; they work fruitfully in the equilibrium theory and present an adequate solution of how the core can shrink to the Lindahl equilibrium.

2.2 Contractual economy with public goods

2.2.1 Fuzzy core vs. Foley core—what is the best?

Let us start from the definition and analysis of the fuzzy core. Recall that a fuzzy coalition is identified with any vector

$$t = (t_1, t_2, \dots, t_n) \neq 0, \quad 0 \leq t_i \leq 1 \quad \forall i \in \mathcal{I},$$

where real t_i is standardly interpreted as a measure of agent i 's participation in the coalition activities. The key property that determines the efficiency of fuzzy coalitions

is their ability to dominate a current allocation of the economy, and it defines the fuzzy core. In a model with public goods, this is an especially peculiar thing.

• Let $(x, x^c) = ((x_i)_{i \in \mathcal{I}}, x^c) \in \prod_{\mathcal{I}} X_i^p \times X^c$ be a family of private and public consumption plans, and $y = (y^p, y^c) \in \sum_{\mathcal{J}} Y_j = Y$ be an aggregated production program. Let triplet $(x, x^c, y) = z$ present a feasible allocation, i.e. $x^c = \mathbf{e}^c + y^c$ and $\sum_{\mathcal{I}} x_i = \sum_{\mathcal{I}} \mathbf{e}_i + y^p$ hold. A **fuzzy coalition** $t = (t_1, \dots, t_n)$ **blocks allocation** z if there is a triplet $((\xi_i)_{\mathcal{I}}, \xi^c, (\zeta^p, \zeta^c))$, such that

$$\sum_{\mathcal{I}} t_i (\xi_i - \mathbf{e}_i) = \zeta^p, \quad \xi^c = \zeta^c, \quad (\zeta^p, \zeta^c) \in Y \quad (2.2.1)$$

and

$$\left(\xi_i, \frac{\xi^c}{t_i} + \mathbf{e}^c \right) \succ_i (x_i, x^c) \quad \forall i \in \mathcal{I} : t_i \neq 0. \quad (2.2.2)$$

The set of all allocations that are dominated by no fuzzy coalition is denoted $\mathcal{C}^f(\mathcal{E}^{pg})$ and is called the **fuzzy core**.

Despite the fact that similar constructions have already appeared in the literature (e.g. see [Vasil'ev \(1996\)](#)), further analysis is quite original.

The meaning of the fuzzy core and the blocking is as follows. Imagine that an index $i \in \mathcal{I}$ specifies only the type of economic agent which is represented by many identical copies (the same number for different types). Then $t_i \in (0, 1]$ is a share of type i individuals, entered in a blocking coalition. Being separated the coalition passes to self-sufficiency of all its needs, which is expressed by relation (2.2.1). In this case, however, agents have to improve their situation and reach a more preferred consumption, which is expressed by (2.2.2), and this is the key peculiarity of (semi)public goods. Indeed, the agents estimate public goods produced within the coalition as the relative proportion of individuals of presented type. [Florenzano, Mercato \(2006\)](#) do not introduce a fuzzy core concept; however, they expressly postulate the appropriate type of domination in a replica of the original model (see Definitions 3.4, 3.5). One can say that an *average* level (for a type) of public goods consumption is crucial for agents instead of a common level of consumption. Thus, one can speak about semi-public goods, just as it is done in [Vasil'ev et al. \(1995\)](#). Moreover, in [Vasil'ev et al. \(1995\)](#), domination similar to that described above is interpreted in the terms of congestion or crowding in its provision (in these terms the results on the equivalence of the core and the equilibria are formulated). For example, (dis)pleasure from the consumption of such a good as the opportunity to attend a public skating-rink (park, road infrastructure, etc.) essentially depends on the number of visitors. Below we will see that the domination and the fuzzy core elements can be interpreted in contractual categories, where the coalition has the opportunity to enter into contracts for the production of public goods for inter-coalitional consumption. Then the elements of the fuzzy core correspond to stable sets of contracts (webs), subject to approval of the possibility of an asymmetric partial breaking.

Below, the first important result on the fuzzy core is presented; it is proven under the following additional assumptions.

(P) For each $j \in \mathcal{J}$ the set Y_j is a convex closed cone with a vertex at zero and public goods can only be produced, their amount cannot be reduced; i.e. $Y_j \subset \mathbb{R}^l \times \mathbb{R}_+^s$.

(M) The set of feasible public goods consumption programs obeys $X^c + \mathbb{R}_+^s \subseteq X^c$ and all public goods are desirable⁶ for each individual; i.e.

$$(x_i, x^c + z) \succ_i (x_i, x^c) \quad \forall (x_i, x^c) \in X_i \times X^c, \quad \forall z \in \text{int}\mathbb{R}_+^s \quad \forall i \in \mathcal{I}.$$

Proposition 2.2.1 Let \mathcal{E}^{pg} obey (P), (M). Then the Lindahl equilibrium belongs to the fuzzy core.

Proof of Proposition 2.2.1. Let the triplet $(x, x^c, y) \in \prod_{\mathcal{I}} X_i^p \times X^c \times Y$, $Y = \sum_{\mathcal{J}} Y_j$ satisfy Definition 2.1.1. Assume that there is a dominating coalition $t = (t_i)_{\mathcal{I}} \neq 0$. Then estimating (2.2.2) by prices and applying (2.1.2), we find: $\forall i \in \text{supp}(t)$

$$\langle p, \xi_i \rangle + \langle q_i, \frac{\xi^c}{t_i} + \mathbf{e}^c \rangle > \langle p, x_i \rangle + \langle q_i, x^c \rangle = \langle p, \mathbf{e}_i \rangle + \langle q_i, \mathbf{e}^c \rangle + \sum_{j \in \mathcal{J}} \theta_i^j (py_j^p + \bar{q}y_j^c).$$

Due to (2.1.1) and assumption (P) in the part of technological sets are convex cones with the vertex at zero, we conclude that $py_j^p + \bar{q}y_j^c \geq 0$, $\forall j \in \mathcal{J}$. Substituting this in the last formula and multiplying inequalities on t_i , one finds that

$$t_i \langle p, \xi_i \rangle + \langle q_i, \xi^c \rangle > t_i \langle p, \mathbf{e}_i \rangle, \quad i \in \text{supp}(t) \quad \Rightarrow \quad \langle p, \sum_{\text{supp}(t)} t_i (\xi_i - \mathbf{e}_i) \rangle + \langle \sum_{\text{supp}(t)} q_i, \xi^c \rangle > 0.$$

However, it follows now from (P), (M) that $\xi^c \geq 0$ and $q_i \geq 0$ for all i , which implies that $\langle \sum_{\mathcal{I}} q_i, \xi^c \rangle \geq \langle \sum_{\text{supp}(t)} q_i, \xi^c \rangle$ and, therefore,

$$\langle p, \sum_{\text{supp}(t)} t_i (\xi_i - \mathbf{e}_i) \rangle + \langle \sum_{\mathcal{I}} q_i, \xi^c \rangle \geq \langle p, \sum_{\text{supp}(t)} t_i (\xi_i - \mathbf{e}_i) \rangle + \langle \sum_{\text{supp}(t)} q_i, \xi^c \rangle > 0.$$

Due to (2.2.1), this means that by equilibrium prices (p, \bar{q}) , $\bar{q} = \sum_{\mathcal{I}} q_i$ production plan $(\sum_{\text{supp}(t)} t_i (\xi_i - \mathbf{e}_i), \xi^c) = (\zeta^p, \zeta^c)$ yields a strictly positive profit, which is impossible in view of (2.1.1) and the right-hand side of (2.2.1) for conical technological sets. ■

In order to better understand the properties of the fuzzy core and to reveal the exact relationship between the core and the equilibrium (equivalence?), we establish the following fuzzy core characterization. Define

$$\Upsilon_i(x_i, x^c) = \text{co}(\mathcal{P}_i(x_i, x^c) \cup \{(\mathbf{e}_i, \mathbf{e}^c)\}), \quad i \in \mathcal{I} \quad (2.2.3)$$

Due to the convexity of $\mathcal{P}_i(x_i, x^c)$, for $\mathcal{P}_i(x_i, x^c) \neq \emptyset$ (we have it by (A) or (M)) we conclude that

$$\text{co}(\mathcal{P}_i(x_i, x^c) \cup \{(\mathbf{e}_i, \mathbf{e}^c)\}) = \bigcup_{0 \leq \lambda \leq 1} [\lambda \mathcal{P}_i(x_i, x^c) + (1 - \lambda)(\mathbf{e}_i, \mathbf{e}^c)] =$$

⁶This is monotonicity of preferences regarding public goods.

$$= \Upsilon_i(x_i, x^c) = \bigcup_{0 \leq \lambda \leq 1} \lambda[\mathcal{P}_i(x_i, x^c) - (\mathbf{e}_i, \mathbf{e}^c)] + (\mathbf{e}_i, \mathbf{e}^c), \quad i \in \mathcal{I}.$$

Next, consider a set $\prod_{\mathcal{I}} \Upsilon_i(x_i, x^c) \times Y$, a vector

$$\tilde{\mathbf{e}} = ((\mathbf{e}_1, \mathbf{e}^c), (\mathbf{e}_2, \mathbf{e}^c), \dots, (\mathbf{e}_n, \mathbf{e}^c), 0) \in \mathbb{R}^{(n+1)(l+s)}$$

and condition $z + \tilde{\mathbf{e}} \in \prod_{\mathcal{I}} \Upsilon_i(x_i, x^c) \times Y$. By construction and analysis, one has a representation

$$z = (\lambda_1(\xi_1 - \mathbf{e}_1, \xi_1^c - \mathbf{e}^c), \lambda_2(\xi_2 - \mathbf{e}_2, \xi_2^c - \mathbf{e}^c), \dots, \lambda_n(\xi_n - \mathbf{e}_n, \xi_n^c - \mathbf{e}^c), \zeta^p, \zeta^c) \quad (2.2.4)$$

considered relative to some $(\xi_i, \xi_i^c) \in \mathcal{P}_i(x_i, x^c)$, $i \in \mathcal{I}$, $(\zeta^p, \zeta^c) \in Y$. Now, let us consider a subspace which corresponds to the material balance conditions for the model with public goods:

$$\begin{aligned} \mathcal{L}^{pg} = \{ & ((z_1^p, z_1^c), (z_2^p, z_2^c), \dots, (z_n^p, z_n^c), (y^p, y^c)) \mid \\ & z_1^p + z_2^p + \dots + z_n^p = y^p + \sum_{\mathcal{I}} \mathbf{e}_i, \quad z_1^c = z_2^c = \dots = z_n^c = y^c + \mathbf{e}^c \}. \end{aligned} \quad (2.2.5)$$

Finally, if $z + \tilde{\mathbf{e}} \in \prod_{\mathcal{I}} \Upsilon_i(x_i, x^c) \times Y \cap \mathcal{L}^{pg}$ then these balance conditions are added to constraint (2.2.4) for z :

$$\lambda_1(\xi_1 - \mathbf{e}_1) + \dots + \lambda_n(\xi_n - \mathbf{e}_n) = \zeta^p \quad \& \quad \lambda_1(\xi_1^c - \mathbf{e}^c) = \dots = \lambda_n(\xi_n^c - \mathbf{e}^c) = \zeta^c,$$

as in (2.2.1). For $\xi_i^c = \frac{\zeta^c}{\lambda_i} + \mathbf{e}^c$, by construction, one has $(\xi_i, \frac{\zeta^c}{\lambda_i} + \mathbf{e}^c) \in \mathcal{P}_i(x_i, x^c)$, which is equivalent to (2.2.2), $i \in \mathcal{I}$.

Hence, fuzzy blocking occurs if and only if there exists *nonzero* z , satisfying all requirements (*i.e.*, $z + \tilde{\mathbf{e}}$ belongs to the intersection). The presented reasonings prove the following characteristic

Lemma 2.2.1 *A feasible allocation $(x, x^c, y^p, y^c) \in \mathcal{C}^f(\mathcal{E}^{pg})$ if and only if*

$$\prod_{\mathcal{I}} \Upsilon_i(x_i, x^c) \times Y \cap \mathcal{L}^{pg} = \{\tilde{\mathbf{e}}\}. \quad (2.2.6)$$

Provided that in (2.2.6) all intersected sets are convex, it allows us to apply the separation theorem for the characterization of the elements of fuzzy core in value categories. One can do this applying the result and the mathematical technique used in Lemma 2.1.1 to intersection (2.2.6), and further apply the following.

Proposition 2.2.2 *Let the allocation $\bar{z} = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c) \in \mathcal{C}^f(\mathcal{E}^{pg})$ and preferences be non-satiated (via (\mathbf{A})). Then $(\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c)$ is Pareto optimal and therefore there exist prices $p \in \mathbb{R}^l$, $q_i \in \mathbb{R}^s$, $i \in \mathcal{I}$, not all equal to zero, such that relations (2.1.6) and (2.1.7) hold. Moreover, if (\mathbf{P}) holds (convex conic production) then in addition*

$$\langle (p, q_i), (\bar{x}_i, \bar{x}^c) \rangle = \langle p, \mathbf{e}_i \rangle + \langle q_i, \mathbf{e}^c \rangle, \quad i \in \mathcal{I}, \quad (2.2.7)$$

i.e., all the budgets are balanced and therefore the allocation is a **quasi-equilibrium**. If the conditions of the second part of Lemma 2.1.1 are true, then it is a genuine Lindahl equilibrium.

Now a theorem on the equivalence of the core and the equilibrium can be obtained as a consequence of the unconditional approval of the latter in conjunction with the Proposition 2.2.1 proved above.

Theorem 2.2.1 *Let \mathcal{E}^{pg} satisfy (P), (M), $(x_i, x^c) \in \text{int}(X_i \times X^c)$, $i \in \mathcal{I}$ and (2.1.5) be true (separated non-satiation). Then allocation $z = ((x_i)_{\mathcal{I}}, x^c, y^p, y^c) \in \mathcal{C}^f(\mathcal{E}^{pg})$ if and only if it is a **Lindahl equilibrium** allocation.*

Remark 2.2.1 In order to avoid such restrictive requirement as an interior point in the consumption of each individual, one can use the *irreducibility* assumption (specified for the public goods) and *non-trivial quasi-equilibrium* property, as in [Florenzano, Mercato \(2006\)](#), [Florenzano \(2009\)](#), and [Marakulin \(2012\)](#) §1.2.4. Here and below, I restrict myself to the case of an interior point, in order to not overburdened the presentation. ■

Proof of Proposition 2.2.2. Let $\bar{z} = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c) \in \mathcal{C}^f(\mathcal{E}^{pg})$, which, due to Lemma 2.2.1, is equivalent to (2.2.6). Now applying separation theorem,⁷ one can find a non-zero linear functional non-strictly separating these sets. Since

$$\prod_{\mathcal{I}} \mathcal{P}_i(x_i, x^c) \times Y \subset \prod_{\mathcal{I}} \Upsilon_i(x_i, x^c) \times Y,$$

it is clear that this functional also separates the sets of (2.1.11); this relation is equivalent to Pareto optimality (a small difficulty with the fact that $Y = \sum_{\mathcal{J}} Y_j$ can be easily bypassed), and therefore all conclusions regarding the separating functional from Lemma 2.1.1 can be applied to our functional. This proves the first part of the statement: there are prices $p \in \mathbb{R}^l$, $q_i \in \mathbb{R}^s$, $i \in \mathcal{I}$, not all zeros, such that the relations (2.1.6) and (2.1.7) are true. We now prove (2.2.7).

The vector representing the functional that separates the sets of (2.2.6) has the following structure:

$$f = (p, q_1, p, q_2, \dots, p, q_n, (-p, -\bar{q})), \quad \bar{q} = \sum_{\mathcal{I}} q_i.$$

Now, from the local non-satiation we conclude that $(\bar{x}_i, \bar{x}^c) \in \overline{\mathcal{P}_i(\bar{x}_i, \bar{x}^c)}$, $i \in \mathcal{I}$. Hence $\text{co}\{(\bar{x}_i, \bar{x}^c), (\mathbf{e}_i, \mathbf{e}^c)\} \subset \Upsilon_i(\bar{x}_i, \bar{x}^c)$, $i \in \mathcal{I}$ and certainly $\text{co}\{(\bar{y}^p, \bar{y}^c), (0, 0)\} \subset Y$. Therefore, functional f non-strictly separates the set

$$\text{co}\{(\bar{x}_1, \bar{x}^c), (\mathbf{e}_1, \mathbf{e}^c)\} \times \dots \times \text{co}\{(\bar{x}_n, \bar{x}^c), (\mathbf{e}_n, \mathbf{e}^c)\} \times \text{co}\{(\bar{y}^p, \bar{y}^c), (0, 0)\}$$

and affine subspace \mathcal{L}^{pg} . Calculating further the value of the functional at the point $\tilde{\mathbf{e}} = ((\mathbf{e}_1, \mathbf{e}^c), \dots, (\mathbf{e}_n, \mathbf{e}^c), 0, 0) \in \mathcal{L}^{pg}$ and for an appropriate element of the product, one finds that

$$\sum_{j \neq i} (p\mathbf{e}_j + q_j \mathbf{e}^c) + p\bar{x}_i + q_i \bar{x}^c \geq \sum_{j \neq i} (p\mathbf{e}_j + q_j \mathbf{e}^c) + p\mathbf{e}_i + q_i \mathbf{e}^c \Rightarrow p\bar{x}_i + q_i \bar{x}^c \geq p\mathbf{e}_i + q_i \mathbf{e}^c.$$

⁷Strictly, to apply the theorem, one has to extract point $\tilde{\mathbf{e}}$ from the set, located on the left side, and notice that it does not imply the loss of convexity. Instead of the set, one can also take its interior, which is convex, and clearly has an empty intersection with a subspace.

Summing inequalities over $i \in \mathcal{I}$ and taking into account the balance relations $\sum_{\mathcal{I}} \bar{x}_i = \sum_{\mathcal{I}} \mathbf{e}_i + \bar{y}^p$ and $\bar{x}^c = \mathbf{e}^c + \bar{y}^c$, we find that $p\bar{y}^p + \bar{q}\bar{y}^c \geq 0$, which can only be executed in the form of equity (since Y is a convex cone with the vertex at zero). Consequently, each added inequality can be fulfilled only in the form of equality. ■

Further, we pass to the contract-based approach.

2.2.2 Partial breaking and properly allocations with public goods

The main thing that is necessary to clarify is what and how a contract is concluded in the production sector and how is it broken. The essential feature of these contracts is that they are carrying out a joint production of collective consumption goods. Formally, the contract is

$$(r_1, \dots, r_n, y^c) \in \mathbb{R}^{ln} \times \mathbb{R}^s : \quad \left(\sum_{\mathcal{I}} r_i, y^c \right) \in Y = \sum_{\mathcal{J}} Y_j.$$

With every production contract $w = (r, y^c)$, $r = (r_i)_{i \in \mathcal{I}}$ one can associate its support:

$$\text{supp}(w) = \{i \in \mathcal{I} \mid r_i \neq 0\} = S(w).$$

It is formed by the agents which are involved to realize the production program $(\sum_{i \in \mathcal{I}} r_i, y^c)$. As follows from the definition, the main specific feature of a production contract for the model with public goods is its cooperative nature. That is, unlike the classical Arrow–Debreu model where production can be individualized, it is a project consisting of some joint activities related to the production of goods $y^c = (y_1^c, y_2^c, \dots, y_s^c)$ from resources $\bar{r} = \sum_{i \in S(w)} r_i$ obtained from the agents from $S(w)$. Notice also that it is not a necessity for private commodities to be consumed in production: private goods can be produced along with public goods if the technology allows it... In other words, the vectors r_i and $\bar{r} = \sum_{i \in S(w)} r_i$ can also have positive components. The breaking of production contract w is possible by any of its members $i \in S(w)$, and this means that all mutual obligations among members of the coalition $S(w)$ are void.

Similarly to barter, production contracts may form a *web*, *i.e.*, a finite set W of contracts, each subset $U \subseteq W$ of which forms a set of agreements, which correspond to a *feasible* production plan:

$$\left(\sum_{w \in U} \sum_{i \in \mathcal{I}} r_i^w, \sum_{w \in U} y_w^c \right) \in \sum_{j \in \mathcal{J}} Y_j.$$

Thus, the specific feature of production webs is that the breaking of a part of the contracts does not directly have any effect on the implementation of other contracts and the corresponding production programs. Notice also that for the *convex conic total technological set* every collection of feasible *contracts forms a feasible web* (and

vice versa). Note that forming a joint web with a family V of barter contracts one needs also to require feasibility in consumption:

$$\mathbf{e}_i + \sum_{w \in U} r_i^w + \sum_{v \in U'} v_i \in X_i, \quad \mathbf{e}^c + \sum_{w \in U} y_w^c \in X^c, \quad \forall U \subseteq W, U' \subseteq V, \forall i \in \mathcal{I}.$$

It is worthwhile to compare this with the similar requirement determining webs of barter contracts in an exchange model, see Section 1.1, p. 20.

Now let us assume that the production contracts of a web can be broken not only fully but also partially—similarly to the case of pure barter contracts for private commodities. Moreover, this analogy is also extended to the specific concepts of web stability: to the concepts of lower stable (the breaking), upper (conclusion of a new contract), and just a stable web (simultaneous breaking and signing of new contracts); in addition, these concepts can be applied to the union of barter (exchange) and production webs. Hereby *admitting only the total breaking* of contracts we arrive at the concept of contractual allocation that likes Definition 1.3.3. A simple analysis of the definitions shows that the contractual allocations of this type in an economy with public goods are exactly the allocations of the Foley core; see the definition above. Indeed, the fact that a coalition $S \subseteq \mathcal{I}$ dominates (blocks) the current allocation can be expressed in contractual terms as follows: this coalition breaks *all* contracts and signs new inter-coalitional contracts (both barter and production ones), in which only members of the coalition and their technological set $Y_S = \sum_{i \in S} \sum_{j \in \mathcal{J}} \theta_i^j Y_j$ are involved. As a result of these activities, each member of the coalition should improve upon its position (get higher utility). Finally, the core is the set of feasible allocations implemented by a web of contracts that no coalition can improve upon. Allowing the possibility to break contracts partially leads to a more qualified type of stability and a number of new concepts, which is reflected in the subsequent definitions.

Now let us consider a specific notion of properly contractual allocation. Having this in mind, I first extend the relation \simeq , see (1.3.13), to the set of all production webs. Here \simeq corresponds to the partial breaking of contracts that can be represented so that instead of the web of contracts its partition into two or more webs is considered, so that exchange or production ratios are not changed, but the total volume of exchange and production flows also is saved. Simplistic, but strict enough, this means that the webs V, W are replaced by (\simeq equivalent) the webs $\alpha V \cup (1 - \alpha)V$, $0 \leq \alpha \leq 1$ and $\beta W \cup (1 - \beta)W$, $0 \leq \beta \leq 1$. For properly contractual allocation, this operation does not imply that the implementing web $V \cup W$ loses stability. Formalism can be as follows.

Let V be a web of barter contracts and W be a web of production ones in the model \mathcal{E}^{pg} . Define $z(V, W) = (x, x^c, y) \in [\mathbb{R}^l]^{\mathcal{I}} \times \mathbb{R}^s \times \mathbb{R}^{(l+s)} = \mathfrak{L}$ as an allocation these webs implement, *i.e.*, for

$$z(V, W) = ((x_i(V, W))_{i \in \mathcal{I}}, x^c(W), y(V, W))$$

determine

$$x_i(V, W) = \mathbf{e}_i + \sum_{v \in V} v_i + \sum_{w \in W} r_i^w, \quad i \in \mathcal{I}, \quad y(V, W) = \left(\sum_{\mathcal{I}} \sum_{w \in W} r_i^w, \sum_{w \in W} y_w^c \right) \quad (2.2.8)$$

$$x^c(W) = \mathbf{e}^c + \sum_{w \in W} y_w^c. \quad (2.2.9)$$

In terms of equivalent webs, an allocation $z \in \mathcal{A}(\mathcal{E}^{pg})$ of the model with public goods is called *properly contractual*, if there exist barter V and production W webs such that $z = z(V, W)$ and for every $V' \simeq V$, $W' \simeq W$ allocation $z = z(V', W')$ is a contractual one. Below, a narrative substantial definition is presented.

Definition 2.2.1 A triplet (x, x^c, y) , where $((x_i)_{i \in \mathcal{I}}, x^c) \in \prod_{\mathcal{I}} X_i^p \times X^c$ is a family of private and public consumption plans, and $y \in \sum_{\mathcal{J}} Y_j$ is an aggregated production program, is called a **properly contractual allocation** if there exists a barter web V and a production web W such that the following hold.

- (i) $x_i = x_i(V, W) = \mathbf{e}_i + \sum_{v \in V} v_i + \sum_{w \in W} r_i^w$, $i \in \mathcal{I}$,
 $x^c = x^c(W) = \mathbf{e}^c + \sum_{w \in W} y_w^c$ & $y = y(V, W) = (\sum_{\mathcal{I}} \sum_{w \in W} r_i^w, \sum_{w \in W} y_w^c)$.
- (ii) There is no coalition $S \subseteq \mathcal{I}$ for which it is profitable (in a separate or simultaneous regime)
 - (α) to partially break barter and production contracts;
 - (β) to sign new barter and production contracts which together with preserved contracts form new feasible webs.

In other words, the allocation is properly contractual if it is implemented by a pair of stable webs (barter and production) that do not lose stability with respect to any of their partial decompositions.

Remark 2.2.2 For some public goods it is adequate to assume the possibility of partial breaking of production contracts while for others it is not. For example, public skating rinks or picture galleries can be viewed as a good of the first type because an individual contribution to the production contract can be understood as the money spent by the individual to visit a rink or a gallery (season tickets, *etc.*). Here, partial breaking of a contract means reducing the number of visits, which implies a decrease (in the same volume) of the consumption of the public good. A contract for goods such as street lighting or national security is fundamentally different: failure to pay taxes to finance these goods (assume such a form of financing) has almost no impact on their consumption. The described theory is related to goods of the first kind. ■

The following results characterize properly contractual allocations in terms of values that allow them to establish their relationship with the equilibria and state the main result of this section: the theorem on coincidence (under appropriate assumptions) of Lindahl equilibria and properly contractual allocations.

Proposition 2.2.3 Let \mathcal{E}^{pg} obey **(P)**, **(M)**. Then Lindahl equilibrium is a properly contractual allocation.

Proof of Proposition 2.2.3. Let the triple $(x, x^c, y) \in \prod_{\mathcal{I}} X_i^p \times X^c \times Y$, $Y = \sum_{\mathcal{J}} Y_j$ obey Definition 2.1.1. One needs to construct two webs of contracts: a barter web V and a production web W , which implement the equilibrium allocation and satisfy Definition 2.2.1. Having this in mind define $\bar{y} = \sum_{\mathcal{J}} y_j = (\bar{r}, \bar{y}^c)$, *i.e.*, one specifies a vector \bar{r} of total production inputs and outputs of private goods as $\bar{r} = \sum_{\mathcal{J}} y_j^p$. Consider a production web W , consisting of a single contract $w = (r_1, \dots, r_n, \bar{y}^c)$, whereby definition $r_i = x_i - \mathbf{e}_i$, $i \in S$. Then as soon as allocation is balanced one has $\sum_{\mathcal{I}} r_i = \sum_{\mathcal{J}} y_j^p = \bar{r}$ and therefore contract w implements production program $y = \sum_{\mathcal{J}} y_j$. Furthermore, by definition $p(x_i - \mathbf{e}_i - r_i) = 0$, $\forall i \in \mathcal{I}$. Next define $v_i = x_i - \mathbf{e}_i - r_i = 0$ and form a formal web $V = \{v\}$, consisting of a single contract $v = 0$. Let us show that the constructed web implements a properly contractual allocation.

First notice that the bundle (r_i, \bar{y}^c) has zero value by equilibrium prices, *i.e.*, $pr_i + q_i \bar{y}^c = 0$. This is implied by budget balance (equality) in the equilibrium and zero profit in production: due to **(P)**, the technological set is a cone. Further one argues going to a contradiction: Assume that there is $0 \leq t \leq 1$ and a production contract $\vartheta = (\vartheta_1, \dots, \vartheta_n, \eta^c)$, $(\sum \vartheta_k, \eta^c) \in Y$, such that, with contract w being broken in an amount $(1 - t)$, a new contract ϑ is concluded by coalition $S = \text{supp}(\vartheta)$, whose members are better off with respect to their consumption plans: $(\xi_i, \xi^c) = (\mathbf{e}_i + tr_i + \vartheta_i, \mathbf{e}^c + t\bar{y}^c + \eta^c) \succ_i (x_i, x^c)$, $i \in S$. Now estimating these consumptions by equilibrium prices and via (2.1.2) we find that $\langle (p, q_i), (\xi_i, \xi^c) \rangle > \langle (p, q_i), (x_i, x^c) \rangle$, *i.e.*, for all $i \in \text{supp}(\vartheta)$, we conclude that

$$p(\mathbf{e}_i + tr_i + \vartheta_i) + q_i(\mathbf{e}^c + t\bar{y}^c + \eta^c) > p(\mathbf{e}_i + r_i) + q_i(\mathbf{e}^c + \bar{y}^c) \Rightarrow p\vartheta_i + q_i\eta^c > 0.$$

Summing these inequalities for $\vartheta \neq 0$ one obtains

$$p \sum_{\text{supp}(\vartheta)} \vartheta_k + \left(\sum_{\text{supp}(\vartheta)} q_i \eta^c \right) > 0 \Rightarrow p \sum_{\text{supp}(\vartheta)} \vartheta_k + \left(\sum_{\mathcal{I}} q_i \eta^c \right) > 0,$$

which due to (2.1.1) for a conic production is impossible. For $\vartheta = 0$ domination is possible only for a singleton coalition and only then the current contract is (partially) broken, *i.e.*, for $t < 1$. Here similar reasonings yield $0 > 0$. Everything leads to a contradiction. The proof is completed. \blacksquare

Characteristic properties of properly contractual allocations are analyzed in the following theorems: they allow one to reverse the last statement.

Theorem 2.2.2 *Let $z(V, W) = ((x_i(V, W))_{i \in \mathcal{I}}, x^c(W), y(W))$ be an allocation implemented by a joint web of contracts: barter web V and production web W . Let $z(V, W) \in \text{int}(\prod_{\mathcal{I}} X_i^p \times X^c) \times \sum_{\mathcal{J}} Y_j$ and let \mathcal{E}^{pg} be a convex model with a smooth consumption sector and each individual be **non-satiated** in private and public goods.*

*Then, if $z(V, W)$ is implemented as a properly contractual allocation, there exist **nonzero** vectors $p \in \mathbb{R}^l$, $q_i \in \mathbb{R}^s$, $i \in \mathcal{I}$ such that*

$$\langle (p, q_i), \mathcal{P}_i(x_i, x^c) \rangle > \langle (p, q_i), (x_i, x^c) \rangle, \forall i \in \mathcal{I}, \quad (2.2.10)$$

$$pv_i = 0, \quad \forall v \in V, \quad \forall i \in \mathcal{I}, \quad (2.2.11)$$

$$pr_i^w + q_i y_w^c \geq 0, \quad \forall w \in W, \quad \forall i \in \mathcal{I}, \quad (2.2.12)$$

$$\langle (p, \sum_{i \in \mathcal{I}} q_i), y(W) \rangle \geq \langle (p, \sum_{i \in \mathcal{I}} q_i), \sum_{j \in \mathcal{J}} Y_j \rangle. \quad (2.2.13)$$

The peculiar circulation of the result of Theorem 2.2.2 gives the following

Theorem 2.2.3 *Let (\mathbf{P}) , (\mathbf{M}) ⁸ and the conditions of Theorem 2.2.2 hold. Then every contractual allocation satisfying (2.2.10)–(2.2.13) is a properly contractual one.*

Equalities (2.2.11) mean that the cost of the consumed private goods bundle x_i is equal to the cost of initial endowments plus the cost of private goods flow directed (derived from) in the production sector, while (2.2.10) means that if the consumption of public goods is valued through individual prices, then there is no other strictly preferred bundle that is cheaper than this. Inequality (2.2.12) says that every production contract is individually profitable, and (2.2.13) shows that the production sector of the economy as a whole operates in a maximum profitable way.

The following corollary of Theorems 2.2.2, 2.2.3 actually presents an equivalent description of Lindahl equilibrium in purely contractual categories. Note that the contractual allocation of any type, as well as a properly contractual one, does not appeal to the value parameters, their stability has a cooperative nature, expressed in terms of product flows and the contractual obligations among agents. Thus our approach eliminates the main theoretical difficulty of the concept of equilibrium with public goods—the presence of individual prices their production mechanism.

Corollary 2.2.1 *Let economy \mathcal{E}^{pg} have a smooth consumption sector, obey (\mathbf{P}) , (\mathbf{M}) and each individual be non-satiated in private and public goods. Then a feasible allocation $z = (x, x^c, y) \in \mathbb{Z}$, $(x, x^c) \in \text{int}(\prod_{i \in \mathcal{I}} X_i^p \times X^c)$ and a bundle of private and individualized prices $(p, q_1, \dots, q_n) \in \mathbb{R}^{l+ns}$ present **Lindahl equilibrium if and only if** there are the webs of barter V and production W contracts, implementing this allocation as a **properly contractual** one.*

Remark 2.2.3 Note that the only difference between the conditions of Corollary 2.2.1 and the assumptions of Theorem 2.2.1 consists of the fact that the consumption sector is smooth. ■

Proof of Corollary 2.2.1. Necessity is due to Proposition 2.2.3. Sufficiency: Under the corollary conditions Theorem 2.2.3 takes place, and hence (2.2.10)–(2.2.13) are fulfilled. Since the production sets are cones, the total profit is zero and according to (2.2.13) the inequalities (2.2.12) can be performed in the form of equalities. This provides budget equalities on the right side of (2.1.2). Other requirements of equilibrium definition are also obviously following from (2.2.10)–(2.2.13). ■

⁸Public goods are desirable, can be only produced but cannot be spent, and the technological set is a convex cone with vertex at zero.

Proof of Theorem 2.2.2. The fact that $z(V, W)$ is a properly contractual allocation implies that allocation $((x_i(V, W))_{i \in \mathcal{I}}, x^c(W), (y^p(W), y^c(W))) = z(V, W)$, where

$$x_i(V, W) = \mathbf{e}_i + \sum_{v \in V} v_i + \sum_{w \in W} r_i^w, \quad i \in \mathcal{I},$$

$$x^c(W) = \mathbf{e}^c + \sum_{w \in W} y_w^c$$

and

$$(y^p(W), y^c(W)) = \left(\sum_{w \in W} \sum_{i \in \mathcal{I}} r_i^w, \sum_{w \in W} y_w^c \right) \in \sum_{j \in \mathcal{J}} Y_j$$

is feasible and Pareto optimal. Hence, by Lemma 2.1.1, there exist non-zero vectors $p \in \mathbb{R}^l$, $q_i \in \mathbb{R}^s$, $i \in \mathcal{I}$, satisfying (2.1.6), where all inequalities are fulfilled in a strict form and therefore (2.2.10) is true, and

$$\left\langle \left(p, \sum_{i \in \mathcal{I}} q_i \right), (y^p(W), y^c(W)) \right\rangle \geq \left\langle \left(p, \sum_{i \in \mathcal{I}} q_i \right), \sum_{j \in \mathcal{J}} Y_j \right\rangle.$$

Next one uses these price vectors and proves (2.2.11)–(2.2.13). The requirement (2.2.13) is true by construction. Let us prove (2.2.11). It is sufficient to show by the definition that for each individual and every contract $v \in V$ one has $pv_i \geq 0$, $i \in \mathcal{I}$. Assuming to the contrary, one finds contract v and an individual i such that $pv_i < 0$. By Remark 2.1.1, it follows that $\langle \nabla u_i(x_i(V, W), x^c(W)), (-v_i, 0) \rangle > 0$, *i.e.*, the derivative of the utility in the direction $-(v_i, 0)$ is *positive*. Hence, it would be advantageous for this individual to partially break contract v in a possibly small volume $\alpha > 0$, because in this case locally his/her change of utility is calculated as $\alpha \langle \nabla u_i(x_i(V, W), x^c(W)), (-v_i, 0) \rangle > 0$. This contradicts to the definition of properly contractual allocation.

To prove (2.2.12), let us consider the value of the consumption bundle of the individual i at prices (p, q_i) after a partial breaking of the production contract $w \in W$, $i \in \text{supp}(w)$ in a volume $\alpha > 0$. It is easy to see that after the break one has:

$$\langle (p, q_i), (x_i(V, W), x^c(W)) \rangle - \alpha \langle (p, q_i), (r_i^w, y_w^c) \rangle.$$

In other words, the value is changed by $-\alpha \langle (p, q_i), (r_i^w, y_w^c) \rangle$, which for $pr_i^w + q_i y_w^c < 0$ is positive. Applying Remark 2.1.1 to this case, one obtains $\langle \nabla u_i(x_i(V, W), x^c(W)), (-r_i^w, -y_w^c) \rangle > 0$, *i.e.*, the derivative of i 's utility in the direction corresponding to the breaking of the contract w is strictly greater than zero. Therefore, it is beneficial for the individual to break this contract at least in a small volume. Hence, the assumption $pr_i^w + q_i y_w^c < 0$ leads us to a contradiction with the definition of a properly contractual allocation. \blacksquare

Proof of Theorem 2.2.3. Let conditions (2.2.10)–(2.2.13) be fulfilled. Arguing by contradiction, it is easy to see that (2.2.10) and (2.2.13) together imply Pareto optimality of the allocation implemented by a web of contracts (one comes to a contradiction by comparing the aggregate cost balance of the current and dominating allocations).

Without loss of generality, assume that there are only one barter contract v and one production contract w . Suppose there exists a coalition $S \subseteq \mathcal{I}$, interested in a partial breaking of existing contracts in amounts $1 \geq 1 - \alpha \geq 0$, $1 \geq 1 - \beta \geq 0$ and in the conclusion of new contracts \tilde{v} and \tilde{w} . Consumption bundles obtained in this manner should be preferred for the agents $i \in S$. These bundles are

$$\tilde{x}_i = \mathbf{e}_i + \alpha v_i + \beta r_i + \tilde{v}_i + \tilde{r}_i, \quad i \in S, \quad \& \quad \tilde{x}^c = \mathbf{e}^c + \beta y^c + \tilde{y}^c,$$

$$x_i = \mathbf{e}_i + v_i + r_i, \quad i \in S, \quad \& \quad x^c = \mathbf{e}^c + y^c.$$

Estimating the bundles by prices (p, q_i) , and applying (2.2.10) and (2.2.11), one finds that

$$p(\beta r_i + \tilde{v}_i + \tilde{r}_i) + q_i(\beta y^c + \tilde{y}^c) > p r_i + q_i y^c.$$

Summing up these inequalities over $i \in S$, via $\sum_S \tilde{v}_i = 0$, one concludes that

$$\begin{aligned} \beta p \sum_S r_i + p \sum_S \tilde{r}_i + \left(\sum_S q_i \right) (\beta y^c + \tilde{y}^c) &> p \sum_S r_i + \left(\sum_S q_i \right) y^c \Rightarrow \\ p \sum_S \tilde{r}_i + \left(\sum_S q_i \right) \tilde{y}^c &> (1 - \beta) \left[p \sum_S r_i + \left(\sum_S q_i \right) y^c \right]. \end{aligned} \quad (2.2.14)$$

On the other hand, the updated production program must be technologically acceptable, *i.e.*, it has to be

$$\left(\sum_S \tilde{r}_i + \beta \sum_{\mathcal{I}} r_i, \tilde{y}^c + \beta y^c \right) \in \sum_{\mathcal{J}} Y_j,$$

which in view of (2.2.13) yields

$$\begin{aligned} p \left(\sum_S \tilde{r}_i + \beta \sum_{\mathcal{I}} r_i \right) + \left(\sum_{\mathcal{I}} q_i \right) (\tilde{y}^c + \beta y^c) &\leq p \sum_{\mathcal{I}} r_i + \left(\sum_{\mathcal{I}} q_i \right) y^c \Rightarrow \\ p \sum_S \tilde{r}_i + \left(\sum_{\mathcal{I}} q_i \right) \tilde{y}^c &\leq (1 - \beta) \left[p \sum_{\mathcal{I}} r_i + \left(\sum_{\mathcal{I}} q_i \right) y^c \right]. \end{aligned}$$

Further, remember that all public goods are desirable for all agents (an assumption of the theorem), *i.e.*, preferences are monotone in this commodity group. Hence, by (2.2.10) it is easy to conclude that $q_i \geq 0$, $\forall i \in \mathcal{I}$. In addition, it was assumed in the theorem that public goods can be produced but not expended, *i.e.*, $\tilde{y}^c \geq 0$, which together with the previous gives $\sum_{\mathcal{I} \setminus S} q_i \tilde{y}^c \geq 0$. Now, applying (2.2.12) (summing the inequalities over $i \in S$) one concludes that the right-hand side of (2.2.14) is non-negative, and finally one concludes that

$$\begin{aligned} 0 &\leq (1 - \beta) \left[p \sum_S r_i + \left(\sum_S q_i \right) y^c \right] < \sum_{\mathcal{I} \setminus S} q_i \tilde{y}^c + p \sum_S \tilde{r}_i + \left(\sum_S q_i \right) \tilde{y}^c \leq \\ &\leq (1 - \beta) \left[p \sum_{\mathcal{I}} r_i + \left(\sum_{\mathcal{I}} q_i \right) y^c \right]. \end{aligned}$$

Now $\beta = 1$ implies $0 < 0$, which is impossible. For $\beta < 1$, one concludes that

$$p \sum_{\mathcal{I}} r_i + \left(\sum_{\mathcal{I}} q_i \right) y^c > 0,$$

which contradicts (2.2.13) and the theorem assumption on production sets which are the cones with the vertex at zero: in this case, firms' profits (the value on the left-hand side of inequality (2.2.13)) have to be zero. ■

2.2.3 Fuzzy contractual allocations

It was shown in previous studies that the contractual approach and especially its methodology and concepts related to the partial breaking of contracts may be considered as a specific way to model perfect competition conditions. Being much simpler than "classical" methods (non-atomic space of economic agents by Aumann, or replicas and Edgeworth equilibria by Debreu, Scarf and Aliprantis, *etc.*), and well-known in the literature, the contractual approach demonstrates high efficiency and leads to the same conclusions as in the previously analyzed situations. However, it is also applicable to many other situations which have not yet been explored. Here, I introduce the concept of fuzzy contractual allocation, which really can be considered as an alternative model of perfect competition. Consider first a meaningful scenario.

Imagine that at some intermediate moment of economic interaction, individuals intend to improve the structure of their contracts, partially breaking the old ones and entering into new contracts. Nobody controls their contractual activities and there is no coordinating body. Therefore, the breaking of contracts may take place asynchronously and secretly, with the result that in an intermediate planning stage individuals may operate with unrealistic asymmetrical agreements which are not contracts at all. However, during the search for a new contract, agents can rely on the resources made through such bogus contractual options. This can motivate them to sign new contracts and to break old ones, but now the contractual system as whole breaks down, as the breaking always occurs at the highest possible option, because all contracts are concluded on a voluntary basis, and they are voluntarily prolonged. The above situation may occur when the current allocation is not fuzzy contractual in the sense described below. Fuzzy contractual allocation is resistant to such perturbations of the contractual agreements. A formalization of this scenario is now presented.

Suppose again that V is a web of barter contracts and W is a web of production contracts of the model \mathcal{E}^{pg} and they implement allocation $z(V, W) = (x, x^c, y)$ according to (2.2.8) and (2.2.9). For simplicity and without loss of generality, one can assume that both webs are singletons, *i.e.*, $V = \{v\}$ and $W = \{w\}$. Suppose that an individual $i \in \mathcal{I}$ intends to partially break the barter contract in the amount of $(1 - g_i^v)$, $0 \leq g_i^v \leq 1$ and production in the amount of $(1 - t_i^w)$, $0 \leq t_i^w \leq 1$. As a result, he/she will have the following bundle of private and public goods:

$$\tilde{x}_i(t_i^w, g_i^v) = \mathbf{e}_i + g_i^v \cdot v_i + t_i^w \cdot r_i^w \quad \& \quad \tilde{x}^c(t_i^w) = \mathbf{e}^c + t_i^w \cdot y_w^c.$$

Further, the individual intends to sign new barter $\varsigma = (s_1, \dots, s_n)$ and production $\vartheta = (\vartheta_1, \dots, \vartheta_n, \eta^c)$ contracts, which together should lead to a preferred consumption:

$(\xi_i, \xi^c) \succ_i (x_i, x^c)$, where

$$\xi_i = \xi_i(t_i^w, g_i^v) = \mathbf{e}_i + g_i^v \cdot v_i + t_i^w \cdot r_i^w + s_i + \vartheta_i, \quad \xi^c = \xi^c(g_i^v) = \mathbf{e}^c + t_i^w \cdot y_w^c + \eta^c.$$

The situation can be further simplified if one notes that due to the definition of barter contract $\sum s_k = 0$ and hence if ϑ was a *feasible* production contract, *i.e.* $(\sum \vartheta_k, \eta^c) \in Y$, then contract $(\vartheta_1 + s_1, \dots, \vartheta_n + s_n, \eta^c)$ is also feasible. Therefore, for *domination* there is no need to use two new contracts; it is sufficient to apply *only one production contract* ϑ . Moreover, it will also be valid for the original allocation: if instead of two contracts v and w one considers only a production contract $(r_1 + v_1, r_2 + v_2, \dots, r_n + v_n, y_w^c)$ then the stability of allocation and the web of contracts can only be strengthened due to the fact that now the partition of contracts is carried out only in *equal amounts*. In other words, without loss of generality, one may always assume that $v = 0$. Below, a formal definition is presented.

Definition 2.2.2 *An allocation $(x, x^c, y^p, y^c) \in \mathcal{A}(\mathcal{E}^{pg})$ is called **fuzzy contractual** if a production contract $w = (r_1, r_2, \dots, r_n, y^c)$ implementing the allocation as*

$$x_i = \mathbf{e}_i + r_i, \quad i \in \mathcal{I}, \quad x^c = \mathbf{e}^c + y^c, \quad y^p = \sum_{\mathcal{I}} r_i, \quad (y^p, y^c) \in Y$$

is such that, for every $t = (t_i)_{i \in \mathcal{I}}$, $0 \leq t_i \leq 1$, $\forall i \in \mathcal{I}$ there is no another feasible contract $\vartheta = (\vartheta_1, \dots, \vartheta_n, \eta^c)$ implementing a new allocation

$$\xi_i^p = \xi_i^p(t, w, \vartheta) = \mathbf{e}_i + t_i r_i + \vartheta_i, \quad \xi_i^c = \xi_i^c(t, w, \vartheta) = \mathbf{e}^c + t_i y^c + \eta^c$$

such that $\forall i : (\xi_i^p, \xi_i^c) = (x_i, x^c)$ implies $t_i = 1$ and

$$(\xi_i^p, \xi_i^c) \succ_i (x_i, x^c) \quad \forall i : (\xi_i^p, \xi_i^c) \neq (x_i, x^c). \quad (2.2.15)$$

With respect to this definition, note that, by applying (2.2.15) with $\vartheta = 0$ it can be seen that no individual may be interested only in a partial breaking of contracts, *i.e.*, in accordance with the terminology of [Marakulin \(2003, 2006b\)](#) the production contract could be called *proper*.⁹ In addition, the definition posits the absence of explicit contractual missense activity, leaving the consumption bundle unchanged. But if the non-trivial current contract is broken, then the broken part is returned to the individual through his/her participation in a new contract. Notice also that unlike the concept of properly contractual allocation, here requirements regarding the joint admissibility of the old (partially asymmetrically broken) contract(s), and a new production contract are not imposed. However, notice again that for conic production sets this happens automatically: any set of admissible contracts constitutes a valid web.

Proposition 2.2.4 *Let \mathcal{E}^{pg} obey **(P)**, **(M)**. Then Lindahl equilibrium is a fuzzy contractual allocation.*

⁹In [Marakulin \(2003, 2006b\)](#) only pure exchange economies were studied.

Proof of Proposition 2.2.4. The proof completely repeats the arguments of the proof of Proposition 2.2.3; the fact that the production contract is broken asymmetrically does not matter. ■

The following lemma characterizes fuzzy contractual allocation in “geometrical” categories.

Lemma 2.2.2 *An allocation $\bar{z} = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c) \in \mathcal{A}(\mathcal{E}^{pg})$ is fuzzy contractual if and only if it is lower stable relative to the partial breaking of contracts and*

$$\mathcal{L}^{pg} \cap \prod_{i \in \mathcal{I}} [(\mathcal{P}_i(\bar{x}_i, \bar{x}^c) + \text{co}\{0, (\mathbf{e}_i - \bar{x}_i, \mathbf{e}^c - \bar{x}^c)\}) \cup \{(\mathbf{e}_i, \mathbf{e}^c)\}] \times Y = \{\tilde{\mathbf{e}}\}. \quad (2.2.16)$$

Here as above

$$\tilde{\mathbf{e}} = ((\mathbf{e}_1, \mathbf{e}^c), (\mathbf{e}_2, \mathbf{e}^c), \dots, (\mathbf{e}_n, \mathbf{e}^c), 0) \in \mathbb{R}^{(n+1)(l+s)},$$

and \mathcal{L}^{pg} is the subspace corresponding to the balance constraints of economy with public goods; see (2.2.5):

$$\begin{aligned} \mathcal{L}^{pg} = \{ & ((z_1^p, z_1^c), (z_2^p, z_2^c), \dots, (z_n^p, z_n^c), (y^p, y^c)) \mid \\ & z_1^p + z_2^p + \dots + z_n^p = y^p + \sum_{\mathcal{I}} \mathbf{e}_i, \quad z_1^c = z_2^c = \dots = z_n^c = y^c + \mathbf{e}^c \}. \end{aligned}$$

By Definition (2.2.3) one has

$$\Upsilon_i(x_i, x^c) \subset (\mathcal{P}_i(x_i, x^c) + \text{co}\{0, (\mathbf{e}_i - x_i, \mathbf{e}^c - x^c)\}) \cup \{(\mathbf{e}_i, \mathbf{e}^c)\}, \quad i \in \mathcal{I}.$$

Hence, applying Lemmas 2.2.1 and 2.2.2 implies the following

Corollary 2.2.2 *Every fuzzy contractual allocation belongs to the fuzzy core.*

Remark 2.2.4 One can purely mathematically note that in general (2.2.16) itself implies that the allocation is lower stable relative to the partial breaking of contracts. This is so (due to corollary) when all elements of the fuzzy core are equilibria. Thus in general (2.2.16) is solely sufficient for an allocation to be fuzzy contractual. ■

Proof of Lemma 2.2.2. Let \bar{z} be a fuzzy contractual allocation according to Definition 2.2.2. Assume that (2.2.16) is false, and therefore that there is $z = ((z_i^p, z_i^c)_{\mathcal{I}}, y^p, y^c) \neq \tilde{\mathbf{e}}$, belonging to the left part of (2.2.16). Consider a coalition $S = \{i \in \mathcal{I} \mid z_i \neq (\mathbf{e}_i, \mathbf{e}^c)\}$. Notice that $\mathcal{P}_i(\bar{x}_i, \bar{x}^c) \neq \emptyset$, $i \in S$ and find $(\xi_i^p, \xi_i^c) \in \mathcal{P}_i(\bar{x}_i, \bar{x}^c)$, $i \in S$ such that $z_i = \xi_i + t_i[(\mathbf{e}_i, \mathbf{e}^c) - (\bar{x}_i, \bar{x}^c)]$ for some $0 \leq t_i \leq 1$, $i \in S$ and $z_i = (\mathbf{e}_i, \mathbf{e}^c)$, $i \notin S$ (because $z \in \mathcal{L}^{pg}$, the possibility $S \neq \mathcal{I}$ occurs only at zero production of public goods, which is one of the options). Define $\vartheta_i = z_i^p - \mathbf{e}_i$, $i \in \mathcal{I}$. Now $\sum_{i \in \mathcal{I}} z_i^p = \sum_{i \in \mathcal{I}} \mathbf{e}_i + y^p$ implies $\sum_{\mathcal{I}} \vartheta_i = y^p$ and, therefore, for $\eta^c = y^c$, the vector $\vartheta = (\vartheta_1, \dots, \vartheta_n, \eta^c)$ presents a feasible production contract. Now for $i \in S \neq \emptyset$ one can write

$$\xi_i^p = z_i^p - \mathbf{e}_i + t_i(\bar{x}_i - \mathbf{e}_i) + \mathbf{e}_i = \vartheta_i + t_i r_i + \mathbf{e}_i$$

and, for the public sector,

$$z_i^c = \xi_i^c + t_i(\mathbf{e}^c - \bar{x}^c) = y^c + \mathbf{e}^c \quad \Rightarrow \quad \xi_i^c = y^c + \mathbf{e}^c + t_i\bar{y}^c = \eta^c + t_i\bar{y}^c + \mathbf{e}^c.$$

We have $(\xi_i^p, \xi_i^c) \succ_i (\bar{x}_i, \bar{x}^c)$, $i \in S$. Thus, in accordance with Definition 2.2.2, one finds a vector $t = (t_1, \dots, t_n)$ and contract $\vartheta = (\vartheta_1, \dots, \vartheta_n, \eta^c)$ satisfying (2.2.15) (here $t_i = 1$ and $\vartheta_i = 0$ for $i \notin S$). One comes to a contradiction.

We show that if a stable allocation $\bar{z} = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c) \in \mathcal{A}(\mathcal{E}^{pg})$ relative to the partial breaking obeys (2.2.16), then it is fuzzy contractual with respect to the production web $V = \{(r_1, r_2, \dots, r_n, \bar{y}^c)\}$, where $r_i = \bar{x}_i - \mathbf{e}_i$, $i \in \mathcal{I}$. Assume to the contrary, and find $t = (t_1, \dots, t_n)$ and contract $\vartheta = (\vartheta_1, \dots, \vartheta_n, \eta^c)$, $(y^p, y^c) = (\sum_{\mathcal{I}} \vartheta_i, \eta^c) \in Y$, $\vartheta \neq 0$ such that, $\forall i$: $(\xi_i^p, \xi_i^c) \neq (\bar{x}_i, \bar{x}^c)$, $\xi_i^p = \mathbf{e}_i + t_i r_i + \vartheta_i$, $\xi_i^c = \mathbf{e}^c + t_i \bar{y}^c + \eta^c$,

$$(\xi_i^p, \xi_i^c) \succ_i (\bar{x}_i, \bar{x}^c) \iff$$

$$z_i = (\mathbf{e}_i, \mathbf{e}^c) + (\vartheta_i, \eta^c) \in \mathcal{P}_i(\bar{x}_i, \bar{x}^c) + t_i((\mathbf{e}_i, \mathbf{e}^c) - (\bar{x}_i, \bar{x}^c)). \quad (2.2.17)$$

Here, either $(\xi_i^p, \xi_i^c) \neq (\bar{x}_i, \bar{x}^c) \forall i \in \mathcal{I}$ or $\exists i \in \mathcal{I}$: $(\xi_i^p, \xi_i^c) = (\bar{x}_i, \bar{x}^c)$.

In the first case, (2.2.17) is realized for all $i \in \mathcal{I}$, and $z = ((z_i)_{\mathcal{I}}, y^p, y^c) \neq \tilde{\mathbf{e}}$ belongs to the intersection on the left side of (2.2.16); this is a contradiction.

In the second case, by Definition 2.2.2 for some i one has $t_i = 1$ and $\mathbf{e}^c + t_i \bar{y}^c + \eta^c = \xi_i^c = \bar{x}^c = \mathbf{e}^c + \bar{y}^c$, which implies that $\eta^c = 0 = y^c$. Take $z_i = (\mathbf{e}_i, \mathbf{e}^c)$ for $i \notin \text{supp}(\vartheta)$. Now via the contract's definition, one can conclude that $\sum_{i \in \mathcal{I}} z_i^p = \sum_{i \in \mathcal{I}} \mathbf{e}_i + y^p$, $z_i^c = \mathbf{e}^c \forall i \in \mathcal{I}$. Thus, there is found an allocation $z = ((z_i)_{\mathcal{I}}, (y^p, y^c)) \neq \tilde{\mathbf{e}}$, belonging to the left side of (2.2.16), which is a contradiction. ■

The central result of the section is the following theorem on the equivalence of Lindahl equilibrium and fuzzy contractual allocation.

Theorem 2.2.4 *Let \mathcal{E}^{pg} satisfy **(P)**, **(M)**, $(x_i, x^c) \in \text{int}(X_i \times X^c)$, $i \in \mathcal{I}$ and (2.1.5) be true (separate non-satiation). Then allocation $z = ((x_i)_{\mathcal{I}}, x^c, y^p, y^c) \in \mathcal{A}(\mathcal{E}^{pg})$ is fuzzy contractual if and only if it is a Lindahl equilibrium allocation.*

Proof of Theorem 2.2.4. The proof of necessity follows immediately from the fact that the conditions of this theorem are identical to the conditions of Theorem 2.2.1 on the coincidence of equilibria with the elements of the fuzzy core: therefore, being an element of the fuzzy core, a fuzzy contractual allocation is an equilibrium one. The proof of sufficiency is due to Proposition 2.2.4 and repeats the arguments of Proposition 2.2.3. ■

The following final statement of the section fully reveals the relationship between the elements of the fuzzy core and fuzzy contractual allocations.

Lemma 2.2.3 *Let $z = (x, x^c, y) \in \mathcal{A}(\mathcal{E}^{pg})$ and $\mathcal{P}_i(x_i, x^c) \neq \emptyset$ for all $i \in \mathcal{I}$. Then $(x, x^c, y) \in \mathcal{C}^f(\mathcal{E}^{pg})$ implies that*

$$\mathcal{L}^{pg} \bigcap \prod_{i \in \mathcal{I}} (\mathcal{P}_i(x_i, x^c) + \text{co}\{0, (\mathbf{e}_i - x_i, \mathbf{e}^c - x^c)\}) \times Y = \emptyset. \quad (2.2.18)$$

Here (as before) \mathcal{L}^{pg} is a subspace corresponding to the balance constraints of the economy with public goods; see (2.2.5).

Comparison of formulas (2.2.16) and (2.2.18) clarifies the difference between the fuzzy core allocations and fuzzy contractual ones. It is evident that this difference is not too large, which allows us to interpret the allocations of the fuzzy core as fuzzy contractual. Moreover, now the fact that every element of the fuzzy core is a quasi-equilibrium (this is the main reason why the fuzzy core is so popular in existence theory) can be easily deduced from formula (2.2.18).

Proof of Lemma 2.2.3. The proof is based on Lemma 2.2.1 and relation (2.2.6), characterizing elements of the fuzzy core. One needs to show that (2.2.6) implies (2.2.18). Assume that $z = (\bar{x}, \bar{x}^c, \bar{y}^p, \bar{y}^c)$ satisfies (2.2.6), but (2.2.18) is false. Then there is a vector $t = (t_1, \dots, t_n)$, $0 \leq t_i \leq 1$, production plan $(y^p, y^c) \in Y$ and bundles $\zeta_i = (\zeta_i^p, \zeta_i^c)$, $\zeta_i^p = \xi_i^p + t_i(\mathbf{e}_i - \bar{x}_i)$, $\zeta_i^c = \xi_i^c + t_i(\mathbf{e}^c - \bar{x}^c)$ satisfying $\xi_i = (\xi_i^p, \xi_i^c) \succ_i (\bar{x}_i, \bar{x}^c)$, $i \in \mathcal{I}$ and such that

$$\sum_{\mathcal{I}} \xi_i^p + \sum_{\mathcal{I}} t_i(\mathbf{e}_i - \bar{x}_i) = \sum_{\mathcal{I}} \mathbf{e}_i + y^p \quad \& \quad \xi_i^c + t_i(\mathbf{e}^c - \bar{x}^c) = \mathbf{e}^c + y^c.$$

Define $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n, y^p, y^c)$. By construction $\zeta \in \mathcal{L}^{pg}$. Further, for a real $0 < \beta \leq \frac{1}{2}$, consider a vector $\beta\zeta + (1 - \beta)z = \rho(\beta) = \rho$, where for $i \in \mathcal{I}$ one has by construction

$$\rho_i^p(\beta) = \beta[\xi_i^p + t_i(\mathbf{e}_i - \bar{x}_i)] + (1 - \beta)\bar{x}_i, \quad \& \quad \rho_i^c(\beta) = \beta[\xi_i^c + t_i(\mathbf{e}^c - \bar{x}^c)] + (1 - \beta)\bar{x}^c.$$

In view of $z \in \mathcal{A}(\mathcal{E}^{pg}) \subset \mathcal{L}^{pg}$, one has $\rho(\beta) \in \mathcal{L}^{pg}$ for every β . Moreover,

$$(\beta y^p + (1 - \beta)\bar{y}^p, \beta y^c + (1 - \beta)\bar{y}^c) \in Y,$$

i.e., a feasible production program corresponds to the vector $\rho(\beta)$. Further, let us present vectors $\rho_i(\beta)$ in the form

$$\rho_i(\beta) = (1 - \beta t_i)(\bar{x}_i, \bar{x}^c) + \beta t_i(\mathbf{e}_i, \mathbf{e}^c) + (1 - \beta t_i) \frac{\beta}{1 - \beta t_i} [(\xi_i^p, \xi_i^c) - (\bar{x}_i, \bar{x}^c)], \quad i \in \mathcal{I},$$

whereby the choice of β one has $\mu_i = \frac{\beta}{1 - \beta t_i} \leq 1$. For $i \in \mathcal{I}$ the last expression due to (A) entails

$$\begin{aligned} \mu_i(\xi_i - (\bar{x}_i, \bar{x}^c)) &\in \mathcal{P}_i(\bar{x}_i, \bar{x}^c) - (\bar{x}_i, \bar{x}^c) \Rightarrow \\ \exists \eta_i \in \mathcal{P}_i(\bar{x}_i, \bar{x}^c) : \mu_i(\xi_i - (\bar{x}_i, \bar{x}^c)) &= \eta_i - (\bar{x}_i, \bar{x}^c). \end{aligned}$$

Hence, from the previous formula, one concludes that

$$\rho_i = (1 - \beta t_i)\eta_i + \beta t_i(\mathbf{e}_i, \mathbf{e}^c),$$

which implies that $\rho_i \in \Upsilon_i(\bar{x}_i, \bar{x}^c)$, $i \in \mathcal{I}$. Now one can apply (2.2.6) and conclude that $\rho = \rho(\beta) = \tilde{\mathbf{e}}$ for all real $0 < \beta \leq \frac{1}{2}$. We write this equation component-wise, and, by definition of $\rho_i(\beta)$, we find that

$$\beta[\xi_i + t_i((\mathbf{e}_i, \mathbf{e}^c) - (\bar{x}_i, \bar{x}^c))] + (1 - \beta)(\bar{x}_i, \bar{x}^c) = (\mathbf{e}_i, \mathbf{e}^c) \Rightarrow$$

$$\xi_i + t_i[(\mathbf{e}_i, \mathbf{e}^c) - (\bar{x}_i, \bar{x}^c)] = (\bar{x}_i, \bar{x}^c) + \frac{(\mathbf{e}_i, \mathbf{e}^c) - (\bar{x}_i, \bar{x}^c)}{\beta},$$

which has to be true for all $i \in \mathcal{I}$ and *all* $0 < \beta \leq \frac{1}{2}$. However, these equalities hold for *different* β , which is possible only if $(\bar{x}_i, \bar{x}^c) = (\mathbf{e}_i, \mathbf{e}^c) = \xi_i$, $i \in \mathcal{I}$, which by the choice of ξ_i implies $(\bar{x}_i, \bar{x}^c) \succ_i (\bar{x}_i, \bar{x}^c)$. This contradicts to **(A)** (preferences are irreflexive). Lemma 2.2.3 is proved. ■

Conclusion to Chapter 2

In the chapter, a contractual approach for an economic model with public goods and convex production was proposed and analyzed. The study revealed the high potential of this approach, which presents a contractual description of the important theoretical concept of the Lindahl equilibrium. Moreover, this is realized in several forms and via a number of new contractual concepts. These concepts characterize equilibria in cooperative terms and without value categories, and this is the basic advantage of the contractual approach. The main results are the following.

- Theorems 2.2.1, 2.2.4, and Corollary 2.2.1 on the equivalence of Lindahl equilibria and fuzzy core and/or specific properly and fuzzy contractual allocations.

These results characterizing Lindahl equilibrium do not involve the individual prices apparatus which is difficult to implement in practice; here, using an appropriate concept of contract, purely cooperative properties of the economic model are embodied into the results, and it does not address such notions as congested public goods and crowding in its provision. However, the results can be applied only for public goods that can be provided by production contracts admitting partial breaking in the process of their reproduction (excludable goods). Not all public goods have these properties. They are exclusive public goods, in fact, see Remark 2.2.2. All presented in Chapter 2 results are original and were published in *Marakulin (2013)*.

All this contributes to the further development of the contractual approach as a universal tool for constructing and analyzing the models of economic processes.

Chapter 3

Contract-based incomplete markets

In the world of real economy, individuals are forced to make decisions under uncertainty arising from incomplete information and the objective uncertainty of future events. As a result, in a modern economy one can observe not only ordinary commodity markets but also a rich array of markets of specific financial tools, so-called assets. The functioning of these markets is directly aimed at solving problems of this kind, problems deeply related to the uncertainty of the future. Examples of these markets are the insurance business, the markets of futures contracts, trade with options¹ of different kinds, *etc.* This problem, related with the uncertainty of the future, was well understood by the classical economic theorists (see survey [Radner \(1982\)](#)), but this subject has received new attention in the literature of the early 80s, when opportunities to develop the classical Arrow–Debreu theory ended. The result was the development, in an extended Arrow–Debreu model framework, of the *incomplete market theory* (*e.g.*, see [Geanakoplos \(1990\)](#), [Magill, Shafer \(1991\)](#) for a general overview). The term *incomplete* refers to the fact that the potentially infinite set of possible realizations of the future is surely wider than those created by people, “insurance variants,” expressed in the form of financial assets. Thus the incomplete market theory models an economic environment in which economic agents live and function under the constraints related to the possible differences in times when goods appear on markets and the objectively defined uncertainty of the future with respect to the present. Moreover, from the big array of ways to model uncertainty and time, this theory chooses those that reflect the specific financial features of real market economies, covering simultaneously the classical theory of resource allocation. However, the modern version of incomplete market theory has one essential gap—there is no satisfactory concept of domination by coalitions (of allocations) and consequently an appropriate core notion is lacking.

In fact, in a classical setting, the competitive equilibrium concept, having descriptive power, is also supported by the fact that there is no group of agents (coalition) that has incentives to form an autonomous subeconomy (it is said that the equilib-

¹The trade of property rights for future optional contracts, where “call” is to buy and “put” is to sell some commodity.

rium belongs to the core, *i.e.*, this is not dominated by any coalition). Moreover, in the conditions of perfect competition, *every allocation* from the core allows price decentralization, *i.e.*, it is an equilibrium relative to some prices—this is Edgeworth’s well-known conjecture. So in the ideal world of an Arrow–Debreu economy, competitive equilibrium, primary defined in a purely descriptive way, obtains the normative foundation as an ideally stable (in a given sense) allocation. This is why it seems quite natural to raise the question about the core definition in an incomplete market environment and to clarify its relations with the financial equilibrium. Moreover, the answer to this question will allow us to better understand what kind of (coalition) stability real observed financial markets have and what are the obstacles to stabilizing them.

The main theoretical goal of this chapter study is to suggest and to investigate a *core concept in incomplete (financial) markets*—a difficult quest for economic theory, which still under search of its satisfactory solution. In the author’s strong opinion, a “correct” core concept has to inherit the main properties of classical markets and has to satisfy the following two requirements:

- Let the economy be described as an incomplete market but in fact be complete, *i.e.*, it is mathematically equivalent to a standard pure exchange model, in which equilibria correspond to financial equilibria. Formally, this means that the rank of the matrix of value returns from assets is equal to the number of future events. Then the classical concept of the core and a new concept, introduced for incomplete markets, can be applied simultaneously. In such a case, the set of allocations for the core of an incomplete market should coincide with the set of standard core allocations.
- Under perfect competition conditions, the core and equilibria have to coincide—for a standard exchange economy this is the coincidence of Edgeworth’s equilibria (the allocations that belong to the core of each replicated economy) with competitive equilibria.

So we take these properties as the main criterion for a correct definition for a core and coalition domination in financial markets.

3.1 Incomplete market model

In the general framework of a pure exchange economy \mathcal{E} , let us consider a model with two periods $t = 0, 1$, in which there are l kinds of physically different (potentially) commodities available either today (with certainty) or tomorrow (contingent on each of a finite number s of possible future states of nature). So for this (market) economy, the *total* space of commodities L is associated with the space $\mathbb{R}^{l(s+1)}$. For convenience, we denote by $\sigma = 0$ the state of nature today. At each state $\sigma = 0, 1, \dots, s$, there is a spot market for each of the l commodities, whose price-vector is $p_\sigma \in \mathbb{R}^l$; at time 0, there exists also a financial market for k assets that deliver a random return across

the states at $t = 1$. Let q_j be a price for j -s asset and $q = (q_1, q_2, \dots, q_k)$ be the price-vector for assets. Let

$$\Pi = \{(p, q) \in \mathbb{R}^{l(s+1)} \times \mathbb{R}^k \mid \forall \sigma \quad \|p_\sigma\| \leq 1, \|q\| \leq 1\}$$

denote the set of admissible prices for commodities and assets, the elements of which will be denoted by $\pi = (p, q)$. In a general setting, the asset structure is given by the map

$$\mathfrak{A}(\cdot) = [a_j(\cdot)]_{j=1, \dots, k},$$

defined on $\mathbb{R}^{l(s+1)+k} \times X$; the image $\mathfrak{A}(\pi, x)$ is a $(s \times k)$ -matrix of which the j -th column vector $a_j(\pi, x)$ denotes, given p, q and x , the financial return of asset j across states of nature at period 1, denominated in units of account. In other words, the vector $a_j(\pi, x)$ is the promised monetary-valued payoff in all future states of nature associated with buying a unit of j -th asset. Now, if we denote

$$\lambda_j(x, \pi) = (-q_j, a_j(x, \pi)), \quad \Lambda = \begin{pmatrix} -q \\ \mathfrak{A}(x, \pi) \end{pmatrix},$$

then the total transfer of wealth across different states of the world, which some agent can obtain from the market of assets with respect to his/her portfolio $z = (z^1, \dots, z^k)$ (trade program for assets), is described by the vector

$$\Lambda \cdot z = z^1 \begin{bmatrix} -q_1 \\ a_1(\pi, x) \end{bmatrix} + \dots + z^k \begin{bmatrix} -q_k \\ a_k(\pi, x) \end{bmatrix}.$$

In a general incomplete market setting, consumer i , being ordinary described by a consumption set $X_i \subset L$ and a preference correspondence $\mathcal{P}_i : X_i \Rightarrow X_i$, is also in addition characterized by a portfolio set $Z_i \subset \mathbb{R}^k$ and vector-function

$$\alpha_i(\cdot) = (\alpha_i^\sigma(\cdot))_{\sigma=0}^{\sigma=s}, \quad \alpha_i^\sigma : \mathbb{R}^{l(s+1)+k} \times X \rightarrow \mathbb{R},$$

giving for each state the wealth of consumer i , given prices $p = (p_\sigma)_{\sigma=0}^{\sigma=s}$, q and actions of the other agents. Therefore, each consumer i can choose the net of his/her consumption bundles under the following budget constraints, having the following vector-inequality form:

$$Px_i \leq \alpha_i(\bar{x}, \pi) + \Lambda z_i, \quad x_i \in X_i, \quad z_i \in Z_i,$$

where the matrix

$$P = \begin{bmatrix} p_0 & & 0 \\ & \ddots & \\ 0 & & p_s \end{bmatrix} = \begin{bmatrix} p_0 & \dots & 0 \\ \vdots & P_1 & \\ 0 & & \end{bmatrix}$$

defines the consumption cost operator. Note that ‘‘squared product’’ (the standard notation $p \square x'$ denoting the vector $(p_\sigma \cdot x'_\sigma)_{\sigma=0}^{\sigma=s}$, $x' \in \mathbb{R}^{l(s+1)}$), commonly used in incomplete market theory, coincides with the ordinary matrix-vector product Px' , $x' \in \mathbb{R}^{l(s+1)}$. The incomplete market model under study is also equipped with the vector of endowment $\bar{e} \in L$ of the whole economy or simply with the vectors of

individualized consumers' initial endowments $\mathbf{e}_i \in X_i$, $i \in \mathcal{I}$ and for this case we put $\bar{\mathbf{e}} = \sum_{i \in \mathcal{I}} \mathbf{e}_i$.

Now let us recall the concepts of equilibrium applied in incomplete market theory.

Let us resume with $Z = \prod_{i \in \mathcal{I}} Z_i$, the list of data concerning the portfolio restrictions. Taking as given the actions of the other agents and a market system of commodity and asset prices, the budget set of i -th consumer is

$$B_i(p, q, x) = \{ x'_i \in X_i \mid \exists z_i \in Z_i : p \square x'_i \leq \alpha_i(p, x) + \Lambda(p, q, x)z_i \}.$$

Definition 3.1.1 *A financial Z -equilibrium is a pair of actions and admissible prices $((\bar{x}_i, \bar{z}_i)_{i \in \mathcal{I}}, (\bar{p}, \bar{q})) \in X \times Z \times \Pi$ such that*

(i) *for each $i \in \mathcal{I}$: $\bar{p} \square \bar{x}_i = \alpha_i(\bar{p}, \bar{x}) + \Lambda(\bar{p}, \bar{q}, \bar{x})\bar{z}_i$ and*

$$\mathcal{P}_i(\bar{x}_i) \cap B_i(\bar{p}, \bar{q}, \bar{x}) = \emptyset,$$

(ii) $\sum_{i \in \mathcal{I}} \bar{x}_i = \bar{\mathbf{e}}$ *and* $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$.

Classically, (i) means that each (\bar{x}_i, \bar{z}_i) is an optimal feasible budget plan for agent i , given $(\bar{p}, \bar{q}, \bar{x})$. Condition (ii) is a couple of market clearing conditions under the assumption that no production or inter-temporal storage is possible¹ and assets are in zero net supply. In (ii), if $\sum_{i \in \mathcal{I}} \alpha_i(\bar{p}, \bar{x}) = \bar{p} \square \bar{\mathbf{e}}$, the condition $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$ is obviously redundant when the rank of $\Lambda(\bar{p}, \bar{q}, \bar{x})$ is equal to k .

The concept of consumption sets of the agents expresses the idea of sociological and physiological restrictions on consumption bundles, independently of any limitation of resources. A similar interpretation for portfolio sets seems more difficult, and this is why economic theory is most interested in the particular case of *financial Z -equilibrium* in which there are *no restrictions on trade with assets*.

Definition 3.1.2 *A financial or GEI equilibrium is a pair of actions and admissible prices, representing financial Z -equilibrium for which $Z_i = \mathbb{R}^k$ for each $i \in \mathcal{I}$.*

Note that if $\sum_{i \in \mathcal{I}} \alpha_i(\bar{p}, \bar{x}) = \bar{p} \square \bar{\mathbf{e}}$, even if the matrix $\Lambda(\bar{p}, \bar{q}, \bar{x})$ has a rank strictly less than k , the condition $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$ is redundant in the following sense: by changing the portfolio of any one agent, it is easy to associate a financial equilibrium with any $((\bar{x}_i, \bar{z}_i)_{i \in \mathcal{I}}, \bar{p}, \bar{q})$ satisfying all the other conditions of Definition 3.1.2 but not necessarily $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$.

There are three basic types of assets which are of practical significance and generally considered in the literature. The first one is described by *real* assets—the vectors: $a^1, a^2, \dots, a^k \in \mathbb{R}^{ls}$, which as vector-columns form the $(sl \times k)$ -matrix $A = [a^j]_{j=1}^{j=k}$, *i.e.*,

$$A = \begin{bmatrix} a_1^1 & a_1^2 & \dots & a_1^k \\ \vdots & \vdots & \ddots & \vdots \\ a_s^1 & a_s^2 & \dots & a_s^k \end{bmatrix}$$

¹More exactly, informally inter-temporal storage abilities are accumulated in agents' initial endowment vectors $\mathbf{e}_i = (\mathbf{e}_i^\sigma)_{\sigma=0}^{\sigma=s}$, and via it, in agents' profit functions.

is commodity returns from assets which defines the matrix of financial returns across the future states of the world by formula

$$\mathfrak{A}(x, p, q) = (p_\sigma \cdot a_\sigma^j)_{\substack{\sigma=1,\dots,s \\ j=1,\dots,k}}.$$

Note that the concept of financial equilibrium with real assets is inflation-proof, *i.e.*, the changes of price levels (the type of its normalization) on future and present markets do not influence resource allocation (due to the homogeneity of budget constraints).

If a consumption bundle $e_\sigma \in \mathbb{R}^l$ is chosen as a unit of “numeraire” for $\sigma \geq 1$, *numeraire* assets are given by $a_\sigma^j = r_\sigma^j e_\sigma$, $r_\sigma^j \in \mathbb{R}$ and in this particular case of real assets, we have the second type of the matrix of returns from assets:

$$\mathfrak{A}(x, p, q) = ((p_\sigma \cdot e_\sigma) r_\sigma^j)_{\substack{\sigma=1,\dots,s \\ j=1,\dots,k}}.$$

With purely financial securities, the matrix $\mathfrak{A}(x, p, q)$ does not depend on p , q , and this is the third type of *nominal* assets.

It has to be clear that it makes sense to consider the notion of the core for an incomplete market economy with only real assets *and* so that the agents’ profit functions are defined through the value of individualized vectors of initial endowments, *i.e.*, for

$$\alpha_i^\sigma(p, q, x) = p_\sigma \mathbf{e}_i^\sigma, \quad \mathbf{e}_i = (\mathbf{e}_i^0, \dots, \mathbf{e}_i^s)$$

on the domain of $\alpha_i^\sigma(\cdot)$ and for all i , σ . Note that in this case the i -th consumer budget constraints have the form

$$Px_i \leq P\mathbf{e}_i + \begin{pmatrix} -q \\ P_1 A \end{pmatrix} z_i, \quad x_i \in X_i, \quad z_i \in \mathbb{R}^k, \quad (3.1.1)$$

where the matrix

$$P_1 = \begin{bmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_s \end{bmatrix}$$

defines the consumption cost operator for future events, *i.e.*, for $t = 1$. This matrix is the submatrix of P , which is formed by the rows $\sigma = 1, \dots, s$ and omitting the first l zero columns. Clearly in this case, we have $\mathfrak{A}(x, p, q) = P_1 A$.

As a result, this model under study may be written in the following short form:

$$\mathcal{E}^{in} = \langle \mathcal{I}, L, (X_i, P_i, \mathbf{e}_i)_{i \in \mathcal{I}}, A \rangle.$$

An important property of *GEI*-equilibrium prices is *no-arbitrage*, *i.e.* the lack of opportunities to make a guaranteed profit by acting in the assets market; in the absence of restrictions on trading portfolios, this profit can be increased infinitely. Mathematically, arbitrage and the *absence of arbitrage of opportunities* is presented as follows

$$\nexists z \in \mathbb{R}^k : [\Lambda(\pi, x) \cdot z \geq 0 \iff \begin{pmatrix} -q \\ \mathfrak{A}(\pi, x) \end{pmatrix} z > 0].$$

If each spot market is non-satiated, then the individual's utility-maximizing problem can be solved only in the presence of no-arbitrage prices, otherwise, the trading assets portfolio will grow to infinity (if there is one arbitrage portfolio $z \in \mathbb{R}^k$, then the ray $\{rz\}_{r \in \mathbb{R}_+}$ also presents arbitrage portfolios). In the important special case of real assets no-arbitrage condition takes the following form

$$\forall z \in \mathbb{R}^k [\langle p_\sigma A_\sigma, z \rangle \geq 0 \quad \forall \sigma \geq 1 \quad \& \quad \exists \sigma \geq 1 \quad \langle p_\sigma A_\sigma, z \rangle > 0 \quad \Rightarrow \quad \langle q, z \rangle > 0].$$

Thus here inequality $qz > 0$ is a corollary of inequalities $(p_\sigma A_\sigma)z \geq 0$, $\sigma = 1, \dots, s$, that allows to application of Farkas' Lemma (*e.g.* see [Rockafellar \(1979\)](#), p. 200) or directly Separation Theorem.

Lemma 3.1.1 *Incomplete market prices $q \in \mathbb{R}^k$, $p_\sigma \in \mathbb{R}^l$, $\sigma = 1, \dots, s$ are **no-arbitrage** iff there are real $\alpha_\sigma > 0$ such that*

$$q = \alpha_1 p_1 A_1 + \dots + \alpha_s p_s A_s.$$

Proof of Lemma 3.1.1. No-arbitrage asset price condition can be written as

$$\mathcal{L} \cap \Delta^s = \emptyset,$$

where $\mathcal{L} \subset \mathbb{R}^{s+1}$ is a linear hull of vector-columns of the matrix $\begin{pmatrix} -q \\ P_1 A \end{pmatrix}$, and $\Delta^s \subset \mathbb{R}^{s+1}$ is a standard simplex: $\Delta^s = \{y \in \mathbb{R}_+^{s+1} \mid \sum_{j=0}^s y_j = 1\}$. Now applying Second Separation Theorem, one finds a linear functional (vector) $\beta = (\beta_0, \beta_1, \dots, \beta_s) \neq 0$, strictly separating these sets:

$$\langle \beta, \mathcal{L} \rangle < \langle \beta, \Delta^s \rangle.$$

Functional $\beta(\cdot)$ is bounded on the space \mathcal{L} , that is possible only if $\langle \beta, \mathcal{L} \rangle = 0$. We obtain

$$\langle (-\beta_0 q + \beta_1 p_1 A_1 + \dots + \beta_s p_s A_s), (y_1, \dots, y_k) \rangle = 0 \quad \forall y \in \mathbb{R}^k.$$

Moreover, for the standard unit vectors $e_j \in \Delta^s$ one has $\langle \beta, e_j \rangle = \beta_j > 0$, $j = 0, \dots, s$. This and the latter one is possible only if

$$-\beta_0 q + \beta_1 p_1 A_1 + \dots + \beta_s p_s A_s = 0 \quad \Leftrightarrow \quad q = \frac{\beta_1}{\beta_0} p_1 A_1 + \dots + \frac{\beta_s}{\beta_0} p_s A_s.$$

Finally, one can specify $\alpha_\sigma = \frac{\beta_\sigma}{\beta_0} > 0$, $\sigma = 1, \dots, s$. Lemma 3.1.1 is proved. ■

Lemma 3.1.1 presents an important property of equilibrium prices, which is a basic concept in finance; of course, the statement of the lemma is true not only for the case of real assets but also in a general case (the proof is the same). However, in the case of real assets, it takes an even more elegant form: due to the homogeneity of budget constraints one can think without loss of generality that the equilibrium prices $\pi = (p, q)$ obey

$$q = \sum_{\sigma=1}^s p_\sigma A_\sigma. \tag{3.1.2}$$

Really, due to Lemma 3.1.1, having equilibrium prices one can readily adjust them as $\bar{p}_0 = p_0$, $\bar{p}_\sigma = \alpha_\sigma p_\sigma$, $\sigma \geq 1$.

Being imposed for \mathcal{E}^{in} , given the above assumption **(A)**, see page 19, we also assume that for an incomplete market model every X_i is “rectangular” over the states of the world, *i.e.*, $X_i = \prod_{\sigma=0}^s X_i^\sigma$, $X_i^\sigma \subset L_\sigma$ and

(S) Preferences are locally non-satiated in each spot market, *i.e.*, for every σ and each $i \in \mathcal{I}$

$$x_i^\sigma \in \overline{\mathcal{P}_i(x_i^\sigma, x_i^{-\sigma}) \cap L_\sigma}, \quad \forall x_i = (x_i^\sigma, x_i^{-\sigma}) \in X_i,$$

holds, where $x_i^{-\sigma} = (x_i^0, \dots, x_i^{\sigma-1}, x_i^{\sigma+1}, \dots, x_i^s)$ is a fragment-vector of x_i , complementing x_i^σ to x_i , and L_σ is a subspace of L , related with the event σ .

The reader can find more on incomplete market theory in [Magill, Quinzii \(2002\)](#), [Geanakoplos \(1990\)](#), [Magill, Shafer \(1991\)](#).

3.1.1 The contractual approach in incomplete markets

Further, in the framework of the incomplete market \mathcal{E}^{in} let us consider the model of a contractual economy, for which we define the set \mathcal{W} of all permissible contracts as follows

$$\mathcal{W} = \mathcal{W}^{in} = \bigcup_{\sigma=0}^{\sigma=s} \mathcal{V}_\sigma,$$

where $\mathcal{V}_\sigma \subset L^\mathcal{I}$ are some subspaces of $L^\mathcal{I}$ corresponding to markets in all possible states of the world. More exactly these subspaces are defined by

$$\mathcal{V}_\sigma = \{v \in L^\mathcal{I} \mid v_i^m = 0, \forall m \neq \sigma, m = 0, \dots, s, \forall i \in \mathcal{I}\}$$

for all $\sigma = 1, \dots, s$, and for the present ($t = 0$) let us put

$$\mathcal{V}_0 = \{v \in L^\mathcal{I} \mid \exists z_i \in \mathbb{R}^k : v_i^\sigma = A_\sigma z_i, \forall i \in \mathcal{I}, \forall \sigma = 1, \dots, s\},$$

where A_σ are the submatrices of matrix A corresponding to the future states of the world, *i.e.*, $A_\sigma = (a_\sigma^j)_{j=1}^{j=k}$. Note that if I were to apply an incomplete market with some portfolio constraints, then I would change the previous formula, and in addition require that $z_i \in Z_i$. Also from the last definition, one can easily check that $v \in \mathcal{V}_0$ is a contract, *i.e.*, $\sum_{i \in \mathcal{I}} v_i = 0$, if and only if there exist such $z_i \in \mathbb{R}^k$, $i \in \mathcal{I}$, that $\sum_{i \in \mathcal{I}} z_i = 0$ and $v_i^\sigma = A_\sigma z_i$ hold for every $i \in \mathcal{I}$ and $\sigma = 1, \dots, s$. This allows us, for the convenience of the below considerations, to transit to the initial incomplete market terms and to work with portfolios for assets instead of with deviations of goods. Namely, to avoid misunderstanding, we shall apply the following specific notion of contract for the present—this is a couple $w = (v, z)$, such that $v = (v_i)_{i \in \mathcal{I}} \in L^\mathcal{I}$, $z = (z_i)_{i \in \mathcal{I}} \in (\mathbb{R}^k)^\mathcal{I}$ and

$$\sum_{i \in \mathcal{I}} v_i = 0, \quad \sum_{i \in \mathcal{I}} z_i = 0 \quad \& \quad v_i^\sigma = 0, \quad \forall i \in \mathcal{I}, \quad \forall \sigma = 1, \dots, s$$

hold. Also, for convenience of notation, we will identify the contract $v \in \mathcal{V}_\sigma$ for $\sigma \geq 1$, which formally belongs to the space $L^{\mathcal{I}}$ with the vector v^σ from $(\mathbb{R}^I)^{\mathcal{I}}$.

It follows from above that in incomplete markets the contracts are classified according to the state of the world to which they belong and therefore each web V may be represented as

$$V = \bigcup_{\sigma=1}^s V^\sigma \cup W,$$

where V^σ is the set of all contracts relative to the state $\sigma \neq 0$, and W is the set of contracts in the present. The structure of permissible contracts in an incomplete market is presented in Figure 3.1.1.

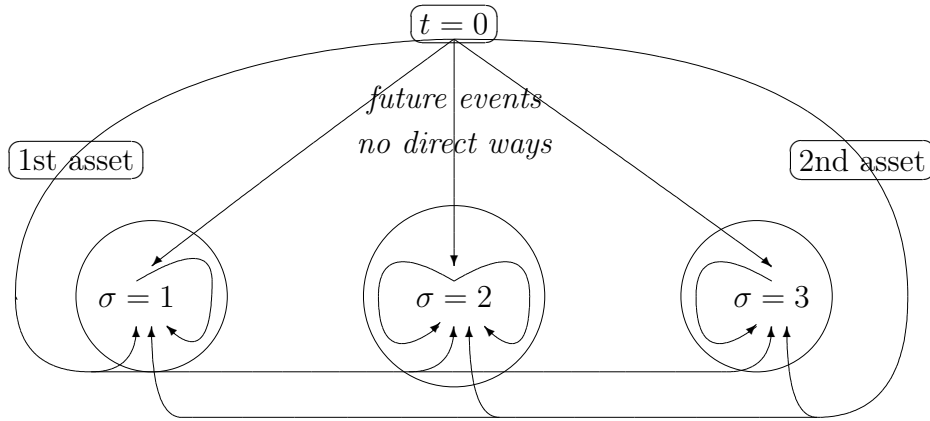


Figure 3.1.1: Contractual structure for an incomplete market

Let a web V be presented. Then by definition consumer i 's consumption bundle y_i corresponding to this web satisfies

$$\begin{aligned} y_i^0(V) &= e_i^0 + \sum_{(u,z) \in W} u_i^0, \\ y_i^\sigma(V) &= e_i^\sigma + \sum_{v \in V^\sigma} v_i^\sigma + \sum_{(u,z) \in W} A_\sigma z_i, \quad \sigma = 1, \dots, s. \end{aligned} \quad (3.1.3)$$

Now, if we denote

$$\Delta_i^0(V) = \Delta_i^0(W) = \sum_{(u,z) \in W} u_i^0,$$

$$\Delta z_i = \Delta z_i(W) = \sum_{(u,z) \in W} z_i, \quad \Delta_i^\sigma(V^\sigma) = \sum_{v \in V^\sigma} v_i^\sigma, \quad \sigma = 1, \dots, s$$

and put $\Delta_i^\sigma(V) = \Delta_i^\sigma(V^\sigma) + A_\sigma \Delta z_i$ for $\sigma \geq 1$, then the total deviation of agent i 's initial endowments may be represented by the vector

$$\Delta_i(V) = (\Delta_i^0(W), \Delta_i^1(V^1) + A_1 \Delta z_i, \dots, \Delta_i^s(V^s) + A_s \Delta z_i),$$

which by definition is the i 's fragment-vector of the total deviation of initial allocation \mathbf{e} . Now relations (3.1.3) may be rewritten in the form

$$\begin{aligned} y_i^0(V) &= \mathbf{e}_i^0 + \Delta_i^0(W), \\ y_i^\sigma(V) &= \mathbf{e}_i^\sigma + \Delta_i^\sigma(V^\sigma) + A_\sigma \Delta z_i, \quad \sigma = 1, \dots, s. \end{aligned} \quad (3.1.4)$$

The definition of a set of all permissible contracts \mathcal{W}^{in} and also the rules of operating with webs, described in Section 1.1, imply the following properties for breaking and signing contracts in an incomplete market:

- the agents can break any contracts;
- for given event $\sigma = 1, \dots, s$ the agents can sign new contracts—commodity exchanges at this event;
- for event $\sigma = 0$ (*i.e.*, in the present) they can sign new contracts by means of assets and by commodity exchanges for date $t = 0$; the agents can do it in a common regime, as well as in a separate style.

Thus the situation with breaking contracts and with signing new ones is *non-symmetrical*, since consumers can break any kind of contract, but they can sign only the contracts relative to a *fixed* state of the world. This non-symmetry appears due to specific incomplete market properties and fits with the item (*iii*) of the definition of $F(V, T)$: the set of possible webs that may be realized by a coalition T after breaking some contracts and signing new ones, see page 20.

Now, in the context of an incomplete market, let us consider some notions of contractual allocations presented in Section 1.1. To characterize the core and equilibrium allocations, we shall use two kinds of complex contractual allocation. Also, one can call them first and second contractual.

Definition 3.1.3 Let $V = \bigcup_{\sigma=1}^s V^\sigma \cup W$ be a weakly stable web such that all contracts from V^σ are perfect for all $\sigma \geq 1$. The allocation $x = x(V)$ is called *semi-perfectly contractual* if, for every virtual $U^\sigma \sim V^\sigma$, $U^\sigma \subseteq \mathcal{V}_\sigma$, $\sigma \geq 1$, there is no $S \subseteq \mathcal{I}$ and $\widehat{V} \subseteq \bigcup_{\sigma=1}^s U^\sigma$, satisfying

$$\text{supp}(\widehat{v}) \subseteq S, \quad \forall \widehat{v} \in \widehat{V} \quad (3.1.5)$$

and such that for $t = \sigma = 0$, there is a contract $w' = (u', z')$, $\text{supp}(w') \subseteq S$ such that

$$y_i(V') \succ_i x_i(V), \quad \forall i \in S$$

is true for web $V' = \{w'\} \cup \widehat{V}$.

The set of all **semi-perfectly contractual** allocations from \mathcal{E}^{in} is denoted

$$\mathcal{D}^{sp}(\mathcal{E}^{in}).$$

In reference to Definition 3.1.3, note that allocation is semi-perfect contractual if there exists such a weakly stable web realizing this allocation, in which all contracts

for the future states of the world are proper. Moreover, every one of these contracts can be changed by any weakly equivalent (virtual) web without the loss of stability in the following sense. In the present, a coalition considers the possibility to create an autonomous sub-economy. To do this, it has to break all contracts in the present, and moreover, the coalition is forced *to break some contracts at every future event*, using a virtual equivalent web, such that *all contracts in which there is a non-trivial exchange with some non-members of the coalition* are broken. The condition (3.1.5) realizes this requirement. When this deal is realized, the coalition can sign some new contracts in the present. In so doing, the breaking of given contracts and the signing of a new one is considered being *simultaneous* procedure. Of course, the fact that a coalition can use virtual contracts for future events is very important.

Notice only that these contracts and the property of contracts from V^σ being perfect, one has to consider relative to contracts from \mathcal{V}^σ —only in the limits of this set of permissible contracts, one may pass to (weakly) equivalent contracts. Let us also be reminded that the fact that we apply perfect contracts for future events implies that there is no coalition that is able to increase its members' utility by breaking a part of equivalent contracts and signing a new one relative to *any given future state of the world*. This is why this ability is not considered in the semi-perfectly contractual allocation definition directly, but of course, it is taken into account in further considerations.

It should be clear that as in the general case, the stability can be increased if, for a given fixed event, one changes the group of contracts in the web by their sum.² It follows that the subsystem of all contracts in the present, W , can be changed by $\sum_{w \in W} w$ and, therefore, condition (3.1.5) and the other requirements of Definition 3.1.3 have to be fulfilled, subject to breaking *the only contract* $w' = \sum_{w \in W} w$.

Definition 3.1.4 Let $V = \bigcup_{\sigma=1}^s V^\sigma \cup W$ be a weakly stable web such that for $\sigma \geq 1$ all contracts from V^σ are perfect and (for $\sigma = 0$) all contracts from W are proper. Then the complex contractual allocation $x = x(V)$ is called *proper-perfectly contractual*, if for every virtual $U^\sigma \sim V^\sigma$, $\sigma \geq 1$, and for every partition $\widetilde{W} \simeq W$ there is no $S \subseteq \mathcal{I}$ and $\widehat{V} \subseteq \bigcup_{\sigma=1}^s U^\sigma \cup \widetilde{W}$ such that for $t = \sigma = 0$ there is a contract $w = (u, z)$, $S(w) \subseteq S$, such that

$$y_i(V') \succ_i x_i(V), \quad \forall i \in S$$

is true for the web $V' = \{w\} \cup \widehat{V}$.

The set of all **proper-perfectly contractual** allocations from \mathcal{E}^{in} is denoted

$$\mathcal{D}^{cp}(\mathcal{E}^{in}).$$

Notice that due to Definition 3.1.4, contracts in the present may be partially broken and, moreover, condition (3.1.5) is not imposed. Each of these differences increases

²Note that it cannot be done for any subsystem of contracts, since one has to be sure the summed contract is permissible.

the requirements for the stability of allocation. However, note (it will be clear later) that condition (3.1.5) does not play a special role in proper-perfectly contractual allocations and may be added to the definition. The coalition of all agents, \mathcal{I} , plays the main role. This situation is similar to ordinary markets, where the most important thing is that the allocation is Pareto optimal and all contracts are proper. Thus a proper-perfectly contractual allocation differs from a semi-perfectly contractual one only in that we can partially break contracts in the present for the first allocation, but for a semi-perfectly contractual allocation non-proper contracts in the present are allowed and they may be broken only as a whole. In particular, note that it is always true

$$\mathcal{D}^{cp}(\mathcal{E}^{in}) \subseteq \mathcal{D}^{sp}(\mathcal{E}^{in}).$$

In the following analysis, we will examine the possibility to treat *GEI*-equilibrium as a contractual allocation stable in some sense. To approach this problem, we first need a system of contracts to be assigned to every equilibrium allocation. Having this in mind we associate a system of contracts $V = \{v^\sigma\}_{\sigma=0}^s$ with an equilibrium allocation (\bar{x}, \bar{z}, p, q) of incomplete market \mathcal{E}^{in} , where

$$v^0 = w = (u, \bar{z}) = (\bar{x}^0 - \mathbf{e}^0, \bar{z}), \quad v^\sigma = (\bar{x}^\sigma - \mathbf{e}^\sigma - A_\sigma \bar{z}), \quad \sigma = 1, \dots, s \quad (3.1.6)$$

and by convention $A_\sigma \bar{z} = (A_\sigma \bar{z}_i)_{i \in \mathcal{I}}$. We shall call this *equilibrium* system of contracts. An important feature of the equilibrium system is the zero value of the contract for each individual in each state of the world.

Proposition 3.1.1 *Suppose assumption (S) holds (preferences are non-satiated for each spot market). Then, without loss of generality, one can consider that for any state of nature and for each agent, the system of contracts (3.1.6) which implements GEI-equilibrium has zero value in the **adjusted** prices:*

$$\langle p_0, \bar{x}_i^0 - \mathbf{e}_i^0 \rangle + \sum_{\sigma=1}^s \langle p_\sigma A_\sigma, \bar{z}_i \rangle = 0; \quad \langle p_\sigma, v_i^\sigma \rangle = 0 \quad \forall \sigma = 1, \dots, s.$$

Proof of Proposition 3.1.1. When preferences are non-satiated in equilibrium, all budget constraints are fulfilled as equality. Hence, in any of the future states of nature, every equilibrium contract has zero value just due to the contract definition:

$$p_\sigma x_i^\sigma = p_\sigma \mathbf{e}_i^\sigma + p_\sigma A_\sigma \bar{z}_i \quad \Rightarrow \quad p_\sigma (x_i^\sigma - \mathbf{e}_i^\sigma - A_\sigma \bar{z}_i) = p_\sigma v_i^\sigma = 0, \quad \forall i \in \mathcal{I}, \quad \forall \sigma = 1, \dots, s.$$

A similar property of a zero value of the contract in the present (presented in terms of consumption goods) follows from $p_0(\bar{x}_i^0 - \mathbf{e}_i^0) = -q\bar{z}_i$, no-arbitrage for assets equilibrium prices and by (3.1.2), that is always true for *adjusted* equilibrium prices $\bar{p}_0 = p_0$, $\bar{p}_\sigma = \alpha_\sigma p_\sigma$, $\sigma \geq 1$, see Lemma 3.1.1. Proposition 3.1.1 is proved. \blacksquare

3.2 Contractual analysis and results

3.2.1 Preliminary analysis and technical results

The analysis and the key properties of complex contractual allocations in incomplete markets are based on the following observation. Let some semi-perfectly contractual allocation $\bar{x}(W, V) \in \mathcal{D}^{sp}(\mathcal{E}^{in})$ be given. Consider and fix some event $\sigma \geq 1$ and fix consumption for other events. Further, let us consider the reduced model \mathcal{E}^σ , the model in which only exchanges and the deviation of consumption bundles in state σ are allowed. If in this model one considers $\mathbf{e}_i^\sigma + A_\sigma \Delta z_i(W)$ to be the vectors of agents' initial endowments, then one can transit to the standard exchange economy in which consumption sets are the appropriate sections of initial sets. Now if one presumes model \mathcal{E}^{in} is smooth, then due to assumption $\bar{x}_i \in \text{int}X_i$ and from the perfectness of contracts from V^σ and their upper stability in the reduced model, one can in a standard manner conclude that³ there is a vector-price p_σ such that

$$\forall i \in \mathcal{I} \quad \exists \lambda_i > 0 : \quad p_\sigma = (p_\sigma^1, \dots, p_\sigma^l) = \lambda_i \nabla_{|x_i^\sigma} u_i(\bar{x}_i) \quad (3.2.1)$$

and $(\bar{x}_i^\sigma)_{i \in \mathcal{I}}$ is equilibrium relative to x_i^σ and subject to fixed $\bar{x}_i^{-\sigma}$, where $\bar{x}_i^{-\sigma} = (\bar{x}_i^0, \dots, \bar{x}_i^{\sigma-1}, \bar{x}_i^{\sigma+1}, \dots, \bar{x}_i^s)$, and $\nabla_{|x_i^\sigma} u_i(\bar{x}_i)$ denotes the subvector of the gradient of utility function, calculated at the point \bar{x}_i , and corresponding to the state $\sigma \geq 1$. Therefore, due to assumptions, the budget equalities are fulfilled for \bar{x}_i^σ , *i.e.*,

$$p_\sigma \bar{x}_i^\sigma = p_\sigma \mathbf{e}_i^\sigma + p_\sigma A_\sigma z_i, \quad \sigma = 1, \dots, s,$$

for $z_i = \Delta z_i$ and each i . Now denote the total vector-price in future markets by

$$p^1 = (p_\sigma)_{\sigma=1}^{\sigma=s}, \quad p_\sigma \in \mathbb{R}^l, \quad \sigma \geq 1.$$

Define

$$H = H(p^1) = \{x \in \mathfrak{L} = \mathbb{R}^{nl(s+1)} \mid \exists z \in \mathbb{R}^{nk} : \sum_{i \in \mathcal{I}} z_i = 0 \quad \& \\ p_\sigma x_i^\sigma - p_\sigma \mathbf{e}_i^\sigma = p_\sigma A_\sigma z_i, \quad \forall \sigma = 1, \dots, s, \quad \forall i \in \mathcal{I}\}.$$

By construction we have $\bar{x} \in H$. Now put

$$\mathcal{H}_i = \mathcal{H}_i(p^1) = \{x_i \in \mathbb{R}^{l(s+1)} \mid \exists z_i \in \mathbb{R}^k : p_\sigma x_i^\sigma - p_\sigma \mathbf{e}_i^\sigma = p_\sigma A_\sigma z_i, \quad \sigma = 1, \dots, s\};$$

this is (in fact) the projection of subspace H onto a subspace corresponding to agent i 's consumption bundles. Clearly,

$$\mathcal{H}_i = \mathcal{H} + \mathbf{e}_i, \quad \mathcal{H} = \{y \in \mathbb{R}^{l(s+1)} \mid \exists z \in \mathbb{R}^k : p_\sigma y = p_\sigma A_\sigma z, \quad \forall \sigma \geq 1\}. \quad (3.2.2)$$

takes place for all i .

The useful properties of incomplete market complex-contractual allocations (more exactly, for semi-perfectly contractual and therefore for proper-perfectly contractual) are stated in the following lemma.

³Note that for this assumption, **(S)** plays an important role.

Lemma 3.2.1 *Let \mathcal{E}^{in} be a smooth incomplete market, an allocation $\bar{x} \in \text{int}X \cap \mathcal{D}^{sp}(\mathcal{E}^{in})$ and $p^1 = (p_\sigma)_{\sigma=1}^{s=s}$ be the prices (3.2.1) found in the above arguments. Then*

- (i) $\bar{x} \in H$ and does not leave space H after an appropriate breaking of contracts of goods and assets,
- (ii) \bar{x} is not Pareto-dominated via an allocation from the space H ,
- (iii) \bar{x} is not Pareto-dominated via an allocation from the space $L_{\bar{x}}^\sigma = \{y = (y_i)_{i \in \mathcal{I}} \in L^\mathcal{I} \mid y_i^{-\sigma} = \bar{x}_i^{-\sigma}, \forall i \in \mathcal{I}, \forall \sigma \geq 0\}$.

Proof of Lemma 3.2.1. The first part of item (i) Lemma 3.2.1 is obvious. To check the second part, recall that for every contractual $x = (x_i)_{i \in \mathcal{I}}$, $x_i = (x_i^0, \dots, x_i^s)$, the representation (3.1.3) or equivalent relation (3.1.4) takes place. Further, as soon as $(\bar{x}_i^\sigma)_{i \in \mathcal{I}}$ is an equilibrium allocation for reduced onto $\sigma = 1, \dots, s$ economy \mathcal{E}^σ equipped with endowments $(\mathbf{e}_i^\sigma + A_\sigma \Delta z_i(W))$ (it was noted above and follows from Theorem 1.2.2), then

$$p_\sigma \bar{x}_i^\sigma = p_\sigma \mathbf{e}_i^\sigma + p_\sigma A_\sigma \Delta z_i, \quad (3.2.3)$$

and $\bar{x}_i \in \mathcal{H}_i$ for all i . Moreover, the equilibrium properties of \bar{x}^σ and Theorem 1.2.2 imply $p_\sigma v_i^\sigma = 0$, $v^\sigma \in V^\sigma$ for all i and $\sigma \geq 1$. Now if one breaks a part of contract $v^\sigma \in V^\sigma$, then the new allocation $(\hat{x}_i^\sigma)_{i \in \mathcal{I}}$ satisfies the system (3.2.3). Therefore $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n) \in H$. Note that breaking of a share of contracts in the present touches the exchanges of assets and the agents just realize the new allocation, for which the condition to be in H is realized for a new $\Delta \tilde{z}_i$. The last one ends with the checking of (i).

To see that statement (ii) is true, let us presume that some semi-perfectly contractual x is Pareto-dominated by allocation $y \in H$. Now since $x, y \in H$, then there are such z, z' , that the following equalities are true:

$$p_\sigma x_i^\sigma - p_\sigma \mathbf{e}_i^\sigma = p_\sigma A_\sigma z_i, \quad \sigma = 1, \dots, s, \quad i \in \mathcal{I},$$

$$p_\sigma y_i^\sigma - p_\sigma \mathbf{e}_i^\sigma = p_\sigma A_\sigma z'_i, \quad \sigma = 1, \dots, s, \quad i \in \mathcal{I}.$$

As soon as all contracts from V^σ , $\sigma \geq 1$ in the web $V = \bigcup_{\sigma=1}^s V^\sigma \cup W$, which realizes $x = x(V)$, are perfect, due to perfect contract definition one may change contracts related to the future states of the world and realize x^σ by the (proper) web, which consists of two contracts— v'^σ and v''^σ , defined by formulas:

$$v'_i{}^\sigma = y_i^\sigma - \mathbf{e}_i^\sigma - A_\sigma z'_i, \quad i \in \mathcal{I},$$

$$v''_i{}^\sigma = x_i^\sigma - y_i^\sigma - A_\sigma(z_i - z'_i), \quad i \in \mathcal{I},$$

for all $\sigma = 1, \dots, s$, and saving “old” contracts in the present. This is a web due to the fact that the consumption sets are rectangular. Now due to Proposition 1.1.1 and its corollaries, to check the properness of v'^σ and v''^σ it is enough to verify that $v'_i{}^\sigma p_\sigma = 0$ and $v''_i{}^\sigma p_\sigma = 0$ for all i , which we already have. In view of Definition 3.1.3, the new web of contracts has to be stable relative to the *simultaneous* procedure of contracts

breaking and signing a new contract in the “*present*.” Now one can break (as a whole) the contracts of the second type and all contracts in the present and sign the new contract $\hat{w}^0 = (y^0 - e^0, z')$ for $\sigma = 0$. In so doing the agents can realize allocation y , which contradicts the definition of semi-perfectly (first) and proper-perfectly (second) contractual allocations.

Item (iii) follows from the definition of semi-perfectly contractual allocation and from Theorem 1.2.2. ■

The items (ii) and (iii) of this lemma and the above considerations induce the following terminology.

An allocation $x \in \mathcal{A}(X)$ is called σ -Pareto optimal, $\sigma = 0, \dots, s$, if it is not Pareto dominated via an allocation $y \in \mathcal{A}(X)$ from the space

$$L_{\bar{x}}^{\sigma} = \{y = (y_i)_{\mathcal{I}} \in L^{\mathcal{I}} \mid y_i^{-\sigma} = \bar{x}_i^{-\sigma}, \forall i \in \mathcal{I}\}.$$

An allocation, which is σ -Pareto optimal for every $\sigma \geq 0$, is called *partially Pareto optimal*.

Let $x = (x_{\sigma})_{\sigma=0}^{\sigma=s} \in \mathcal{A}(X)$ be a σ -Pareto optimal allocation. The *nonzero* vector (functional) $p_{\sigma} \in \mathbb{R}^l$ is called σ -Pareto prices if

$$p_{\sigma} y_i^{\sigma} > p_{\sigma} \bar{x}_i^{\sigma}, \quad \forall (y_i^{\sigma}, \bar{x}_i^{-\sigma}) \in \mathcal{P}_i(\bar{x}_i), \quad \forall i \in \mathcal{I}. \quad (3.2.4)$$

Notice that for smooth preferences and if $x \in \text{int}X$, relation (3.2.4) is equivalent to the existence of $\gamma_i^{\sigma} > 0$, satisfying

$$\nabla_{|x_i^{\sigma}} u_i(x_i) = \gamma_i^{\sigma} p_{\sigma}, \quad \forall i \in \mathcal{I}. \quad (3.2.5)$$

A collection of vectors $(p_{\sigma})_{\sigma=0}^{\sigma=s}$, $p_{\sigma} \in \mathbb{R}^l$ is called (partial) *Pareto prices* if (3.2.4) is true for all $\sigma = 0, \dots, s$.

An allocation from H is called *Pareto H -optimal* if it cannot be Pareto-dominated via an allocation from $H = H(p^1)$.⁴ Using (3.2.2) in a standard manner, one can see that an allocation is Pareto H -optimal if and only if it cannot be Pareto-dominated via an allocation $y \in \mathcal{A}(X)$, for which $y - e \in \mathcal{H}^{\mathcal{I}}$, and this is the specific form of *constrained Pareto optimality*.

The following lemma is the most technically difficult and important for the consequent analysis, the lemma states the key properties of H -optimal allocations. The “involved” proof is presented at the end of this subsection.

Lemma 3.2.2 *Let \mathcal{E}^{in} be an incomplete market, $\bar{x} \in \text{int}X$ be an allocation and $p^1 = (p_{\sigma})_{\sigma=1}^{\sigma=s}$ be prices for future states of the world. Let $i_0 \in \mathcal{I}$ be an arbitrarily chosen and fixed agent. Then $\bar{x} = (\bar{x}_i)_{\mathcal{I}} \in H(p^1) \cap \mathcal{A}(\mathcal{E})$ is Pareto $H(p^1)$ -optimal if and only if the following property is true.*

There exists $\bar{p} = (\bar{p}_0, \bar{p}_1, \dots, \bar{p}_s)$ such that $\bar{p}_{\sigma} \neq 0$ for all $\sigma = 0, \dots, s$ and

$$\bar{p}_i y_i > \bar{p}_i \bar{x}_i \quad \forall y_i \in \mathcal{P}_i(\bar{x}_i) \mid \exists z_i \in \mathbb{R}^k : p_{\sigma}(y_i^{\sigma} - e_i^{\sigma}) = p_{\sigma} A_{\sigma} z_i, \quad \forall \sigma \geq 1 \quad (3.2.6)$$

⁴Notice that now p^1 may not be partially Pareto prices.

is true for all $i \in \mathcal{I}$, $i \neq i_0$. For i_0 , a stronger property is true:

$$\bar{p}y_{i_0} > \bar{p}\bar{x}_{i_0}, \quad \forall y_{i_0} \in \mathcal{P}_{i_0}(\bar{x}_{i_0}). \quad (3.2.7)$$

Note that the analysis of the lemma proof shows that $\bar{x} \in \text{int}X$ is essential just to obtain the strict inequality in relation (3.2.6). The next corollary gives us a convenient reformulation of Lemma 3.2.2 for a smooth case.

Corollary 3.2.1 *In Lemma 3.2.2 conditions, let us assume that \mathcal{E}^{in} is a smooth market and let $u_i(\cdot)$ be a utility function for $i \in \mathcal{I}$. Then for allocation \bar{x} to be Pareto $H(p^1)$ -optimal, the following property is necessary and sufficient. Let*

$$\bar{p} = \nabla u_{i_0}(\bar{x}_{i_0})$$

for some i_0 . Then for all $i \neq i_0$ and each $\sigma \geq 1$, there exist real $\alpha_i > 0$ and λ_i^σ , $\sigma \geq 1$, such that

$$\begin{aligned} \nabla_{|_{\bar{x}_i}} u_i(\bar{x}_i) &= \alpha_i \bar{p}_0, \\ \nabla_{|_{x_i^\sigma}} u_i(\bar{x}_i) &= \alpha_i \bar{p}_\sigma + \lambda_i^\sigma p_\sigma, \quad \forall \sigma \geq 1 \end{aligned}$$

hold and, moreover, $\sum_{\sigma=1}^{\sigma=s} \lambda_i^\sigma p_\sigma A_\sigma = 0$ is fulfilled.

Proof of Corollary 3.2.1. We have to consider the smooth case in the context of Lemma 3.2.2. On the necessary side, for the existence of values $\alpha_i > 0$ and $\lambda_i^\sigma \forall \sigma \geq 1$, one can state it directly from relations (3.2.6) and (3.2.7), applying separation theorem (or simply from the necessary conditions of a convex programming problem). However, the easiest way to see it may be found from condition $\bar{x} \in \text{int}X$ and relations (3.2.10), stated in the proof of Lemma 3.2.2. From this, we conclude in a standard way the existence of such $\alpha_i > 0$, that $\nabla u_i(\bar{x}_i) = \alpha_i f_i$ ($\alpha_i \neq 0$ due to $\nabla u_i(\bar{x}_i) \neq 0$). Finally, one needs to apply (3.2.11).

To state the sufficiency, let us show that relations (3.2.6) and (3.2.7) are true. For some i and $y_i \in \mathcal{P}_i(\bar{x}_i)$, assume $\exists z_i \in \mathbb{R}^k : p_\sigma(y_i^\sigma - \mathbf{e}_i^\sigma) = p_\sigma A_\sigma z_i \forall \sigma \geq 1$. Due to gradient's properties for interior points we have

$$\langle \nabla u_i(\bar{x}_i), y_i \rangle > \langle \nabla u_i(\bar{x}_i), \bar{x}_i \rangle \quad \forall i \in \mathcal{I}.$$

Now substituting the gradient presentation given in the corollary conditions, one can conclude

$$\alpha_i \bar{p}y_i + \sum_{\sigma=1}^s \lambda_i^\sigma p_\sigma y_i^\sigma > \alpha_i \bar{p}\bar{x}_i + \sum_{\sigma=1}^s \lambda_i^\sigma p_\sigma \bar{x}_i^\sigma.$$

However, there are $z_i, \bar{z}_i \in \mathbb{R}^k$, such that $p_\sigma y_i^\sigma = p_\sigma \mathbf{e}_i^\sigma + p_\sigma A_\sigma z_i$ & $p_\sigma \bar{x}_i^\sigma = p_\sigma \mathbf{e}_i^\sigma + p_\sigma A_\sigma \bar{z}_i \forall \sigma \geq 1$. Substituting these expressions under summation in the formula, one can find

$$\alpha_i \bar{p}y_i + \sum_{\sigma=1}^s p_\sigma \mathbf{e}_i^\sigma + \left(\sum_{\sigma=1}^s \lambda_i^\sigma p_\sigma A_\sigma \right) z_i > \alpha_i \bar{p}\bar{x}_i + \sum_{\sigma=1}^s p_\sigma \mathbf{e}_i^\sigma + \left(\sum_{\sigma=1}^s \lambda_i^\sigma p_\sigma A_\sigma \right) \bar{z}_i,$$

that due to $\sum_{\sigma=1}^s \lambda_i^\sigma p_\sigma A_\sigma = 0$ and $\alpha_i > 0$ gives the result. \blacksquare

Let us consider a smooth economy when $\bar{x} \in \text{int}X$ is also Pareto optimal in each future market and (nonzero) spot prices satisfy $p_\sigma = \gamma_i^\sigma \nabla_{|x_i^\sigma} u_i(\bar{x}_i)$, $\gamma_i^\sigma > 0$, *i.e.*, prices are (uniquely) derived from necessary optimal conditions (hence they are Pareto prices). Now applying Lemma 3.2.2 and its corollary, one can immediately conclude

Corollary 3.2.2 *Let \mathcal{E}^{in} be a smooth incomplete market and $\bar{x} \in \text{int}X$. Suppose \bar{x} be a partially Pareto optimal and let $p^1 = (p_\sigma)_{\sigma=1}^{\sigma=s}$ be a bundle of σ -Pareto prices (*i.e.*, (3.2.4) is true for $\sigma \geq 1$). Presume also that $\bar{x} = (\bar{x}_i)_{\mathcal{I}} \in H(p^1)$ and is Pareto $H(p^1)$ -optimal. Then there exists a vector $\bar{p} = (\bar{p}_0, \bar{p}_1, \dots, \bar{p}_s)$ such that $\bar{p}_\sigma = \beta_\sigma p_\sigma$ for some $\beta_\sigma > 0$ and all $\sigma \geq 1$, so that for $\bar{q} = \sum_{\sigma=1}^{\sigma=s} \bar{p}_\sigma A_\sigma$ and every $i \in \mathcal{I}$*

$$\bar{p}_0 y_i^0 - \bar{p}_0 \mathbf{e}_i^0 + \bar{q} z_i > \bar{p}_0 \bar{x}_i^0 - \bar{p}_0 \mathbf{e}_i^0 + \bar{q} \bar{z}_i, \quad \forall y_i \in \mathcal{P}_i(\bar{x}_i) \quad (3.2.8)$$

is true for all $z_i, \bar{z}_i \in \mathbb{R}^k$, which satisfy

$$\bar{p}_\sigma (y_i^\sigma - \mathbf{e}_i^\sigma) = \bar{p}_\sigma A_\sigma z_i \quad \& \quad \bar{p}_\sigma (\bar{x}_i^\sigma - \mathbf{e}_i^\sigma) = \bar{p}_\sigma A_\sigma \bar{z}_i, \quad \sigma = 1, \dots, s.$$

Proof of Corollary 3.2.2. To verify this corollary first note that since $(p_\sigma)_{\sigma=1}^{\sigma=s}$ is a bundle of σ -Pareto prices, in corollary conditions (3.2.5) is true. Now let us take vector $\bar{p} = \nabla u_{i_0}(x_{i_0})$ for $i_0 \in \mathcal{I}$ from the statement of Lemma 3.2.2 and via (3.2.5) put $\beta_\sigma = \gamma_{i_0}^\sigma > 0$. Now it is easy to see that in these lemma conditions we have $\mathcal{H}_i(p^1) = \mathcal{H}_i(\bar{p}^1)$, *i.e.*, in the right-hand side of (3.2.6) one can equivalently change vector p_σ by \bar{p}_σ for all $\sigma \geq 1$. Now rewrite the inequality from the left-hand side of (3.2.6) in the form $\bar{p} y_i - \bar{p} \mathbf{e}_i > \bar{p} \bar{x}_i - \bar{p} \mathbf{e}_i$ and substitute the following representations:

$$\sum_{\sigma=1}^s (\bar{p} y_i^\sigma - \bar{p} \mathbf{e}_i^\sigma) = \sum_{\sigma=1}^s \bar{p}_\sigma A_\sigma z_i = \bar{q} z_i \quad \& \quad \sum_{\sigma=1}^s (\bar{p} \bar{x}_i^\sigma - \bar{p} \mathbf{e}_i^\sigma) = \sum_{\sigma=1}^s \bar{p}_\sigma A_\sigma \bar{z}_i = \bar{q} \bar{z}_i.$$

This proves the result. \blacksquare

Proof of Lemma 3.2.2. Let us write in matrix form the conditions that define the allocations from H . In fact for *feasible* $x \in H$, there are $z_i \in \mathbb{R}^k$, $i \in \mathcal{I}$ such that

$$\begin{cases} \sum_{i=1}^n x_i = \sum_{i=1}^n \mathbf{e}_i; \\ \sum_{i=1}^n z_i = 0; \\ p_\sigma x_i^\sigma - p_\sigma A_\sigma z_i = p_\sigma \mathbf{e}_i^\sigma, \quad \sigma = 1, \dots, s \end{cases}$$

holds. Notice that if balance relations and budget constraints $p_\sigma x_i^\sigma - p_\sigma A_\sigma z_i = p_\sigma \mathbf{e}_i^\sigma$ are satisfied for some fixed $\sigma \geq 1$ and all $i \in \mathcal{I} \setminus \{i_0\}$, then the last budget constraint is also true automatically. This is why all agent i_0 's budget constraints, being linear dependent, may be removed from the system of linear equations defining the space H . One may think without loss of generality that $i_0 = n$. Denote by B the matrix

$$\begin{pmatrix} E_l & 0 & \dots & 0 & \dots & E_l & 0 & \dots & 0 & E_l & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & E_l & \dots & 0 & \dots & 0 & E_l & \dots & 0 & 0 & E_l & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & E_l & \dots & 0 & 0 & \dots & E_l & 0 & 0 & \dots & E_l & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & E_k & \dots & E_k & E_k \\ 0 & p_1 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -p_1 A_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & p_s & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -p_s A_s & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & p_1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & -p_1 A_1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & p_s & 0 & 0 & \dots & 0 & 0 & \dots & -p_s A_s & 0 \end{pmatrix}$$

Here in the standard manner E_l and E_k denote the unit matrices of an appropriate size and $p_1, p_s, p_1 A_1$ and $p_s A_s$ are row-vectors. Clearly, we have the following equivalence: *feasible* $x \in H \iff$ *there exists* $z = (z_1, \dots, z_n)$ *such that*

$$B * \begin{pmatrix} x_1^0 \\ \vdots \\ x_1^s \\ \vdots \\ x_n^0 \\ \vdots \\ x_n^s \\ z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \bar{e}^0 \\ \vdots \\ \bar{e}^s \\ \mathbf{0}_k \\ p_1 \mathbf{e}_1^1 \\ \vdots \\ p_s \mathbf{e}_1^s \\ \vdots \\ p_1 \mathbf{e}_{n-1}^1 \\ \vdots \\ p_s \mathbf{e}_{n-1}^s \end{pmatrix}.$$

Further let us consider the subspace

$$H^Z = \{((x_1, z_1), \dots, (x_n, z_n)) \in (\mathbb{R}^{l(s+1)} \times \mathbb{R}^k)^{\mathcal{I}} \mid B(x_1, \dots, x_n, z_1, \dots, z_n) = 0\};$$

this is the *kernel* of operator $B(\cdot)$, in which the order of components is changed for the convenience of the below considerations. Due to Lemma 3.2.1, the allocation $\bar{x} \in H$ and is Pareto-optimal relative to H . Therefore,

$$\prod_{\mathcal{I}} [(\mathcal{P}_i(\bar{x}_i) - \bar{x}_i) \times \mathbb{R}^k] \cap H^Z = \emptyset$$

takes place (note that via **(S)** each of these sets is nonempty). Now by the separation theorem we may find such linear functional $f = (f_1, \dots, f_n) \neq 0, f_i = (f_i^x, f_i^z) \in \mathbb{R}^{l(s+1)+k}$ that

$$\langle f, \prod_{\mathcal{I}} [(\mathcal{P}_i(\bar{x}) - \bar{x}_i) \times \mathbb{R}^k] \rangle \geq \langle f, H^Z \rangle$$

holds. Notice the functional f is constant (and hence is equal to zero) onto subspace H^Z , since the right-hand side of the last inequality is bounded. Therefore,

$$\langle f, \prod_{\mathcal{I}} [(\mathcal{P}_i(\bar{x}) - \bar{x}_i) \times \mathbb{R}^k] \rangle \geq 0 \quad (3.2.9)$$

is true. Let us show further that $f_i^z = 0$, $i \in \mathcal{I}$, *i.e.*,

$$f_i = (f_i^0, \dots, f_i^s, \underbrace{0, \dots, 0}_k)$$

holds for every $i \in \mathcal{I}$. In fact, consider fixed $\hat{x} = (\hat{x}_1, \dots, \hat{x}_I)$, $\hat{x}_i \in (\mathcal{P}_i(\bar{x}) - \bar{x}_i)$, $i \in \mathcal{I}$, $\hat{u}_j \in \mathbb{R}^k$, $j \neq i_0$ for some $i_0 \in \mathcal{I}$. In view of (3.2.9), for any $u \in \mathbb{R}^k$ we have

$$\sum_{j \neq i_0} \langle f_j, (\hat{x}_j, \hat{u}_j) \rangle + \langle f_{i_0}, (\hat{x}_{i_0}, u) \rangle \geq 0,$$

which is possible only if $f_{i_0}^z = 0$ and in view of the arbitrariness of i_0 , for all $i_0 \in \mathcal{I}$. For the convenience of the below notations I will identify the functional f_i with f_i^x , *i.e.*, by convention let us put $f_i = (f_i^x, 0) = (f_i, 0)$.

Let us show further that

$$\langle f_i, (\mathcal{P}_i(\bar{x}) - \bar{x}_i) \rangle \geq 0, \quad i \in \mathcal{I}, \quad (3.2.10)$$

and moreover, if $f_i \neq 0$ and $\bar{x}_i \in \text{int}X_i$, then the inequality is strict. Indeed, due to the preferences are locally non-satiated, we have $\bar{x}_j \in \text{cl}(\mathcal{P}_j(\bar{x}_j))$; now the substitution of \bar{x}_j instead of $\mathcal{P}_j(\bar{x}_j)$ in (3.2.9) for all $j \neq i$ and due to $f_j^z = 0$, $j \in \mathcal{I}$ immediately gives us the result. So, (3.2.10) is true in the non-strict form of inequalities. Now **(A)** and $\bar{x}_i \in \text{int}X_i$ for $f_i \neq 0$ standardly implies the strict inequalities.

Further, the fact that functional $f = (f_1, \dots, f_n)$ is constant onto subspace H^Z implies that this functional can be represented as a linear combination of the vector-rows of matrix B. Now using the structure of matrix B, one can conclude the existence of such real λ_i^σ , $\sigma \geq 1$ $i \in \mathcal{I}$, $i \neq n$ and such vectors $q \in \mathbb{R}^k$, $\bar{p} = (\bar{p}_0, \dots, \bar{p}_s) \in L'$, that for all $i \neq n$ the following system of linear equations is true:

$$\begin{cases} f_i^0 = \bar{p}_0, \\ f_i^\sigma = \bar{p}_\sigma + \lambda_i^\sigma p_\sigma, & \sigma \geq 1, \\ f_i^z = 0 = -q + \sum_{\sigma=1}^s \lambda_i^\sigma p_\sigma A_\sigma, \end{cases} \quad (3.2.11)$$

and for $i = n$ we have $f_n^z = 0 = -q + 0$ and $f_n^x = \bar{p}$. Putting $\lambda_n^\sigma = 0$ for all $\sigma \geq 1$, one may think (3.2.11) is true for all $i \in \mathcal{I}$. Moreover, system (3.2.11) implies that $q = \sum_{\sigma=1}^s \lambda_i^\sigma p_\sigma A_\sigma = 0$ for all i . Note also that assumption $\bar{p}_0 = 0$ contradicts **(S)**, the local-nonsatiation in each spot market (and therefore for $\sigma = 0$)⁵. Therefore, $f_i \neq 0$

⁵Due to \mathcal{H}_i specification, if $y_i \in \mathcal{H}_i$ then for $\bar{y}_i = (\bar{y}_i^\sigma)_{\sigma=0}^s$, where $\bar{y}_i^\sigma = y_i^\sigma$ for $\sigma \geq 1$, we have $\bar{y}_i \in \mathcal{H}_i$ for every \bar{y}_i^0 , and therefore $\mathcal{P}_i(\bar{x}) \cap \mathcal{H}_i \neq \emptyset$ for all i .

and on the right-hand side of (3.2.10) we have a *strict inequality for all i* . Moreover, as soon as $f_n = \bar{p} \neq 0$, due to the same arguments—from the local-non-satiation agent n in each future spot market—we conclude that $\bar{p}_\sigma \neq 0$ for all σ . It is also clear that due to (3.2.10) and (S), we have $f_i^\sigma \neq 0$ for all i and σ .

Now let us show that \bar{p} satisfies the other requirements of Lemma 3.2.2. First note that by subspaces \mathcal{H}_i specification for every $x_i \in \mathcal{H}_i$ we have

$$p_\sigma(x_i^\sigma - \mathbf{e}_i^\sigma) = p_\sigma A_\sigma z_i, \quad \sigma = 1, \dots, s$$

for some $z_i \in \mathbb{R}^k$. Now multiplying equalities on λ_i^σ and then summing them by $\sigma = 1, \dots, s$, one obtains

$$\sum_{\sigma=1}^s \lambda_i^\sigma p_\sigma (x_i^\sigma - \mathbf{e}_i^\sigma) = \sum_{\sigma=1}^s \lambda_i^\sigma p_\sigma A_\sigma z_i = \left(\sum_{\sigma=1}^s \lambda_i^\sigma p_\sigma A_\sigma \right) z_i = q z_i. \quad (3.2.12)$$

Next let us recall that due to (3.2.10) and the above considerations we also have

$$\langle f_i, (\mathcal{P}_i(\bar{x}) - \bar{x}_i) \rangle > 0 \implies \langle f_i, (y_i - \bar{x}_i) \rangle > 0 \quad \forall y_i \in \mathcal{P}_i(\bar{x}) \cap \mathcal{H}_i \neq \emptyset \quad (3.2.13)$$

for all $i \in \mathcal{I}$. Now substituting the representation of f_i from (3.2.11), we obtain

$$\bar{p}_0(y_i^0 - \bar{x}_i^0) + \sum_{\sigma=1}^s \langle (\bar{p}_\sigma + \lambda_i^\sigma p_\sigma), (y_i^\sigma - \bar{x}_i^\sigma) \rangle > 0 \quad \forall y_i \in \mathcal{P}_i(\bar{x}) \cap \mathcal{H}_i.$$

Subtracting from the left and right-hand sides of the inequality the value $\sum_{\sigma=1}^s \lambda_i^\sigma (p_\sigma \mathbf{e}_i^\sigma)$, after transformations, we obtain

$$\bar{p}_0 y_i^0 + \sum_{\sigma=1}^s \bar{p}_\sigma y_i^\sigma + \sum_{\sigma=1}^s \lambda_i^\sigma p_\sigma (y_i^\sigma - \mathbf{e}_i^\sigma) > \bar{p}_0 \bar{x}_i^0 + \sum_{\sigma=1}^s \bar{p}_\sigma \bar{x}_i^\sigma + \sum_{\sigma=1}^s \lambda_i^\sigma p_\sigma (\bar{x}_i^\sigma - \mathbf{e}_i^\sigma).$$

Since $\bar{x}_i, y_i \in \mathcal{H}_i$, there are such $\bar{z}_i = \bar{z}_i(\bar{x}_i)$ $z_i = z_i(y_i)$ that relations (3.2.12) are true. Now due to the previous inequality, we get

$$\langle \bar{p}, y_i \rangle + q z_i > \langle \bar{p}, \bar{x}_i \rangle + q \bar{z}_i.$$

However, from the last equation of (3.2.11) we have $q = f_n^z = 0$, which gives

$$\langle \bar{p}, y_i \rangle > \langle \bar{p}, \bar{x}_i \rangle \quad \forall y_i \in \mathcal{P}_i(\bar{x}) \cap \mathcal{H}_i \iff \langle \bar{p}, ((\mathcal{P}_i(\bar{x}) \cap \mathcal{H}_i) - \bar{x}_i) \rangle > 0. \quad (3.2.14)$$

The sufficiency of relations (3.2.6) and (3.2.7) for an allocation $\bar{x} \in \mathcal{A}(X) \cap H(p^1)$ to be Pareto $H(p^1)$ -optimal is stated quite standardly. In fact, let $y = (y_i)_{\mathcal{I}} \in \mathcal{A}(X) \cap H(p^1)$ be such that $y_i \succ_i \bar{x}_i$ is true for all i . Then as soon as the right-hand side in (3.2.6) is fulfilled for $y = (y_i)_{\mathcal{I}} \in H(p^1)$ due to $H(p^1)$ determination, via the left-hand part of (3.2.6), we can conclude $\bar{p} y_i > \bar{p} \bar{x}_i$ for all i . Now, summing inequalities over i one finds $\bar{p} \sum_{i \in \mathcal{I}} y_i > \bar{p} \sum_{i \in \mathcal{I}} \bar{x}_i$. Since $\sum_{i \in \mathcal{I}} y_i = \sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$, we are coming to a contradiction. Lemma 3.2.2 is proved. \blacksquare

3.2.2 Contractual characterization of GEI -equilibria

The next theorem presents one of the most meaningful results of this paper. This theorem states the equivalence between proper-perfectly contractual allocations of an incomplete market and GEI -equilibria.

Theorem 3.2.1 *Let \mathcal{E}^{in} be a smooth incomplete market. Then*

$$\text{int}X \cap \mathcal{D}^{cp}(\mathcal{E}^{in}) = W(\mathcal{E}^{in}) \cap \text{int}X$$

holds, where $\mathcal{D}^{cp}(\mathcal{E}^{in})$ denotes the set of all proper-perfectly contractual allocations and $W(\mathcal{E}^{in})$ is the set of GEI -equilibrium allocations.

Using the individual rationality of equilibrium allocations one directly yields

Corollary 3.2.3 *If \mathcal{E}^{in} is a smooth incomplete market, and $\bar{\mathcal{P}}_i(\mathbf{e}_i) \subset \text{int}X_i$ for all $i \in \mathcal{I}$, then*

$$\mathcal{D}^{cp}(\mathcal{E}^{in}) = W(\mathcal{E}^{in}),$$

i.e., an allocation is a GEI -equilibrium if and only if this allocation is proper-perfectly contractual.

Proof of Theorem 3.2.1. To check the inclusion

$$W(\mathcal{E}^{in}) \cap \text{int}X \subset \text{int}X \cap \mathcal{D}^{cp}(\mathcal{E}^{in}),$$

take some $x \in W(\mathcal{E}^{in}) \cap \text{int}X$ and consider the web $V = \{v^\sigma\}_{\sigma=0}^{\sigma=s}$, where

$$v^0 = (x^0 - \mathbf{e}^0, z), \quad v^\sigma = (x^\sigma - \mathbf{e}^\sigma - A_\sigma z), \quad \sigma = 1, \dots, s$$

and by convention $A_\sigma z = (A_\sigma z_i)_{i \in \mathcal{I}}$ for the appropriate portfolios z_i existing due to the GEI -definition. Due to Theorem 1.2.2, Proposition 3.1.1, and the GEI -equilibrium specification, it is easy to see that v^0 is proper and v^σ are perfect contracts for all $\sigma \geq 1$. Also, due to the equilibrium specification, one can easily see that this web satisfies the condition of contracts' common stability by Definition 3.1.4. In fact, due to Proposition 3.1.1 in adjusted prices contract v^0 has zero value and it will be so also for every contract $v^\sigma = \hat{v}^\sigma + \check{v}^\sigma$; moreover, if \hat{v}^σ and \check{v}^σ are proper, then $p_\sigma \hat{v}_i^\sigma = p_\sigma \check{v}_i^\sigma = 0 \forall i \in \mathcal{I}$ also hold. Thus an allocation $y(\hat{V})$, implemented by a web \hat{V} , obtained via a partial loss $0 \leq \lambda \leq 1$ of a contract volume in present and the break of some proper contracts for future states, has the same value as an initial allocation:

$$px_i(V) = \sum_{\sigma=0}^s p_\sigma x_i^\sigma = \sum_{\sigma=0}^s p_\sigma y_i^\sigma = py_i(\hat{V}) = p\mathbf{e}_i, \quad i \in \mathcal{I}.$$

Suppose coalition $S \subseteq \mathcal{I}$ signs a new contract (w, z') , $\sum_S w_i = 0$, $\sum_S z'_i = 0$ such that plans

$$\xi_i^0 = y_i^0 + w_i, \quad \xi_i^\sigma = y_i^\sigma + A_\sigma z'_i, \quad \sigma \geq 1,$$

satisfy $\xi_i \succ_i x_i$, $i \in S$. Now by equilibrium properties, ξ_i cannot be budget feasible in spite of $p_\sigma \xi_i^\sigma = p_\sigma y_i^\sigma + p_\sigma A_\sigma z'_i = p_\sigma \mathbf{e}_i^\sigma + p_\sigma A_\sigma (z'_i + (1 - \lambda)z_i)$. Therefore,

consumption program $(\xi_i, z'_i + (1 - \lambda)z_i)$ violates budget constraint in present, *i.e.* $p_0\xi_i^0 > p_0\mathbf{e}_i^0 - q(z'_i + (1 - \lambda)z_i)$. Now due to budget equalities for future states and no-arbitrage for assets prices in equilibrium (3.1.2), one concludes

$$p\xi_i = \sum_{\sigma=0}^s p_0\xi_i^{\sigma} > \sum_{\sigma=0}^s p_0\mathbf{e}_i^{\sigma} + \left(\sum_{\sigma=1}^s p_{\sigma}A_{\sigma} - q\right)z'_i + (1 - \lambda)\left(\sum_{\sigma=1}^s p_{\sigma}A_{\sigma} - q\right)z_i = p\mathbf{e}_i.$$

Summing these inequalities over $i \in S$ one obtains

$$\begin{aligned} \sum_S p\xi_i &= \sum_S py_i + \sum_S p_0w_i + \sum_S \left(\sum_{\sigma=1}^s p_{\sigma}A_{\sigma}\right)z'_i = \\ &= p \sum_S \mathbf{e}_i + p_0 \sum_S w_i + \left(\sum_{\sigma=1}^s p_{\sigma}A_{\sigma}\right)\left(\sum_S z'_i\right) > p \sum_S \mathbf{e}_i. \end{aligned}$$

However, this contradicts the definition of a contract, because we have $\sum_S w_i = 0$ & $\sum_S z'_i = 0$. This contradiction proves \subseteq .

Let us check the inverse inclusion. Let $\bar{x} \in \text{int}X \cap \mathcal{D}^{cp}(\mathcal{E}^{in})$. By definition, there is a weak stable web of contracts $V = \bigcup_{\sigma=1}^s V^{\sigma} \cup W$, realizing the allocation $\bar{x} = x(V)$, so that all contracts from V^{σ} are perfect, all contracts from W are proper, and the web is stable relative to the simultaneous procedure of breaking (corresponding to the type of contract) and signing new ones in the “present”.

Since we always have $\mathcal{D}^{cp}(\mathcal{E}^{in}) \subset \mathcal{D}^{sp}(\mathcal{E}^{in})$, then due to Lemma 3.2.1 and condition $\bar{x} \in \text{int}X \cap \mathcal{D}^{cp}(\mathcal{E}^{in})$, one can conclude that the conditions of Lemma 3.2.2 and its Corollary 3.2.2 are satisfied. Now let $\bar{p} = (\bar{p}_0, \dots, \bar{p}_s)$ be a vector which due to Corollary 3.2.2 corresponds (uniquely) to the allocation \bar{x} and satisfies (3.2.8). We have to show that there exists \bar{z} such that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$, where $\bar{q} = \sum_{\sigma=1}^s \bar{p}_{\sigma}A_{\sigma}$, is the GEI-equilibrium allocation of model \mathcal{E}^{in} . Having this in mind first let us state that $\bar{x} \in \text{int}X \cap \mathcal{D}^{cp}(\mathcal{E}^{in})$ implies

$$\langle \bar{p}, \bar{x}_i \rangle = \langle \bar{p}, \mathbf{e}_i \rangle, \quad i \in \mathcal{I}.$$

Assuming to the contrary, suppose there is consumer i_0 such that $\bar{p}\mathbf{e}_{i_0} > \bar{p}\bar{x}_{i_0}$. Then, via the smoothness of preferences, we obtain

$$\bar{x}_{i_0} + \mu(\mathbf{e}_{i_0} - \bar{x}_{i_0}) \succ_{i_0} \bar{x}_{i_0}$$

for some real $\mu > 0$ small enough. However, now using (3.1.4), one can write

$$\begin{aligned} \bar{x}_{i_0}^0 + \mu(\mathbf{e}_{i_0}^0 - \bar{x}_{i_0}^0) &= \mathbf{e}_{i_0}^0 + \Delta_{i_0}^0(V) - \mu\Delta_{i_0}^0(V) = \mathbf{e}_{i_0}^0 + (1 - \mu)\Delta_{i_0}^0(W), \\ \bar{x}_{i_0}^{\sigma} + \mu(\mathbf{e}_{i_0}^{\sigma} - \bar{x}_{i_0}^{\sigma}) &= \mathbf{e}_{i_0}^{\sigma} + \Delta_{i_0}(V^{\sigma}) + A_{\sigma}\Delta z_{i_0} - \mu\Delta_{i_0}(V^{\sigma}) - \mu A_{\sigma}\Delta z_{i_0} = \\ &= \mathbf{e}_{i_0}^{\sigma} + (1 - \mu)\Delta_{i_0}(V^{\sigma}) + (1 - \mu)A_{\sigma}\Delta z_{i_0}, \quad \sigma = 1, \dots, s. \end{aligned}$$

Clearly by the choice of μ one can think $0 \leq (1 - \mu) < 1$, $\mu > 0$, and the participant i_0 can partially break all contracts in a share μ , increasing utility, which contradicts

the lower stability of the proper contractual allocation \bar{x} . Therefore $\langle \bar{p}, \bar{x}_i \rangle \leq \langle \bar{p}, \mathbf{e}_i \rangle$ for all $i \in \mathcal{I}$. Now via the feasibility of \bar{x} , one obtains the result.

Further, from Lemma 3.2.1, Lemma 3.2.2 and its Corollary 3.2.2 we have $\bar{x} \in H(\bar{p}^\perp)$, that means

$$\bar{p}_\sigma \bar{x}_i^\sigma = \bar{p}_\sigma \mathbf{e}_i^\sigma + \bar{p}_\sigma A_\sigma \bar{z}_i \quad \forall \sigma \geq 1$$

for some $\bar{z}_i \in \mathbb{R}^k$ and all $i \in \mathcal{I}$, and also $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$. Let us take this $\bar{z} = (\bar{z}_i)_{i \in \mathcal{I}}$ as a net trade portfolio for allocation \bar{x} . From the above relations, using

$$\bar{p} \bar{x}_i = \sum_{\sigma=0}^s \bar{p}_\sigma \bar{x}_i^\sigma = \bar{p}_0 \bar{x}_i^0 + \sum_{\sigma=1}^s \bar{p}_\sigma \mathbf{e}_i^\sigma + \sum_{\sigma=1}^s \bar{p}_\sigma A_\sigma \bar{z}_i = \sum_{\sigma=0}^s \bar{p}_\sigma \mathbf{e}_i^\sigma$$

one can easily conclude that $\bar{p}_0 \bar{x}_i^0 = \bar{p}_0 \mathbf{e}_i^0 - \bar{q} \bar{z}_i$ takes place for $\bar{q} = \sum_{\sigma=1}^s \bar{p}_\sigma A_\sigma$. Now applying Corollary 3.2.2 and (3.2.8), one can conclude that for each i the vector \bar{x}_i is the maximal element of \succsim_i on the set $\mathcal{B}_i(\bar{p}, \bar{q})$ of all $x_i \in X_i$, satisfying the conditions

$$\exists z_i \in \mathbb{R}^k : \bar{p}_0 x_i^0 = \bar{p}_0 \mathbf{e}_i^0 - \bar{q} z_i \quad \& \quad \bar{p}_\sigma x_i^\sigma = \bar{p}_\sigma \mathbf{e}_i^\sigma + \bar{p}_\sigma A_\sigma z_i \quad \forall \sigma \geq 1.$$

However, using (S), local non-satiation in each of the spot markets, and following along a standard line of augmentation, one can state that if \succsim_i attains a maximal point⁶ on the set of all $x_i \in X_i$ such that

$$\exists z_i \in \mathbb{R}^k : \bar{p}_0 x_i^0 \leq \bar{p}_0 \mathbf{e}_i^0 - \bar{q} z_i \quad \& \quad \bar{p}_\sigma x_i^\sigma \leq \bar{p}_\sigma \mathbf{e}_i^\sigma + \bar{p}_\sigma A_\sigma z_i \quad \forall \sigma \geq 1$$

is true (in fact equal to i 's budget set for incomplete market), then this point undoubtedly has to belong to the set $\mathcal{B}_i(\bar{p}, \bar{q})$ (*i.e.*, for this point all inequalities are realized in the form of equality). So we have proven condition (i) of Definition 3.1.1 for $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$.

As the requirement (ii) of this definition is also obviously true, Theorem 3.2.1 is proved. \blacksquare

3.2.3 GEI-equilibrium as a fuzzy contractual allocation

In this section, we study the fuzzy-contractual characterization of GEI-equilibria of an incomplete market. Now, first of all, we will present a proper adaptation of the concept of fuzzy contractual allocation.

The structural description of the contractual allocation remains the same. Here the allocation $x = (x_i)_{i \in \mathcal{I}}$ is defined by a web of contracts

$$V = \bigcup_{\sigma=1}^s V^\sigma \cup W,$$

where V^σ is the set of contracts specified for $\sigma \geq 1$, and W is the bundle of contracts concluded in the present. Without loss of generality, we assume that in each event

⁶To be sure, such a point does exist, one may assume, in addition, strict monotonicity for at least one agent's preferences, and every consumption set is bounded from below. This is so provided that budget sets are compact, which due to the continuity of preferences, gives the result.

exactly one contract is concluded, i.e., $W = \{(v^0, z)\}$, $V^\sigma = \{v^\sigma\}$, $\sigma = 1, \dots, s$. By specification (3.1.3) one has:

$$\begin{aligned} x_i^0(V) &= \mathbf{e}_i^0 + v_i^0, \\ x_i^\sigma(V) &= \mathbf{e}_i^\sigma + v_i^\sigma + A_\sigma z_i, \quad \sigma = 1, \dots, s. \end{aligned}$$

Definition 3.2.1 *An allocation $x = (x_i)_{\mathcal{I}} \in \mathcal{A}(\mathcal{E})$ is called fuzzy contractual if for every $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$, $0 \leq \lambda_i \leq 1$, $i \in \mathcal{I}$:*

- (i) $t = 0$ (present) for $y_i^0 = x_i^0(V) - \lambda_i v_i^0$, $y_i^\sigma = x_i^\sigma - \lambda_i A_\sigma z_i$, $\sigma = 1, \dots, s$ for the allocation $(y_i)_{\mathcal{I}}$ there is no coalition that is able to conclude a new mutually beneficial contract at the state $t = \sigma = 0$.
- (ii) $t = 1$ (future) $\forall \sigma \geq 1$ for $y_i^\sigma = x_i^\sigma - \lambda_i v_i^\sigma$ for the allocation $(x_i^{-\sigma}, y_i^\sigma)_{i \in \mathcal{I}}$ there is no coalition that is able to conclude a new mutually beneficial contract at the state $\sigma = 1, \dots, s$.

The set of all **fuzzy contractual allocations** is denoted as

$$\mathcal{D}^f(\mathcal{E}^{in}).$$

Substantially the claims (i), (ii) say that there is no coalition, the members of which (each!) would be beneficial partially breaking the current contract (each $i \in \mathcal{I}$ in any state of the world in an own measure $\lambda_i \in [0, 1]$), and then signing a new contract at this state. Subsequent analysis reveals the relationship between equilibria and fuzzy contractual allocations. It allows us to assert (with some reservations) that they coincide.

Theorem 3.2.2 *Let economy \mathcal{E}^{in} obey (S) (non-satiation). Then*

$$W(\mathcal{E}^{in}) \subseteq \mathcal{D}^f(\mathcal{E}^{in}), \quad (3.2.15)$$

where $W(\mathcal{E}^{in})$ is the set of all GEI-allocations.

So the theorem states that every GEI-equilibrium is a fuzzy contractual allocation.

Proof of Theorem 3.2.2. Consider any GEI-equilibrium (x, z, p, q) . Let us prove that this equilibrium is fuzzy contractual with respect to the standard system of contracts. To this end, consider the “equilibrium” system of contracts $V = \{v^\sigma\}_{\sigma=0}^s$, where

$$v^0 = w = (u, z) = (x^0 - \mathbf{e}^0, z), \quad v^\sigma = (x^\sigma - \mathbf{e}^\sigma - A_\sigma z), \quad \sigma = 1, \dots, s$$

and $A_\sigma z = (A_\sigma z_i)_{i \in \mathcal{I}}$ by convention. Due to Theorem 1.2.2 and GEI-equilibrium definition, one can easily see that this web of contracts satisfies the condition of their joint stability by Definition 3.2.1. Indeed, for example, let in the state $\sigma \geq 1$ an (asymmetric) partial break of the contract v^σ occur and a coalition $S \subseteq \mathcal{I}$ enter

into a new agreement such that each member of the coalition improves its position: $\lambda_i \in [0, 1]$, $i \in S$, $w = (w_i)_{i \in S} \in \mathbb{R}^{|S|}$, $\sum_S w_i = 0$

$$x'_i \succ_i x_i, \quad x'_i = (x_i^{-\sigma}, y_i^\sigma), \quad y_i^\sigma = x_i^\sigma - \lambda_i v_i^\sigma + w_i, \quad \forall i \in S.$$

Calculating further the value of the new and old consumption bundle in the state σ , via $\langle p_\sigma, v_i^\sigma \rangle = 0$ (from the budget constraints in the definition of GEI), we find

$$\langle p_\sigma, x_i'^\sigma \rangle = \langle p_\sigma, x_i^\sigma \rangle + \langle p_\sigma, w_i \rangle > \langle p_\sigma, x_i^\sigma \rangle \Rightarrow \langle p_\sigma, w_i \rangle > 0, \quad \forall i \in S.$$

Summarizing the inequalities obtained, we come to a contradiction with the definition of the contract w : $\sum_S p_\sigma w_i > 0 = p_\sigma \sum_S w_i$.

The case of the break of the current contract and the conclusion of a new one in the present is hereby analyzed in a similar way. Really, after the breaking of the contract in the volume $\lambda_i \in [0, 1]$ one has

$$y_i^0 = x_i^0 - \lambda_i(x_i^0 - \mathbf{e}_i^0), \quad y_i^\sigma = x_i^\sigma - \lambda_i A_\sigma z_i, \quad \sigma \geq 1.$$

Earlier, by virtue of Proposition 3.1.1, we saw that the non-arbitrage property of equilibrium prices implies the zero value of the contract in the present, i.e.,

$$\langle (x_i^0 - \mathbf{e}_i^0, (A_\sigma z_i)_{\sigma=1}^{\sigma=s}), p \rangle = p_0(x_i^0 - \mathbf{e}_i^0) + \sum_{\sigma=1}^s p_\sigma A_\sigma z_i = -qz_i + \sum_{\sigma=1}^s p_\sigma A_\sigma z_i = 0.$$

It follows that under a partial break of the contract in the present “full” value of the equilibrium consumption bundle remains unchanged:

$$py_i = px_i = \sum_{\sigma=0}^s p_\sigma x_i^\sigma = p\mathbf{e}_i \quad \forall i \in \mathcal{I}.$$

Now, if a coalition $S \subseteq \mathcal{I}$ enters into a new agreement (w, z') , $\sum_S w_i = 0$, $\sum_S z'_i = 0$, then consumptions reached by the coalition members are

$$\xi_i^0 = y_i^0 + w_i, \quad \xi_i^\sigma = y_i^\sigma + A_\sigma z'_i, \quad \sigma \geq 1,$$

and being dominating they obey $\xi_i \succ_i x_i$, $i \in S$. Consequently, consumption plans ξ_i are not budget acceptable. However, at the same time $p_\sigma \xi_i^\sigma = p_\sigma y_i^\sigma + p_\sigma A_\sigma z'_i = p_\sigma \mathbf{e}_i^\sigma + p_\sigma A_\sigma (z'_i + (1 - \lambda_i)z_i)$, whence we conclude that consumption program $(\xi_i, z'_i + (1 - \lambda_i)z_i)$ does not satisfy budget constraint in the present, i.e., it is true $p_0 \xi_i^0 > p_0 \mathbf{e}_i^0 - q(z'_i + (1 - \lambda_i)z_i)$. Hence, taking into account budget equalities in the future states and the non-arbitrage of equilibrium prices (3.1.2), one concludes

$$p\xi_i = \sum_{\sigma=0}^s p_\sigma \xi_i^\sigma > \sum_{\sigma=0}^s p_\sigma \mathbf{e}_i^\sigma + \left(\sum_{\sigma=1}^s p_\sigma A_\sigma - q \right) z'_i + (1 - \lambda_i) \left(\sum_{\sigma=1}^s p_\sigma A_\sigma - q \right) z_i = p\mathbf{e}_i.$$

Finally, we summarize these inequalities regarding $i \in S$, obtaining

$$\sum_S p\xi_i = \sum_S py_i + \sum_S p_0 w_i + \sum_S \left(\sum_{\sigma=1}^s p_\sigma A_\sigma \right) z'_i =$$

$$= p \sum_S \mathbf{e}_i + p_0 \sum_S w_i + \left(\sum_{\sigma=1}^s p_\sigma A_\sigma \right) \left(\sum_S z'_i \right) > p \sum_S \mathbf{e}_i.$$

However, this contradicts the contractual specification, since $\sum_S w_i = 0$ & $\sum_S z'_i = 0$. Theorem 3.2.2 is proved. ■

In further analysis, in order to establish the inclusion inverse of (3.2.15), we need to reveal certain mathematical properties of fuzzy contractual allocations presented in the following Lemma. Recall that $\mathcal{V}_\sigma \subset L^\mathcal{I}$ denotes the subspace of valid contracts in an incomplete market in a state $\sigma \geq 0$, see page 125. We specify also

$$\mathcal{P}_i^\sigma(x_i) = \{y_i^\sigma \in X_i^\sigma \mid (y_i^\sigma, x_i^{-\sigma}) \succ_i x_i\}, \quad \sigma = 0, \dots, s, \quad i \in \mathcal{I}.$$

Lemma 3.2.3 *An allocation $x \in \mathcal{A}(X)$ in \mathcal{E}^{in} is fuzzy contractual if and only if*

$$\prod_{\mathcal{I}} (\mathcal{P}_i^\sigma(x_i) \cup \{x_i^\sigma\} - [x_i^\sigma, \mathbf{e}_i^\sigma + A_\sigma z_i]) \cap \{(v_1^\sigma, \dots, v_n^\sigma) \in \mathcal{V}_\sigma \mid \sum_{\mathcal{I}} v_i^\sigma = 0\} = \{0\}, \quad \sigma \geq 1,$$

$$\prod_{\mathcal{I}} (\mathcal{P}_i(x_i) \cup \{x_i\} - [x_i, (\mathbf{e}_i^0, (x_i^\sigma - A_\sigma z_i)_{\sigma=1}^{\sigma=s})]) \cap \{(v_1^0, \dots, v_n^0) \in \mathcal{V}_0 \mid \sum_{\mathcal{I}} v_i^0 = 0\} = \{0\}.$$

Here $[x_i^\sigma, \mathbf{e}_i^\sigma + A_\sigma z_i] \subset \mathbb{R}^l$ this is a linear segment connecting the vectors corresponding to consumption x_i^σ and $\mathbf{e}_i^\sigma + A_\sigma z_i$, where contracts in the state $\sigma \geq 1$ are completely broken. Analogously, $(\mathbf{e}_i^0, (x_i^\sigma - A_\sigma z_i)_{\sigma=1}^{\sigma=s})$ is the consumption vector in the case when i -th agent breaks all contracts in the present, keeping all other contracts unchanged.

The following corollary is applied in further analysis.

Corollary 3.2.4 *If the allocation $x = (x_i)_{\mathcal{I}}$ is fuzzy contractual, then*

$$\prod_{\mathcal{I}} (\mathcal{P}_i^\sigma(x_i) - [x_i^\sigma, \mathbf{e}_i^\sigma + A_\sigma z_i]) \cap \{(v_1^\sigma, \dots, v_n^\sigma) \in \mathcal{V}_\sigma \mid \sum_{\mathcal{I}} v_i^\sigma = 0\} = \emptyset, \quad \sigma \geq 1,$$

$$\prod_{\mathcal{I}} (\mathcal{P}_i(x_i) - [x_i, (\mathbf{e}_i^0, (x_i^\sigma - A_\sigma z_i)_{\sigma=1}^{\sigma=s})]) \cap \{(v_1^0, \dots, v_n^0) \in \mathcal{V}_0 \mid \sum_{\mathcal{I}} v_i^0 = 0\} = \emptyset.$$

Note that, unlike the Lemma, the components of the current consumption bundle $(x_i^\sigma)_{\sigma=0}^{\sigma=s}$ are excluded from the left side of the intersections described in this corollary.

Proof of Lemma 3.2.3. Consider the case of the contract in the present. After a partial break in volumes $(\lambda_i)_{\mathcal{I}}$, one has $y_i^0 = x_i^0(V) - \lambda_i v_i^0$, $y_i^\sigma = x_i^\sigma - \lambda_i A_\sigma z_i$, $\sigma \geq 1$, $i \in \mathcal{I}$. By definition in allocation $(y_i)_{\mathcal{I}}$ no coalition can conclude a new mutually beneficial contract.

If a coalition $S \subseteq \mathcal{I}$ enters into a new agreement (w, z') , $\sum_S w_i = 0$, $\sum_S z'_i = 0$, $\text{supp}(w, z') = S$, then the level of consumption achieved by the members of the coalition is

$$\xi_i^0 = y_i^0 + w_i, \quad \xi_i^\sigma = y_i^\sigma + A_\sigma z'_i, \quad \sigma \geq 1$$

and the domination means $\xi_i \in \mathcal{P}_i(x_i)$, $i \in S$. Hence, coalition domination is equivalent to

$$\exists \xi_i \in \mathcal{P}_i(x_i), i \in S : (\xi_i)_S - (y_i)_S \in \mathcal{V}_0(S),$$

where

$$\mathcal{V}_0(S) = \{(v_i^0)_S \in L^S \mid \sum_S v_i^0 = 0, \forall i \in S \exists z_i \in \mathbb{R}^k : v_i^\sigma = A_\sigma z_i, \sigma \geq 1\}$$

is the space of contracts that coalition S is able to conclude in the present. A contract concluded in the state $\sigma = 0$ is presented as $v^0 = [(x_i^0 - \mathbf{e}_i^0, (A_\sigma z_i)_{\sigma=1}^{\sigma=s})]_{i \in \mathcal{I}}$. So, after a partial break in the volume $0 \leq \lambda_i \leq 1$ the following consumptions are realized:

$$y_i(\lambda_i) = x_i - \lambda_i(x_i^0 - \mathbf{e}_i^0, (A_\sigma z_i)_{\sigma=1}^{\sigma=s}) \iff y_i(\lambda_i) \in [x_i, (\mathbf{e}_i^0, (x_i^\sigma - A_\sigma z_i)_{\sigma=1}^{\sigma=s})].$$

To complete the picture, the allocations $(y_i)_S$ and $(\xi_i)_S$ for $i \in \mathcal{I} \setminus S$ specified for coalition S can be supplemented with bundles $\xi_i = y_i = x_i$. As a result, we conclude: contractual dominance in the state $\sigma = 0$ is possible if and only if the intersection

$$\prod_{\mathcal{I}} (\mathcal{P}_i(x_i) \cup \{x_i\} - [x_i, (\mathbf{e}_i^0, (x_i^\sigma - A_\sigma z_i)_{\sigma=1}^{\sigma=s})]) \cap \{(v_1^0, \dots, v_n^0) \in \mathcal{V}_0 \mid \sum_{\mathcal{I}} v_i^0 = 0\}$$

contains a non-zero element. Thus, the required characterization is obtained in the state $\sigma = 0$. For the states $\sigma = 1, \dots, s$, the characterization described in the Lemma is proved in a similar way. Lemma 3.2.3 is proved. \blacksquare

The main result of the section is the following theorem which, together with Theorem 3.2.2, characterizes fuzzy contractual allocations as equilibria.

Theorem 3.2.3 *Let incomplete market economy \mathcal{E}^{in} obey (S). Then*

$$\text{int}X \cap \mathcal{D}^f(\mathcal{E}^{in}) \subseteq W(\mathcal{E}^{in}) \cap \text{int}X,$$

where $\mathcal{D}^f(\mathcal{E}^{in})$ is the set of all fuzzy contractual and $W(\mathcal{E}^{in})$ is the set of all GEI-equilibrium allocations of the model \mathcal{E}^{in} .

The proof of this theorem is ultimately reduced to the specific application of the separation theorem.

Proof of Theorem 3.2.3. Let us use the characteristic property of a fuzzy contractual allocation presented in Corollary 3.2.4. For this purpose, it is necessary to describe correctly the space of valid contracts in all possible states of the world. Let us do it.

For $\sigma = 0$ (present), the presentation of the new contract is rather cumbersome here because it is possible to make exchanges in future markets, using “standard” contracts for real assets. However, in this case, contracts can be associated with the kernel of the operator specified by the matrix

$$B_0 = \begin{bmatrix} E_l & 0 & \cdots & 0 & E_l & 0 & \cdots & 0 & \cdots & E_l & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & [& E_{ls} &] & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -A & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & [& E_{ls} &] & \cdots & 0 & 0 & \cdots & 0 & 0 & -A & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & [& E_{ls} &] & 0 & 0 & \cdots & -A & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & E_k & E_k & \cdots & E_k & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

We have

$$(v_1^0, \dots, v_n^0) \in \mathcal{V}_0, \quad \sum_{\mathcal{I}} v_i^0 = 0 \quad \Longleftrightarrow$$

$$v^0 = (v_1^0, \dots, v_n^0) \in \mathbb{R}^{l(s+1)n} \ \& \ \exists (z_1, \dots, z_n) \in \mathbb{R}^{kn} : B_0[v^0, z]^t = 0.^7$$

Here, the first l equations correspond to the requirement $\sum_{\mathcal{I}} v_i^{00} = 0$, i.e. vector $v^{00} = (v_i^{00})_{\mathcal{I}}$ is a contract. Next ls equations realize $v_1^{01} - Az_1 = 0$ and so on, i.e., $l s n$ equations provide the condition $v_i^{01} = Az_i, i \in \mathcal{I}$, where $v_i^{01} = (v_i^{0\sigma})_{\sigma=1, \dots, s} \in \mathbb{R}^{ls}$. The last group of k equations meets the requirement $\sum_{\mathcal{I}} z_i = 0$, which specifies the couple (v^0, z) as a contract. Thus, $(v^0, z) \in \ker B_0(\cdot) \Longleftrightarrow v^0 \in \mathcal{V}_0$.

For $\sigma = 1, \dots, s$ the space of contracts can be represented as the kernel of the operator defined on $L_{\sigma}^{\mathcal{I}} = \mathbb{R}^{ln}$ and defined by the matrix

$$B_{\sigma} = [E_l \ E_l \ \dots \ E_l].$$

Here E_l of the identity matrix of size $l \times l$. We obtain

$$v^{\sigma} = (v_1^{\sigma}, \dots, v_n^{\sigma}) \in \mathbb{R}^{ln}, \quad \sum_{\mathcal{I}} v_i^{\sigma} = 0 \quad \Longleftrightarrow \quad B_{\sigma} v^{\sigma} = 0.$$

Let's consider the final part of the construction, which involves the preferences of individuals. By virtue of Lemma 3.2.3, for any fuzzy contractual allocation (\bar{x}, \bar{z}) the conditions

$$\prod_{\mathcal{I}} (\mathcal{P}_i(\bar{x}_i) - [\bar{x}_i, (\mathbf{e}_i^0, (\bar{x}_i^{\sigma} - A_{\sigma} \bar{z}_i)_{\sigma=1}^{\sigma=s})]) \times \mathbb{R}^{kn} \bigcap \ker(B_0) = \emptyset, \quad (3.2.16)$$

$$\prod_{\mathcal{I}} (\mathcal{P}_i^{\sigma}(\bar{x}_i) - [\bar{x}_i^{\sigma}, \mathbf{e}_i^{\sigma} + A_{\sigma} \bar{z}_i]) \bigcap \ker(B_{\sigma}) = \emptyset, \quad \sigma = 1, \dots, s \quad (3.2.17)$$

are satisfied. Note that the last factor \mathbb{R}^{kn} in the left part of the intersection (3.2.16) corresponds to the area of change of trading portfolios (z_1, \dots, z_n) by assets.

The first of the intersecting sets in (3.2.16), (3.2.17) has a nonempty interior, and the second one is a subspace. Applying the separability theorem, we conclude the existence of a linear functional separating these sets, whose characteristic vector F_{σ} has a structure:

$$F_0 = (\pi_1, \dots, \pi_n, g_1, \dots, g_n) \neq 0, \quad \pi_i \in \mathbb{R}^{l(s+1)}, \quad g_i \in \mathbb{R}^k, \quad i \in \mathcal{I}$$

$$F_{\sigma} = (h_1^{\sigma}, \dots, h_n^{\sigma}) \neq 0, \quad h_i^{\sigma} \in \mathbb{R}^l, \quad i \in \mathcal{I}, \quad \sigma = 1, \dots, s.$$

From the separability of a subspace and a convex set, we conclude that $\langle F_{\sigma}, \ker(B_{\sigma}) \rangle = 0, \sigma \geq 0$, that standardly implies

$$\pi_i^0 = \pi_j^0 = p_0, \quad g_i = g_j = g = 0 \ \& \ h_i^{\sigma} = h_j^{\sigma} = p_{\sigma} \quad \forall i, j \in \mathcal{I}, \quad \sigma = 1, \dots, s.$$

⁷The upper symbol t indicates a transpose operation. Thus, $[v^0, z]^t = \begin{bmatrix} (v^0)^t \\ z^t \end{bmatrix}$ is a column vector.

In fact, $\langle F_0, \ker(B_0) \rangle = 0$ implies $\langle (g_1, \dots, g_n), \mathbb{R}^{kn} \rangle \geq 0$, that gives $g_i = 0, \forall i$. Another consequence can be done due to the structure of contracts in the future states, presented by the kernel of B_σ : if $y \in \mathbb{R}^l$, then $(0, \dots, 0, y, 0, \dots, 0, -y, 0, \dots, 0) \in \ker B_\sigma$ is a permissible contract in the state σ , that corresponds to a commodity flow y of agent i and $-y$ for agent j . We have $\langle h_i^\sigma, y \rangle + \langle h_j^\sigma, -y \rangle = 0 \Rightarrow \langle h_i - h_j, y \rangle = 0 \forall y \in \mathbb{R}^l$, that is possible only if $h_i^\sigma = h_j^\sigma$. Similar conclusions also can be easily done for the space of contracts in “present” and for vectors π_i^0 .

Further, from separability we have

$$\sum_{\mathcal{I}} \langle (\mathcal{P}_i(\bar{x}_i) - [\bar{x}_i, (\mathbf{e}_i^0, (\bar{x}_i^\sigma - A_\sigma \bar{z}_i)_{\sigma=1}^{\sigma=s})]) , \pi_i \rangle \geq 0,$$

$$\sum_{\mathcal{I}} \langle (\mathcal{P}_i^\sigma(\bar{x}_i) - [\bar{x}_i^\sigma, \mathbf{e}_i^\sigma + A_\sigma \bar{z}_i]) , p_\sigma \rangle \geq 0, \quad \sigma = 1, \dots, s.$$

By virtue of (S)—local non-satiation of preferences in each of the spot markets—we have $\bar{x}_i^\sigma \in \bar{\mathcal{P}}_i^\sigma(\bar{x}_i) \forall i \in \mathcal{I}$ and, therefore, zero belongs to the closure of each of the terms under summation. Therefore, these inequalities entail: $\forall i \in \mathcal{I}$

$$\langle (\mathcal{P}_i(\bar{x}_i) - [\bar{x}_i, (\mathbf{e}_i^0, (\bar{x}_i^\sigma - A_\sigma \bar{z}_i)_{\sigma=1}^{\sigma=s})]) , \pi_i \rangle \geq 0,$$

$$\langle (\mathcal{P}_i^\sigma(\bar{x}_i) - [\bar{x}_i^\sigma, \mathbf{e}_i^\sigma + A_\sigma \bar{z}_i]) , p_\sigma \rangle \geq 0, \quad \sigma = 1, \dots, s.$$

Moreover, from the last inequalities we can immediately conclude that $\forall \sigma = 1, \dots, s$

$$\langle \bar{x}_i^\sigma, p_\sigma \rangle \geq \langle [\bar{x}_i^\sigma, \mathbf{e}_i^\sigma + A_\sigma \bar{z}_i], p_\sigma \rangle \Rightarrow \langle \bar{x}_i^\sigma - \mathbf{e}_i^\sigma - A_\sigma \bar{z}_i, p_\sigma \rangle \geq 0, \quad \forall i.$$

By virtue of $\sum_{\mathcal{I}} \bar{x}_i^\sigma = \sum_{\mathcal{I}} \mathbf{e}_i^\sigma$ and $\sum_{\mathcal{I}} \bar{z}_i = 0$, we conclude that all inequalities are performed in the form of equality. Therefore,

$$\langle \mathcal{P}_i^\sigma(\bar{x}_i), p_\sigma \rangle \geq \langle \bar{x}_i^\sigma, p_\sigma \rangle = \langle \mathbf{e}_i^\sigma + A_\sigma \bar{z}_i, p_\sigma \rangle \quad \forall i \in \mathcal{I}, \quad \sigma \geq 1. \quad (3.2.18)$$

Similarly, for the “present”, the first group of inequalities gives

$$\langle \bar{x}_i, \pi_i \rangle \geq \langle [\bar{x}_i, (\mathbf{e}_i^0, (\bar{x}_i^\sigma - A_\sigma \bar{z}_i)_{\sigma=1}^{\sigma=s})] , \pi_i \rangle \iff \langle \bar{x}_i^0 - \mathbf{e}_i^0, p_0 \rangle + \sum_{\sigma=1}^s \langle A_\sigma \bar{z}_i, \pi_i^\sigma \rangle \geq 0 \quad \forall i \in \mathcal{I}.$$

Moreover, the vector (functional) F_0 obtained from the separability of sets is also represented as a linear combination of the vector of rows of the matrix B_0 (since it turns to zero by $\ker(B_0)$) and, since $g_i = 0 \forall i$, then

$$\sum_{\sigma=1}^{\sigma=s} \pi_j^\sigma A_\sigma = q = \sum_{\sigma=1}^{\sigma=s} \pi_i^\sigma A_\sigma \quad \forall i, j \in \mathcal{I}$$

for some $q \in \mathbb{R}^k$. So, $\langle \sum_{\sigma=1}^{\sigma=s} \pi_i^\sigma A_\sigma, \bar{z}_i \rangle = \langle q, \bar{z}_i \rangle$ and, summing up the above inequalities, we conclude that all of them can be performed only in the form of equality (the sum of the values in the left part of the inequalities is zero). So there should be: $\forall i \in \mathcal{I}$

$$\langle \mathcal{P}_i(\bar{x}_i), \pi_i \rangle \geq \langle \bar{x}_i, \pi_i \rangle = \langle (\mathbf{e}_i^0, (\bar{x}_i^\sigma - A_\sigma \bar{z}_i)_{\sigma=1}^{\sigma=s}) , \pi_i \rangle = \langle \mathbf{e}_i^0, p_0 \rangle + \sum_{\sigma=1}^s \langle \bar{x}_i^\sigma, \pi_i^\sigma \rangle - \langle q, \bar{z}_i \rangle. \quad (3.2.19)$$

Carrying out reductions, from the received equalities we conclude

$$\langle \bar{x}_i^0, p_0 \rangle = \langle e_i^0, p_0 \rangle - \langle q, \bar{z}_i \rangle. \quad (3.2.20)$$

The formulas (3.2.18)–(3.2.19) practically give the desired result, but we still need to exclude the trivial case. First of all, we note that by virtue of $F_\sigma = (p_\sigma, \dots, p_\sigma) \neq 0$, $\sigma \geq 1$ we have

$$p_1 \neq 0, \dots, p_s \neq 0.$$

Consider the case of $\sigma = 0$ and assume $p_0 = 0$. Find $\pi_i \neq 0$ (it exists for $F_0 \neq 0$) and take the vector $w = (w^\sigma)_{\sigma=1}^{\sigma=s} \in \mathbb{R}^{ls}$ such that $\sum_{\sigma=1}^{\sigma=s} \langle \pi_i^\sigma, w^\sigma \rangle < 0$. Next, chose $w^0 \in \mathbb{R}^l$ from the condition $\bar{x}_i^0 + w^0 \in \text{int} \mathcal{P}_i^0(\bar{x}_i)$ ($\text{int} \mathcal{P}_i^0(\bar{x}_i) \neq \emptyset$ by virtue of **(S)**) and consider the vector $w(t) = (w^0, tw^1, \dots, tw^s)$, $t \in \mathbb{R}$. Since $\bar{x}_i \in \text{int} X_i$, for a sufficiently small $t > 0$ we have $x_i(t) = \bar{x}_i + w(t) \in \mathcal{P}_i(\bar{x}_i)$. However, then $\langle \pi_i, x_i(t) \rangle = \langle \pi_i, \bar{x}_i \rangle + \langle \pi_i, w(t) \rangle = \langle \pi_i, \bar{x}_i \rangle + t \sum_{\sigma=1}^{\sigma=s} \langle \pi_i^\sigma, w^\sigma \rangle < \langle \pi_i, \bar{x}_i \rangle$, which contradicts (3.2.19). Similar reasoning proves $\pi_i^\sigma \neq 0$. As a result we have

$$p_0 \neq 0, \quad \pi_i^\sigma \neq 0, \quad \forall i \in \mathcal{I}, \quad \forall \sigma = 1, 2, \dots, s.$$

For differentiable utilities from (3.2.18) and (3.2.19), we further conclude the existence of real $\alpha_i > 0$ and $\beta_i^\sigma > 0$ such that

$$\pi_i = \alpha_i \nabla u_i(\bar{x}_i), \quad p_\sigma = \beta_i^\sigma \nabla_{|x_i^\sigma} u_i(\bar{x}_i) \quad \forall i \in \mathcal{I}, \quad \sigma = 1, \dots, s.$$

Therefore, for some $\lambda_i^\sigma > 0$ it is executed

$$\pi_i^\sigma = \lambda_i^\sigma p_\sigma, \quad \forall i \in \mathcal{I}, \quad \sigma = 1, \dots, s. \quad (3.2.21)$$

These identities allow us to accept $p = (p_0, p_1, \dots, p_s)$ and $q = \sum_{\sigma=1}^s \lambda_i^\sigma p_\sigma A_\sigma$ as price vectors corresponding to a fuzzy contractual allocation of an incomplete market. The general non-smooth case is more nontrivial and is analyzed in the following Lemma.

Lemma 3.2.4 *Under the conditions of Theorem 3.2.3, for every $i \in \mathcal{I}$ and $\sigma = 1, 2, \dots, s$, there are real $\lambda_i^\sigma > 0$ such that*

$$\begin{aligned} \langle \mathcal{P}_i(\bar{x}_i), (p_0, \lambda_1^i p_1, \dots, \lambda_s^i p_s) \rangle &\geq \langle \bar{x}_i, (p_0, \lambda_1^i p_1, \dots, \lambda_s^i p_s) \rangle = \\ &= \langle (e_i^0, (\bar{x}_i^\sigma - A_\sigma \bar{z}_i)_{\sigma=1}^{\sigma=s}), (p_0, \lambda_1^i p_1, \dots, \lambda_s^i p_s) \rangle. \end{aligned}$$

Moreover, $\sum_{\sigma=1}^s \lambda_i^\sigma p_\sigma A_\sigma \bar{z}_i = \langle q, \bar{z}_i \rangle$.

Thus, this Lemma states that there are real $\lambda_i^\sigma > 0$ such that the vectors $\lambda_i^\sigma p_\sigma$ can be used instead of π_i^σ while preserving all their properties significant for the analysis. Therefore, further without loss of generality, we use the specification (3.2.21).

Next, we prove that (\bar{x}, \bar{z}, p, q) is an equilibrium in \mathcal{E}^{in} . To do this, it is sufficient to show that \bar{x}_i is the solution to the consumer problem for each $i \in \mathcal{I}$. Let us do it.

From the above considerations, it follows that the pair (\bar{x}_i, \bar{z}_i) satisfies all budget (in)equalities. Suppose $y_i \in X_i$ is such that $y_i \succ_i \bar{x}_i$ and for some $z_i \in \mathbb{R}^k$, the plan y_i satisfies all budget constraints in future states:

$$p_\sigma y_i^\sigma \leq p_\sigma \mathbf{e}_i^\sigma + p_\sigma A_\sigma z_i, \quad \sigma = 1, \dots, s \quad \Rightarrow \quad \sum_{\sigma=1}^s \pi_i^\sigma y_i^\sigma \leq \sum_{\sigma=1}^s \pi_i^\sigma \mathbf{e}_i^\sigma + \sum_{\sigma=1}^s \pi_i^\sigma A_\sigma z_i.$$

From here, using (3.2.19), we conclude

$$p_0 y_i^0 + \sum_{\sigma=1}^s \pi_i^\sigma \mathbf{e}_i^\sigma + \sum_{\sigma=1}^s \pi_i^\sigma A_\sigma z_i \geq \pi_i y_i > \pi_i \bar{x}_i = p_0 \mathbf{e}_i^0 + \sum_{\sigma=1}^s (\bar{x}_i^\sigma - A_\sigma \bar{z}_i) \pi_i^\sigma.$$

However, by virtue of (3.2.18), we have $(\bar{x}_i^\sigma - A_\sigma \bar{z}_i) p_\sigma = \mathbf{e}_i^\sigma p_\sigma \Rightarrow (\bar{x}_i^\sigma - A_\sigma \bar{z}_i) \pi_i^\sigma = \mathbf{e}_i^\sigma \pi_i^\sigma \forall \sigma \geq 1$, which, after reducing the identical members and taking into account the representation of q , allows us to conclude

$$y_i^0 p_0 > \mathbf{e}_i^0 p_0 - q z_i.$$

Thus, the pair (y_i, z_i) is not budget feasible in the present and, therefore, cannot belong to the budget set, which proves the budget optimality of (\bar{x}_i, \bar{z}_i) . Theorem 3.2.3 is proved. \blacksquare

Proof of Lemma 3.2.4. Let us consider for the space $E = \prod_{\sigma=0}^s \mathbb{R}^{l_\sigma}$ the value operator $P : E \rightarrow \mathbb{R}^{s+1}$, specified by the matrix

$$P = \begin{bmatrix} p_0 & & 0 \\ & \ddots & \\ 0 & & p_s \end{bmatrix}.$$

Suppose $C = \mathcal{P}_i(\bar{x}_i)$ and consider the image $P(C) = D$ of the set C under the linear mapping $P(y) = Py$, $y \in E$, and the image of the hyperplane

$$H_i^\pi = \{y \in E \mid \langle y, \pi_i \rangle = \langle \bar{x}_i, \pi_i \rangle\}.$$

By virtue of (3.2.19), we have $(\mathbf{e}_i^0, (\bar{x}_i^\sigma - A_\sigma \bar{z}_i)_{\sigma=1}^s) \in H_i^\pi$, but the analysis and (S) imply

$$y_i^\sigma \in \text{int} \mathcal{P}_i^\sigma(\bar{x}_i) \quad \Rightarrow \quad \langle y_i^\sigma, \pi_i^\sigma \rangle > \langle \bar{x}_i^\sigma, \pi_i^\sigma \rangle \quad \& \quad \langle y_i^\sigma, p_\sigma \rangle > \langle \bar{x}_i^\sigma, p_\sigma \rangle.$$

Hence, the dimension of the affine manifold $P(H_i^\pi)$ is no more than s and, moreover, reasoning from the opposite, we conclude $P(H_i^\pi) \cap \text{int} D = \emptyset^8$. Obviously, the set $D \subset \mathbb{R}^{s+1}$ is convex and $P(\bar{x}_i) \in \text{cl} D$. Now we can apply the separability theorem and, if necessary, continue $P(H_i^\pi)$ to the hyperplane in \mathbb{R}^{s+1} while preserving the separability property from $\text{int} D$. Because (by virtue of (S)) for some real $r > 0$

$$\{y \in \mathbb{R}^{s+1} \mid y > 0 \ \& \ \|y\| \leq r\} + P(\bar{x}_i) \subset \text{int} D,$$

⁸Assuming the existence of $y \in P(H_i^\pi) \cap \text{int} D$ consider $P^{-1}(y) \subset P^{-1}(P(H_i^\pi)) = S$. By virtue of (S), the affine manifold S has a dimension of no more than $l(s+1) - 1$ and, since $H_i^\pi \subset S$, then necessarily $H_i^\pi = S$, i.e., S is a hyperplane in E . Since $y \in \text{int} D$, the hyperplane S intersects with $P^{-1}(\text{int} D)$, which contradicts the construction of H_i^π .

then we can conclude the existence of a vector representing a separating linear functional as a scalar product, having strictly positive components. As a result, we found a vector $h^i = h = (h_0, h_1, \dots, h_s) \gg 0$ such that

$$\langle h, D \rangle \geq \langle h, P(\bar{x}_i) \rangle = \langle h, P(\mathbf{e}_i^0, (\bar{x}_i^\sigma - A_\sigma \bar{z}_i)_{\sigma=1}^{\sigma=s}) \rangle.$$

Next, consider an arbitrary $y = (y_\sigma)_{\sigma=0}^{\sigma=s} \in C$. Using the operator P , from the previous one we have

$$\langle h^i, P(y) \rangle \geq \langle h^i, P(\bar{x}_i) \rangle = h_0^i \langle p_0, \bar{x}_0 \rangle + h_1^i \langle p_1, \bar{x}_1 \rangle + \dots + h_s^i \langle p_s, \bar{x}_s \rangle.$$

Now, specifying $\lambda_\sigma^i = \frac{h_\sigma^i}{h_0^i}$, $\sigma = 1, 2, \dots, s$, we conclude

$$\begin{aligned} \langle \mathcal{P}_i(\bar{x}_i), (p_0, \lambda_1^i p_1, \dots, \lambda_s^i p_s) \rangle &\geq \langle \bar{x}_i, (p_0, \lambda_1^i p_1, \dots, \lambda_s^i p_s) \rangle = \\ &= \langle (\mathbf{e}_i^0, (\bar{x}_i^\sigma - A_\sigma \bar{z}_i)_{\sigma=1}^{\sigma=s}), (p_0, \lambda_1^i p_1, \dots, \lambda_s^i p_s) \rangle. \end{aligned}$$

The last equality after reductions and taking into account (3.2.20) gives

$$\mathbf{e}_i^0 p_0 - \sum_{\sigma=1}^s \lambda_\sigma^i p_\sigma A_\sigma \bar{z}_i = \langle \bar{x}_i^0, p_0 \rangle = \langle \mathbf{e}_i^0, p_0 \rangle - \langle q, \bar{z}_i \rangle \Rightarrow \sum_{\sigma=1}^s \lambda_\sigma^i p_\sigma A_\sigma \bar{z}_i = \langle q, \bar{z}_i \rangle. \quad \blacksquare$$

3.2.4 Core for incomplete markets

Now let us pass on the analysis of the core concept for incomplete markets. Let us identify, by definition, an allocation from the core of \mathcal{E}^{in} with semi-perfectly contractual allocation by Definition 3.1.3, *i.e.*, put

$$\mathcal{C}(\mathcal{E}^{in}) = \mathcal{D}^{sp}(\mathcal{E}^{in}).$$

The first important fact immediately follows from Theorem 3.2.1 and core specification.

Proposition 3.2.1 *In Theorem 3.2.1 conditions, every GEI-equilibrium belongs to an incomplete market core.*

The proof is via inclusion:

$$W(\mathcal{E}^{in}) \cap \text{int}X = \text{int}X \cap \mathcal{D}^{cp}(\mathcal{E}^{in}) \subset \mathcal{D}^{sp}(\mathcal{E}^{in}) = \mathcal{C}(\mathcal{E}^{in}).$$

Below the main properties of $\mathcal{C}(\mathcal{E}^{in})$ are investigated and, in particular, it is shown that under some assumptions, which are not too strong in the context of incomplete market theory, the set $\mathcal{C}(\mathcal{E}^{in})$ fits with the ordinary notion of the core as soon as the market becomes complete.

Consider $p^1 = (p_\sigma)_{\sigma=1}^{\sigma=s}$, $p_\sigma \in \mathbb{R}^l$, $\sigma = 1, \dots, s$ as a fixed vector of prices in the spot markets of the future states of the world.

Definition 3.2.2 An allocation $x = (x_1, \dots, x_n) \in X$ is called p^1 -feasible if there are portfolios $z = (z_1, \dots, z_n)$, $z_i \in \mathbb{R}^k$, $\sum_{i \in \mathcal{I}} z_i = 0$ such that equalities

$$p_\sigma x_i^\sigma = p_\sigma \mathbf{e}_i^\sigma + p_\sigma A^\sigma z_i, \quad \forall i \in \mathcal{I}, \forall \sigma = 1, \dots, s$$

hold.

The definition of a p^1 -feasible allocation $x \in X$ can be written in an equivalent form:

$$P_1(x_i^1 - \mathbf{e}_i^1) \in \mathcal{L}(P_1A), \quad i \in \mathcal{I}, \quad (3.2.22)$$

where $\mathcal{L}(P_1A)$ is the linear hull of the vector-columns of the matrix of returns in future markets from assets P_1A under prices p^1 .

Analogously one can define the notion of a p^1 -feasible allocation for an arbitrary (nonempty) coalition $S \subset \mathcal{I}$, substituting in Definition 3.2.2 the set \mathcal{I} by means of S .

Let us denote by $\mathcal{A}_{p^1}(S)$ (or by $\mathcal{A}_p(S)$) the set of all p^1 -feasible via coalition S allocations. Note that the set $\mathcal{A}_{p^1}(S) \neq \emptyset$ for every $S \subset \mathcal{I}$ since the vector of initial endowments $\mathbf{e}^S = (\mathbf{e}_i)_{i \in S}$ always belongs to $\mathcal{A}_{p^1}(S)$. Moreover, it has to be clear from the above definitions that $\mathcal{A}_{p^1}(\mathcal{I}) = H(p^1) \cap X$.

Definition 3.2.3 p -core is the set $C_p(\mathcal{E}^{in})$ of all p^1 -feasible allocations which cannot be dominated via coalitions, i.e.,

$$x \in C_p(\mathcal{E}^{in}) \iff x \in \mathcal{A}_p(\mathcal{I}) \ \& \ \nexists S \subset \mathcal{I} : \exists y \in \mathcal{A}_p(S) \mid y_i \succ_i x_i \ \forall i \in S.$$

Let agents' preferences be defined via utility functions, (which are presumed to be concave and continuous), and let a price-vector p^1 for future markets be fixed. Then for an incomplete market one can put into correspondence some cooperative game with non-transferable utility (to be short, the NTU-game). Recall that formally NTU-game (for details, e.g. see [Moulin \(1988\)](#)) is a couple $(\mathcal{I}, (V(S))_{S \subset \mathcal{I}})$, described by the set of players (agents) $\mathcal{I} = \{1, \dots, n\}$, ($n \geq 2$) and the sets of permissible vector-payoffs $V(S) \subseteq \mathbb{R}^S$ for every (nonempty) coalition $S \subset \mathcal{I}$, which have to satisfy the following properties:

- $V(S)$ is the nonempty closed subset in \mathbb{R}^S ;
- $V(S)$ is comprehensive from below, i.e., $x \in V(S)$ and $y \leq x$ imply $y \in V(S)$;
- every singleton coalition has nonempty and bounded from above possibilities, i.e., $V(\{i\}) \neq \emptyset$ and $V(\{i\}) < +\infty$, $\forall i \in \mathcal{I}$;
- the set of all individual-rational vector-payoffs from $V(S)$, this is by definition the set

$$Q(S) := \{v \in V(S) \mid v_i \geq V(\{i\}) \ \forall i \in S\},$$

is bounded from above in \mathbb{R}^S .

In our case, the set of all permissible vector-payoffs for coalition S is determined by formula

$$V_p(S) = \bigcup_{x \in \mathcal{A}_p(S)} V_p^x(S),$$

where

$$V_p^x(S) = \{(v_i)_{i \in S} \leq (u_i(x_i))_{i \in S} \mid (x_i)_{i \in S} \in \mathcal{A}_p(S)\}.$$

Clearly that the sets $V_p(S)$ satisfy all the above described necessary conditions, it can be checked easily due to the compactness of $\mathcal{A}(X)$ and the continuity of utilities in the initial incomplete market model.

Recall that the family \mathcal{B} of subsets in \mathcal{I} is said to be *balanced* if for every $S \in \mathcal{B}$ there is a real $\lambda_S \geq 0$, such that

$$\sum_{S \in \mathcal{B}: i \in S} \lambda_S = 1 \quad \forall i \in \mathcal{I}$$

holds or, in an equivalent form,

$$\sum_{S \in \mathcal{B}} \lambda_S \mathbf{1}_S = \mathbf{1}_{\mathcal{I}}$$

takes place where, by definition, $\mathbf{1}_S \in \mathbb{R}^{\mathcal{I}}$ is such a vector that $\mathbf{1}_S(i) = 1$ for $i \in S$ and $\mathbf{1}_S(i) = 0$ if $i \notin S$, *i.e.*, this is the indicator function of the set $S \subseteq \mathcal{I}$.

A game $(\mathcal{I}, (V(S))_{S \subseteq \mathcal{I}})$ is said to be *balanced* if for every balanced family of coalitions \mathcal{B}

$$\bigcap_{S \in \mathcal{B}} \text{pr}_{|S}^{-1}(V(S)) \subseteq V(\mathcal{I}).$$

Here $\text{pr}_{|S}(\cdot)$ is the projection map of space $\mathbb{R}^{\mathcal{I}}$ onto \mathbb{R}^S .

The famous Scarf's theorem (see Theorem 8.1.1 p. 301, for the proof see Theorem 8.1.2 p. 302) states that the core of a balanced game $(\mathcal{I}, (V(S))_{S \subseteq \mathcal{I}})$ is nonempty. Applying this theorem and using standard arguments, one can prove the following

Proposition 3.2.2 *Let the set of all individually rational feasible allocations⁹ of \mathcal{E}^{in} be bounded and let agents' preferences be defined via concave continuous utility functions. Then $C_p(\mathcal{E}^{in}) \neq \emptyset$.*

Proof of Proposition 3.2.2. To apply Scarf's theorem, we need to show that the game (\mathcal{I}, V_p) , determined via an incomplete market under fixed prices p^1 , is balanced. With this in mind, let us consider a balanced family of coalitions \mathcal{B} . It needs to be shown that for every utility-vector v , corresponding to some p^1 -feasible allocation, the following

$$[\forall S \in \mathcal{B} \quad \text{pr}_{|S}(v) \in V_p(S)] \Rightarrow v_{\mathcal{I}} \in V_p(\mathcal{I})$$

is true. In fact, due to the game definition, for each $S \in \mathcal{B}$ there is $x^S \in \mathcal{A}_p(S)$ such that $v_S = (v_i)_{i \in S} \leq (u_i(x_i^S))_{i \in S} = u_S(x^S)$. Using the balanced family definition, for each $S \in \mathcal{B}$ one can find a real $\lambda_S \geq 0$. Now, multiplying inequalities on λ_S and

⁹This is $\mathcal{A}(X) \cap \prod_{\mathcal{I}} \mathcal{P}_i(e_i)$.

summing then by S from \mathcal{B} , due to the concavity of the utility functions we yield $(v_i)_{i \in \mathcal{I}} \leq (u_i(\bar{x}_i))_{i \in \mathcal{I}}$ for some $\bar{x} \in \mathcal{A}_p(\mathcal{I})$, which proves $v_{\mathcal{I}} \in V_p(\mathcal{I})$.

Finally, if $\bar{v}_{\mathcal{I}}$ is the vector realizing the maximum of $\sum_{i \in \mathcal{I}} v_i$ subject to $v_{\mathcal{I}} = (v_i)_{\mathcal{I}}$ from the core of (\mathcal{I}, V_p) (which is compact), then obviously the allocation corresponding to it is from $C_p(\mathcal{E})$. \blacksquare

The following lemma presents a convenient tool for the study of an incomplete market core.

Lemma 3.2.5 *Let \mathcal{E}^{in} be a smooth economy and $x \in \text{int}X$. Then $x \in \mathcal{C}(\mathcal{E}^{in})$ if and only if*

- (i) *x is a partially Pareto optimal allocation, i.e., for every $\sigma \geq 0$ it cannot be dominated via allocations from $L_x^\sigma = \{y = (y_i)_{\mathcal{I}} \in L^{\mathcal{I}} \mid y_i^{-\sigma} = x_i^{-\sigma} \forall i \in \mathcal{I}\}$, and*
- (ii) *$x \in C_p(\mathcal{E}^{in})$, where $(p_\sigma)_{\sigma=1}^{\sigma=s}$ is a bundle of σ -Pareto prices that corresponds to item (i).*

Note that if, in addition, $\overline{\mathcal{P}_i(\mathbf{e}_i)} \subset \text{int}X_i$ for all $i \in \mathcal{I}$, then in the previous lemma the assumption $x \in \text{int}X$ can be omitted. So in this case this lemma gives the full description of $\mathcal{C}(\mathcal{E}^{in})$.

Proof of Lemma 3.2.5. To state the conclusion of the lemma on the side of necessity, let us consider some $x \in \mathcal{C}(\mathcal{E}^{in}) \cap \text{int}X$. Clearly, every allocation from the core of \mathcal{E}^{in} is a relative equilibrium for the markets of future states of the world in reduced economies \mathcal{E}_σ^{in} relative to endowments $\mathbf{e}_i^\sigma + A_\sigma \Delta z_i(W)$, $i \in \mathcal{I}$ and subject to the fixed consumption in all other markets (since every contract for future events is perfect, then it follows from Theorem 1.2.2). This proves (i) for $\sigma \geq 1$; clearly it is true for $\sigma = 0$. Now let us prove (ii). Due to $x \in \text{int}X$ and all utilities being differentiable, the vectors of equilibrium prices \bar{p}_σ , $\sigma \geq 1$ are uniquely determined (up to normalization). Show that $x \in C_p(\mathcal{E}^{in})$. The proof parallels the proof of item (ii) from Lemma 3.2.1. Presuming the contrary, find $y \in \mathcal{A}_{\bar{p}}(S)$ such that $y \succ_S x$ relative to \bar{p}^1 . Since $x \in \mathcal{A}_{\bar{p}}(\mathcal{I})$, $y \in \mathcal{A}_{\bar{p}}(S)$, there are such z, z' that the equalities

$$\bar{p}_\sigma x_i^\sigma - \bar{p}_\sigma \mathbf{e}_i^\sigma = \bar{p}_\sigma A_\sigma z_i, \quad \sigma = 1, \dots, s, \quad i \in \mathcal{I},$$

$$\bar{p}_\sigma y_i^\sigma - \bar{p}_\sigma \mathbf{e}_i^\sigma = \bar{p}_\sigma A_\sigma z'_i, \quad \sigma = 1, \dots, s, \quad i \in S$$

are fulfilled. As soon as all contracts for future states are assumed to be perfect, we can substitute the initial web V^σ , which realizes the allocation $(x_i^\sigma)_{i \in \mathcal{I}}$ by the web consisting of two *proper* contracts v'^σ and v''^σ :

$$v_i'^\sigma = y_i^\sigma - \mathbf{e}_i^\sigma - A_\sigma z'_i, \quad i \in S, \quad v_i''^\sigma = 0, \quad i \in \mathcal{I} \setminus S,$$

$$v_i''^\sigma = (x_i^\sigma - y_i^\sigma) - A_\sigma(z_i - z'_i), \quad i \in S, \quad v_i''^\sigma = x_i^\sigma - \mathbf{e}_i^\sigma - A_\sigma z_i, \quad i \in \mathcal{I} \setminus S.$$

The last one can be done for all $\sigma = 1, \dots, s$. In the present, one can save “old” contracts. Due to consumption sets being rectangular and due to the $\mathcal{A}_{\bar{p}}(S)$ specification, one can easily see that all of them are *contracts* forming a *web*. Recall that to

see if these contracts are proper, it is enough to check $v_i^{\prime\sigma} \bar{p}_\sigma = 0$ $v_i^{\prime\prime\sigma} \bar{p}_\sigma = 0$ for all i . In so doing, by definition, the new web has to be stable relative to the *simultaneous* procedure of breaking and signing a new contract in the “*present*.” However, breaking all contracts of a second kind now and all contracts in the present and signing for $t = \sigma = 0$ the new contract $w = [(y_i^0 - \mathbf{e}_i^0, z_i^0)]_{i \in S}$, the members of coalition S are able to realize the allocation $y \in \mathcal{A}_{\bar{p}}(S)$, which contradicts the incomplete market core definition.

Further, let us prove the sufficiency of items (i) and (ii) of Lemma 3.2.5. By the assumption, x^σ is Pareto optimal in a σ -reduced model for a fixed $x^{-\sigma}$ relative to endowments $\mathbf{e}_i^\sigma + A_\sigma z_i$, $\sigma \geq 1$. And moreover, $x \in C_{\mathbf{p}}(\mathcal{E}^{in}) \cap \text{int}X$ for partial Pareto prices $p^1 = (p_1, \dots, p_s)$. Therefore, there are $z_i \in \mathbb{R}^k$ satisfying the condition

$$P_1 x_i^1 = P_1 \mathbf{e}_i^1 + P_1 A z_i.$$

Now let us consider the web $V = \bigcup_{\sigma \geq 1} \{v^\sigma\} \cup \{w\}$, where

$$w = (x^0 - \mathbf{e}^0, z), \quad v^\sigma = (x^\sigma - \mathbf{e}^\sigma - A_\sigma z), \quad \sigma = 1, \dots, s$$

for $A_\sigma z = (A_\sigma z_i)_{i \in \mathcal{I}}$. We have to prove that the allocation $x = x(V)$, realized via this web, belongs to the incomplete market core, *i.e.*, $x = x(V) \in \mathcal{C}(\mathcal{E}^{in})$.

First, we show that each contract v^σ , $\sigma \geq 1$ is perfect in fact. Really, for fixed $\sigma \geq 1$ due to a specification, we have $p_\sigma v_i^\sigma = 0$ for all i . Since the prices p^1 are partial Pareto, then for all i we also have

$$p_\sigma y_i^\sigma > p_\sigma x_i^\sigma \quad \forall y_i = (y_i^\sigma, x_i^{-\sigma}) \in \mathcal{P}_i(x_i).$$

Now one can apply Proposition 1.1.1 and using the sufficiency of (1.1.1) conclude that contract v^σ is coherent. But applying Theorem 1.2.2 for a Pareto optimal allocation $x \in \text{int}X$ realized by coherent web (since (ii) implies (iv)) we can conclude that the web is perfect. Therefore, contract v^σ is perfect, and this is true for all $\sigma \geq 1$.

Let a coalition $S \subset \mathcal{I}$, virtual proper webs $V^\sigma \sim \{v^\sigma\}$ and their subwebs $U^\sigma \subset V^\sigma$, which are realized after this coalition breaks a part of its virtual contracts, be given for all $\sigma \geq 1$, and let $\text{supp}(u^\sigma) \subseteq S$ for every $u^\sigma \in U^\sigma$ and each $\sigma \geq 1$, *i.e.*, (3.1.5) is true.

In the present let the members of S sign a new contract (u^0, z^0) , breaking contract w . In such a case, the members of S realize the following allocation $y = (y_i)_{i \in S}$:

$$y_i^0 = \mathbf{e}_i^0 + u_i^0, \quad i \in S,$$

$$y_i^\sigma = \mathbf{e}_i^\sigma + \Delta_i(U^\sigma) + A_\sigma z_i^0, \quad \sigma \geq 1, i \in S.$$

Now, since contracts from U^σ are proper and in view of Theorem 1.2.2, we obtain $p_\sigma u_i^\sigma = 0$, $u^\sigma \in U^\sigma$, which implies

$$p_\sigma y_i^\sigma = p_\sigma \mathbf{e}_i^\sigma + p_\sigma A_\sigma z_i^0, \quad \sigma \geq 1, i \in S.$$

From (3.1.5) we have $\sum_S \Delta_i(U^\sigma) = 0$ for all $\sigma \geq 1$, and by contract specification $\sum_S u_i^0 = 0$ and $\sum_S z_i^0 = 0$; this proves $\sum_S y_i = \sum_S \mathbf{e}_i$. So as a result, we have $y \in \mathcal{A}_p(S)$ and see that domination by coalition S is impossible.

So, it is proved that contracts v^σ , $\sigma = 1, \dots, s$ are perfect and the web is stable by Definition 3.1.3. Thus x is a semi-perfectly contractual allocation, as we wanted to prove. ■

In considerations and results immediately below, we always presume incomplete market \mathcal{E}^{in} satisfies the strict monotonicity assumption for every spot market as follows:

(M) For some $i \in \mathcal{I}$ and every $x \in \mathcal{A}(\mathcal{E})$

$$(\{x_i\} + L^+) \setminus \{x_i\} \subset \mathcal{P}_i(x_i).^{10}$$

Let us call a market (*i.e.*, model \mathcal{E}^{in}) *complete relative to prices* $p^1 = (p_\sigma)_{\sigma=1}^{\sigma=s}$ if the rank of matrix P_1A is equal to s , the total number of possible future states of the world.

A market is *complete* if it is complete relative to every bundle of spot prices $p_1, \dots, p_s \in \mathbb{R}^l$ such that $p_\sigma \gg 0$, $\sigma = 1, \dots, s$.

Clearly, this property of completeness (uniform relative to $p^1 \gg 0$) is a kind of restriction for matrix A , more exactly for financial markets of assets, the number of which under this hypothesis has to be no less than s .¹¹ An example of an incomplete market, satisfying this completeness assumption, is the described above market of *numeraire* assets for $e_\sigma > 0$,¹² $\sigma \geq 1$, in which the matrix $R = (r_\sigma^j)_{\substack{\sigma=1, \dots, s \\ j=1, \dots, k}}$ has rank equal to s .

Slightly strengthening the assumptions of the model, we arrive at the description of the incomplete market core in familiar terms when the model is complete. This is stated in the following important corollary of Lemma 3.2.5.

Corollary 3.2.5 *Let \mathcal{E}^{in} be a complete smooth economy satisfying (M). Then*

$$\text{int}X \cap \mathcal{C}(\mathcal{E}^{in}) = \bigcup_{p^1 \gg 0} (C_p(\mathcal{E}^{in}) \cap \text{int}X)$$

takes place. If, in addition, \mathcal{E}^{in} is such that $\overline{\mathcal{P}_i(e_i)} \subset \text{int}X_i$ for all $i \in \mathcal{I}$, then

$$\mathcal{C}(\mathcal{E}^{in}) = \bigcup_{p^1 \gg 0} C_p(\mathcal{E}^{in}).$$

Proof of Corollary 3.2.5. Applying Lemma 3.2.5 on the side of necessity for $x \in \text{int}X \cap \mathcal{C}(\mathcal{E}^{in})$, due to (i) one can conclude that the allocation x is partially Pareto optimal. Therefore, via assumptions (S) and (M) there exists (and unique) σ -Pareto

¹⁰Remember L denotes the *commodity space* of the economy, where $L = \mathbb{R}^{l(s+1)}$ and $L^+ = \mathbb{R}_+^{l(s+1)}$.

¹¹Of course this condition cannot be sufficient.

¹²This is a consumption bundle $e_\sigma \in \mathbb{R}^l$ chosen as a unit of “numeraire” for assets and for future spot market σ .

prices $\bar{p}^1 = (\bar{p}_\sigma)_{\sigma=1}^{\sigma=s}$, which satisfy $\bar{p}^1 \gg 0$. Now applying item (ii) of Lemma 3.2.5, we obtain

$$x \in C_{\bar{p}}(\mathcal{E}^{in}) \subset \bigcup_{p^1 \gg 0} C_p(\mathcal{E}^{in}).$$

To prove the inverse inclusion for *complete* model \mathcal{E}^{in} , let us chose any $x \in C_{\bar{p}}(\mathcal{E}^{in}) \cap \text{int}X$ for fixed $\bar{p}^1 \gg 0$. Now let us note that due to the completeness of \mathcal{E}^{in} for every $p^1 \gg 0$ the next system of linear equations

$$P_1 \hat{x}_i^1 = P_1 \mathbf{e}_i^1 + P_1 A \hat{z}_i \quad (3.2.23)$$

has a solution relative to \hat{z}_i and other parameters of any kind. Note that these solutions satisfy $\sum_{i \in \mathcal{I}} \hat{z}_i = 0$ when $(\hat{x}_i)_{i \in \mathcal{I}}$ is feasible; in (3.2.23) instead of $P_1 A$, one can take any square non-singular submatrix whose dimension is $s \times s$. Therefore, every feasible $(\hat{x}_i)_{\mathcal{I}}$ is p^1 -feasible for every $p^1 \gg 0$. Moreover, the condition $x \in C_{\bar{p}}(\mathcal{E}^{in}) \cap \text{int}X$ for $\bar{p}^1 \gg 0$ implies that x is Pareto optimal, which entails its partial Pareto optimality. From this, via (S) and (M), we can conclude the existence of (unique up to normalization) σ -Pareto prices $\bar{p}^1 = (\bar{p}_\sigma)_{\sigma=1}^{\sigma=s}$, which also satisfy $\bar{p}^1 \gg 0$. Now having in mind the application of Lemma 3.2.5 in the part of sufficiency, we have to show only that $x \in C_{\bar{p}}(\mathcal{E}^{in})$. Let $y \in \mathcal{A}_{\bar{p}}(S)$ for $S \subset \mathcal{I}$. Once again, using the completeness of the market, we can conclude that system (3.2.23) may be solved with respect to \hat{z}_i for all $i \in S$ when one substitutes y_i for \hat{x}_i and after the substitution of \bar{p}^1 by \hat{p}^1 . Thus we obtain $\mathcal{A}_{\bar{p}}(S) \subset \mathcal{A}_{\hat{p}}(S)$ and, using $x \in C_{\bar{p}}(\mathcal{E}^{in})$, may conclude that coalition S cannot dominate allocation x under prices \bar{p} . Now the application of Lemma 3.2.5 finishes the proof. ■

The characterization of an incomplete market core for *complete* exchange economies gives the following

Theorem 3.2.4 *Let \mathcal{E}^{in} be a smooth economy satisfying (M) such that $\overline{\mathcal{P}_i(\mathbf{e}_i)} \subset \text{int}X_i$ for each $i \in \mathcal{I}$. If \mathcal{E}^{in} is complete, then $\mathcal{C}(\mathcal{E}^{in}) = \mathcal{C}(\mathcal{E})$.*

Proof of Theorem 3.2.4. The proof is based on the simple application of Corollary 3.2.5 of Lemma 3.2.5, however, let us consider this in a detailed form.

First, let us remark that assumption $\overline{\mathcal{P}_i(\mathbf{e}_i)} \subset \text{int}X_i \forall i \in \mathcal{I}$, implies that $x \in \text{int}X$ as for $x \in \mathcal{C}(\mathcal{E})$ and also for $x \in \mathcal{C}(\mathcal{E}^{in})$. This is why in the below considerations we can always think $x \in \text{int}X$.

Let us show the inclusion $\mathcal{C}(\mathcal{E}) \subseteq \mathcal{C}(\mathcal{E}^{in})$. Let $x \in \mathcal{C}(\mathcal{E})$. Then for each σ the allocation x_σ is partially Pareto optimal, which implies the existence of price-vector $\bar{p}^1 \gg 0$ ¹³ such that condition $(\tilde{x}_i^\sigma, x_i^{-\sigma}) \succ_i (x_i^\sigma, x_i^{-\sigma})$ (relative to fixed $x^{-\sigma}$) implies $\bar{p}^\sigma \tilde{x}_i^\sigma > \bar{p}^\sigma x_i^\sigma$ for all i . Now let us represent x as a \bar{p}^1 -feasible allocation (to apply Corollary 3.2.5). For this we need to find a feasible trade net of portfolios $(z_i)_{i \in \mathcal{I}}$, satisfying the system of linear equations

$$P_1 x_i^1 = P_1 \mathbf{e}_i^1 + P_1 A z_i. \quad (3.2.24)$$

¹³The property $\bar{p}^1 \gg 0$ follows from the strict monotonicity of utilities, which is guaranteed due to assumption (M).

Since the market is complete, this system is solvable relative to z_i . Therefore, $x \in \mathcal{A}_{\bar{p}}(\mathcal{I})$. Now let us presume that $x \notin \mathcal{C}(\mathcal{E}^{in})$. Then due to Corollary 3.2.5, one obtains $x \notin C_{\bar{p}}(\mathcal{E}^{in})$ for each $p^1 \gg 0$, and hence for a given $\bar{p}^1 \gg 0$. Thus there is a coalition $S \subset \mathcal{I}$ and a \bar{p}^1 -feasible for S allocation y such that

$$y_i \succ_i x_i, \quad \forall i \in S.$$

However this contradicts $x \in \mathcal{C}(\mathcal{E})$, which ends the check for inclusion. It is easy to see that the web realizing the given complex contractual allocation $x \in \mathcal{C}(\mathcal{E}^{in})$ is the collection

$$V = \{(u, z)\} \bigcup_{\sigma=1}^s \{v^\sigma\},$$

where $u = x^0 - e^0$ and $z = (z_i)_{i \in \mathcal{I}}$ are such that z_i satisfy the system (3.2.24),¹⁴ and $v_i^\sigma = x_i^\sigma - e_i^\sigma - A^\sigma z_i$ for every $i \in \mathcal{I}$ and $\sigma = 1, \dots, s$.

Let us prove the inverse inclusion $\mathcal{C}(\mathcal{E}^{in}) \subseteq \mathcal{C}(\mathcal{E})$. Consider $x \in \mathcal{C}(\mathcal{E}^{in})$. Due to Corollary 3.2.5, there exists a price-vector $\bar{p}^1 \gg 0$ such that $x \in C_{\bar{p}}(\mathcal{E}^{in})$. Note if allocation y dominates via S allocation x (in an ordinary sense), *i.e.*, if there are $y_i \in X_i$ such that $y_i \succ_i x_i$ for all $i \in S$ and $\sum_{i \in S} y_i = \sum_{i \in S} e_i$, then in view of market completeness one can find such $\tilde{z}_i \in \mathbb{R}^k$, $i \in S$ that system (3.2.24) has a solution relative to \tilde{z}_i and for fixed y_i (substitute \tilde{z}_i instead of z_i and y_i instead of x_i). Since these \tilde{z}_i satisfy $\sum_S \tilde{z}_i = 0$, then $y \in \mathcal{A}_{\bar{p}}(S)$, and therefore $x \notin C_{\bar{p}}(\mathcal{E}^{in})$, which is a contradiction. ■

3.2.5 Fuzzy contractual core in incomplete markets

An alternative way to introduce a core in \mathcal{E}^{in} is an approach based on a fuzzy-contractual characterization of equilibrium, described in Section 3.2.3: The behavior of the agents for the future states has to be specified as a fuzzy contractual one, but the present is the opposite case. Here it is necessary to allow agents to break off all current contracts in “present” and in all future events, as well as allow them the opportunity to conclude new contracts in all states of the world, but so that in future states the new contracts **do not improve** the preference of anyone involved individual even if they are implemented partially. So, to present the idea simpler, we consider the following concept of an ineffective contract.

The contract $v \in W$ is called **ineffective** if $\sum_{i \in \mathcal{I}} v_i = 0$ and

$$\forall i \in \text{supp}(v) \quad x_i \succeq_i x_i + \lambda v_i \quad \forall 0 \leq \lambda \leq 1.$$

Thus, contracts are inefficient if with the minus sign they are coherent by Definition 1.1.4. In an incomplete market, the concept of an ineffective contract is associated with the state of the world; for $\sigma \geq 1$, *i.e.*, in future events, it means: $v^\sigma \in V^\sigma$, $\sum v_i^\sigma = 0$ and

$$\forall i \in \text{supp}(v^\sigma) \quad x_i \succeq_i (x_i^\sigma + \lambda v_i^\sigma, x_i^{-\sigma}) \quad \forall 0 \leq \lambda \leq 1.$$

As a result, we come to the following definition.

¹⁴More exactly, it is the system defined via the square non-degenerate submatrix of $P_1 A$, whose dimension is equal to s .

Definition 3.2.4 An allocation $x(V) \in \mathcal{A}(X)$, implemented by the web of contracts $V = \{(v^0, z)\} \cup \{v^\sigma\}_{\sigma=1}^{\sigma=s}$ belongs to the core of the incomplete market if the web V is weakly stable and

- (i) $t = 0$ (present) there is no coalition $S \subseteq \mathcal{I}$ for which it would be beneficial: break all contracts in the present and future states, conclude a family $(w^\sigma)_{\sigma=1}^{\sigma=s}$ of ineffective contracts in the future and a new contract in the present $w = (w^0, z') \in \mathcal{V}_0$;
- (ii) $t = 1$ (future) $\forall \sigma \geq 1$ and any $0 \leq \lambda_i \leq 1, i \in \mathcal{I}, \lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ for $y_i^\sigma = x_i^\sigma - \lambda_i v_i^\sigma$ allocation $(x_i^{-\sigma}, y_i^\sigma)_{i \in \mathcal{I}}$ is upper stable regarding the conclusion of a new contract in the state σ .

Item (i) of this definition indicates that no coalition $S \subseteq \mathcal{I}$ can conclude a family of ineffective contracts in future events and a (profitable) contract in the present so that $\text{supp}(u), \text{supp}(w^\sigma) \subseteq S$ and

$$\forall i \in S, \quad \xi_i = (\mathbf{e}_i^0 + w_i^0, (\mathbf{e}_i^\sigma + A_\sigma z'_i + v_i^\sigma)_{\sigma=1}^{\sigma=s}) \succ_i x_i.$$

Thus, no matter what the ineffective contracts are concluded by the coalition members after the break of current contracts: it is impossible to compensate them by concluding a new contract in the present.

Item (ii) suggests that the allocation x is fuzzy contractual in each of the future states of nature. The set of all allocations of the core by Definition 3.2.4 we denote

$$\mathcal{C}^f(\mathcal{E}^{in}).$$

To study the suggested concept of the core, we need some auxiliary notions are introduced and investigated.

The following lemma is similar to Lemma 3.2.5 and allows us to reduce the analysis of core by Definition of 3.2.4 to the previous results of Section 3.3.2 (core by Definition 3.1.3).

Lemma 3.2.6 Let \mathcal{E}^{in} be a smooth economy and $x \in \text{int}X$. Then $x \in \mathcal{C}^f(\mathcal{E}^{in})$ if and only if

- (i) x is a partially Pareto optimal allocation, i.e., for every $\sigma \geq 0$ it cannot be dominated via allocations from $L_x^\sigma = \{y = (y_i)_{\mathcal{I}} \in L^{\mathcal{I}} \mid y_i^{-\sigma} = x_i^{-\sigma} \forall i \in \mathcal{I}\}$, and
- (ii) $x \in C_{\mathbf{p}}(\mathcal{E}^{in})$, where $(p_\sigma)_{\sigma=1}^{\sigma=s}$ is a bundle of σ -Pareto prices that corresponds to item (i).

So, for a smooth model

$$\mathcal{C}(\mathcal{E}^{in}) \cap \text{int}X = \text{int}X \cap \mathcal{C}^f(\mathcal{E}^{in})$$

takes place.

Proof of Lemma 3.2.6. By Definition 3.2.4 for each $\sigma \geq 1$, the allocation $(x^\sigma, x^{-\sigma})$ from the core of the model \mathcal{E}^{in} is fuzzy contractual one for the economy \mathcal{E}_σ^{in} reduced to state σ future market relative to the endowments $\mathbf{e}_i^\sigma + A_\sigma \Delta z_i(W)$, $i \in \mathcal{I}$ and for fixed consumption in all other markets. Therefore, in the model \mathcal{E}_σ^{in} the allocation $x^\sigma = (x_i^\sigma)_{i \in \mathcal{I}}$ is a quasi-equilibrium. Since $x^\sigma \in \text{int}X^\sigma$ according to the conditions of the lemma, this allocation is an equilibrium in fact and, since the model is smooth, its equilibrium prices p_σ satisfy:

$$\forall i \in \mathcal{I} \quad \exists \lambda_i > 0 : \quad p_\sigma = (p_\sigma^1, \dots, p_\sigma^l) = \lambda_i \nabla_{|x_i^\sigma} u_i(\bar{x}_i).$$

Under these conditions, for an ineffective contract $v^\sigma = (v_i^\sigma)_{i \in \mathcal{I}}$ we have

$$\forall i \in \mathcal{I} \quad \langle p_\sigma, v_i^\sigma \rangle \leq 0,$$

whence, summing up the inequalities and from the contract property $\sum_{i \in \mathcal{I}} v_i^\sigma = 0$, we conclude

$$0 \geq \sum_{i \in \mathcal{I}} \langle p_\sigma, v_i^\sigma \rangle = \langle p_\sigma, \sum_{i \in \mathcal{I}} v_i^\sigma \rangle = 0 \quad \Rightarrow \quad \langle p_\sigma, v_i^\sigma \rangle = 0 \quad \forall i \in \mathcal{I}.$$

The converse is also true: under the conditions of the lemma, this property fully characterizes ineffective contracts.

Let us consider an arbitrary $x \in \mathcal{C}^f(\mathcal{E}^{in}) \cap \text{int}X$ and establish the necessity of Lemma 3.2.6. Now (i) follows from the upper stability of the web V (the allocation $x(V)$ is weakly stable, see Definition 1.1.2). Let us establish further (ii).

By virtue of $x \in \text{int}X$ and due to the differentiability of utility functions, the equilibrium price vectors p_σ , $\sigma \geq 1$ corresponding to partial equilibria are uniquely determined up to normalization. We show that $x \in C_p(\mathcal{E}^{in})$. Assuming the contrary, we find $y \in \mathcal{A}_p(S)$ such that $y \succ_S x$. By definition, we have $\sum_S y_i = \sum_S \mathbf{e}_i$ and there exists $z' = (z'_i)_{i \in S} \in [\mathbb{R}^k]^S$ such that¹⁵

$$p_\sigma y_i^\sigma - p_\sigma \mathbf{e}_i^\sigma = p_\sigma A_\sigma z'_i, \quad \sigma = 1, \dots, s, \quad i \in S.$$

Now define

$$v_i^\sigma = y_i^\sigma - \mathbf{e}_i^\sigma - A_\sigma z'_i, \quad i \in S.$$

We have $p_\sigma v_i^\sigma = 0$, $i \in S$ and $\sum_S v_i^\sigma = 0$. Therefore, in the state σ the contract $(v_i^\sigma)_{i \in S}$ is ineffective. Now members of coalition S can conclude contract $w = (w^0, z') \in \mathcal{V}_0$, in which $w_i^0 = y_i^0 - \mathbf{e}_i^0$, $i \in S$ and $z' = (z'_i)_{i \in S}$. The latter one is the exchange contract for assets of an incomplete market we found above. As a result, members of the coalition S realize the allocation

$$(\mathbf{e}_i^0 + w_i^0, (\mathbf{e}_i^\sigma + A_\sigma z'_i + v_i^\sigma)_{\sigma=1}^{\sigma=s}) = y_i \succ_i x_i \quad \forall i \in S$$

However, this contradicts (i) of Definition 3.2.4.

¹⁵As usual, by virtue of the condition below, we can assume without loss of generality that $\sum_S z'_i = 0$.

Let us prove the sufficiency of the conditions (i), (ii) of Lemma 3.2.6. By assumption, x^σ is Pareto optimal in the σ -reduced model for fixed $x^{-\sigma}$ and initial endowments $\mathbf{e}_i^\sigma + A_\sigma z_i$, $\sigma \geq 1$. Now we have $x \in C_{\mathbf{p}}(\mathcal{E}^{in}) \cap \text{int}X$ for partial Pareto prices $p^1 = (p_1, \dots, p_s)$. Therefore, there are $z_i \in \mathbb{R}^k$, such that

$$P_1 x_i^1 = P_1 \mathbf{e}_i^1 + P_1 A z_i.$$

Now let us consider the web of contracts $V = \bigcup_{\sigma \geq 1} \{v^\sigma\} \cup \{w\}$, where for $A_\sigma z = (A_\sigma z_i)_{\mathcal{I}}$ we set

$$w = (x^0 - \mathbf{e}^0, z), \quad v^\sigma = (x^\sigma - \mathbf{e}^\sigma - A_\sigma z), \quad \sigma = 1, \dots, s$$

and show that $x = x(V) \in \mathcal{C}^f(\mathcal{E}^{in})$. Obviously, the constructed web of contracts is weakly stable.

Next, we first show that $(x^\sigma, x^{-\sigma})$ is fuzzy contractual in the markets of the future states in the reduced models \mathcal{E}_σ^{in} . Indeed, by construction for given $\sigma \geq 1$ we have $p_\sigma v_i^\sigma = 0$ for all i . Since $p^1 = (p_1, p_2, \dots, p_s)$ are Pareto prices, then for all i we also have

$$p_\sigma y_i^\sigma > p_\sigma x_i^\sigma \quad \forall y_i = (y_i^\sigma, x_i^{-\sigma}) \in \mathcal{P}_i(x_i).$$

This immediately implies the equilibrium properties of $(x^\sigma, x^{-\sigma})$ in the σ -reduced model. However, every equilibrium is fuzzy contractual allocation, which proves item (ii) of Definition 3.2.4.

Next, let a coalition $S \subset \mathcal{I}$ and a family of inefficient contracts $\{u^\sigma\}_{\sigma=1}^s$, where $\text{supp}(u^\sigma) \subseteq S$ for every $\sigma \geq 1$, be given. It was shown above that in presented conditions

$$p_\sigma u_i^\sigma = 0 \quad \forall i \in S.$$

Therefore, if in the present the members of coalition S enter into a contract $(t_i, z'_i)_{i \in S}$, $\sum_S t_i = \sum_S \mathbf{e}_i$, $\sum_S z'_i = 0$, then for

$$y_i = (\mathbf{e}_i^0 + t_i, \mathbf{e}_i^\sigma + u_i^\sigma + A_\sigma z'_i)_{\sigma=1}^s, \quad i \in S$$

one has

$$P_1 y_i^1 = P_1 \mathbf{e}_i^1 + P_1 A z'_i.$$

Thus, with $y_i \succ_i x_i$ $i \in S$ this would mean domination and $x \notin C_{\mathbf{p}}(\mathcal{E}^{in})$. So domination is impossible, and this proves the point (i) of Definition 3.2.4. Lemma 3.2.6 is proved. \blacksquare

3.2.6 Replicas and asymptotic properties of core

The asymptotic analog of Theorem 3.2.1 continues the analysis of the incomplete core concept. This result is expressed in the form of a replicated incomplete market, that better fits with the classical representation of perfect competition conditions. The proof is based on the reduction of the domination in replicas to the study of domination via fuzzy coalitions with the succeeding fuzzy core consideration. Of course, the concept of the fuzzy core has to be adopted to the incomplete markets in a proper way. In what follows, the mathematical problem is reduced to the separation

theorem being applied to separate some convex set from zero (zero cannot belong to the set due to the fuzzy core property). In so doing, our analysis is essentially based on the characteristic Lemma 3.2.5 and on the fact that rational numbers are dense in the set of all real ones.

An incomplete market replica of volume $r \in \mathbb{N}$ is called the economy \mathcal{E}_r^{in} , in which r exact copies of each consumer from initial model \mathcal{E}^{in} is put into correspondence in \mathcal{E}_r^{in} . The agents from \mathcal{E}_r^{in} are numbered by double index (i, m) , $i \in \mathcal{I}$, $m = 1, \dots, r$, and it is put $X_{im} = X_i$, $\mathbf{e}_{im} = \mathbf{e}_i$. Agents' preferences are defined and take values in X_{im} due to identification $\mathcal{P}_{im} = \mathcal{P}_i$. An assets structure for a replica exactly repeats the structure of the initial model. To an initial economy \mathcal{E}^{in} allocation $x = (x_i)_{\mathcal{I}}$, we can put into correspondence the replicated economy allocation $x^r = (x_{im}^r)$ by the rule $x_{im} = x_i, \forall i, m$.

Definition 3.2.5 *An allocation x is called incomplete market Edgeworth equilibrium if $x^r \in \mathcal{C}(\mathcal{E}_r^{in})$ for every natural $r = 1, 2, \dots$. $\mathcal{C}^e(\mathcal{E}^{in})$ denotes the set of all Edgeworth equilibria for the model \mathcal{E}^{in} .*

Now let us consider the most characteristic properties of the Edgeworth equilibria. This analysis is convenient to realize under assumptions for which Lemma 3.2.5 is true. So let \mathcal{E}^{in} be a smooth economy and $x \in \text{int}X$. Then due to Lemma 3.2.5, the property $x \in \mathcal{C}^e(\mathcal{E}^{in})$ is equivalent to the fact that allocation x is partially Pareto optimal, and for partial Pareto prices $p^1 = (p_\sigma)_{\sigma \geq 1}$, the allocation belongs to the p -core of \mathcal{E}_r^{in} for every natural r . Consider the last requirement in more detail. It is very important that domination is admitted via *any coalitions and via any inter-coalition allocation*.

Presume that for some r a coalition $S \subseteq \mathcal{I} \times \{1, \dots, r\}$ dominates the allocation x^r . Let $\mathcal{I}(S) \subseteq \mathcal{I}$ be the set of all agent types non-trivially presented in the coalition S . Due to p -core specification, this domination means that for every $(i, m) \in S$ there is $y_{im} \in \mathcal{P}_i(x_i)$ such that for some $z_{im} \in \mathbb{R}^k$

$$P_1 y_{im} = P_1 \mathbf{e}_i + P_1 A z_{im}$$

and, in addition,

$$\sum_{(i,m) \in S} y_{im} = \sum_{(i,m) \in S} \mathbf{e}_{im}.$$

Now if we “average out” the dominating consumption bundles and portfolios for each given type of agents, *i.e.*, if we put

$$y_i = \left(\sum_{m|(i,m) \in S} y_{im} \right) / s_i \quad \& \quad z_i = \left(\sum_{m|(i,m) \in S} z_{im} \right) / s_i \quad \forall i \in \mathcal{I}(S),$$

where s_i is the number of elements (capacity) in the set $S^i = \{m \mid (i, m) \in S\}$ (we have $i \in \mathcal{I}(S) \iff S^i \neq \emptyset$), then former equalities yield

$$P_1 y_i = P_1 \mathbf{e}_i + P_1 A z_i \quad \& \quad \sum_{\mathcal{I}(S)} s_i y_i = \sum_{\mathcal{I}(S)} s_i \mathbf{e}_i.$$

Since $\mathcal{P}_i(x_i)$ is a convex set, we also obtain $y_i \in \mathcal{P}_i(x_i)$ for all $i \in \mathcal{I}(S)$. Next define a vector $t = (t_1, \dots, t_n)$ by putting

$$t_i = s_i/r, \quad i \in \mathcal{I}(S) \quad \& \quad t_i = 0, \quad i \in \mathcal{I} \setminus \mathcal{I}(S).$$

Now it has to be clear that in the previous equality *natural* numbers s_i can be equivalently substituted by *rational* t_i . Moreover, under imposed assumptions the described logical chain can be inverted, *i.e.*, one can show the sufficiency of described properties for some partially Pareto optimal allocation to be dominated via a coalition in a replica. Resuming the described arguments, we are going to a fuzzy core concept for incomplete markets, which is described below.

Recall that any n -dimension vector $t = (t_1, \dots, t_n) \neq 0$, $0 \leq t_i \leq 1 \quad \forall i \in \mathcal{I}$ is said to be a fuzzy coalition. Let $p^1 = (p_\sigma)_{\sigma \geq 1}$ be some fixed bundle of spot prices for future states of the world. Introduce now the notion of fuzzy p -domination.

A fuzzy coalition t is called p -dominating p^1 -feasible allocation $x \in \mathcal{A}_p(\mathcal{I})$, if there is $y^t \in \prod_{i \in \mathcal{I}} X_i$ such that

$$\sum_{i \in \mathcal{I}} t_i y_i^t = \sum_{i \in \mathcal{I}} t_i e_i \quad (3.2.25)$$

and for $\text{supp}(t) = \{i \in \mathcal{I} \mid t_i > 0\}$

$$y_i^t \succ_i x_i \quad \& \quad \exists z_i \in \mathbb{R}^k : P_1 y_i^t = P_1 e_i^1 + P_1 A z_i, \quad \forall i \in \text{supp}(t) \quad (3.2.26)$$

holds.

Notice that if \mathcal{E}^{in} is a smooth economy and allocation $x \in \text{int}X$, then the fact $x \notin \mathcal{C}^e(\mathcal{E}^{in})$ is equivalent to the ability of its p -domination via a fuzzy coalition with *rational* components relative to *partial Pareto prices*, corresponding to this allocation.

Definition 3.2.6 *The set $C_p^f(\mathcal{E}^{in})$ of all p^1 -feasible allocations $x \in \mathcal{A}_p(\mathcal{I})$, for which there is no p -dominating fuzzy coalition, is called a fuzzy p -core.*

In accordance with this definition, the concept of the p -fuzzy core differs from ordinary requirements only on the right-hand side of (3.2.26), where the potential financial marketability of consumption bundles relative to the given prices is additionally required. If, moreover, one requires these prices to be partially Pareto, then one achieves the notion of the incomplete market fuzzy core.

Definition 3.2.7 *The fuzzy core is the set $\mathcal{C}^f(\mathcal{E}^{in})$ of all feasible allocations satisfying the following properties:*

- (i) *x is partially optimal by Pareto, *i.e.*, for every $\sigma \geq 0$ it cannot be Pareto dominated via an allocation from subspace $L_x^\sigma = \{y = (y_i)_{\mathcal{I}} \in L^{\mathcal{I}} \mid y_i^{-\sigma} = x_i^{-\sigma}, \forall i \in \mathcal{I}\}$,*
- (ii) *$x \in \mathcal{A}_p(\mathcal{I})$, *i.e.*, it is p^1 -feasible, where $p^1 = (p_\sigma)_{\sigma=1}^{\sigma=s}$ is a bundle of σ -Pareto prices, existing due to item (i),*
- (iii) *$x \in C_p^f(\mathcal{E}^{in})$, *i.e.*, it belongs to the fuzzy p -core of an incomplete market.*

The following lemma states the key properties of a fuzzy p -core.

Lemma 3.2.7 *Let $p^1 = (p_\sigma)_{\sigma \geq 1}$ be a bundle of spot prices for future events and x be a p^1 -feasible allocation. Let $x \in C_{\mathbf{p}}^f(\mathcal{E}^{in})$ and $x_{i_0} \in \text{int}X_{i_0}$ for some i_0 . Then there is a vector $\bar{p} = (\bar{p}_0, \bar{p}_1, \dots, \bar{p}_s)$, such that $\bar{p}_\sigma \neq 0$ for all $\sigma \geq 0$ and*

$$\bar{p}y_i \geq \bar{p}e_i, \quad \forall y_i \in \mathcal{P}_i(\bar{x}_i) \mid \exists z_i \in \mathbb{R}^k : p_\sigma(y_i^\sigma - e_i^\sigma) = p_\sigma A_\sigma z_i, \quad \forall \sigma \geq 1 \quad (3.2.27)$$

is true for all $i \in \mathcal{I}$. Moreover, stronger property

$$\bar{p}y_{i_0} > \bar{p}e_{i_0}, \quad \forall y_{i_0} \in \mathcal{P}_{i_0}(\bar{x}_{i_0}) \quad (3.2.28)$$

is true for agent i_0 .

It is useful to compare the statement of this lemma with the statement of Lemma 3.2.2. The first difference is that the inequalities in Lemma 3.2.7 are non-strict. The second one is that on the right-hand side of the inequalities, the value of initial endowments is applied. One can find similarities between these facts with classical market case, when Pareto optimality is compared with quasi-equilibrium.

Proof of Lemma 3.2.7. The proof of lemma is reduced to the application of the separation theorem to a convex set properly constructed; this set correspond to the possibilities of fuzzy coalitions to dominate an allocation.

Analogously to formula (3.2.2), let us determine the subspaces

$$\mathcal{H}_i = \mathcal{H} + e_i, \quad \mathcal{H} = \{y \in \mathbb{R}^{l(s+1)} \mid \exists z \in \mathbb{R}^k : p_\sigma y^\sigma = p_\sigma A_\sigma z, \quad \forall \sigma \geq 1\}.$$

Next let us take any consumer $i_0 \in \mathcal{I}$, let $i_0 = 1$, and determine the following set

$$G = G(x) = \text{co} \left[(\mathcal{P}_1(x_1) - e_1) \cup \left(\bigcup_{i=2}^n [(\mathcal{P}_i(x_i) - e_i) \cap \mathcal{H}] \right) \right].$$

Now we show that if $0 \in G$, then there is a fuzzy coalition p -dominating given allocation x in the incomplete market. In fact, $0 \in G$ implies the existence of $t = (t_1, \dots, t_n) \geq 0$, $\sum t_i = 1$ such that for some $y_i \in \mathcal{P}_i(x_i)$

$$\sum_{i \in \mathcal{I}} t_i (y_i - e_i) = 0 \iff \sum_{i \in \mathcal{I}} t_i y_i = \sum_{i \in \mathcal{I}} t_i e_i \quad (3.2.29)$$

is true and moreover for $i = 2, \dots, n$

$$\exists z_i \in \mathbb{R}^k : P_1 y_i = P_1 e_i + P_1 A z_i$$

takes place. To check the fuzzy domination definition in part (3.2.26), it is sufficient to state the last relation for $i = 1 \in \text{supp}(t)$. To realize this, multiply (3.2.29) on matrix P_1 , which after transformations due to $t_1 \neq 0$ yields

$$P_1 y_i = P_1 e_i + P_1 A \left(- \sum_{i=2}^n \frac{t_i}{t_1} z_i \right).$$

Thus one can take $z_1 = -\sum_{i=2}^n \frac{t_i}{t_1} z_i$ as the necessary solution (portfolio) for agent 1. As a result, we conclude that coalition t can p -dominate the allocation.

Therefore, for every element x of a fuzzy core of incomplete market it has to be that $0 \notin G$ and since $\text{int } G \neq \emptyset$ (because of $\text{int } X_1 \neq \emptyset$ due to **(A)** and $\text{int } \mathcal{P}_1(x_1) \neq \emptyset$), then one can apply the separation theorem and find such *non-zero* \bar{p} that

$$\langle \bar{p}, G \rangle \geq 0.$$

Since $\mathcal{P}_1(x_1) - \mathbf{e}_1$ and $(\mathcal{P}_i(x_i) - \mathbf{e}_i) \cap \mathcal{H}$, $i = 2, \dots, n$ are the subsets of G , we conclude

$$\langle \bar{p}, \mathcal{P}_1(x_1) \rangle \geq \bar{p}\mathbf{e}_1 \quad \& \quad \langle \bar{p}, \mathcal{P}_i(x_i) \cap (\mathbf{e}_i + \mathcal{H}) \rangle \geq \bar{p}\mathbf{e}_i, \quad \forall i = 2, \dots, n.$$

Moreover, the first of these inequalities due to **(S)**, $x_1 \in \text{int } X_1$ and $\bar{p} \neq 0$ implies $\bar{p}_\sigma \neq 0$, $\forall \sigma$ in a standard way (presuming the contrary, one can conclude $\bar{p} = 0$, that is impossible). Lemma is proved. \blacksquare

Furthermore, notice that applying the local non-satiation assumption for agents' preferences in the present ($\sigma = 0$ and **(S)**) and passing to limits in the left-hand side of inequalities from (3.2.27), one can state $\bar{p}x_i \geq \bar{p}\mathbf{e}_i \quad \forall i \in \mathcal{I}$, that due to $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$ eventually yields

$$\bar{p}x_i = \bar{p}\mathbf{e}_i \quad \forall i \in \mathcal{I}.$$

Finally, let $x \in \text{int } X$, the economy be smooth and prices $p^1 = (p_\sigma)_{\sigma \geq 1}$ be partially Pareto optimal. Then the first, non-strict inequalities from the left side of (3.2.27) are turned into strict ones. Therefore, now the conditions of Lemma 3.2.2 and its Corollary 3.2.2 are true. This is why, due to similar arguments applied in Corollary 3.2.2 proof, one can state the following

Corollary 3.2.6 *Let \mathcal{E}^{in} be a smooth incomplete market and $\bar{x} \in \text{int } X \cap \mathcal{C}^f(\mathcal{E}^{in})$. Then there is a vector $\bar{p} = (\bar{p}_0, \bar{p}_1, \dots, \bar{p}_s)$ such that $\bar{p}_\sigma \neq 0$ for all $\sigma \geq 0$, and for $\bar{q} = \sum_{\sigma=1}^s \bar{p}_\sigma A_\sigma$ and each $i \in \mathcal{I}$*

$$\bar{p}_0 y_i^0 > \bar{p}_0 \mathbf{e}_i^0 + \bar{q} z_i \quad \forall y_i \in \mathcal{P}_i(\bar{x}_i) \mid \exists z_i \in \mathbb{R}^k : \bar{p}_\sigma y_i^\sigma = \bar{p}_\sigma \mathbf{e}_i^\sigma + \bar{p}_\sigma A_\sigma z_i \quad \forall \sigma \geq 1 \quad (3.2.30)$$

is true.

Applying this corollary one can easily prove the asymptotic theorem which states that every incomplete market Edgeworth equilibrium is a *GEI*-equilibrium.

Theorem 3.2.5 *Let \mathcal{E}^{in} be a smooth incomplete market. Then*

$$\mathcal{C}^f(\mathcal{E}^{in}) = \mathcal{C}^e(\mathcal{E}^{in}) \quad \& \quad \text{int } X \cap \mathcal{C}^e(\mathcal{E}^{in}) = W(\mathcal{E}^{in}) \cap \text{int } X.$$

Proof of Theorem 3.2.5. First let us show $\mathcal{C}^e(\mathcal{E}^{in}) = \mathcal{C}^f(\mathcal{E}^{in})$. To do it let us state the inclusion $\mathcal{C}^e(\mathcal{E}^{in}) \subseteq \mathcal{C}^f(\mathcal{E}^{in})$ (the inverse inclusion is true due to definitions). Assuming to the contrary find an allocation $x \in \mathcal{C}^e(\mathcal{E}^{in})$, which is dominated via a fuzzy coalition $t \neq 0$. By definition this means the existence of $y^t \in \prod_{\mathcal{I}} X_i$, satisfying relations (3.2.25) and (3.2.26). We can show then that the allocation x is dominated via fuzzy coalition $q = (q_1, \dots, q_n)$ with *rational* components q_i , $i \in \mathcal{I}$. With this in mind for $t_i > 0$, put

$$x'_i = (t_i/q_i)y_i + (1 - t_i/q_i)\mathbf{e}_i \implies q_i(x'_i - \mathbf{e}_i) = t_i(y_i - \mathbf{e}_i),$$

where *rational* q_i satisfies the condition $t_i \leq q_i \leq 1$, and for $t_i = 0$ define $q_i = 0$ and $x'_i = y_i^t$. Since $\mathbf{e}_i \in X_i$, then $x' = (x'_i)_{\mathcal{I}} \in \prod_{\mathcal{I}} X_i$ and

$$\sum_{i \in \mathcal{I}} q_i(x'_i - \mathbf{e}_i) = 0.$$

However due to (A), the scalars q_i can be chosen in such a way that $x'_i \in \mathcal{P}_i(x_i)$ is true for all i , satisfying $q_i > 0$. Moreover, for these i

$$\exists z'_i \in \mathbb{R}^k : P_1 x'_i = P_1 \mathbf{e}_i + P_1 A z'_i$$

holds relative to σ -Pareto prices, corresponding to x , as soon as similar relations are true for y^t (put $z'_i = \frac{q_i}{t_i} z_i$). We obtain a contradiction with the choice of $x \in \mathcal{C}^e(\mathcal{E}^{in})$.

So, the coincidence of a fuzzy core with the set of Edgeworth equilibria has been proved for an incomplete market. Now let us apply Corollary 3.2.6 and using arguments fully equivalent to those described in the second part of Theorem 3.2.1, we can state $\text{int}X \cap \mathcal{C}^f(\mathcal{E}^{in}) = W(\mathcal{E}^{in}) \cap \text{int}X$. Theorem 3.2.5 is proved. \blacksquare

3.3 Contractual incomplete core in examples

In finishing this section, let us consider some examples of incomplete market and describe our core concept in their context.

3.3.1 Incomplete market with one asset

Example 3.3.1 (A MARKET WITH ONE ASSET) Let us consider an economic model with two consumers, two states of the world in the future and no present. Note, that the last feature (there is no present) is not an essential factor, since formally the present can always be added to the model description and moreover, to save the non-satiation assumption, one can presume that agents are full antagonists in the present—let agents' preferences be separable and in the present let them be defined via linear monotonic and equal utility functions for $\sigma = 0$. Also in future events, $\sigma = 1, 2$, there are two commodities and let $x = (x^{\sigma=1}, x^{\sigma=2})$ correspond to the consumption of the 1st agent, but $y = (y^{\sigma=1}, y^{\sigma=2})$ be the consumption program for the 2nd one. Let $X_i = \mathbb{R}_+^4$, $i = 1, 2$, a total vector of initial endowments $\mathbf{e} = (\mathbf{e}_i)_{i=1,2} \in \mathbb{R}_+^8$ satisfy $\mathbf{e}_i \gg 0$ $i = 1, 2$, and let utilities be described by functions

$$u_i(x) = \rho_{\sigma=1}^1 U_1^{\sigma=1}(x^{\sigma=1}) + \rho_{\sigma=2}^1 U_1^{\sigma=2}(x^{\sigma=2}),$$

$$u_2(y) = \rho_{\sigma=1}^2 U_2^{\sigma=1}(y^{\sigma=1}) + \rho_{\sigma=2}^2 U_2^{\sigma=2}(y^{\sigma=2}),$$

where for $i, \sigma = 1, 2$ real $\rho_{\sigma}^i > 0$, and U_i^{σ} are (logarithmic) Cobb–Douglas functions:

$$U_1^{\sigma}(z) = \alpha_{\sigma} \ln(z_1) + (1 - \alpha_{\sigma}) \ln(z_2), \quad 0 < \alpha_{\sigma} < 1,$$

$$U_2^{\sigma}(z) = \beta_{\sigma} \ln(z_1) + (1 - \beta_{\sigma}) \ln(z_2), \quad 0 < \beta_{\sigma} < 1.$$

The analysis of an incomplete market core will be based on key Lemma 3.2.5, which for Cobb–Douglas functions gives a complete description of core allocations. Now to apply item (i) of this lemma, we first need to give the constructive description of partially Pareto optimal allocations. As soon as utilities are separable relative to events, the σ -Pareto optimality of allocation (x, y) is completely determined by consumption bundles (x^{σ}, y^{σ}) for this event (in the general case it may depend on consumption at other events), *i.e.*, by functions $U_1^{\sigma}(\cdot)$, $U_2^{\sigma}(\cdot)$ and via total initial endowments $\bar{\mathbf{e}}^{\sigma} = \mathbf{e}_1^{\sigma} + \mathbf{e}_2^{\sigma}$. In other words, when utilities are separable, the set of partially Pareto optimal allocations may be represented as the Cart's product (by σ) of Pareto boundaries which correspond to spot markets. Therefore, we first have to describe the Pareto boundary for a model reduced to σ .

Let us calculate in general form the Pareto boundary for a classical economy with Cobb–Douglas utility functions. Let there be two goods and two consumers, whereas above x denotes the 1st agent consumption and y is the consumption of 2nd one. Due to individual rationality we are interested in allocations from the interior of consumption sets, *i.e.*, $(x, y) \gg 0$. In such a case, for each Pareto optimal allocation one can put into correspondence (non-zero) price vector p , which has to be collinear to the gradients of the utility functions. This vector can be found unambiguously up to normalization; this is why the existence conditions of $p = (p_1, p_2) \gg 0$ and $\lambda > 0$ such that

$$p = \nabla U_1 = \left(\frac{\alpha}{x_1}, \frac{1 - \alpha}{x_2} \right) \iff x = \left(\frac{\alpha}{p_1}, \frac{1 - \alpha}{p_2} \right) \quad \& \quad \langle p, x \rangle = 1,$$

$$p = \lambda \nabla U_2 = \lambda \left(\frac{\beta}{y_1}, \frac{1 - \beta}{y_2} \right) \iff y = \lambda \left(\frac{\beta}{p_1}, \frac{1 - \beta}{p_2} \right) \quad \& \quad \langle p, y \rangle = \lambda,$$

are necessary and sufficient for allocation (x, y) to be Pareto optimal. Taking into account $x + y = \bar{\mathbf{e}} = (\bar{\mathbf{e}}^1, \bar{\mathbf{e}}^2)$, from the right-hand side of last relations one can find

$$\left(\frac{\alpha}{p_1}, \frac{1 - \alpha}{p_2} \right) + \lambda \left(\frac{\beta}{p_1}, \frac{1 - \beta}{p_2} \right) = \bar{\mathbf{e}} \implies p = \left(\frac{\alpha}{\bar{\mathbf{e}}^1}, \frac{1 - \alpha}{\bar{\mathbf{e}}^2} \right) + \lambda \left(\frac{\beta}{\bar{\mathbf{e}}^1}, \frac{1 - \beta}{\bar{\mathbf{e}}^2} \right).$$

Thus real $\lambda > 0$ parametrizes the Pareto boundary unambiguously (since x, y can be unambiguously found by p and λ). Moreover, it is clear that this analysis can be easily extended to a more general case, that is, for any (finite) number of goods and consumers. Notice only that then the number of (positive) parameters determining Pareto boundary is equal to the number of agents minus one.

Further, let us turn to an incomplete economy and initially consider the case of an *unique* real asset. Let, for example, this asset a have the form

$$a = (a_{\sigma=1}, a_{\sigma=2}), \quad a_{\sigma=1} = -a_{\sigma=2} = (1, 0).$$

Then for given spot prices, financial returns matrix P_1A for trade portfolios has the form

$$P_1A = \begin{pmatrix} p_{\sigma=1}^1 \\ -p_{\sigma=2}^1 \end{pmatrix}.$$

Now let us turn to item (ii) from Lemma 3.2.5, which requires a current partial Pareto optimal allocation (x, y) to be p^1 -feasible and to belong to the p -core of the economy for partially Pareto prices p corresponding to this allocation.

To simplify further our analysis, let us assume without loss of generality that the total endowment of each commodity in every state of world is equal to 1, *i.e.*, we put $\bar{\mathbf{e}}_{\sigma=1} = \bar{\mathbf{e}}_{\sigma=2} = (1, 1)$. Then partial Pareto optimality for prices means that for some $\lambda > 0$ and $\gamma > 0$ we have

$$\begin{aligned} p_{\sigma=1} &= \left(\frac{\alpha_{\sigma=1}}{\mathbf{e}_{\sigma=1}^1}, \frac{1 - \alpha_{\sigma=1}}{\mathbf{e}_{\sigma=1}^2} \right) + \lambda \left(\frac{\beta_{\sigma=1}}{\mathbf{e}_{\sigma=1}^1}, \frac{1 - \beta_{\sigma=1}}{\mathbf{e}_{\sigma=1}^2} \right) \quad \bar{\mathbf{e}}_{\sigma=(1,1)} \begin{matrix} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{matrix} \\ p_{\sigma=1} &= (\alpha_1 + \lambda\beta_1, 1 - \alpha_1 + \lambda(1 - \beta_1)), \end{aligned} \quad (3.3.1)$$

$$\begin{aligned} p_{\sigma=2} &= \left(\frac{\alpha_{\sigma=2}}{\mathbf{e}_{\sigma=2}^1}, \frac{1 - \alpha_{\sigma=2}}{\mathbf{e}_{\sigma=2}^2} \right) + \gamma \left(\frac{\beta_{\sigma=2}}{\mathbf{e}_{\sigma=2}^1}, \frac{1 - \beta_{\sigma=2}}{\mathbf{e}_{\sigma=2}^2} \right) \quad \bar{\mathbf{e}}_{\sigma=(1,1)} \begin{matrix} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{matrix} \\ p_{\sigma=2} &= (\alpha_2 + \gamma\beta_2, 1 - \alpha_2 + \gamma(1 - \beta_2)). \end{aligned} \quad (3.3.2)$$

The condition of p^1 -feasibility states that there is such real z that $P_1x = P_1\mathbf{e}_1 + P_1Az$ that via the structure of P_1A yields

$$p_{\sigma=1}x^{\sigma=1} = p_{\sigma=1}\mathbf{e}_1^{\sigma=1} + p_{\sigma=1}^1z, \quad p_{\sigma=2}x^{\sigma=2} = p_{\sigma=2}\mathbf{e}_1^{\sigma=2} - p_{\sigma=2}^1z.$$

This, due to $p_{\sigma}x^{\sigma} = 1$, is equivalent to

$$p_{\sigma=2}^1 + p_{\sigma=1}^1 = p_{\sigma=2}^1 \langle p_{\sigma=1}, \mathbf{e}_1^{\sigma=1} \rangle + p_{\sigma=1}^1 \langle p_{\sigma=2}, \mathbf{e}_1^{\sigma=2} \rangle.$$

Now applying (3.3.1), (3.3.2), we find

$$\begin{aligned} \alpha_2 + \gamma\beta_2 + \alpha_1 + \lambda\beta_1 &= \\ &= (\alpha_2 + \gamma\beta_2) \langle (\alpha_1 + \lambda\beta_1), 1 - \alpha_1 + \lambda(1 - \beta_1) \rangle + \\ &\quad + (\alpha_1 + \lambda\beta_1) \langle (\alpha_2 + \gamma\beta_2, 1 - \alpha_2 + \gamma(1 - \beta_2)), \mathbf{e}_1^{\sigma=2} \rangle. \end{aligned} \quad (3.3.3)$$

Thus an allocation (x, y) is p^1 -feasible if and only if, when determining parameters, $\lambda > 0$, $\gamma > 0$ satisfy equation (3.3.3).

Next let us study the property of an allocation (x, y) being dominated by no coalition. Since the list of coalitions contains only singleton and grand coalitions, then this property is equivalent to

$$(i) \quad u_1(x) \geq u_1(\mathbf{e}_1) \quad \& \quad u_2(y) \geq u_2(\mathbf{e}_2),$$

(ii) the allocation (x, y) is Pareto $H(p^1)$ -optimal relative to partial Pareto prices p^1 .

Condition (ii) requires subsequent analysis. To do this, apply Corollary 3.2.1. In our context Corollary 3.2.1 states that an allocation is H -optimal iff there is such $\mu > 0$ that vector $\tilde{p} = \nabla u_2(y) - \mu p$ satisfies

$$\tilde{p}_{\sigma=1} a_{\sigma=1} + \tilde{p}_{\sigma=2} a_{\sigma=2} = 0 \implies \tilde{p}_{\sigma=1}^1 - \tilde{p}_{\sigma=2}^1 = 0,$$

that due to the relationship between $\nabla u_2(y)$ and p gives

$$\left(\frac{1}{\lambda} - \mu\right) p_{\sigma=1}^1 - \left(\frac{1}{\gamma} - \mu\right) p_{\sigma=2}^1 = 0 \iff \mu(p_{\sigma=2}^1 - p_{\sigma=1}^1) = \frac{1}{\gamma} p_{\sigma=2}^1 - \frac{1}{\lambda} p_{\sigma=1}^1.$$

The last relation is disintegrated into the following variants:

- a) $p_{\sigma=2}^1 = p_{\sigma=1}^1 \implies \lambda = \gamma,$
- b) $p_{\sigma=2}^1 > p_{\sigma=1}^1 \implies \lambda p_{\sigma=2}^1 > \gamma p_{\sigma=1}^1,$
- c) $p_{\sigma=2}^1 < p_{\sigma=1}^1 \implies \lambda p_{\sigma=2}^1 < \gamma p_{\sigma=1}^1.$

So, taking into account relations (3.3.1) and (3.3.2), an allocation is H -optimal iff one of the below relations

$$\alpha_2 + \gamma\beta_2 = \alpha_1 + \lambda\beta_1 \quad \& \quad \lambda = \gamma, \quad (3.3.4)$$

$$\alpha_2 + \gamma\beta_2 > \alpha_1 + \lambda\beta_1 \quad \& \quad \lambda(\alpha_2 + \gamma\beta_2) > \gamma(\alpha_1 + \lambda\beta_1), \quad (3.3.5)$$

$$\alpha_2 + \gamma\beta_2 < \alpha_1 + \lambda\beta_1 \quad \& \quad \lambda(\alpha_2 + \gamma\beta_2) < \gamma(\alpha_1 + \lambda\beta_1) \quad (3.3.6)$$

is true.

Let us resume our given analysis. We supposed without loss of generality that $\bar{\mathbf{e}}_{\sigma=1} = \bar{\mathbf{e}}_{\sigma=2} = (1, 1)$. The allocations from the core are unambiguously determined via real parameters $\lambda > 0, \gamma > 0$, which have to satisfy (3.3.3) and one of the relations (3.3.4)–(3.3.6). Then the first agent's consumption is determined due to

$$x^{\sigma=1} = \left(\frac{\alpha_1}{p_{\sigma=1}^1}, \frac{1 - \alpha_1}{p_{\sigma=1}^2} \right), \quad x^{\sigma=2} = \left(\frac{\alpha_2}{p_{\sigma=2}^1}, \frac{1 - \alpha_2}{p_{\sigma=2}^2} \right),$$

where $p_{\sigma=1}$ and $p_{\sigma=2}$ are determined by λ, γ due to formulas (3.3.1), (3.3.2), and then the 2nd agent's consumption y is

$$y = (1, 1, 1, 1) - x.$$

Moreover, the following relations

$$u_1(x) \geq u_1(\mathbf{e}_1) \quad \& \quad u_2(y) \geq u_2(\mathbf{e}_2)$$

have to be true too. These requirements are not conflicting, since the equilibrium allocation, which does exist, satisfies all of them. In the general case, a core is represented as an image of all determining parameters, obtained as the intersection of some hyperbola defined by (3.3.3) and a set defined as the union of three sets, defined by (3.3.4)–(3.3.6). The properties of individual rationality of allocation have to be fulfilled in addition.

In conclusion, let me say some words about the set that is determined via relations (3.3.4)–(3.3.6). It is clear that (3.3.4) can be true only for special parameters $\alpha_\sigma, \beta_\sigma, \sigma = 1, 2$ (either both $\alpha_1 - \alpha_2$ and $\beta_1 - \beta_2$ are not zero simultaneously and have different signs, or $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ simultaneously), and in the general case, defines the empty or a singleton set. Constraints (3.3.5) and (3.3.6) are more involved. In fact, consider a straight line $\alpha_2 + \gamma\beta_2 = \alpha_1 + \lambda\beta_1$ and hyperbola $\lambda(\alpha_2 + \gamma\beta_2) = \gamma(\alpha_1 + \lambda\beta_1)$. The line intersects the positive orthant by some ray with a positive directing vector and separates the plane into two open half-planes, a left and right one. The hyperbola has asymptotes paralleled to coordinate axes which are intersected at point $(\lambda_0, \gamma_0) = (\frac{\alpha_1}{\beta_2 - \beta_1}, \frac{-\alpha_2}{\beta_2 - \beta_1})$. Since $(0, 0)$ satisfies the hyperbola equation, then a path going across this point, and only this path intersects the orthant. Moreover, note that the point of intersection of hyperbola and the line exactly corresponds to condition (3.3.4) (for $(\lambda, \gamma) \gg 0$). Next, let for example $\beta_2 - \beta_1 > 0$. Then the set determined by relation (3.3.5) can be described as the intersection of an open epigraph left hyperbola path with the left upper half-plane, defined by our line. Relation (3.3.6) is true at the points of the interior of a set, which supplements the epigraph of the left hyperbola path up to the positive orthant, being intersected with the right lower half-plane. The union of these two sets with the point of intersection of hyperbola and line completely describes the collection of all points $(\lambda, \gamma) \gg 0$, satisfying conditions (3.3.4)–(3.3.6). The case $\beta_2 - \beta_1 < 0$ is considered in a similar way. However, now the point of hyperbola asymptotes intersection has a negative first component and only the right path of the hyperbola intersects the orthant's interior (it goes across the origin). This is why the set we are interested in is represented as a union of three sets. The first one is the intersection of the right open half-plane, defined due to a line with a part of the orthant's interior restricted by the right hyperbola path (a part of epigraph). The second one is the intersection of the left open half-plane with a part of the orthant's interior, from which one has to remove the subgraph of the right hyperbola path. Finally, one needs to add the point of hyperbola intersection with the line if it does exist. ■

3.3.2 Hart's example

Further, let us consider a more complex incomplete market example, in which utilities are described in the same manner as in Example 3.3.1; however, there are two real assets. The particular case of this market is known in literature as Hart's example, in which *GEI*-equilibrium may not exist. Notice conditions quarantining existence *Anderson, Raimondo (2007)*.

Example 3.3.2 (HART'S EXAMPLE) In the context of the economy described in Example 3.3.1, let us consider a financial market with *two* assets having the following structure. Let

$$a^1 = (a_{\sigma=1}^1, a_{\sigma=2}^1), \quad a_{\sigma=1}^1 = a_{\sigma=2}^1 = (1, 0)$$

be the first asset and let

$$a^2 = (a_{\sigma=1}^2, a_{\sigma=2}^2), \quad a_{\sigma=1}^2 = a_{\sigma=2}^2 = (0, 1)$$

be the second one. Thus the buying of the 1st asset unit promises the delivery of a unit of commodity 1 for the future (at every event). Analogous delivery of the second asset is a unit of the 2nd commodity for every future event. As a whole, the matrix A of real returns has the form

$$A = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array} \right] = A_{\sigma=1} \\ = A_{\sigma=2}$$

From this one can find the matrix of financial returns P_1A for the trade portfolios of the financial sector relative to given prices p^1 for spot markets:

$$P_1A = \left[\begin{array}{c|c} p_{\sigma=1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & p_{\sigma=1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \hline p_{\sigma=2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & p_{\sigma=2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array} \right] = \left[\begin{array}{c|c} p_{\sigma=1}^1 & p_{\sigma=1}^2 \\ \hline p_{\sigma=2}^1 & p_{\sigma=2}^2 \end{array} \right] = \left[\begin{array}{c} p_{\sigma=1} \\ p_{\sigma=2} \end{array} \right].$$

Similar to Example 3.3.1, the analysis of the core is based on Lemma 3.2.5. Due to item (i) of this lemma, a partial Pareto prices $p^1 = (p_{\sigma})_{\sigma=1,2}$ may be put into correspondence to every core allocation (unambiguously). Now in view of Example 3.3.1 analysis for some partially Pareto prices, conveniently normalized by (3.3.1) and (3.3.2) (here $\bar{e}_{\sigma=1} = \bar{e}_{\sigma=2} = (1, 1)$ without loss of generality), we obtain

$$P_1A = \left[\begin{array}{c|c} \alpha_1 + \lambda\beta_1 & 1 - \alpha_1 + \lambda(1 - \beta_1) \\ \hline \alpha_2 + \gamma\beta_2 & 1 - \alpha_2 + \gamma(1 - \beta_2) \end{array} \right],$$

where $\lambda > 0$ and $\gamma > 0$ are some real parameters, which unambiguously determine the partial Pareto boundary. Due to item (ii) of Lemma 3.2.5 in order to current partially Pareto optimal allocation to be an element of the core, it is also necessary (and sufficient) that the allocation be p^1 -feasible and be an element of p -core for its partially Pareto prices p^1 . The condition of p^1 -feasibility says that there is a *vector* $z = (z_1, z_2)$ such that $P_1x = P_1e_1 + P_1Az$. This, for the chosen normalization of prices (it implies $p_{\sigma=1}x_{\sigma=1} = p_{\sigma=2}x_{\sigma=2} = 1$), is equivalent to the fact that the system of linear equations

$$P_1Az = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - P_1e_1, \quad (3.3.7)$$

has a solution relative to z . First note that if square matrix P_1A is non-degenerated, then a solution of system (3.3.7) does exist for every right-hand side, and therefore for a given one. On the contrary, if the matrix columns are linearly dependent (degeneration), then a solution of the system may exist only if there is a solution for the system with only one unknown variable, where instead of matrix P_1A one can take a matrix consisting of one (any) column of the initial matrix with the same right-hand side. In other words, if the matrix is degenerated, then we obtain a model with the only asset. Thus for non-degenerated matrix P_1A , the condition of p^1 -feasibility is true automatically, for the degenerated one it is not the case and it turns to be a

non-trivial condition. A matrix P_1A is degenerated if and only if its determinant is zero, *i.e.*, $\det(P_1A) = 0 \iff$

$$(\alpha_1 + \lambda\beta_1)(1 - \alpha_2 + \gamma(1 - \beta_2)) = (\alpha_2 + \gamma\beta_2)(1 - \alpha_1 + \lambda(1 - \beta_1)). \quad (3.3.8)$$

In view of (3.3.7) the condition of p^1 -feasibility takes place only if for some real z

$$1 = p_{\sigma=1}e_1^{\sigma=1} + p_{\sigma=1}^1z, \quad 1 = p_{\sigma=2}e_1^{\sigma=2} + p_{\sigma=2}^1z$$

is true, which is equivalent to

$$p_{\sigma=2}^1 - p_{\sigma=1}^1 = p_{\sigma=2}^1 \langle p_{\sigma=1}, e_1^{\sigma=1} \rangle - p_{\sigma=1}^1 \langle p_{\sigma=2}, e_1^{\sigma=2} \rangle.$$

Now applying (3.3.1) and (3.3.2), this equation may be standardly rewritten as an equation relative to $\lambda > 0$ and $\gamma > 0$, similar to (3.3.3), which I omitted. It is important that this relation has to be fulfilled together with (3.3.8). Moreover, due to the fact that the allocation has to belong to the p -core, it is necessary to require the following relations to be true:

$$(i) \quad u_1(x) \geq u_1(e_1) \quad \& \quad u_2(y) \geq u_2(e_2),$$

(ii) allocation (x, y) is Pareto $H(p^1)$ -optimal for partial Pareto prices p^1 .

For the analysis of (ii) one can apply Corollary 3.2.1 which taking $i_0 = 1$ and $p = \nabla u_1(y)$, now states that an allocation is $H(p^1)$ -optimal iff there is $\mu > 0$ such that (here $\bar{p} = p$ is a vector of partial Pareto prices) $\tilde{p} = \nabla u_2(y) - \mu p$ satisfies

$$\tilde{p}_{\sigma=1}a_{\sigma=1}^1 + \tilde{p}_{\sigma=2}a_{\sigma=2}^2 = 0 \implies \tilde{p}_{\sigma=1}^1 + \tilde{p}_{\sigma=2}^1 = 0.$$

Now from partial Pareto optimality we have $\nabla u_2(y) = (\lambda p_{\sigma=1}, \gamma p_{\sigma=2})$ for some $\lambda > 0$, $\gamma > 0$, that gives

$$\left(\frac{1}{\lambda} - \mu\right)p_{\sigma=1}^1 + \left(\frac{1}{\gamma} - \mu\right)p_{\sigma=2}^1 = 0 \iff \mu(p_{\sigma=2}^1 + p_{\sigma=1}^1) = \frac{1}{\gamma}p_{\sigma=2}^1 + \frac{1}{\lambda}p_{\sigma=1}^1.$$

Clearly, for positive prices and other parameters $\mu > 0$ that we need do exist. It means that (ii) is always true and only p^1 -feasibility and individual rationality (i) are essential ones.

Now let us turn to the case when (3.3.8) is false, *i.e.*, the *matrix of financial returns* for partial Pareto prices is *non-degenerated*. In this case an allocation belongs to the incomplete core only if it belongs to the classical core, *i.e.*, requirements (i) and (ii) are true, where (ii) is transformed into *ordinary Pareto optimality*. The last requirement can be expressed in a standard way as the requirement of utilities' gradients to be collinear. Due to their relationship with the partial Pareto prices and according to chosen normalization, we have

$$\nabla u_1(x) = (\rho_{\sigma=1}^1 p_{\sigma=1}, \rho_{\sigma=2}^1 p_{\sigma=2}) \quad \& \quad \nabla u_2(y) = \left(\frac{\rho_{\sigma=1}^2}{\lambda} p_{\sigma=1}, \frac{\rho_{\sigma=2}^2}{\gamma} p_{\sigma=2}\right)$$

that via the collinearity of vectors yields

$$\lambda \rho_{\sigma=1}^1 / \rho_{\sigma=1}^2 = \gamma \rho_{\sigma=2}^1 / \rho_{\sigma=2}^2 \iff \lambda = \gamma \frac{\rho_{\sigma=1}^2 \rho_{\sigma=2}^1}{\rho_{\sigma=1}^1 \rho_{\sigma=2}^2}. \quad (3.3.9)$$

Thus if parameters $\lambda > 0$, $\gamma > 0$ satisfy (3.3.9) and simultaneously *do not satisfy* (3.3.8), then the generated individually rational allocation belongs to the incomplete core.

Further, let us consider properly *Hart's example*, which corresponds to our model with two assets under an additional condition:

$$\rho_{\sigma}^1 = \rho_{\sigma}^2 = \rho_{\sigma}, \quad \sigma = 1, 2 \quad \& \quad \alpha_{\sigma=1} = \alpha_{\sigma=2} = \alpha, \quad \beta_{\sigma=1} = \beta_{\sigma=2} = \beta.$$

Now one can note that the first part of this requirement and (3.3.9) imply $\lambda = \gamma$, *i.e.*, an allocation is optimal by Pareto iff $\lambda = \gamma$. Initial endowments for Hart's example are determined as

$$\mathbf{e}_1^{\sigma=1} = (1 - \varepsilon, 1 - \varepsilon), \quad \mathbf{e}_1^{\sigma=2} = (\varepsilon, \varepsilon), \quad \mathbf{e}_1^{\sigma} + \mathbf{e}_2^{\sigma} = (1, 1), \quad \sigma = 1, 2,$$

where real $0 < \varepsilon < 1$.

Let us show that for Hart's example the set of allocations from the incomplete core which corresponds to the *non-degenerated* matrix of financial returns, forms the empty set. In fact, if $(x, y) \in \mathcal{C}(\mathcal{E}^{in})$ and $\det(\mathbf{P}_1 \mathbf{A}) \neq 0$, then (x, y) is Pareto optimal and $\lambda = \gamma$. However then from $\alpha_{\sigma} = \alpha$, $\beta_{\sigma} = \beta$, $\sigma = 1, 2$ one can conclude the coincidence of matrix $\mathbf{P}_1 \mathbf{A}$ rows and therefore $\det(\mathbf{P}_1 \mathbf{A}) = 0$ —contradiction.

Next consider the second possibility: $(x, y) \in \mathcal{C}(\mathcal{E}^{in})$ and $\det(\mathbf{P}_1 \mathbf{A}) = 0$. It may be realized only if system (3.3.7) is solvable. The last one for this case is equivalent to the solvability of system

$$\begin{pmatrix} \alpha + \lambda\beta \\ \alpha + \gamma\beta \end{pmatrix} z = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} (\alpha + \lambda\beta, 1 - \alpha + \lambda(1 - \beta))\mathbf{e}_1^{\sigma=1} \\ (\alpha + \gamma\beta, 1 - \alpha + \gamma(1 - \beta))\mathbf{e}_1^{\sigma=2} \end{pmatrix};$$

substituting for the value of initial endowments and realizing some elementary transformations we find

$$\begin{pmatrix} \alpha + \lambda\beta \\ \alpha + \gamma\beta \end{pmatrix} z = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} (1 - \varepsilon)(1 + \lambda) \\ \varepsilon(1 + \gamma) \end{pmatrix}. \quad (3.3.10)$$

Next, substituting $\alpha_{\sigma} = \alpha$, $\beta_{\sigma} = \beta$, $\sigma = 1, 2$ in (3.3.8) and doing transformations, we obtain

$$\det(\mathbf{P}_1 \mathbf{A}) = 0 \iff (\gamma - \lambda)(\alpha - \beta) = 0.$$

Thus the matrix of financial returns is degenerated only if $\gamma = \lambda$ (as seen above) or when $\alpha = \beta$. In the first case, (3.3.10) may be fulfilled only if $\varepsilon = 1/2$. In the second case, (3.3.10) is reduced to

$$\alpha \begin{pmatrix} 1 + \lambda \\ 1 + \gamma \end{pmatrix} z = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} (1 - \varepsilon)(1 + \lambda) \\ \varepsilon(1 + \gamma) \end{pmatrix} = \begin{pmatrix} \lambda(\varepsilon - 1) + \varepsilon \\ 1 + \varepsilon + \varepsilon\gamma \end{pmatrix}.$$

This system is solvable only if

$$\frac{\lambda(\varepsilon - 1) + \varepsilon}{1 + \varepsilon + \varepsilon\gamma} = \frac{1 + \lambda}{1 + \gamma} \iff 1 + 2\lambda + \lambda\gamma = 0.$$

However, the last equation cannot be solved for $\lambda > 0$ and $\gamma > 0$.

Let us resume our analysis. For Hart's example the core of an incomplete market is a *nonempty* set *only* for $\varepsilon = 1/2$, and for this case the incomplete core coincides with the classical market core (since then the solvability of (3.3.10) is equivalent to an allocation be Pareto optimal). For $\varepsilon \neq 1/2$, the *core is empty*, which can be explained via the specific features of given model parameters: preferences and real assets. This peculiarity is such that contracting each other at every nature event and applying real assets (in a given structure) in the present, the agents are not able to arrive at Pareto optimal allocation, regardless of the fact that potentially there are enough assets (so much as there are many future states of the world). In other words any feasible net of contracts is unstable in the sense that coalition $\{1, 2\}$ of all market operators is able to find an opportunity to sign a new exchange contract, taking into account the ability to break some of the given contracts (remember that for future events one can break virtual contracts). Figure 3.3.2 illustrates the case. In

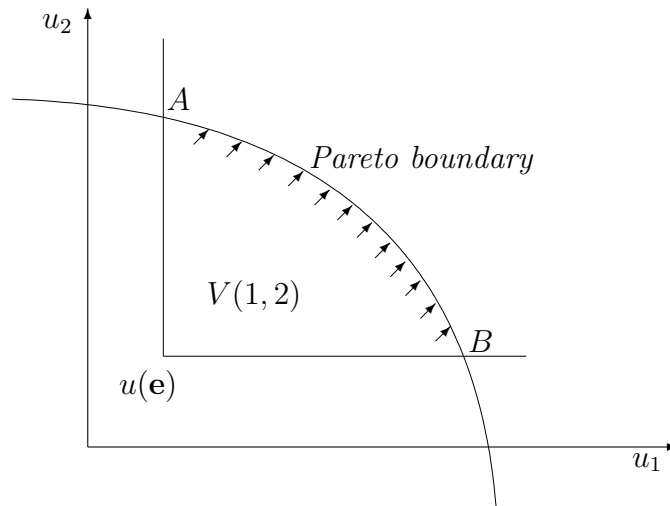


Figure 3.3.2: *Classical and incomplete core in Hart's example*

this figure the possibilities $V(1, 2) = u[\mathcal{A}(X)]$ of coalition $\{1, 2\}$ are described in the criteria space of “utilities” in a standard manner. The abilities of singleton coalitions are presented via vector $u(\mathbf{e}) = (u_1(\mathbf{e}_1), u_2(\mathbf{e}_2))$. Curve AB , representing a part of the Pareto boundary, corresponds to the standard core of the market. For an incomplete market the points of this curve are not available for consumers since a bundle of utilities from the curve is realized via an allocation which is not p^1 -feasible. Notice that one can infinitesimally closely approach the points of this curve via allocations, which are *partially* Pareto optimal and p^1 -feasible relative to partial Pareto prices. In fact, partial Pareto boundary is completely parameterized by couples $(\lambda, \gamma) \gg 0$,

and in doing so a point belongs to the (classical) Pareto boundary only if $\lambda = \gamma$. Moreover, the matrix of financial returns is also degenerated only if $\lambda = \gamma$ (let for simplicity $\alpha \neq \beta$). Thus for every point $(\lambda, \gamma) \gg 0$, $\lambda \neq \gamma$ a partially optimal and simultaneously p^1 -feasible allocation may be put into correspondence, which for λ/γ being near enough to 1 realizes a utility vector which is close enough to the Pareto optimal one. In this economy, Pareto optimal allocations may be attained *only in a limit*, and a *sequence of contracts*, reflecting the exchange of assets in the p^1 -feasibility condition in this passing to a limit which is an *unbounded* one. The last observation is rather important, and we are going to discuss it more detailed below.

A contract of this kind for some $(\lambda, \gamma) \gg 0$, $\lambda \neq \gamma$ can be calculated as a solution of system (3.3.7), that for $\det(P_1A) \neq 0$ (true for $\lambda \neq \gamma$) yields

$$z = [P_1A]^{-1} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} - P_1\mathbf{e}_1 \right].$$

However, for $\gamma \rightarrow \lambda$ we have $\det(P_1A) \rightarrow 0$ and, since $\det([P_1A]^{-1}) = 1/\det(P_1A)$, then $\det([P_1A]^{-1}) \rightarrow \infty$. Therefore, $\|[P_1A]^{-1}\|$ (the norm of operator $[P_1A]^{-1}$, considered as a function of parameters λ, γ), is unbounded for $\gamma \rightarrow \lambda > 0$. Moreover, one can show that exactly for the vectors of form $y^{\lambda\gamma} = (1, 1) - P_1\mathbf{e}_1$ (prices depend on λ, γ), operator $[P_1A]^{-1}$ values are norm-unbounded for $\gamma \rightarrow \lambda > 0$. It seems to be true that exactly this fact is the main cause of the potential non-stability of financial market. Speaking substantially in terms of contracts, one may see that during the contracting and recontracting process, market operators may realize “a race to infinity”, *i.e.*, the total volume of contracts for one of the agents may rise with no limit. The problem can be solved if one imposes some constraints on the total volume of contracts from the asset market for each agent; it is enough to restrict only the volumes of sales or purchase, but with the same style for all agents. Notice that choosing some finite but big enough constraints, one can realize allocations that are near enough to the allocations (or to the utilities bundles) of a classical core. ■

Conclusion to Chapter 3

The main goal of presented analysis was to study the possibility to correctly introduce domination via coalitions and subsequently a core for incomplete markets. This concept was introduced in this chapter, and it is based on our contractual approach. In so doing, it was proved that a suggested incomplete market core satisfies the following (two) requirements:

- If an economy is described as an incomplete market, but is mathematically equivalent to a classical pure exchange model (*i.e.*, it is a complete economy in fact), then the core in the context of an incomplete market has to coincide with the classical core of a pure exchange economy.
- In perfect competition conditions, the core of an incomplete market coincides with the set of equilibrium allocations.

Exactly these two properties allow one to assert that the correct concept of core is truly introduced. This concept seems to be a natural generalization of the classical approach and preserves its most meaningful properties. It is important that the suggested concept being based on the notion of a contract does not address to current allocation value characterizations—attention is concentrated on exchange (barter) contracts. Some kind of value characterizations (p -core, fuzzy p -domination) have appeared in the analysis of core allocations, but it is a more technical element of investigation than its foundation. The mathematical generality of the proved theorems seems to be reasonable in the incomplete market context. Further, let us consider the detailed results and conclusions that have been achieved in this investigation.

A contract-based model, which is put into correspondence to an incomplete market, is such that only contracts implementing the *exchange* of commodities for some *fixed elementary event* are permissible ones, where exchanges via *real assets* are allowed in the *present*: the trade via these fixed standard contracts is realized in initial model. Applying only these standard contracts, one can realize the commodity exchange between different states of nature. In so doing, one can also realize an exchange of commodities in the present for the consumption at future events. Let the model be smooth and a point (allocation) be taken from the interior of the consumption sets.

For the allocations of an *incomplete market core*, it is suggested to take complex contractual allocations, which can be implemented via webs of permissible contracts that are stable in the following sense. The web as a whole is weak stable, and all contracts corresponding to future events are perfect. The web also has to satisfy an additional stability property: there is no coalition for which it is profitable to break all contracts with non-members of the coalition; for future events they can break contracts from virtual webs, but contracts of present may be broken only as a whole, and moreover the coalition can sign a new contract in the present.

Theorem 3.2.4 states that if the structure of assets in a (incomplete) market is complete, then the incomplete market core coincides with the classical one. Here an *assets structure is complete* if all commodities are desirable (goods) and for all positive prices in future events markets the matrix of financial returns, defined via real assets matrix, has a rank equal to the number of future events. Simply, the latter means that there are enough independent assets — enough to realize any value transfer from one to any other future event without a loss of value at other non-true events. This means that the considered core concept satisfies the first of suggested criteria. It also has proven that the concept satisfies the second criteria—under perfect competition conditions, the core and equilibria coincides. Moreover, this fact is presented in two versions.

The first (Theorem 3.2.1) states the description of *GEI*-equilibria in pure contractual terms, where a stability relative to the partial breaking of contracts is required in addition to all forms of stability applied for core allocation. In other words, for every event, a subweb relative to this event has to be stable therewith for future events in the strong sense (since then contracts are perfect), and the web as a whole has to satisfy the condition of joint stability described above.

The second (Theorem 3.2.5) follows to the classical tradition of perfect competition conditions modeling—being replicated, an allocation can be considered as an

allocation of a replicated economy, and it has to belong to the core of the replicated model. Then such an allocation may be decentralized.

The investigation of Section 3.3 also contains the analysis of examples including well known in incomplete market theory *Hart's example*. Under specific model parameters, there is no equilibrium in this example. It was clarified that the *core* in the described sense *may also not exist*. The cause of this are the specific properties of financial market. Namely, if the number of assets is limited, the situation may occur when market operators tend to raise contract volumes for assets with no limit. It seems to be true that this is a degenerate case, which may happen only under a specific relationship between preferences and assets (it is well known in the theory that financial equilibria, which always are in a core, generically do exist). Of course, there are more chances for core non-emptiness relative to equilibria. Moreover, the example shows that an incomplete market core may be nonclosed set. In particular, this is why one cannot apply the classical scheme of the equilibrium existence proof, in which (quasi)equilibria allocations are the elements of a symmetric part of core allocations of the replicated model—the intersection of nonempty, bounded, embedded, but nonclosed sets may be empty. In my opinion for the incomplete market core to be nonempty, it is necessary that grand coalition abilities be supplemented by marginal variants of consumption bundles, *i.e.*, one needs in fact to pass to the closure of an appropriate set. The problem of *core emptiness* can be solved if one imposes some institutional constraints on trade volumes for a financial market. It is known that when some constraints of this kind take place, then equilibria exist under rather weak model assumptions. Moreover, relaxing constraints to infinity, one can also consider limit equilibrium allocations, *e.g.* see *Marakulin* (1999, 2012). These arguments may be applied to the core concept also, and in such a case one can pass to the consideration of approximating and marginal dominating variants and the incomplete market core.

Part II

Dynamics

Classical presentation on the mechanism of market functioning (how are prices settled or how do individuals, having chosen preferred consumption bundles, transit to final resource allocation?) is that equilibrium prices are realized as a result of little by little permanently going *tâtonnement* process, which corrects current prices in accordance with excess demand law: price for a commodity increases if demand exceeds supply; when supply exceeds demand price decreases. Economic intuition says us that moving in this manner economic system as a whole has to find, to grope toward equilibrium prices. Applying mathematical terms this means that if one describes price change process by differential equation (inclusion), having in right hand side excess demand, then every solution of this equation converges to equilibrium prices¹. However what is this demand and can we observe it in reality? In a mathematical model by definition demand is the summation of optimal individual solutions in consumer problems, which are defined by current non-equilibrium prices and by agents' preferences. How is it possible to observe demand under *non-equilibrium* prices, if it is the sum of *unrealized* wishes to buy commodity bundles? One can observe the total volume of purchases or the volume of sellings, supply and its excess, but we think that demand is, evidently, fundamentally an unobservable category.² Moreover, classical view on prices change in accordance with excess demand rule is commonly based on a fictitious auctioneer hypothesis. This auctioneer conducts prices, but he/she is not a revealed economic agent, more likely this is an impersonal being, realizing a market power.

In the modern literature there are also available other approaches, different from classical *tâtonnement*, aimed at modeling of dynamics of market processes and the analysis of their convergence to equilibrium: processes of the prices changes, using Jacobi matrix of excess demand function (Smale's approach and other); disequilibrium models of trade (Hahn's process, Fisher's approach and other); Edgeworth's processes *etc.* The following chapter contains an extensive review of the literature on this theme, where comparative analysis of the approaches and directions is presented. However all approaches have the shortcomings, partially the same as in Walrasian *tâtonnement* (auctioneer and *etc.*), partially new, as, for example, high information requirement of processes with Jacobian and others. So, it allows us to make a conclusion: classical and other modern views on the market and laws of its homeostasis are not quite satisfactory from the modern point of view.

To clarify the case and to answer questions we seemingly need to reconsider our views of what really occurs in the market. In our opinion there are a lot of commodity exchange dealings and for all involved individuals these dealings are mutually beneficial at the moment of their realization. The current resource allocation is generated as a summation of all the accomplished dealings and of an initial endowment allocation. During a time some new dealings are realized, some of them reiterate those made earlier, other ones do not (probably this can be treated as a form of the rejection of signed in the past dealings which are non-beneficial at the moment). It is extremely important, that such "natural" process of a barter exchange goes itself, here there

¹This result holds only under additional strong assumptions (gross substitutability and *etc.*).

²If one knows supply and excess supply is *positive*, then demand for a commodity can be calculated, but what can be done when excess demand is positive?

is no demand with the supply, or prices. The investigations presented in this part of the monography are aimed to formally describe and to study properties of these processes. So further contractual approach will be analyzed that seems to be closer to an intuitive imagination on real processes of prices and consumed resources forming and to deliver better understanding of cooperative and individual features of agents' behavior in a market. In particular, applying contractual approach one can suggest the clearer description of transition processes to stable (non-dominated) allocations and reveal a specific cooperative tâtonnement process, which formally-mathematical description is one of the goals of this study. This cooperative tâtonnement supposes that coalitions of agents are able to sign new mutually beneficial contracts (exchange commodity dealings) and also each agent can partially break contracts signed in the past if it is beneficial for him. The process of the signing of new and breaking of old contracts is going in simultaneous regime (although, it is possible to consider the separate version), and is extended over time. The last means, that in fact process deals with momentary contracts, which together with signed in the past define the process derivative.

Formally contractual process can be described via differential inclusion $\dot{x}(t) \in F(x)$, where $x(t)$ is current allocation of resources. The right hand side of this inclusion is formed via mutually beneficial contracts for various coalitions where abilities of singleton coalitions are realized by means of partial breaking of contracts.

All feasible solutions of this inclusion form a set of feasible contractual trajectories, which can or cannot converge to (potentially) final allocations. It is known from the theory of contracts presented in the first part of monography (see Theorem 1.2.2 and also *Marakulin* (2003, 2011)) that under some assumptions (interior point, differentiable concave utilities) every proper contractual allocation (this is an allocation which can be realized by a web of contracts stable relative to the signing of new contracts and relative to partial breaking of old ones) is equilibrium allocation. The converse implication is always true: every equilibrium can be presented as a proper contractual allocation. Thus equilibria and only they (under assumptions) are *stationary points* for cooperative proper contractual tâtonnement. However a key question is: when this process, starting at initial endowments, is *converging* and which stationary points are *stable*. The investigation of convergency of contractual processes and related questions is the main goal of this part of the monography.

The main difficulties of the study, as for determination and so for stating of process convergence, are caused by an opportunity of the individuals partially to break contracts (because along a trajectory utilities may change non-monotonically), however it reflects the substance of market processes and otherwise equilibrium relative to initial endowments cannot be attained. It is also to analyze stability of equilibria, but the form of stability actually depends on the type of processes and it may differ for classical and cooperative tâtonnement. Interesting their comparison, and it is possible, that there are equilibria which are stable in one sense but unstable in another one. As a whole the analysis of convergence and stability of process also seems to be a necessary step for developing the theory of barter contracts, this is important for economic theory and rather complicated problem.

Chapter 4

Processes driving economy to equilibrium

There is a vast economic literature, devoted to the research of processes driving a multiproduct economy to competitive equilibrium. By now one can mark out at least five approaches to explain market dynamics, they have own comparative advantages and shortcomings. These approaches are:

- (i) Tâtonnement processes of equilibrium prices of Walrasian¹ type. This is tâtonnement, where a current disequilibrium prices change by the law of excess demand: if it is positive the price increases, if negative then price decreases.
- (ii) Processes, in which the law of change of prices is defined due to Jacobi matrix (differential) of excess demand function. The first process of this type was suggested in *Smale (1981)*.
- (iii) Disequilibrium models of trade processes among consumers; among them Hahn–Negishi process (*Hahn, Negishi, 1962*) and Edgeworth processes by Uzawa (*Uzawa, 1962*).
- (iv) Edgeworth processes. They are the processes of commodity exchange without prices; they are based on a mutually beneficial barter (irrevocable) among the members of any coalition of consumers. As time elapsed the coalitions of agents participated in exchange may vary and run some class of permitted coalitions (some coalitions can be forbidden, maybe because of that the formation of them is incredible from the essential point of view and exchanges are not realized).
- (v) Strategic approach, where equilibrium and competition are examined from purely game theoretical point of view.

Below we consider the specified approaches in more details in (limited) frameworks of well-known Arrow–Debreu type *economy of pure exchange*. For convenience of an exposition we begin with formal descriptions of model, introduction of notations and reminder of concepts and notions.

¹In this process market prices on different goods are changed simultaneously, this is a modification of original Walrasian idea suggested in *Samuelson (1941)*.

4.1 Survey of the literature

Let us consider a typical exchange economy in which $L = \mathbb{R}^l$ denotes the *space of commodities* (l is the number of commodities). Let $\mathcal{I} = \{1, \dots, n\}$ be a set of agents (traders or consumers). A consumer $i \in \mathcal{I}$ is characterized by a consumption set $X_i = L_+ = \mathbb{R}_+^l$, initial endowments $\mathbf{e}_i \in X_i$, and a preference relation described by a utility function $u_i : X_i \rightarrow \mathbb{R}$, where $u_i(x_i) > u_i(y_i)$ means that agent i strictly prefers a bundle x_i to y_i . This may be also standardly denoted as $x_i \succ_i y_i$. So, the pure exchange model under study may be represented as a triplet:

$$\mathcal{E} = \langle \mathcal{I}, L, (X_i, u_i(\cdot), \mathbf{e}_i)_{i \in \mathcal{I}} \rangle.$$

A pair of vectors (x, p) , $x = (x_1, \dots, x_n) \in \prod_{\mathcal{I}} X_i$, $p \in \mathbb{R}^l$, is said to be a *Walrasian or competitive equilibrium* of model \mathcal{E} , if the *price vector* $p \neq 0$ and the following conditions are satisfied:

$$(i) \quad \forall i \in \mathcal{I}, \quad px_i \leq p\mathbf{e}_i \text{ \& \; } \forall y_i \in X_i, \quad y_i \succ_i x_i \Rightarrow py_i > p\mathbf{e}_i;$$

$$(ii) \quad \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i.$$

Condition (i) says that for each i the bundle x_i is an optimal budget-admissible consumption plan, while (ii) means that these plans can be jointly realized (all markets are balanced). Traditionally the concept of competitive equilibrium usually is expressed in the terms of equality between demand and supply or simply as the equality to zero of excess demand.

Individual demand $d^i(p)$ of agent i for prices $p = (p_1, \dots, p_l) \neq 0$ is the solution of utility maximization problem

$$u_i(y) \rightarrow \max, \quad \text{subject to } py \leq p\mathbf{e}_i, \quad y \in X_i.$$

If $p \gg 0$ and utility is continuous, this problem always has a solution and therefore the (multi-valued) map $d^i(\cdot)$ is well defined on \mathbb{R}_{++}^l . Together with individual demand, it is sometimes convenient to consider *individual excess demand* function, determined by formula $z^i(p) = d^i(p) - \mathbf{e}_i$.

In a context of assumptions for Arrow-Debreu model the map (function) of excess demand for a given prices is defined as a *sum* of individual solutions of consumer problems for all agents of economy (total demand) minus the total supply. In such a way the vector $D(p) = \sum_{\mathcal{I}} d^i(p)$ is called *total or aggregated demand*. As soon as in a context of exchange model supply is fixed and equal to $\bar{\mathbf{e}} = \sum_{\mathcal{I}} \mathbf{e}_i$, then *excess demand* under the prices p is

$$Z(p) = D(p) - \bar{\mathbf{e}} = \sum_{\mathcal{I}} d^i(p) - \sum_{\mathcal{I}} \mathbf{e}_i = \sum_{\mathcal{I}} z^i(p).$$

Obviously, the maps of demand and excess demand are correctly defined on area of change of the prices \mathbb{R}_{++}^l . Moreover, a vector $p \gg 0$ is the prices of equilibrium if and only if $Z(p) = 0$ (or, equivalently, $Z(p) \leq 0$). In general, the excess demand map

can be a point-to-set mapping,² however, for simplicity we shall assume below that demand is single-valued.³

Besides, under the natural model assumptions (classical convex, continuous preferences) function $Z(\cdot)$ is continuous on its domain \mathbb{R}_{++}^l . Moreover, where it is necessary, we shall assume without special mentioning that it is differentiable in an appropriate degree. By construction, the function of excess demand $Z(p)$ is homogeneous of degree 0 and satisfies Walras law⁴: $\langle p, Z(p) \rangle = 0$, $\forall p \gg 0$. For the existence of equilibrium and also to provide the convergence of a number of processes of price changes to equilibrium, in addition, it is usually assumed *boundary conditions*. The basic example of this kind condition is the following: if $p^m \rightarrow p^0 \in \partial\mathbb{R}_+^l$, $p^0 \neq 0$ for $m \rightarrow \infty$, then $Z_j(p^m) \rightarrow +\infty$ for $p_j^0 = 0$, $j = 1, \dots, l$. In the limits of this section we always shall think that it is true everywhere, where it is necessary.

Further, we proceed to the direct description of specified above processes and results.

4.1.1 Walrasian tâtonnement process of equilibrium prices

There is the vast literature, devoted to the study this type of processes, now we note only two reviews *Hahn (1982)*, *Polterovich, Spivak (1982)*, where one can find the detailed description of problems and results. Further first of all we describe process formally. The economy is described in most aggregated form via to excess demand function $Z : p \rightarrow Z(p)$, defined for all positive prices $p \gg 0$, $p \in \mathbb{R}^l$.

In a general case “tâtonnement” is described as a process of prices $p(t) = (p_j(t))_{j=1, \dots, l}$, $t \geq 0$ changes, as the solution of the following system of the differential equations:

$$\dot{p}_j(t) = F^j(Z_j(p)), \quad j = 1, \dots, l. \quad (4.1.1)$$

Here it is always assumed that all functions $F^j(\cdot)$ are *sign-preserving*, *i.e.*, we have $Z_j(p)F^j(Z_j(p)) > 0 \iff Z_j(p) \neq 0$. Exactly this property of right-hand side of (4.1.1) implies that the price of j -th commodity adjusts in the same direction as excess demand for that commodity: it increases when excess demand is positive and vice versa. It is commonly assumed in addition (*Hahn, 1982*) that $\frac{d}{dx}F^j(x) > 0$ for points from an appropriate area for process (4.1.1). It is known (*Hahn, 1982*) that the solution of our system exists and is unique for every initial data $p(0) \gg 0$. In classical tâtonnement it is presumed a simplest form of functions $F^j(Z_j(p)) = Z_j(p)$, *i.e.*, in classic approach $F^j(\cdot)$ is identity map of real numbers into itself. In this case, the system of differential equations (4.1.1) turns into $\dot{p} = Z(p)$. Some “intermediate” variants are also considered in literature where, for example, it is supposed $F^j(Z_j(p)) = k_j Z_j(p)$ for some real $k_j > 0$, $j = 1, \dots, l$.

Essential treatment of tâtonnement process (4.1.1) is commonly based on a hypothesis of fictitious auctioneer, which, similar to how it occurs in real auctions, raises

²For example, in *Polterovich, Spivak (1982)* it was investigated convergence to equilibrium of Walrasian processes and its stability under point-to-set excess demand mapping.

³It is provided by strictly quasi-concave utilities.

⁴For Arrow–Debreu model it is provided due to local non-satiated preferences.

the price when demand exceeds supply and, accordingly, reduces price, if the demand is less than supply. An auctioneer is not revealed in model (Arrow–Debreu type) economic agent, this is some impersonal being, whose actions reflect an invisible hand of the market. It is not quite clear in fact how the markets really work in a disequilibrium situation, during the search of equilibrium, because there are no revealed *microeconomic* models of process and of agents' behavior out of equilibrium. In the literature hypothesis on the existence of auctioneer and similar constructions are criticized and are recognized as unrealistic, for example, see *Kreps (1977)* p. 195–198; *Fisher (1983)*, p. 19–26. Fisher (*Fisher, 1983*) have noted the following difficulties in the process (4.1.1) interpretation: First, "... It has nothing to do with the question of whether or not trade, consumption, or production takes place out of equilibrium" (actually, exchange is possible only when equilibrium is attained!). Second, "...we know very little about how individuals do or ought to behave when equilibrium is not presented; hence, the resort to an aggregate equation". Finally, "...in the unrealistic world of no trading out of equilibrium ... individuals take action to make their excess demand effective. ...they *can* take such action which ... implies that they have something of value which they can and do sell so as to have something to offer when they buy..." So, out of equilibrium individuals have to buy and sell to reveal their excess demand, for the price changes to be going, but actually they can do it only under equilibrium prices.

Summarizing, one can conclude that equilibrium theory needs an adequate dynamic theory, in which framework the process of equilibrium prices searching has to be revealed.

The subject of criticism is also rather rigorous conditions, under which one can guarantee the convergence of process (4.1.1) to equilibrium. Results on convergence and stability of process (4.1.1) mainly are based on the property of *gross substitutability* of excess demand functions, in various forms of generality, or just on the *axiom of revealed preferences* and others. Let us consider these assumptions in more details, see *Mas-Colell et al. (1995)*.

- A function⁵ $Z(\cdot)$ has the property of *gross substitutability* (GS–property), if for any prices p' and p such that $p'_m > p_m$ is true for some m and $p'_k = p_k$ for $k \neq m$, then $Z_k(p') > Z_k(p)$ takes place for all $k \neq m$, $k = 1, \dots, l$.

For a *differentiable* function $Z(\cdot)$ the condition of gross substitutability takes the form $\partial Z_k(p)/\partial p_m > 0$, $\forall p \gg 0$, $\forall k \neq m$.

- An excess demand function $Z(\cdot)$ satisfies weak *axiom* (WARP) of *revealed preference*, if for any couple of vector-prices, p and p' ,

$$Z(p) \neq Z(p') \ \& \ pZ(p') \leq 0 \ \Rightarrow \ p'Z(p) > 0$$

takes place.

⁵In a context of an exchange model the gross substitutability of excess demand function and demand function are equivalent.

Being applied to (aggregated) excess demand function, the properties of gross substitutability and (weak) revealed preference are, in general, *non-equivalent* and rather strong requirements. However, both of them have common important corollaries:

- (i) The set of equilibrium prices is *convex*.
- (ii) If p^* is equilibrium price, then $p^*Z(p) > 0$ for all $p \gg 0$ which are *not proportional* to p^* .

In particular, property (ii) allows us easily to understand why process (4.1.1) is converged, if one of the specified conditions is carried out. Really, in the simplest case of (4.1.1), when $\dot{p} = Z(p)$, it is enough to differentiate by t the function of squared Euclidean distance between a current prices and equilibrium prices, $\|p(t) - p^*\|^2 = \sum_{j=1}^l (p_j(t) - p_j^*)^2$. By virtue of Walras law, we obviously have $\frac{d}{dt}\|p(t) - p^*\|^2 = 2(p(t) - p^*)\dot{p} = -2p^*Z(p) < 0$. So, we see that the distance between a current vector of prices and equilibrium ones decreases when time is going.

Finally, we would like to mention one more classical condition, providing the local convergence of tâtonnement process, this is the property of *diagonal domination* of Jacobi matrix $D_p Z^-(p^*) = A$, without the last row and column, of excess demand function $Z(p)$ in a point of equilibrium p^* . Formally, diagonal domination means

$$\exists h = (h_1, \dots, h_{l-1}) \geq 0 : \forall j \quad h_j a_{jj} < - \sum_{k \neq j} h_k |a_{jk}|.$$

Gross substitutability implies this property, but the opposite is false. Moreover, in literature there are unknown other examples of diagonal domination. The diagonal domination and other similar requirements, *e.g.* see Theorem 1.7 in [Hahn \(1982\)](#), implies that eigenvalues of A have negative real parts and this provides the local stability of price adjustment process.

4.1.2 Processes of the prices change, using Jacobi matrix of excess demand function

First, we would like to note that a large part of the critical remarks, made relative to Walrasian processes, can be also addressed to the processes of this type.

Smale ([Smale, 1976](#)) investigated the convergence of prices changes process, based on (global) Newton method,⁶ which is usually applied to find a solution of a system of the nonlinear equations. The process is determined as:

$$[D_p Z^-(p)]\dot{p} = -\lambda(p)Z^-(p). \quad (4.1.2)$$

Here $Z^-(p)$ is an excess demand for all goods excepting (for example) the last one, and $D_p Z^-(p)$ is Jacobi matrix of excess demand function, excepting the last row and column.⁷ It is supposed, that the *sign* of functions $\lambda(p)$, entering as a factor in the

⁶For the first time the method was offered and partially investigated in [Arrow, Hahn \(1991\)](#).

⁷Here the last row and column correspond to a commodity, which is used as a numeraire good. The elimination of a row and a column is necessary, to allocate square nonsingular submatrix in $J[Z(p)]$; since excess demand is homogeneous, $J[Z(p)]p = 0$ and, therefore, matrix $J[(p)]$ is always singular.

right part of (4.1.2),⁸ coincides with the sign of $(-1)^{l-1}\det[D_p Z^-(p)]$. If $D_p Z^-(p)$ is non-singular matrix in the domain of p changes, then process (4.1.2) can be rewritten in an explicit form

$$\dot{p} = -\lambda(p)[D_p Z^-(p)]^{-1} Z^-(p).$$

Smale (1976) has proved that this process converges to equilibrium for any aggregated excess demand function,⁹ if the initial prices $p(0) \neq 0$ are on the boundary of \mathbb{R}_+^l (area of prices change), except for a set of zero measure (with the account of normalization), and under an additional requirement that $D_p Z^-(p)$ is non-singular in effective area of prices change (these are positive and normalized by the last component). Certainly, this is a remarkable result, however its weak side is too large informational requirements. At each time moment the process of prices change requires a knowledge not only excess demand, but also Jacobi matrix, *i.e.*, the change of price in the market explicitly depends on how the prices on other markets are changed.

Kamiya (Kamiya, 1990), developed Smale's approach, and has offered process, defined as

$$\left[\frac{D_p Z^-(p)}{\|D_p Z^-(p)\|} - \frac{I}{\|p - p(0)\|} \right] \dot{p} = -\lambda(p) Z^-(p). \quad (4.1.3)$$

Here, as well as in Smale's process, $D_p Z^-(p)$ is Jacobi matrix of excess demand without the last column and row, $p(t)$ is $(l-1)$ -dimensional vector-function of the prices *without* last component $p_l(t)$, such that $\|p(t)\| \leq 1$ and $p(0)$ is an initial vector of the prices of the same dimension. It is assumed the *sign* of real-valued function $\lambda(p)$, placed in the right part of (4.1.3), is *opposite* to the sign of the determinant of matrix, entered in the left part of (4.1.3), *i.e.*, the sign coincides with the sign of $\det \left[\frac{I}{\|p-p(0)\|} - \frac{D_p Z^-(p)}{\|D_p Z^-(p)\|} \right]$. Using methods suggested by Smale, Kamiya proves, that the process (4.1.3) converges to equilibrium for almost all initial data $p(0)$ from the interior of \mathbb{R}_+^{l-1} .

Mukherji (Mukherji, 1995) investigated another process, using Jacobi matrix of aggregated excess demand function:

$$\dot{p} = -J[Z(p)]^t Z(p). \quad (4.1.4)$$

He has shown in Mukherji (1995), that the process (4.1.4) belongs to a group of so-called locally effective processes (LEPM): the processes of this type converge to any (regular¹⁰) equilibrium locally (*i.e.*, for the equilibrium prices there exists a neighborhood, such that if $p(0)$ in the neighborhood, the process converges to the equilibrium).

Concerning all described above processes, and also other processes from this group,¹¹ it is possible to state one common remark: all of them require too much information. Moreover, Saari, Simon (1978) has proved that this is an unavoidable property of any LEPM-process, *i.e.*, actually, it is a necessary condition for the process to be locally effective for (almost) any function of aggregated excess demand. In

⁸Certainly, it is necessary also to postulate other properties of $\lambda(p)$, ensuring existence and uniqueness of solution (4.1.2).

⁹Here the economy *completely* is set by function $Z(p)$.

¹⁰Jacobi matrix of excess demand at equilibrium point has the maximal rank equal to $l-1$.

¹¹For example, for orthogonal Newton process, described in Jordan (1983).

relationship we would like to recall Sonnenschein–Debreu–Mantel results, see survey *Shaher, Sonnenschein (1982)*, about representation of a general aggregated excess demand function as an excess demand function for Arrow–Debreu model. They show that any continuous, homogeneous and obeying Walras law function allows representation in the specified for Arrow–Debreu model form, the model where the number of the agents is equal to the number of commodities. In so doing the utilities of individuals may be classical: continuous, strictly concave, monotonous and, moreover, homogeneous (degree 1). Thus, one can go to the following conclusion: *any locally effective mechanism of prices change* based on excess demand function for Arrow–Debreu model is *informational requiring* and, with necessity, has to use (whole!) Jacobi matrix $J[Z(p)]$ of excess demand. In particular, making comments to Smale’s process, Hahn (*Hahn (1982)*, p. 767) replies: “Obviously these results are interesting as algorithms and not as models of invisible hand.”

4.1.3 Disequilibrium models of trade processes

There are at least two disequilibrium processes known in literature in a context of a pure exchange model, see *Fisher (1983)*, *Mukherji (2003)*. These are Edgeworth process by Uzawa (*Uzawa, 1962*), and Hahn’s process (*Hahn, Negishi, 1962*), so-named in *Negishi (1962)*. It is a common feature of both processes, that *endowments* are varied at the time, *i.e.*, initial endowments $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n) \in \mathbb{R}_+^{ln}$ are the *function of time*, $\mathbf{e} : [0, +\infty) \rightarrow \mathbb{R}_+^{ln}$ (n is the number of agents). As well as in Walrasian processes (tâtonnement), real consumption comes only at the end of the process where it is described by a limiting point of $\mathbf{e}(\cdot)$. Further, we consider other specific features of processes.

Common properties. The prices change according to excess demand:

$$\dot{p}_j(t) = \begin{cases} F^j(Z_j(p, \mathbf{e}(t))), & \text{unless } p_j = 0 \ \& \ Z_j(p) < 0; \\ 0, & \text{if } p_j = 0 \ \& \ Z_j(p) < 0. \end{cases} \quad (4.1.5)$$

Here the functions $F^j(\cdot)$ satisfy the usual requirements: continuity and sign-preservation.

There is a *law of change* of the initial endowments, which can be also treated as (current) allocation of consumed resources $\mathbf{e} : [0, +\infty) \rightarrow \mathbb{R}_+^{ln}$, and this map has to obey the requirements:

$$\forall i \in \mathcal{I}, \quad \dot{\mathbf{e}}_i(t) = g_i(p(t), \mathbf{e}(t)) - \mathbf{e}_i(t), \quad \sum_{i=1}^n \mathbf{e}_i(t) = \sum_{i=1}^n \mathbf{e}_i(0), \quad \forall t \in [0, +\infty), \quad (4.1.6)$$

where all functions $g_i(p(t), \mathbf{e}(t))$ are assumed to be continuous and, in addition, to satisfy “No Swindling” condition:

$$\forall i \in \mathcal{I}, \quad p(t)\dot{\mathbf{e}}_i(t) = 0 \iff p(t)g_i(p(t), \mathbf{e}(t)) = p(t)\mathbf{e}_i(t), \quad \forall t \geq 0. \quad (4.1.7)$$

Essentially, in both processes the functions $g_i(p(t), \mathbf{e}(t))$, $i \in \mathcal{I}$ set a *rule of trade* (trading or transaction rule). Other requirements in processes differ.

Hahn's process. A specific requirement is the assumption that markets are *orderly*: $\forall t \geq 0$

$$z_j^i(p(t), \mathbf{e}_i(t))Z_j(p(t), \mathbf{e}_i(t)) > 0, \quad j = 1, 2, \dots, l, \quad (4.1.8)$$

unless the case $z_j^i(p(t), \mathbf{e}_i(t)) = 0$ for all $i = 1, 2, \dots, n$. Here $z_j^i(p(t), \mathbf{e}_i(t))$ is *individual excess* demand of i 's agent for the commodity j under current prices and endowments $\mathbf{e}_i(t)$. This requirement means, that if the market of a product j is not balanced, then *all* agents have positive excess demand or supply (there are hence only unsatisfied demanders or suppliers for any given good).

Edgeworth's process by Uzawa. It is supposed that endowments (here it is a current consumption) are changing so that monotonous growth of utility of each individual goes, at least for one in strictly form, *if it is possible in general*; everything under constraints (4.1.6), (4.1.7). Formally it is defined as: $\forall t \geq 0$

$$u_i[g_i(p(t), \mathbf{e}(t))] \geq u_i[\mathbf{e}_i(t)], \quad \forall i, \quad (4.1.9)$$

$$u_i[g_i(p(t), \mathbf{e}(t))] = u_i[\mathbf{e}_i(t)], \quad \forall i \iff g_i(p(t), \mathbf{e}(t)) = \mathbf{e}_i(t), \quad \forall i \ \&$$

$$\forall (x_1, \dots, x_n) \in \mathbb{R}_+^{ln} : p(t)x_i = p(t)\mathbf{e}_i(t) \quad \forall i \ \& \quad \sum x_i = \sum \mathbf{e}_i(t),$$

$$\exists k : u_k(x_k) > u_k(\mathbf{e}_k(t)) \Rightarrow \exists i : u_i(x_i) < u_i(\mathbf{e}_i(t)).$$

Thus, at each current moment of time, a state of economy changes if and only if it appears possible mutually beneficial exchange within the framework of budget constraints.¹²

Both described processes (Hahn's process and Edgeworth's process by Uzawa) are converged to some Pareto optimal allocation under more or less standard assumptions, including boundary condition (provides a movement of a trajectory in the limits of \mathbb{R}_{++}^{ln}), and additional assumption about strict concavity of utility functions, see [Hahn, Negishi \(1962\)](#), [Uzawa \(1962\)](#). In so doing price processes are also converged (for Edgeworth's process by Uzawa see [Mukherji \(1974, 2003\)](#)) and the limiting prices are the prices of equilibrium for the given limiting resources allocation (here this is initial and final allocation simultaneously), this is a situation of no-trade.

The detailed description of specified disequilibrium processes one can find in [Arrow, Hahn \(1991\)](#) (part 13), [Hahn \(1982\)](#), [Fisher \(1983\)](#), [Mukherji \(2003\)](#), where their criticism is also contained. For example, Fisher ([Fisher, 1983](#)), being an advocate of Hahn's process, criticized Edgeworth's process by Uzawa in the following way. First, it is not clear, why Pareto improving trade actually will takes place whenever such a situation arises. The reason is that it is possible that *all* coalitions of the agents, capable to do such mutually beneficial exchange, can have too large size, and small coalitions (bilateral or trilateral or quadrilateral trade and so on) are unable to carry out Pareto-improving exchange. Admitting an opportunity of exchange in the huge

¹²Uzawa also assumes that functions $g_i(\cdot)$, determining barter process along a trajectory, take values equal to the demand of individuals, if aggregated demand is equal to supply (price of equilibrium for current initial endowments). I did not identify a place where we need it. In my opinion this is excessive assumption, though I cannot disagree with it.

coalitions, we impose “very heavy requirements on the dissemination of information and to assume away the costs of coalition formation.” Moreover, the inclusion into the model of money as means of exchange actually does not change a situation. Second, Edgeworth’s processes do not admit a revealed opportunity of production and consumption in disequilibrium situations, now it is still open quest.¹³ Third, the assumption that a trade takes place only when the utilities of individuals are increasing, is not as harmless as it seems. In a true disequilibrium world, the individuals “trade even then there is no direct utility (or profit) gain from so doing because they wish to take advantage of arbitrage opportunities, speculating on their ability later to retrade at more advantage prices.” Moreover, Fisher writes: “Yet a crucial aim of stability theory must be to examine the question of whether arbitrage drives a competitive economy to equilibrium.”

On the other hand, Hahn’s process also does not avoid shortcomings. One of them is that before purchasing something, one needs to sell something. Therefore, many of the potentially interesting bargains may be not realized. In order to solve this problem, Arrow and Hahn (*Arrow, Hahn*, 1991) directly introduced money into the model, using it as the intermediary-goods in any barter bargain, and imposed other additional assumptions. Mukherji (*Mukherji*, 2003) also criticizes Hahn’s process and specifies a main its shortcoming: the absence of revealed *voluntary* nature of barter bargains in process.

If there is no specific model, explaining in an microeconomic way how and why (non-mutually beneficial) barter bargain is realized, voluntary can be understood only as a condition, attracting monotonous growth of utility of the individuals along trajectory, *i.e.*, realizing bargain all its participants should win.

Finally, both processes are indirectly based on a hypothesis of auctioneer, since due to definitions they satisfy (4.1.5).

4.1.4 Edgeworth’s processes

They are the processes of change of current resources allocation, described in continuous or discrete time, which are going *without prices* and, accordingly, there are no budget constraints. Essentially, the processes of this type are close to contractual processes without breaking of the contracts, see *Hahn* (1982) p. 772–777, and also the sections below. The basic sense of process is that the process generates a trajectory in space of allocations such that along trajectory there is the monotonous growth of individual utilities, and at least for one strictly, if a current point still is not Pareto optimum. A process of this type can be set, for example, via a rule of trade, described by functions $g_i(\mathbf{e}(t))$, $i \in \mathcal{I}$, similarly as it was made in the previous paragraph (however here there are no prices): $\mathbf{e} : [0, +\infty) \rightarrow \mathbb{R}_+^{ln}$,

$$\forall i \in \mathcal{I}, \quad \dot{\mathbf{e}}_i(t) = g_i(\mathbf{e}(t)) - \mathbf{e}_i(t), \quad \sum_{i=1}^n \mathbf{e}_i(t) = \sum_{i=1}^n \mathbf{e}_i(0), \quad \forall t \in [0, +\infty).$$

¹³*Hahn* (1982) noted the work of Hurwicz–Radner–Reiter (*Hurwicz et al.*, 1978), in which the process is considered in a stochastic context and, moreover, it is also shown that the production can be incorporated into model.

The functions $g_i(\cdot)$ are continuous and satisfy conditions (4.1.9), *with the exception* of budget constraints.¹⁴ It is simply to prove that every limiting point of such a process is Pareto optimal.¹⁵

There are a number of papers where Edgeworth's processes are considered in stochastic context (see survey *Hahn, 1982*; and also *Hurwicz et al., 1978, Graham, Weintraub, 1975*), where, in our terms, on the set of all mutually beneficial contracts, some reasonable probability distribution is defined. The appropriate stochastic process converges to a Pareto optimum with probability 1 (we omit other specific features and assumptions).

There are also papers, in which Pareto boundary is attained by the efforts of coalitions of limited size (agents are not more than the number of commodities). The first result of this type was received in *Polterovich (1970)*, see also *Feldman (1973), Graham et al. (1976), Madden (1975), Green (1974)*. However, in so doing each active in barter process coalition carries out transition on intra-coalitional Pareto boundary (relative to current allocation) and all permissible coalitions are incorporated in a cycle which is repeated infinite times (compare with (4.1.9)). Thus, essentially these processes are discrete in time.

There are also papers, in which the transition to core allocations are realized, see *Green (1974)*. Here a stochastic context is also available, where the reaction of current blocking coalition replenishes with the reaction of supplementing coalition, that forms the transition from a current allocation in subsequent one. In such a way an allocation from core (if necessary the procedure is repeated infinitely) is attained with probability 1.

It is clear that in all of these directions contractual context is available and, moreover, the contract-based language is simpler, more convenient and it can be better interpreted.

4.1.5 Strategic approach

This direction began to develop in the economic theory from the middle of 1980s of last century and was aimed to clarify the basic hypotheses of competitive equilibrium theory in a context of a strategic game.¹⁶ An idea was to apply game theoretical methods to give the answers on such questions as: whence the prices are undertaken and who defines them, why the agents should accept the prices as given and why they cannot change them (a consumer is said to be a "price-taker"), that is equilibrium and

¹⁴This assumption is important for the process converges to Pareto optimum. In *Hahn (1982)* p. 773 there is another description of process, requiring only a growth of utilities without restrictions on the derivative of process. Such a process can be finished at an "irredundant" point, which is not Pareto optimal, an elementary example can be constructed: Let $\alpha : [0, +\infty) \rightarrow [0, 1]$, $\text{supp}(\alpha) = [0, \infty)$, $\int_0^\infty \alpha(t)dt = 1$ (one can take $\alpha(t) = \frac{1}{(1+t)^2}$, $\int_0^t \alpha(s)ds = 1 - \frac{1}{(1+t)}$) and let v be a mutually beneficial contract such that $u_i(x_i + \lambda v_i)$ strictly increases in $\lambda \in [0, 1]$, $\forall i \in \text{supp}(v)$, but $z^* = x + v$ is not Pareto optimum. Then a trajectory of process $z(t) = x + v \int_0^t \alpha(s)ds$ obeys all Hahn's conditions and $z(t) \rightarrow x + v$ for $t \rightarrow +\infty$.

¹⁵We can not give exact reference, but it seems that this (elementary) fact was clear to economists for a long time ago, and Uzawa is one of them.

¹⁶It is also called as a game in normal form.

perfect competition? The answers to these and other important theoretical questions are given in the analysis of some game in extensive form. These games belong to a class of *DMBG*-games (dynamical matching and bargaining games), constructed by a model of an economy in a special way. For lack of an opportunity to enter in detailed explanations, we specify only two sources of the literature: [Gale \(2000\)](#), [Kunimoto, Serrano \(2004\)](#) and we briefly describe only the basic ideas of the approach¹⁷ in a context of one possible game model.

In the best-known version ([Gale, 2000](#)) in economy, there is a continuum of agents, presented by a finite number of types. Each type is characterized by initial endowments vector and by von Neumann-Morgenstern utility functions. Commodities are infinite divisible, time is discrete and is indexed by the natural numbers. During each time period, each agent can meet the partner with some fixed probability. If some pair of the agents has met, then it is chosen with equal probably an agent who makes an *offer* to another agent, wanting to get the product vector $w \in \mathbb{R}^l$ to be transferred to him from his opponent. If the agent accepts the offer, her/his consumption bundle changes on $-w$, and the partner's one on $+w$ (only the allowable offers are under consideration, they are not to allow consumption bundles to leave the limits of consumption sets).¹⁸ If the agent does not accept the offer, the consumptions do not change. The individual who does not accept the offer made to him, can leave the market at a next moment in time, no other individual (accepting the offer or not taking part in the given round) can leave market. An agent who never leaves market receives utility equal to $-\infty$ (thus, the consumption is possible only after leaving). A player's strategy is a plan that prescribes her/his bargaining behavior in different trade situations for each period, depending on current consumption, type of partner, her/his current consumption and the offer made by him (if it happens). The strategy of the agent depends on the realized (by him) earlier bargains and, if the offer is made to him, it may takes the values: "accept the offer", "reject and stay", "reject and exit". It is supposed that the agents of the same type use common strategy. However, in view of the previous acts of trade, the different agents can have different current consumption bundles, but for each type only a finite number.

Further, for constructed *DMBG*-game a concept of market equilibrium is introduced, which is in fact a specialized kind of perfect Bayesian equilibrium and in discrete (equivalent) variant of game is presented as a sequential equilibrium. The basic result is the theorem which states that in every market equilibrium each player leaves the market with probability 1 if her/his consumption bundle is equal to a bundle according to Walrasian equilibrium.

Thus these studies suggest a specific answer to the question of how economic system arrive at equilibrium and are aimed to solve similar problems of equilibrium theory, which are related to the validity and possibility to be correctly realize basic theoretical hypotheses. In our opinion, the most attractive part of strategic approach is the clear description of what and how the individuals make trades in uncertain market circumstances. In literature there are many generalizations of this approach,

¹⁷Apparently, it was Douglas Gale's idea, but it seems that it is a suitable adaptation and development of ideas of previous researchers, see [Gale \(2000\)](#).

¹⁸In contractual terms w is a proposition to conclude the barter contract $(w, -w)$.

see *e.g.* [Dagan et al. \(2000\)](#), [Yildiz \(2003\)](#). Finally, notice that comparing strategic approach with contractual one can find an analog of possibility to break contracts: the key element of contract-based approach. Really players are applying selected equilibrium strategies, *i.e.*, inappropriate strategic variants were culled (broken) somewhere in the past.

Conclusion to Chapter 4

In this chapter we described the known in the literature results offering and justifying the processes driving an economy to equilibria.

There are at least five different competing approaches which have been elaborated so far to explain how real markets operate, and how economy attains equilibrium, namely: Walrasian tâtonnement, the disequilibrium dynamic model, Edgeworth's processes, Smale-type processes and Strategic bargaining approach. Basically, they are the processes of price changes, but among them there are also processes close to the contractual one: it concerns Edgeworth processes, Edgeworth processes by Uzawa and Strategic approach.

Each of the presented processes has some advantages and shortcomings, but none is completely successful and convincing. The known historian of economic thought, Mark Blaug, commented on this puzzle in an interview for "Challenge" (May-June 1998). To the question "What are the major issues on which we have not made progress?" the answer was: "Markets and how they actually function; that is, how they adjust to match demand and supply. We in economics know a hell of a lot about equilibrium, but we really don't know how markets actually get to equilibrium." We would add that, in reality, we hardly see equilibrium itself, but some infinite convergence process involving reaction to shocks. This study suggests a sixth approach to this puzzle.

Chapter 5

Dynamical contractual process as cooperative tâtonnement

We begin with the description of contractual economy and main contract-based concepts in an appropriate form of generality. Further, the basis of dynamical contractual processes is described.

5.1 On the definition of contractual trajectory

Let us consider a typical exchange economy, described in Section 4.1 and presented as a triplet:

$$\mathcal{E} = \langle \mathcal{I}, L, (X_i, u_i(\cdot), \mathbf{e}_i)_{i \in \mathcal{I}} \rangle.$$

Let us denote by $\mathfrak{L} = L^{\mathcal{I}}$ the space of economy allocations, let $\mathbf{e} = (\mathbf{e}_i)_{i \in \mathcal{I}}$ be the vector of initial endowments of all traders of the economy. Denote $X = \prod_{i \in \mathcal{I}} X_i$ and define

$$\mathcal{A}(X) = \{x = (x_i)_{i \in \mathcal{I}} \in X \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\},$$

the set of all *feasible allocations* in \mathcal{E} .

Everywhere below we shall assume that model \mathcal{E} satisfies the following *smoothness* assumption **(D)**.

(D) All utilities $u_i(\cdot)$ are concave and twice continuously differentiable functions, such that $\forall x_i \in X_i = \mathbb{R}_+^l$, $\forall i \in \mathcal{I}$, $\nabla u_i(x_i) \neq 0$ and matrices $\nabla^2 u_i(x_i)$ are negative definite.

Further, we briefly recall different contractual concepts, see Chapter 1.

By the formal definition, any reallocation of commodities $v = (v_i)_{i \in \mathcal{I}} \in L$, where $v_i \in E$, $i \in \mathcal{I}$, i.e., any vector $v \in L$ satisfying $\sum v_i = 0$, is called a (barter) *contract*. In this project context, we assume that *every* contract is permissible.

A finite collection V of permissible contracts is called a *web of contracts relative to* $y \in X$ if

$$z(U, y) = y + \sum_{v \in U} v \in X \quad \forall U \subset V.$$

A web of contracts V relative to \mathbf{e} is called a *web of contracts* or simply a *web*. Note that $V = \emptyset$ is a web relative to every $y \in X$. Notation $x(V) = \mathbf{e} + \sum_{v \in V} v$ denotes the feasible allocation sustained by V relative to \mathbf{e} . For any contract $v \in V$, let us set

$$S(v) = \text{supp}(v) = \{i \in \mathcal{I} \mid v_i \neq 0\}.$$

the support of the contract v . It is assumed that contract $v \in V$ may be *broken* by any trader in $S(v)$, since he/she simply may not keep his/her contractual obligations. Also, a nonempty group (coalition) of consumers can *sign* any number of new contracts. Being applied jointly, *i.e.*, as a simultaneous procedure, these operations allow coalition $T \subseteq \mathcal{I}$ to yield new webs of contracts.

In contract-based approach, the notion of domination via a coalition is extended onto webs of contracts which can be transformed during contractual process. Let us recall terminology (see Definitions 1.1.1, 1.1.2).

A *web* of contracts V is called *stable* if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \succ_T V$.

A *web* of contracts V is called *lower stable* if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \succ_T V$ and $U \subseteq V$.

A *web* of contracts V is called *upper stable* if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \succ_T V$ and $V \subseteq U$.

An allocation x is called *contractual* (lower, upper contractual) if $x = x(V)$ for a stable (lower, upper stable) web V .

It can be directly deduced from definitions that in any standard market every core allocation allows an alternative description as a contractual one; accordingly, Pareto optimal allocations correspond to upper contractual ones, and individual rational allocations are lower contractual ones *etc.*, see Theorem 1.2.1. The concept of properly contractual allocation is also very important, this concept provides an alternative description of equilibria (assumptions: interior point, smooth preferences, see Theorem 1.2.2).

The simplest way to introduce the concept of a properly contractual allocation can be done in the following manner. Let V be a web of contracts. For a real α define $\alpha V = \{\alpha \cdot v \mid v \in V\}$, *i.e.* αV is a web, yielded from V by multiplying contracts on α . For $0 \leq \alpha \leq 1$ consider web $U = \alpha V \cup (1 - \alpha)V$, which obviously implements the same allocation $z(U, y) = z(V, y)$. The web $U = \alpha V \cup (1 - \alpha)V$ is called α -partition of the web V . An allocation $z = z(V, y)$ is properly contractual if α -partition of V is stable for every $\alpha \in [0, 1]$.

The economic meaning of proper contractual stability of an allocation is that we allow the agents not only to sign new contracts but also partially break contracts if exchange proportions remain constant. This extends agents' operating potentialities and approaches contractual processes to market processes under perfect competition conditions, see details in Chapter 1, 2.

Further, let us turn to the main subject of this study: the investigation of the stability of trajectories which correspond to the proper contractual behavior of traders. However, first we would like to specify one possible interpretation of proper-contractual behavior, driving economy to properly contractual allocations.

Suppose that an economy is not static and lives during a long-duration interval of time. As time elapsed individuals sign the rather short-term contracts on an exchange of commodities. The contract assumes mutual deliveries of goods among agents and, after its execution, an opportunity of renewal, *i.e.*, the same contract can be signed again, but now it is realized during *another* time period. The agents can agree with the contract's renewal (prolongation) or disagree, first studying an opportunity to prolong contract in smaller volumes. Thus, instead of *breaking of the contract*, even if partial, for economy in dynamics living a long time period one *can speak about renewal and non-renewal* of the contracts. Notice that if resources are renewed then according to this interpretation agents can consume goods as time goes on, notwithstanding the fact that current situation is a disequilibrium one. It seems natural to assume that stable in time contracts, *i.e.*, regularly renewed contracts have to take out economy to equilibrium performance (there is no production!). However, the convergence to such a state is not clear and requires careful research.

We believe that suggested description of economic exchange processes—due to rules of proper-contractual behavior—essentially closer to intuitive representations about their character in the real economic environment in comparison with processes considered in items (i)–(iv) of the previous section.

5.1.1 Contractual trajectory: preliminary analysis

Formally, a trajectory is a map $x(\cdot)$, operating from $[0, +\infty)$ into the set of all feasible allocations, *i.e.*, into $\mathcal{A}(X)$,

$$x(\cdot) : [0, +\infty) \rightarrow \mathcal{A}(X).$$

Here the vector $x(t) = (x_i(t))_{\mathcal{I}}$ is a feasible bundle of consumption plans, realized at the moment $t \geq 0$. It is presumed that $t = 0$ is the initial time point, the process 'starts' at this point from initial endowments allocation, *i.e.*, we set $x(0) = \mathbf{e}$.

We are interested in not arbitrary trajectories of this type, but trajectories which can be realized during contractual processes via commodity exchange among agents. Presume $\Delta t > 0$ is the time period during which a contract v is realized, and presume that other *exchange operations* with commodities (the signing of new contracts or the breaking of existing ones) *were not realized*. Then at the moment $t' = t + \Delta t$ trajectory takes value $x(t') = x(t) + v$, wherefore $v = x(t') - x(t)$. As soon as other contractual operations in interval $[t, t']$ were not conducted, one may think that at the point $t'' = \lambda t' + (1 - \lambda)t$, $\lambda \in [0, 1]$ the trajectory value is produced from values of endpoints, which are mixed in the same proportions, *i.e.*, one can postulate $x(t'') = \lambda x(t') + (1 - \lambda)x(t)$.¹ This can be rewritten in the form $x(t'') = x(t) + \lambda v \Rightarrow x(t + \lambda \Delta t) - x(t) = \lambda v$ and therefore,

$$\dot{x}(t) = \lim_{\lambda \rightarrow +0} \frac{x(t + \lambda \Delta t) - x(t)}{\lambda \Delta t} = \frac{v}{\Delta t} \Rightarrow v = \dot{x}(t) \Delta t.$$

¹Applying 'physical' interpretation, one can say that the uniform (constant) speed of contract realization in interval $[t, t']$ is postulated.

Further, let us assume that during time interval $[t, t']$ there was a (finite) sequence of signed contracts, such that their *time periods* of realization *are not overlapping*. Let m be a number of contracts. One can think that the final time point of one contract is simultaneously the starting point of another contract: if not, we can always replenish system with an appropriate number of zero contracts. So, interval $[t, t']$ is divided into m intervals, determined by points $t = t_0 < t_1 < \dots < t_m = t'$, such that $[t_{k-1}, t_k]$ are time intervals of contracts $v_k = x(t_k) - x(t_{k-1})$ realization, $k = 1, \dots, m$. Put $\Delta t_k = t_k - t_{k-1}$ and due to previous formula, find

$$x(t') = x(t) + \sum_{k=1}^m v_k = x(t) + \sum_{k=1}^m \dot{x}(t_{k-1}) \Delta t_k = x(t) + \int_t^{t'} \dot{x}(s) ds.$$

As soon as by assumption contractual process starts at the moment $t = 0$ at the point \mathbf{e} , we have

$$x(t) = \mathbf{e} + \int_0^t \dot{x}(s) ds, \quad \dot{x}(s) = \frac{v_k}{\Delta t_k}, \quad \forall s \in [t_{k-1}, t_k].$$

Further holding ourselves away from the latter (simple) deduction or in other words, if we allow ourself to consider a limit variant of last formula then the number of contacts is passing to infinite and the realization time of each contract is passing to zero, one can draw the following conclusions.

(i) *Contractual trajectory*, which for a finite number of contracts is represented as integral of some step function, in general case is the integral of some *integrable* on every finite interval function $\dot{x}(\cdot)$ and is defined via formula

$$x(t) = \mathbf{e} + \int_0^t \dot{x}(s) ds. \quad (5.1.1)$$

In other words, *contractual trajectory* is an *absolutely continuous*² on every interval $[0, t]$, $t > 0$ map

$$x(\cdot) : [0, +\infty) \rightarrow \mathcal{A}(X).$$

So, we have provided the first property of contractual trajectory definition.

(ii) *Derivative* $\dot{x}(\cdot)$ of a contractual trajectory in general case is defined almost everywhere on $[0, +\infty)$ and the value $\dot{x}(t)$ defines a (momentary) contract, signed at the moment $t \in [0, +\infty)$. If the time $\Delta t > 0$ of contract realization is known, that formally means $\dot{x}(t') = \dot{x}(t'')$, $\forall t', t'' \in [t, t + \Delta t]$, the resulting (gross) contract can be found from $v(t) = \dot{x}(t) \Delta t$. In other words, the derivative of contractual trajectory can be understood as a barter contract per time unit. Notice also the obvious corollary: the *range* of derivative is the subspace of contracts, *i.e.*,

$$\dot{x}(\cdot) : [0, +\infty) \rightarrow \mathfrak{L}^c, \quad \mathfrak{L}^c = \{v \in \mathfrak{L} \mid v = (v_i)_{\mathcal{I}} : \sum v_i = 0\}. \quad (5.1.2)$$

²A function $f(\cdot)$ with domain $[a, b]$ is said to be absolutely continuous if $\forall \varepsilon > 0 \exists \delta > 0$ such that $\sum_{k=1}^m |f(b_k) - f(a_k)| < \varepsilon$ holds for every finite system of pairwise non-overlapping intervals $(a_k, b_k) \subset (a, b)$, $k = 1, 2, \dots, m$, which obeys $\sum_{k=1}^m (b_k - a_k) < \delta$.

One more remark in addition. What is a contract for a given trajectory? By definition of contractual trajectory (curve), we can not determine it in general because we do not know the duration of the contract's realization (what does zero duration mean?). This is why one can correctly say only about *momentary contracts*, or about the summation of contracts, signed during a non-zero time interval. Keeping this point of view, one can say about the summation of contracts for a “measurable time” $\Theta \subseteq [0, \tau]$, where Θ is any measurable subset of interval. In such a case, we have

$$\sum_{\Theta} v(s) = \int_{\Theta} \dot{x}(s) ds.$$

Surely, items (i), (ii) do not describe all properties of contractual trajectory related to contractual processes; these are only initial, unconditional requirements. Further, let us consider other features of trajectory which correspond to contractual processes. In addition, it is necessary to take into account conditions, at which contracts are signed, and also character of a trajectory changes under the breaking of contracts.

For the constructive description of contractual processes, related with the breaking of contracts, it is convenient to consider an extended understanding of a trajectory described below, we shall call this a *coalitional trajectory*.

Suppose that for each coalition $S \subseteq \mathcal{I}$ with at least two elements, $\text{card}(S) \geq 2$, an (absolutely continuous) map

$$v^S : [0, +\infty) \rightarrow \mathfrak{L}_S^c, \quad \mathfrak{L}_S^c = \{v \in \mathfrak{L} \mid v = (v_i)_{\mathcal{I}} : \sum_{i \in S} v_i = 0 \ \& \ v_i = 0, \forall i \notin S\}. \quad (5.1.3)$$

is determined. Essentially, $v^S(t)$ is gross (total) contract, achieved by the members of a coalition S at a moment $t \geq 0$. A collection of all such maps $\{v^S(t)\}_{S \in \mathbf{K}} = V(t)$, related to a set of *permissible* coalitions $\mathbf{K} \subset \mathbf{2}^{\mathcal{I}}$, obviously determines a trajectory in a previous sense by formula

$$x(t) = \mathbf{e} + \sum_{S \in \mathbf{K}} v^S(t), \quad t \geq 0. \quad (5.1.4)$$

Notice that in this description of contractual trajectory we actually describe not only a current allocation, but a set of *varying with time contracts*, where each *coalition* has *the only* current gross contract (for forbidden coalitions—zero). As time elapsed, this set can be transformed according to the rules of proper-contractual behavior. Therefore, to ensure that an allocation realized after partial breaking of the contracts from $V(t)$ is feasible, it is necessary in addition to require that $V(t)$ is a *web* of contracts (relative to \mathbf{e}).

We would like to finish by noting one important thing. Basically, the trajectory $x(t)$, $t \in [0, +\infty)$ comprises the whole information on the contracts made by the agents, their volumes and the time moments of signing. This information is contained in the derivative of trajectory $\dot{x}(t)$. Therefore a coalitional-contractual trajectory is not a new object, but just a convenient form for representation of information in adequate aggregated kind. Really, for each map the value $v^S(t)$ at a point $t \geq 0$ can

be determined by the formula

$$v^S(t) = \int_{\Theta_S^t} \dot{x}(s) ds, \quad \Theta_S^t = \{s \in [0, t] \mid \dot{x}_i(s) \neq 0, i \in S \ \& \ \dot{x}_i(s) = 0, i \in \mathcal{I} \setminus S\}.$$

However, there is one nuance here, which can appear in the case when two or more pairwise non-intersected coalitions are independently signing new contracts at the same moment of time, or simply the time intervals of contracts' realizations are overlapping. Essentially, the consideration of such situations is consistent, especially in coalitional-contractual context. To avoid some collisions, related to appearing now ambiguity in the restoration of gross coalitional contract via derivative of trajectories, the easiest way is to conduct analysis in the terms of a coalitional trajectory.

5.1.2 Contracting and recontracting processes

A coalition can sign a new contract only if all members of coalition have relevant motives in signing, *i.e.*, after contract's realization (up to current moment) the utility of every member has to increase. In the last section, we have seen that contract per time unit is the derivative of trajectory at time point. Thus for smooth preferences one can think that contract v will be signed by coalition S , *i.e.*, trajectory moves along vector $v = \dot{x}(t)$ only if $\text{supp}(v) = S$ and

$$\langle \dot{x}_i(t), \nabla u_i(x(t)) \rangle > 0, \quad \forall i \in S.$$

Since $v_i = 0$ if $i \notin S$, then we can write a determining condition:

$$\dot{x}_i(t) \neq 0 \Rightarrow \langle \dot{x}_i(t), \nabla u_i(x(t)) \rangle > 0, \quad \forall i \in \mathcal{I}, \forall t \geq 0. \quad (5.1.5)$$

This condition characterizes moment t as the case of the contract signing. Now let us consider the case of *contracts' breaking*.

The description of the contractual process with the partial breaking of contracts is possible in a rather general framework. However, in such a case the formal-mathematical analysis of process, with the purpose to prove its convergence, looks very difficult, at least at this stage of research. This is why further we shall make several simplifying hypotheses. These hypotheses determine basic parameters: which *contracts*, in which *time moment* and in which *volume* are broken off, *i.e.*, all vagueness of contractual process related with the breaking of contracts are revealed.

The decision on partial break of the contracts is accepted by each agent individually, in conditions of sufficient information for myopic-rational breaking of the contracts. We conceive that, as opposed to signing of the new contract, where an individual needs to find the partners and to pass a stage of negotiations about the future contract, the breaking of contracts is a simpler decision and, therefore, can be accepted and is realized without temporary delays, as soon as there is the suitable opportunity. This motivates the following hypothesis.

(IB) *Instantaneous Breaking of the contracts.* In each time moment each individual instantly (for zero time) partially breaks the signed earlier contracts in an

optimum volume.

This hypothesis does not say anything about what contracts and in which volume can be broken off. In a general case pertinently to think, that each individual has an opportunity (right) to break in any volume any contract, signed earlier current time moment $t \geq 0$. However, to simplify the subsequent analysis, it is possible to consider for the beginning some particular cases, a little bit limiting opportunities on break of the contracts.

For the *aggregated* contractual trajectory, defined in (5.1.1), (5.1.2) we shall postulate:

(UB) *Uniform Breaking of all contracts.* At each time moment each individual can partially break *all* contracts signed in economy to the given moment, but just in *identical measure* (proportion).

This hypothesis assumes that at a current moment $t \in [0, +\infty)$ each agent makes a decision on break of the contracts. This decision is based on minimum of the *information*, extracted only from *current allocation* $x(t)$ and not accepting in attention the values $x(t')$, “passed” by a trajectory in previous time moments $t' \in (0, t)$. Apparently, such point of view for an opportunity to break contracts is acceptable for an economy with a few agents, where it is possible to assume that the contracts are signed only by a coalition of all agents (grand coalition). However, if economy consists of many agents, this assumption is problematic. It is not clear why the effect of breaking contracts with involved persons has to influence in such crucial manner—the break in the same measure—on uninvolved contract individuals? However, to carry out break only for a part of the contracts in which agent is involved, it is necessary that this part be explicit because it is better for essence of contractual process. One of simple variants of revealing this information is to consider the trajectory in coalitional-contractual form described in the previous section.

For the *coalitional* trajectory, described in (5.1.3), (5.1.4) we shall assume:

(CUB) *Coalitional Uniform Breaking of contracts.* At each moment of time each individual can partially break *all* contracts, signed by any coalition, in which she/he participates, and in limits of a coalition *in an identical measure*, but, probably, in different proportions for different coalitions.

From the informational point of view, this hypothesis means that each agent stores (remembers) the aggregated information about intra-coalition exchanges, in the form of “gross” contract. Thus, now the results of breaking of contracts by an individual will influence only the agents directly involved in the barter contract with this individual by means of gross coalitional contract, and it does not concern exchanges in other coalitions. As the special case of this hypothesis, it is possible to examine variant when *breaking* and the *signing* of new contracts occurs in frameworks of *the same coalition*.

5.2 Proper-contractual UB-processes: uniform breaking of contracts

In the previous section we have discussed contractual processes without breaking of contracts, and also have considered some properties and hypotheses, related with the partial breaking of contracts, signed earlier current time moment. Now we are going to consider processes, in which the signing and partial break of the contracts go in *simultaneous mode*.

At first, with the purpose to simplify the subsequent analysis, we shall consider the case of *aggregated* proper-contractual trajectory, for which it is admitted only partial breaking of *all* contracts signed to the current moment, and all in an equal measure.

Further, we have to clarify that means the fact that an individual i at the moment t *did not want*, but after contract v signing, at some moment $\tau > t$ he/she *wants* partially break contracts. The first means that $\langle \nabla u_i(x(t)), x_i(t) - \mathbf{e}_i \rangle \geq 0$, the second one that $\langle \nabla u_i(x_i(t) + (\tau - t)v_i), x_i(t) + (\tau - t)v_i - \mathbf{e}_i \rangle < 0$. Therefore, does exist a moment $t + \Delta t \in [t, \tau]$ such that

$$\langle \nabla u_i(x_i(t) + \Delta t v_i), x_i(t) + \Delta t v_i - \mathbf{e}_i \rangle = 0.$$

Notice also, that by virtue of **(IB)** the effect of contracts' breaking can influence the change of a trajectory at a moment t if and only if in *each* neighborhood of t there is a moment $\tau > t$ with the specified above properties. Passing $\tau \rightarrow t$ we obtain

$$\langle \nabla u_i(x_i(t)), x_i(t) - \mathbf{e}_i \rangle = 0, \quad (5.2.1)$$

and this is the first condition, which defines the motion of trajectory with contracts' breaking. Notice that the point defined by equation (5.2.1) is the maximal point of utility $u_i(y_i)$ on the ray starting at the point \mathbf{e}_i in direction $x_i - \mathbf{e}_i$ (here $y_i = \mathbf{e}_i + \lambda(x_i - \mathbf{e}_i)$, $\lambda \geq 0$).

Further, let us consider another condition. Primary, at the time t of signing, contract v was mutually beneficial at the point $x(t)$. The fact that at the moment $\tau > t$ an individual $i \in \text{supp}(v)$ partially breaks gross contract $x(t) + (\tau - t)v - \mathbf{e}$ in a volume $1 - \alpha$ means that from the point $x(t)$ the trajectory moves to point $z = \mathbf{e} + \alpha(x(t) + (\tau - t)v - \mathbf{e}) = (1 - \alpha)\mathbf{e} + \alpha(x(t) + (\tau - t)v)$, $0 \leq \alpha < 1$. Once again this is a maximum point for agent i 's utility on the linear segment, linking $x(t) + (\tau - t)v$ with initial endowments vector (thus there is a projection along straight line going through two points). Therefore, new point of trajectory has to satisfy the equation

$$\langle \nabla u_i(x_i(\tau)), x_i(\tau) - \mathbf{e}_i \rangle = 0.$$

Setting $\Delta t = \tau - t$ and substituting in equation expressions

$$x_i(t + \Delta t) = x_i(t) + \Delta t \dot{x}_i(t) + o(\Delta t),$$

$$\nabla u_i(x_i(t + \Delta t)) = \nabla u_i(x_i(t)) + \nabla^2 u_i(x_i(t))(\Delta t \dot{x}_i(t) + o(\Delta t)) + o(\Delta t \dot{x}_i(t) + o(\Delta t)),$$

which are true because of Taylor's formula,³ and taking into account (5.2.1), we find

$$\begin{aligned} \Delta t \langle \nabla u_i(x_i(t)), \dot{x}_i(t) \rangle + \Delta t \langle \nabla^2 u_i(x_i(t)) \dot{x}_i(t), x_i(t) - \mathbf{e}_i \rangle + \\ \Delta t^2 \langle \nabla^2 u_i(x_i(t)) \dot{x}_i(t), \dot{x}_i(t) \rangle + o(\Delta t) = 0. \end{aligned}$$

Now one can divide this on Δt and pass to limit over $\Delta t \rightarrow 0$. As a result we are coming to equation

$$\begin{aligned} \nabla u_i(x_i(t)) \dot{x}_i(t) + \langle \nabla^2 u_i(x_i(t)) \dot{x}_i(t), x_i(t) - \mathbf{e}_i \rangle = 0 \iff \\ \langle h_i(x_i(t)), \dot{x}_i(t) \rangle = 0, \quad h_i(x_i(t)) = \nabla u_i(x_i(t)) + \nabla^2 u_i(x_i(t))(x_i(t) - \mathbf{e}_i). \end{aligned} \quad (5.2.2)$$

Equations (5.2.1), (5.2.2) describe important properties of contractual trajectory but still do not completely define process. It is necessary also to take into account the dependence of $\dot{x}(t)$ from initially mutually beneficial contract v , signing which agents are coming to beneficial breaking of contracts for one of individuals. The situation is illustrated in Figure 5.2.1 (in Edgeworth box style), which reflects a character of transition and objects involved in the analysis.

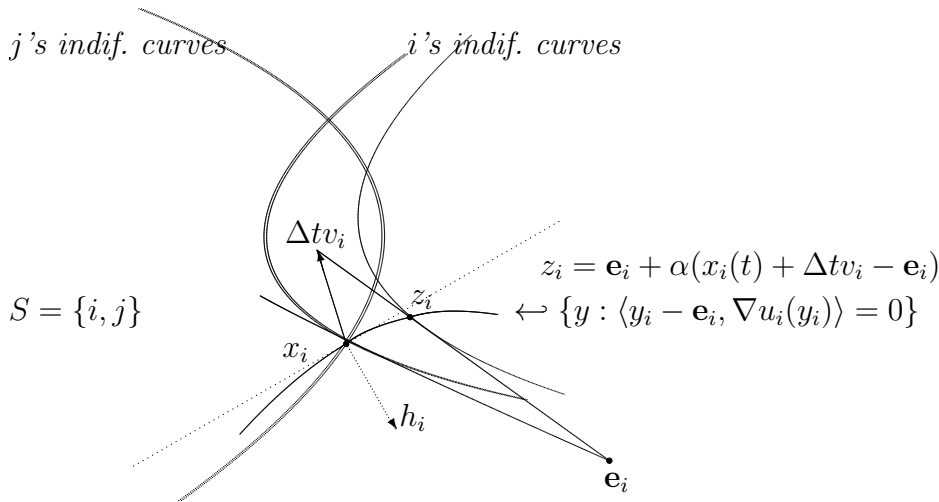


Figure 5.2.1: Proper-contractual transition

Recall that from the point $x(t)$ trajectory moves to the point $x(t + \Delta t) = \mathbf{e} + \alpha_i(x(t) + \Delta t v - \mathbf{e})$, $0 \leq \alpha_i < 1$ at the moment $t + \Delta t$. In general the value α_i depends on current consumption $x_i(t)$, (momentary) contract v and duration $\Delta t > 0$ of its realization. By virtue of assumption **(D)** the model is smooth, and it is easy to see, that $\alpha_i(x, v, \Delta t)$ is *differentiable* function (in general locally), implicitly defined from the equation

$$\langle \nabla u_i(x_i(t + \Delta t)), x_i(t + \Delta t) - \mathbf{e}_i \rangle = 0, \quad x_i(t + \Delta t) = (1 - \alpha_i)\mathbf{e}_i + \alpha_i(x_i(t) + \Delta t v_i). \quad (5.2.3)$$

Here parameter $\alpha_i \geq 0$ determines a point $(1 - \alpha_i)\mathbf{e}_i + \alpha_i(x_i(t) + \Delta t v_i)$ of i 's utility maximum on the ray, parameterized as: $\mathbf{e}_i + \lambda(x_i(t) + \Delta t v_i - \mathbf{e}_i)$, $\lambda \geq 0$. If $\alpha_i < 1$,

³ $o(\cdot)$ is the standard notation of infinitesimal value.

then at a point $x_i(t) + \Delta t v_i$ the breaking of contracts is realized in volume $1 - \alpha_i$, and, there is no the break if $\alpha_i \geq 1$.

From representation $x_i(t + \Delta t)$ in the right part of (5.2.3), we have

$$\frac{x_i(t + \Delta t) - x_i(t)}{\Delta t} = \frac{(\alpha_i(x, v, \Delta t) - 1)}{\Delta t} (x_i(t) - \mathbf{e}_i) + \alpha_i(x, v, \Delta t) v_i,$$

whence, passing to a limit on $\Delta t \rightarrow 0$, with the account $\alpha_i(x(t), v, \Delta t)|_{\Delta t=0} = 1$ (by virtue of (5.2.1)), we obtain

$$\dot{x}_i(t) = \lambda_i(x_i(t) - \mathbf{e}_i) + v_i, \quad \lambda_i = \frac{\partial \alpha_i(x(t), v, \Delta t)}{\partial \Delta t} \Big|_{\Delta t=0}.$$

Further, value λ_i is possible to find from the equation (5.2.2),

$$\langle h_i(x_i(t)), \lambda_i(x_i(t) - \mathbf{e}_i) + v_i \rangle = 0 \quad \Rightarrow \quad \lambda_i = \frac{\langle h_i(x_i(t)), v_i \rangle}{\langle h_i(x_i(t)), (\mathbf{e}_i - x_i(t)) \rangle}. \quad (5.2.4)$$

Thus, if at a moment t there exists *only one* agent, satisfying (5.2.1), with number i , the trajectory locally will change under the law

$$\dot{x}(t) = \lambda_i(x, v)(x(t) - \mathbf{e}) + v, \quad \lambda_i(x, v) = \frac{\langle h_i(x_i(t)), v_i \rangle}{\langle h_i(x_i(t)), (\mathbf{e}_i - x_i(t)) \rangle}.$$

Moreover, presented considerations allow us to reveal complete conditions, describing a moment t as a situation of break of the contracts at a current allocation $x(t)$ and when (momentary) a contract v is signing. The break is realized if the individual i satisfies (5.2.1), value $\alpha_i(x(t), v, \Delta t)$ is locally *decreased* in Δt at a point $\Delta t = 0$, *i.e.*, if derivative with respect to Δt is negative. So, for breaking it is necessary and enough⁴ that $\lambda_i(x, v) < 0$. Further, by virtue of the assumption **(D)** the matrix of second partial derivatives $\nabla^2 u_i(x_i(t))$ is negatively defined, whence by virtue of (5.2.1) and (5.2.2) for $x_i(t) - \mathbf{e}_i \neq 0$ we conclude

$$\langle h_i(x_i(t)), \mathbf{e}_i - x_i(t) \rangle = -\langle \mathbf{e}_i - x_i(t), \nabla^2 u_i(x_i(t))(\mathbf{e}_i - x_i(t)) \rangle > 0.$$

Thus, denominator in (5.2.4) is always positive and, therefore, the situation of breaking of contracts by the individual i is completely characterized by condition (5.2.1) and additional condition

$$\langle h_i(x_i(t)), v_i \rangle < 0.$$

What will take place in the case of several agents, desiring to break off contracts, when a new contract v starts to be realized? In other words, how will the contractual process go, if *more than one* individual satisfies (5.2.1)? For all these individuals values $\alpha_i(x(t), v, \Delta t)|_{\Delta t=0} = 1$, but the character of process is determined by their derivatives. It is clear that the break will happen only if *at least one* derivative relative to Δt is negative, and the measure of breaking is defined by greatest absolute value from negative derivatives. Thus, it is proved the following

⁴When $\lambda_i(x, v) = 0$ the point $x = x(t)$ can be or cannot be a limit point of breaking contracts points of a trajectory.

Lemma 5.2.1 *Consider contractual process with partial breaking of barter contracts, satisfying hypotheses (IB), (UB)—Instant Uniform Breaking of all contracts. Let (x, v) be a couple achieved in process at a moment $t \geq 0$, where $x = x(t) = (x_1, \dots, x_n)$ is allocation and $v = (v_1, \dots, v_n)$ is a momentary mutually beneficial barter contract, signed among individuals at the moment t . A pair (x, v) sets a situation of breaking of contracts if and only if for some $i \in \mathcal{I}$*

$$\langle \nabla u_i(x_i(t)), x_i(t) - \mathbf{e}_i \rangle = 0 \quad \& \quad \langle h_i(x_i(t)), v_i \rangle < 0, \quad (5.2.5)$$

$$h_i(x_i(t)) = \nabla u_i(x_i(t)) + \nabla^2 u_i(x_i(t))(x_i(t) - \mathbf{e}_i)$$

takes place. In this case, the local law of contractual process is defined as:

$$\dot{x}(t) = \lambda(x, v)(x(t) - \mathbf{e}) + v, \quad (5.2.6)$$

where $\lambda(x, v)$ is minimum of $\lambda_i(x_i, v_i)$, calculated for individuals $i \in \mathcal{I}$, satisfying condition $\langle \nabla u_i(x_i(t)), x_i(t) - \mathbf{e}_i \rangle = 0$; here

$$\lambda_i(x_i, v_i) = \frac{\langle h_i(x_i(t)), v_i \rangle}{\langle h_i(x_i(t)), (\mathbf{e}_i - x_i(t)) \rangle}.$$

As it already was noted, if the break of contracts does not occur, the local law of change of a contractual trajectory is set by a rule $\dot{x}(t) = v$. Hence, one can apply the law (5.2.6) in a general case, if for $\lambda(x, v) > 0$ replace this value by zero. Combining this fact with the result of previous Lemma 5.2.1, we come to the following definition of proper contractual trajectories. Let's define

$$\lambda^{\min}(x, v) = 0 \bigwedge \min \left\{ \frac{\langle h_i(x_i), v_i \rangle}{\langle h_i(x_i), (\mathbf{e}_i - x_i) \rangle} \mid i : \langle \nabla u_i(x_i), x_i - \mathbf{e}_i \rangle = 0 \right\}. \quad (5.2.7)$$

Definition 5.2.1 *An absolutely continuous map $x(\cdot) : [0, +\infty) \rightarrow \mathcal{A}(X)$ is called proper contractual trajectory under hypotheses (IB), (UB), if the following conditions are satisfied:*

(i) $\langle \nabla u_i(x_i(t)), x_i(t) - \mathbf{e}_i \rangle \geq 0, \forall i \in \mathcal{I};$

(ii) *Derivative of the trajectory obeys the law*

$$\dot{x}(t) = \lambda^{\min}(x, v)(x(t) - \mathbf{e}) + v, \quad (5.2.8)$$

where $v \in L^c$ is mutually beneficial contract, i.e., $\langle \nabla u_i(x_i(t)), v_i \rangle > 0, \forall i \in \text{supp}(v)$, and value $\lambda^{\min}(x, v)$ is defined by (5.2.7).

Notice that due to this definition proper contractual trajectory is actually described as a solution of some differential inclusion

$$\dot{x}(t) \in F(x), \quad x(0) = \mathbf{e}$$

⁵Here standardly $a \wedge b = \min\{a, b\}$ and $\min\{\lambda_i(x, v) \mid i \in \emptyset\} = +\infty$ by convention.

on interval $[0, +\infty)$, where the right-hand side obeys (i), (ii).

The law of change of proper contractual trajectory (5.2.8) can have another form, but, certainly, the law should satisfy restrictions (5.2.1), (5.2.2). Really, (5.2.8) postulates the certain form of projection of current allocation $x = (x_1, \dots, x_n)$ on area, defined by constraints

$$\langle \nabla u_i(x_i(t)), x_i(t) - \mathbf{e}_i \rangle \geq 0, \quad \forall i \in \mathcal{I}.$$

Essentially, this projection corresponds to a procedure of partial breaking of all contracts, and the area, on which allocation is projected, is the set of all allocations stable relative to the partial breaking of all contracts (or, alternatively, realized by a proper contractual web $V = \{x - \mathbf{e}\}$, see Chapter 1, Def. 1.1.3). In doing so, the projection is realized along the ray $(\mathbf{e}_i - x_i)$, where i is the individual, satisfying (5.2.1), onto hyperplane $\{z \in \mathbb{R}^l \mid \langle h_i(x_i), z \rangle = \langle h_i(x_i), x_i \rangle\}$, where the vector $h_i(x_i)$ is determined by (5.2.2). Such kind of projection assumes that only contracts signed before the moment t may be broken and that the contract v is unchanged. However, it is possible to postulate another law of breaking, in which the breaking of a current new contract is possible:

$$\dot{x}(t) = \beta(x, v)v + (1 - \beta(x, v))(\mathbf{e} - x(t)). \quad (5.2.9)$$

We can treat this process as a break with delay, *i.e.*, the break occurs only after realization of (momentary) contract v . A value $\beta(x, v)$ can be found from the equation (5.2.2), whence, if i satisfies (5.2.1), we find

$$\beta(x, v) = \beta_i(x_i(t), v_i) = \frac{\langle h_i, x_i(t) - \mathbf{e}_i \rangle}{\langle h_i, x_i(t) - \mathbf{e}_i \rangle + \langle h_i, v_i \rangle}. \quad (5.2.10)$$

Notice that numerator in (5.2.10) is always negative and that the denominator is a sum of numerator with the value $\langle h_i, v_i \rangle$, which in case of contracts' breaking is negative, and, therefore, if the break of the contracts is beneficial for i , the whole ratio is a value between zero and unit. The value $\beta_i(x_i(t), v_i) \leq 1$ defines for the process (5.2.9) a measure of contracts' breaking: if it is less, the break is more. Moreover, for $\beta_i(x_i(t), v_i) \geq 1$ should be $\langle h_i, v_i \rangle \geq 0$ and the break does not occur. In case when several individuals satisfy (5.2.1) the break should come true at a maximum level. Therefore, the law of a trajectory change has the following form. Let's define

$$\beta^{min} = \beta^{min}(x(t), v) = 1 \wedge \min\{\beta_i \mid i \in \mathcal{I} : \langle \nabla u_i(x_i(t)), x_i(t) - \mathbf{e}_i \rangle = 0\},$$

where values $\beta_i(x(t), v)$ are set by the formula (5.2.10). Then our process obeys the law

$$\dot{x}(t) = \beta^{min}v + (1 - \beta^{min})(\mathbf{e} - x(t)), \quad (5.2.11)$$

where $v \in \mathfrak{L}^c$ is a mutually beneficial contract, *i.e.*, $\langle \nabla u_i(x_i(t)), v_i \rangle > 0, \forall i \in \text{supp}(v)$.

How do the described processes correspond among themselves? In essence, they are equivalent. Really, if only one agent is ready to break contracts and carries out it in the process (5.2.8) then locally this law can be written as

$$\dot{x}(t) = \frac{\langle h_i(x_i(t)), v_i \rangle}{\langle h_i(x_i(t)), (\mathbf{e}_i - x_i(t)) \rangle} (x(t) - \mathbf{e}) + v = \frac{1}{\beta_i} [\beta_i v + (1 - \beta_i)(\mathbf{e} - x(t))],$$

where β_i is defined by (5.2.10). It is easy to see that in general case (few agents satisfy (5.2.1)) will be the similar connection, *i.e.*, we have

$$\dot{x}(t) = \lambda^{\min}(x, v)(x(t) - \mathbf{e}) + v = \frac{1}{\beta^{\min}(x, v)}[\beta^{\min}(x, v)v + (1 - \beta^{\min}(x, v))(\mathbf{e} - x(t))].$$

Thus, the distinction in processes is reduced to some positive factor, defined by function $\beta^{\min}(x, v)$, that influences only the speed of the contractual process and can be eliminated by the arbitrariness in a choice of mutually beneficial contract v .⁶

5.3 CUB-processes: coalitional breaking of contracts

Further, we shall consider the concept of *coalitional* proper contractual trajectories, *i.e.*, a trajectory, satisfying hypothesis **(CUB)** (instead of **(UB)**). For this one can apply a significant part of the above analysis, but there are also essential differences:

- (i) The partial breaking of gross intra-coalitional contract for different coalitions can be realized in a different degree (measure).
- (ii) Breaking the gross coalitional contract the individuals are guided not on initial endowments allocation, available in the beginning of a trajectory, but on their sum with a flow of goods, received from participation in contracts of other coalitions.
- (iii) The signing of a new barter contract by the members of a coalition with, probably, subsequent breaking of intra-coalitional contract, potentially can also initiate the break of contracts in other coalitions (if the individual participates in contract).

The difficulties which may happen here are caused by item (iii), which in general case can entail hardly predictable character of contractual process under hypothesis **(IB)**—instantaneous breaking of the contracts. A problem, which we often encounter, can be illustrated by the following example.

Let us consider an economy with three agents in which all coalitions are permitted. Let the coalition $\{1, 2, 3\}$ be active and its members sign a new contract, which being realized creates the following situation. All three agents can wish partially break off the contracts in bilateral coalitions. However, the break of the contracts in coalition changes consumption bundles of its members and, thus, influences a measure of a desirable break of the contracts in other coalition. For example, in a coalition $\{i, j\}$ the agent i would like to break off intra-coalitional contract in volume $\frac{1}{2}$, if a coalition $\{i, k\}$ does not break anything. However, if the agent k will break the coalitional contract in volume $\frac{1}{3}$, the agent i changes opinion on volume of break of contract

⁶Since the law (5.2.8) for a contract $w = \beta^{\min}(x, v)v$ is equivalent to the law (5.2.11) relative to v .

in $\{i, j\}$ on the break in a measure $\frac{1}{3}$. However, the break of contract in coalition $\{k, j\}$, initiated by the agent j , can affect the opinion of individual k but the opinion of j is influenced by a measure of break of contract in $\{i, j\}$, it may be $\frac{1}{2}$ or *e.g.* $\frac{1}{3}$, and *etc.* So, what will happen as a result of such cyclic breaking of the contracts? Before the study a general case, it is reasonable to analyze some particular variants of proper-contractual process.

One of the opportunities is to try to modify the assumption **(IB)**. One can assume that, similarly to the signing of a new contract, a *breaking* of contracts in a coalition occurs only *if* this *coalition is active*. The passive coalition neither able to sign a new contract nor break signed earlier contracts. Further, this behavioral hypothesis is designated as **(IBA)** (*i.e.* **(IB)** only for active coalitions).

The second opportunity consists of restriction of a class **K** of permitted coalitions. For example, it can be a class of all paired (two elements) coalitions. Then, if the coalition $\{i, j\}$ is active in contractual process, it can entail breaks of contracts only in coalitions $\{i, k\}$ and $\{k, j\}$, where $k \neq i, j$. Moreover if *only one coalition* is active at a current moment of time then only its members, i and j , can initiate the breaking of contracts and the transition is unequivocally determined.

5.3.1 Breaking of contracts in active coalitions

It is natural to assume in addition, that only *non-intersected* coalitions can be *active* when time is passing, of course, for different time moments there may be the different sets of active coalitions. Besides, we shall assume that if in some active coalition there is an agent, aspiring to break contracts *without signing of new contract*, then *only the breaking* is realized in contractual process for this coalition at a current moment of time. Further, we shall describe this process in details.

Let some coalition-contractual trajectory be given, described by a set of maps $\{v^S(t)\}_{S \in \mathbf{K}}$, adequate for some set of *permitted* coalitions $\mathbf{K} \subset \mathbf{2}^I$. These maps (see (5.1.3)) are absolute continuous on $[0, \infty)$ and satisfy: $\forall t \geq 0$

$$v^S(t) = (v_i^S(t))_I : \sum_{i \in S} v_i^S(t) = 0 \ \& \ v_i^S(t) = 0, \ \forall i \notin S.$$

The trajectory is related with these maps by equality

$$x(t) = \mathbf{e} + \sum_{S \in \mathbf{K}} v^S(t), \quad t \geq 0.$$

Notice that now, in difference with *IUB*-trajectory by Definition 5.2.1, a current state of trajectory $x(t)$ can already not satisfy condition of absence of desire to partially break intra-coalitional gross contracts $v^S(t)$ for the agents from each of permitted coalitions $S \in \mathbf{K}$, *i.e.*, item (i) in coalitional context of Definition 5.2.1 can be broken. This is the basic difference of *CUB*-process with breaking only in active coalitions.

Let in a moment $t \geq 0$ there is a list of active coalitions $\Psi(t) = \{S_1, \dots, S_k\} \subset \mathbf{K}$, which are *pairwise non-intersected*. At the given moment t the exchange processes

can go only in coalitions from this list. Thus for each of active coalitions (we shall denote a current one as S) where one of two opportunities can be realized:

(i) The members of coalition S do not desire to break gross coalitional contract $v^S(t)$, and there is an opportunity to sign a new mutually beneficial contract (if not, then coalition is idle, *i.e.*, the zero contract is signed).

(ii) There is an individual in S , aspiring to break the coalitional contract $v^S(t)$.

Further, we consider specified opportunities consistently. Let at a moment t the first opportunity is realized and coalition S signs a momentary mutually beneficial contract $w^S = (w_i^S)_S = \mathfrak{L}_S^c$ (this is a space of the possible contracts for coalition $S \subseteq \mathcal{I}$). Now conditions determining a situation of contracts' breaking are similar to described in Lemma 5.2.1 and are adapted to coalition-contractual trajectory. Really, the favourable situation for the breaking of gross contract $v^S(t)$ as a result of the new contract realization can happen only if for some $i \in S$

$$\langle \nabla u_i(x_i(t)), v_i^S(t) \rangle = 0$$

takes place, this is an analogue of condition (5.2.1). Similarly, the breaking of contract $v^S(t) \neq 0$ occurs if and only if for one of agents, satisfying the previous condition

$$\langle h_i^S(x, v^S(t)), w_i^S \rangle < 0, \quad h_i^S(x, v^S(t)) = \nabla u_i(x_i(t)) + \nabla^2 u_i(x_i(t)) v_i^S(t) \quad (5.3.1)$$

takes place. Now, if only one agent $i \in S$ aspires to break contracts, for the agents from S the law of change of a trajectory can look like

$$\dot{v}^S(t) = \lambda_i^S(x, v^S, w^S) v^S(t) + w^S, \quad \lambda_i(x, v^S, w^S) = -\frac{\langle h_i^S(x_i, v_i^S(t)), w_i^S \rangle}{\langle h_i^S(x_i, v_i^S(t)), v_i^S(t) \rangle}.$$

If more than one agent of the coalition aspires to break coalitional contract, instead of $\lambda_i^S(x, v^S, w^S)$ in the law of a trajectory one have to take a minimum from values of this type relative to the set of all such agents. For $S \in \Psi(t)$ let us determine

$$\lambda_S^{min}(x, v^S(t), w^S) = 0 \bigwedge \min \left\{ \frac{-\langle h_i^S(x, v^S(t)), w_i^S \rangle}{\langle h_i^S(x, v^S(t)), v_i^S(t) \rangle} \mid i \in S : \langle \nabla u_i(x_i), v_i^S(t) \rangle = 0 \right\}. \quad (5.3.2)$$

Let (ii) of above possibilities be realized and let for some $i \in S$ the breaking of gross coalitional contract be favorable, *i.e.*,

$$\langle \nabla u_i(x_i), v_i^S(t) \rangle < 0$$

takes place. We may think that in such a case the new contract is not signed, and there is *only the break* of current contract $v_i^S(t)$. The latter means that derivative of coalitional contract should be proportional to a vector $-v^S(t)$, that allows to postulate the law⁷

$$\dot{v}^S(t) = -v^S(t).$$

⁷This law of trajectory's change can be deduced analytically, if in a basis of the analysis one takes a trajectory by (5.2.11), for which specified situation is realized in limit when $\beta^{min} \rightarrow 1$.

As a result, we are coming to the following definition. Recall that

$$\mathfrak{L}_S^c = \{w \in \mathfrak{L} \mid w = (w_i)_{\mathcal{I}} : \sum_{i \in S} w_i = 0 \ \& \ w_i = 0, \ \forall i \notin S\}.$$

Definition 5.3.1 A set $\{v^S(t)\}_{S \in \mathbf{K}}$ of absolutely continuous maps $v^S(\cdot) : [0, +\infty) \rightarrow \mathfrak{L}_S^c$ is called coalitional proper contractual trajectory under hypotheses **(IBA)**, **(CUB)**, if the following conditions are satisfied:

- (i) For every $t \geq 0$ a list $\Psi(t) = \{S_1, \dots, S_k\} \subset \mathbf{K}$ of active pairwise non-intersected coalitions is specified, in which and only in them, contractual processes may go on;
- (ii) The derivative of trajectory is defined by a set of momentary mutually beneficial contracts $\{w^S\}_{S \in \Psi(t)}$, signed among the members of active at the moment $t \geq 0$ coalitions $\Psi(t) \subset \mathbf{K}$, and obeys the law

$$\dot{v}^S(t) = \lambda_S^{\min}(x, v^S(t), w^S) v^S(t) + w^S, \quad S \in \Psi(t), \quad (5.3.3)$$

where if $\langle \nabla u_i(x_i(t)), v_i^S(t) \rangle \geq 0, \forall i \in S \in \Psi(t)$ then the value $\lambda_S^{\min}(x, v^S(t), w^S)$ is defined by formula (5.3.2). If

$$\langle \nabla u_i(x_i(t)), v_i^S(t) \rangle < 0, \text{ for some } i \in S \in \Psi(t)$$

then $\lambda_S^{\min}(x, v^S(t), w^S) = -1$ and $w^S = 0$ takes place.

- (iii) For $T \in \mathbf{K} \setminus \Psi(t)$, i.e., if coalition T at the moment t is passive (non-involved in contractual process), then $\dot{v}^T(t) = 0$ and gross intra-coalitional contract does not change.

5.3.2 Proper contractual bilateral CUB-process

In this section we shall assume that the set \mathbf{K} of permitted coalitions in contractual process is exhausted by all paired coalitions, i.e., we consider the case when

$$\mathbf{K} = \{\{i, j\} \mid i, j \in \mathcal{I} \ \& \ i \neq j\}.$$

In addition, we shall assume that at any current time moment $t \geq 0$ only one from paired coalitions $\{i, j\}$ can sign a new mutually beneficial contract w^{ij} .

Under these hypotheses one can describe proper-contractual process, adequate to postulates **(IB)** and **(CUB)**. As only paired coalitions are permitted, then the breaking of contracts initiated by the signing of a new contract may proceed in all coalitions in a predictable way and in conformity with **(IB)**—instantaneous breaking of all unprofitable contracts. Let us describe this process.

Let to a moment $t \geq 0$ for any permitted coalition $S = \{i, j\}$ a state of a trajectory $x(t) = \sum v^{ij}(t) + \mathbf{e}$, which is defined via a family of gross coalitional contracts $v^{ij}(t)$,

obey a condition of absence of desire partially break them. This can be expressed in a form

$$\langle \nabla u_k(x_k(t)), v_k^{ij}(t) \rangle \geq 0, \quad \forall k \in \{i, j\}, \quad \forall i, j \in \mathcal{I}.$$

Fix further coalition $S = \{i, j\}$, which we think to be active one at a moment t , and let its members sign a new contract w^{ij} . Then at moment $\tau = t + \Delta t > t$ the current consumption bundles will look like:

$$x_k(\tau) = \mathbf{e}_k + \sum_{m \neq i, j, k} v_k^{km}(t) + v_k^{ki}(t) \alpha_k^i(w^{ij}, \Delta t) + v_k^{kj}(t) \alpha_k^j(w^{ij}, \Delta t), \quad k \neq i, j;$$

$$x_i(\tau) = \mathbf{e}_i + \sum_{k \neq i, j} v_i^{ik}(t) \alpha_k^i(w^{ij}, \Delta t) + \alpha^{ij}(w^{ij}, \Delta t) [v_i^{ij}(t) + \Delta t w_i^{ij}];$$

$$x_j(\tau) = \mathbf{e}_j + \sum_{k \neq i, j} v_j^{jk}(t) \alpha_k^j(w^{ij}, \Delta t) + \alpha^{ij}(w^{ij}, \Delta t) [v_j^{ij}(t) + \Delta t w_j^{ij}].$$

Here all values $\alpha_k^i(w^{ij}, \Delta t)$, $\alpha_k^j(w^{ij}, \Delta t)$ and $\alpha^{ij}(w^{ij}, \Delta t)$ are between zero and one and define saved volumes of gross coalitional contracts depending on the new contract w^{ij} and time interval of its realization $\Delta t \approx 0$. Thus for $\Delta t = 0$ all of them are equal to one. Notice that only individuals i and j can wish to break off the signed earlier contracts. Moreover, only contracts between i and agents from

$$I(i) = \{k \in \mathcal{I} \mid k \neq i, \langle \nabla u_i(x_i(t)), v_i^{ik}(t) \rangle = 0\},$$

can be broken, similarly between j and agents from

$$I(j) = \{k \in \mathcal{I} \mid k \neq j, \langle \nabla u_j(x_j(t)), v_j^{jk}(t) \rangle = 0\}.$$

If simultaneously $j \notin I(i)$ and $i \notin I(j)$, then only the contracts between one of the members of a coalition $\{i, j\}$ and non-members of the coalition can be broken off. In this case $\alpha^{ij}(w^{ij}, \Delta t) = 1$ for small Δt . In opposite case it is possible $\alpha^{ij}(w^{ij}, \Delta t) < 1$, *i.e.* coalition $\{i, j\}$ can also be involved in the process of contracts' breaking.

Further, as well as earlier, by virtue of the Taylor's formula

$$\nabla u_i(x_i(t + \Delta t)) = \nabla u_i(x_i(t)) + \nabla^2 u_i(x_i(t))(\Delta t \dot{x}_i(t) + o(\Delta t)) + o(\Delta t \dot{x}_i(t) + o(\Delta t)),$$

and similar representation for j . Besides, by virtue of previous considerations we have

$$\dot{x}_i(t) = \sum_{k \neq i, j} v_i^{ik}(t) \lambda_k^i(w^{ij}) + \lambda_i^j(w^{ij}) v_i^{ij}(t) + w_i^{ij};$$

where

$$\lambda_k^i(w^{ij}) = \frac{\partial \alpha_k^i(w^{ij}, \Delta t)}{\partial \Delta t} \Big|_{\Delta t=0}, \quad \lambda_i^j(w^{ij}) = \frac{\partial \alpha^{ij}(w^{ij}, \Delta t)}{\partial \Delta t} \Big|_{\Delta t=0}.$$

Here values $\lambda_k^i(w^{ij})$ and $\lambda_i^j(w^{ij})$ one can find solving a system of the equations, received from the first order necessary conditions (now they are also sufficient). Really,

if $I(i)^b \subseteq I(i)$ is the set of contractors for i , for which the break of contracts actually occurs, formally for them $\lambda_k^i(w^{ij}) < 0$, then it should be

$$\langle \nabla u_i(x_i(t + \Delta t)), v_i^{ki} \rangle = 0, \quad k \in I(i)^b.$$

Using further the representation of a gradient, after necessary transformations, division on Δt , with the subsequent passing to limit in $\Delta t \rightarrow 0$, we come to a linear system of the equations for $\lambda_k^i = \lambda_k^i(w^{ij})$:

$$\begin{aligned} \langle \nabla^2 u_i(x_i(t)) \dot{x}_i(t), v_i^{ki} \rangle = 0, \quad k \in I(i)^b & \iff \\ \sum_{m \in I(i)^b} v_i^{ki} [-\nabla^2 u_i(x_i(t))] v_i^{mi} \lambda_m^i = v_i^{ki} \nabla^2 u_i(x_i(t)) w_i^{ij}, \quad k \in I(i)^b. & \quad (5.3.4) \end{aligned}$$

Let us show now that due to assumption **(D)** this system has a unique solution only if the system of vectors $\{v_i^{ki}\}_{k \in I(i)^b}$ is linearly independent. From **(D)** the matrix $A = -\nabla^2 u_i(x_i(t))$ is positive definite and, therefore, it has a square root, *i.e.*, there exists a symmetric positive definite matrix $B = \sqrt{A}$, satisfying $B^2 = A$. Now the matrix of coefficients of system is represented as a Grama matrix $\|\langle b_k, b_m \rangle\|_{k, m \in I(i)^b}$ for the system of vectors $b_k = B v_i^{ki}$, $k \in I(i)^b$, and system (5.3.4) can be rewritten in a form

$$\sum_{m \in I(i)^b} \langle b_k, b_m \rangle \lambda_m^i = -\langle b_k, B w_i^{ij} \rangle, \quad k \in I(i)^b. \quad (5.3.5)$$

It is known from linear algebra that Grama matrix for a system of vectors is nonsingular one only when the system is linearly independent. However, as soon as B is nonsingular matrix, which determines nonsingular linear transformation, the system of vectors $\{b_k\}_{k \in I(i)^b}$ is linearly independent if and only if the system $\{v_i^{ki}\}_{k \in I(i)^b}$ is linearly independent.

Recall, that in general it is not enough that system (5.3.4) has a unique solution, it is also necessary that the solution is non-positive in each component. Actually, in some sense a solution of system (5.3.4) always exists, because values λ_m^i can be found as a limiting solution of a problem of convex optimization for a given (fixed) $\Delta t > 0$ on the compact set of variables $0 \leq \alpha_m^i \leq 1$. However, if the system of vectors $\{v_i^{ki}\}_{k \in I(i)}$ is linearly dependent, this solution may be not unique (in spite of the fact that functions u_i are strictly concave).

It is not easy to take into account all specified difficulties, which may appear in described context of proper contractual process—*CUB*-process with paired coalitions, it is too cumbersome for a first sight. This is why we specify below only the law of change of trajectories in non-degenerated case: it is a situation, when system (5.3.4) or, accordingly, (5.3.5), has a unique non-positive solution for both agents, i and j . There are two variants of this situation.

First. Assume that in a coalition $\{i, j\}$ no agent aspire to break coalitional contract $v^{ij}(t)$ after realization of momentary contract w^{ij} . It means that for i either $\langle \nabla u_i(x_i(t)), v_i^{ij} \rangle > 0$, or $\langle \nabla u_i(x_i(t)), v_i^{ij} \rangle = 0$ and in the solution of system (5.3.5) (which is unique!) takes place $\lambda_i^j(w^{ij}) = 0$. Similar properties should be carried out

for the individual j . Thus, in this case $j \notin I(i)^b$ and $i \notin I(j)^b$ simultaneously. Then at the point t the law looks like:

$$\begin{cases} \dot{v}^{ij}(t) &= w^{ij}, \\ \dot{v}^{ik}(t) &= \lambda_k^i(w^{ij})v^{ik}(t), & k \in I(i)^b, \\ \dot{v}^{jk}(t) &= \lambda_k^j(w^{ij})v^{jk}(t), & k \in I(j)^b, \\ \dot{v}^{km}(t) &= 0, & \text{for other pairs } (k, m). \end{cases}$$

Here all values $\lambda_k^i(w^{ij})$ and $\lambda_k^j(w^{ij})$ are found as a solution of system (5.3.5) (for j its own similar).

Second. Let in the coalition $\{i, j\}$ one of agents *aspire* to break coalitional contract $v^{ij}(t)$ after realization of momentary contract w^{ij} . It is possible only if, for example for agent i , $\langle \nabla u_i(x_i(t)), v_i^{ij} \rangle = 0$ and if in the solution of system (5.3.5) we have $\lambda_i^j(w^{ij}) < 0$ (or similar conditions for j). Now define $\lambda_{min}^{ij} = \lambda_i^j(w^{ij}) \wedge \lambda_j^i(w^{ij}) < 0$ and assume that individual i wishes to break off contract $v^{ij}(t)$ in the greatest measure, i.e., $\lambda_i^j(w^{ij}) \leq \lambda_j^i(w^{ij})$. It is a situation, in which the individual i imposes to the agent j the greater break of gross barter contract, than j wishes. However, it is nothing to do for j , he/she can only accept such rules of game, in so doing changing the initial plans of breaking of the contracts with other individuals. These new volumes of break should be found as a solution of system similar to (5.3.4), but for the agent j and with fixed unknown variable at $v_j^{ij}(t)$, where one has to take λ_{min}^{ij} . Thus, volumes λ_m^j of breaking of bilateral contracts between j and other agents now can be found from system

$$\sum_{m \in I(j)^b, m \neq i} v_j^{kj} [-\nabla^2 u_j(x_j)] v_j^{mj} \lambda_m^j = v_j^{kj} [\nabla^2 u_j(x_j)] (w_j^{ij} + \lambda_{min}^{ij} v_j^{ij}), \quad k \in I(j)^b, k \neq i.$$

Of course, everything said above can be correct only if all values to be found are non-positive (otherwise one needs to replace positive values by zero and to continue the search of a solution.) As a result, in this case we come to a system of the type

$$\begin{cases} \dot{v}^{ij}(t) &= \lambda_{min}^{ij} v^{ij} + w^{ij}, \\ \dot{v}^{ik}(t) &= \lambda_k^i(w^{ij})v^{ik}(t), & k \in I(i)^b, \\ \dot{v}^{jk}(t) &= \lambda_k^j(w^{ij})v^{jk}(t), & k \neq i, k \in I(j)^b, \\ \dot{v}^{km}(t) &= 0, & \text{for other pairs } (k, m). \end{cases}$$

We finish the description of bilateral contractual process at this point: basic character of this should be clear, but there is a wide variety of arising variants, which are rather cumbersome. This essentially reduces prospects for hereinafter effective analysis of convergence in a more-or-less general case.

5.4 Contractual processes: final specifications and associated price process

In the previous sections, there were considered basic hypotheses about behavior of the individuals, adequate to contractual processes with partial breaking of the contracts.

Besides, there were developed some variants of proper contractual processes (trajectories), which correspond to different combinations of these behavioral hypotheses. The elaborated processes actually were described in the terms of differential inclusions with an autonomous right part, having form

$$\dot{x}(t) \in F(x), \quad x(0) = \mathbf{e}, \quad t \in [0, +\infty).$$

However, for the analysis of convergence of contractual processes this form is not quite convenient, and already for a process without breaking of the contracts, for the convergence (to Pareto boundary) there are required some additional assumptions, ensuring the contractual process is going fast enough (see Section 4.1.4 and footnote 14)). We could formulate the necessary hypotheses in general terms of process, however more convenient form is simply to add to the description of process some *trade rule*, which unequivocally determines process as a whole. Doing so, we actually fix some selector of point-to-set mapping, described via differential inclusion. Thus, some additional conditions on a contractual trajectory will be made out as the requirements for a rule of trade. Further, we shall consider a rule of trade close to that stated in Sections 4.1.3, 4.1.4, but in an adequate form for contractual processes.

5.4.1 Trade rule for a contractual trajectory

Let us consider a coalitional proper-contractual trajectory, corresponding to *CUB*-process by Definition 5.3.1. The trajectory is set by collection $\{v^S(t)\}_{S \in \mathbf{K}}$ absolutely continuous maps $v^S(\cdot) : [0, +\infty) \rightarrow \mathfrak{L}_S^c$, satisfying conditions (i)–(iii). Now we have to specify two things.

First. In item (i) it is necessary to postulate the law, revealing for a current moment $t \geq 0$ a set of active coalitions $\Psi(t) = \{S_1, \dots, S_k\} \subset \mathbf{K}$. Apparently, the simplest way to solve this problem is, that for each permitted coalition $S \in \mathbf{K}$ to set open, with *infinite measure*⁸ a subset $U_S \subset [0, +\infty)$ of time moments, in which the coalition is active. Further let's define

$$\Psi(t) = \{S \in \mathbf{K} \mid t \in U_S\}$$

and also postulate, that for each $t \geq 0$ set $\Psi(t)$ (can be empty) consists of pairwise non-intersected coalitions.

Second. In item (ii), it is necessary to specify *which* of the mutually beneficial contracts is signed by the members of an active coalition. And now a map occurs that is possible to name a *trade rule*. As soon as the fact that a contract is mutually beneficial for the members of a coalition depends only on current consumption bundles, we can assume that for each coalition $S \in \mathbf{K}$ a *continuous* map

$$w^S : \mathcal{A}(X) \rightarrow \mathfrak{L}_S^c = \{w \in \mathfrak{L} \mid w = (w_i)_{\mathcal{I}} : \sum_{i \in S} w_i = 0 \ \& \ w_i = 0, \ \forall i \notin S\}$$

is determined such that

$$u_i(x_i + w_i^S(x)) > u_i(x_i) \iff \exists \nu \in \mathfrak{L}_S^c : u_j(x_j + \nu_j) > u_j(x_j) \ \forall j \in S, \quad (5.4.1)$$

⁸This property is necessary so that the coalition was able to realize the interests at least in infinity.

and

$$\nexists \nu \in \mathfrak{L}_S^c : u_j(x_j + \nu_j) > u_j(x_j) \quad \forall j \in S \quad \Rightarrow \quad w^S(x) = 0.$$

The vector value $w^S(x) \in \mathfrak{L}_S^c$ unequivocally specifies a contract, which will be signed by the coalition S members at moment t under two conditions:

- (1) if $S \in \Psi(t)$, *i.e.*, the coalition has to be active, and
- (2) if $x(t) = x$, *i.e.*, if allocation achieved by a trajectory at moment t coincides with x .

The stated requirements to the map $w^S(\cdot)$, $S \in \mathbf{K}$ mean, that at each time moment when the coalition is active some mutually beneficial contract is signed, if there is such opportunity at all; if not, the contract is not concluded (more precisely, the zero contract is signed). Notice that when utilities are differentiable instead of (5.4.1) it is appropriate to apply condition presented for gradients (derivatives by direction):

$$\langle \nabla u_i(x_i), w_i^S(x) \rangle > 0 \quad \iff \quad \exists \nu \in \mathfrak{L}_S^c : u_j(x_j + \nu_j) > u_j(x_j) \quad \forall j \in S.$$

Requirement (5.4.1) seems quite correct for discrete time presentation, however when time is continuous $w^S(x(t))$ is a momentary contract which defines not only beneficial exchange proportions but also the speed of exchange processes (multiplying $w^S(x(t))$ by a number more of unit we increase the speed of exchange and for positive but less than unit numbers a speed is decreasing and *etc.*). Integrating $w^S(x(t))$ over time on interval $[t, t + \Delta t]$ one can find a gross contract, obeying (5.4.1) at the moment t and when a time of its realization is $\Delta t > 0$ (see comments from §5.1.1).

So, up to the moment *coalitional proper-contractual process* and the trajectory are completely described, for it is enough in Definition 5.3.1 to add the additional factors: sets U_S , $S \in \mathbf{K}$, determining current structure of active coalitions $\Psi(t)$, and map $w^S(\cdot)$, $S \in \mathbf{K}$, specifying contracts signed by active coalitions $w^S(x(t))$, which is necessary to use in the law (5.3.3).

Actually, we have described not only coalitional but also *aggregated* proper contractual trajectory, determined in Definition 5.2.1. Aggregated trajectory turns out from coalitional one if $\mathbf{K} = \{\mathcal{I}\}$. Certainly, in this case only a trade rule, formalized by map $w^{\mathcal{I}}(\cdot)$ will be actually applied (requirements to this map can be relaxed a little).

In closing of this section, we would like to make an important statement. The *right-hand side* of equation determining the law of contractual trajectory (5.2.8), (5.3.3), is *discontinuous* in general—in spite of it is defined unambiguously and is formulated via continuous functions! Of course this problem (perhaps imperceptible for a first view) is arisen because of parameter $\lambda^{min}(x, v)$ defined by formula (5.2.7) (or analogous $\lambda_S^{min}(x, v^S, w^S)$ by (5.3.2)) which vanishes in an open area

$$x \in \mathcal{A}(X) : \quad \langle \nabla u_i(x_i), x_i - \mathbf{e}_i \rangle > 0 \quad \forall i$$

but in general it is non-zero on its boundary, *i.e.*, at the points $(x_1, x_2, \dots, x_n) \in \mathcal{A}(X)$ where $\langle \nabla u_i(x_i), x_i - \mathbf{e}_i \rangle = 0$ at least for one i . Thus the law of proper contractual trajectory is described via differential equation with a discontinuous right-hand part. Moreover, in such a case already, the concept of solution requires an accurate definition. Solution is a *continuous* function of time, which satisfies the law of trajectory

change for *almost all* time moments. The solution of this kind defines a contractual trajectory in an appropriate way.

Notice that classical theorems on existence, uniqueness and continuous dependence over initial data cannot be applied to equations with discontinuous right-hand parts since their right parts do not obey Lipschitz condition. However, by now there is a theory of equations of this type in which appropriate theorems (on existence, uniqueness and continuous dependence) have proven and these theorems are applicable to equations describing contractual processes, see *Filippov (1985)*.

5.4.2 Price process, associated with a contractual trajectory

The purpose of this paragraph is to discuss available opportunities to associate with a contractual trajectory some parallel dual price process. This idea seems to be rather tempting, since being successfully realized, it would allow us, on the one hand, better to reveal interrelations and peculiarities of contractual trajectories with processes known in the literature (first of all with tâtonnement). On the other hand, it is impossible to move away from the fact that in a real economy the barter processes are going mainly with use of money, which are exchanged on goods and services in proportions given by prices. A discussion about an appropriate price dynamics is also important for better understanding of how markets function. However, the realization (successful) of this idea encounters some difficulties. A possible approach to realize the idea will be discussed below.

The most natural way to define current prices $p(t)$ is to take the prices as a vector, specifying exchange proportions in the current barter contract. It would be not bad, if the economy has only two goods and, if in the bargain only two individuals have been participated. Then if the first agent receives from the contract a vector (v^1, v^2) , where $v^1 > 0$ and $v^2 < 0$, it means that 2nd good exchanges on 1st in proportion $-v^1/v^2$, *i.e.*, (normalized) vector of prices can be chosen as $p = (1, -\frac{v^1}{v^2})$. This vector is possible to determine via an equivalent way, from equations:

$$\langle (v^1, v^2), (p_1, p_2) \rangle = 0, \quad p_1 = 1.$$

This property, equality to zero of scalar product of a vector of the prices on vectors of commodities flows v_i , received by the individuals from contract $v = (v_i)_{\mathcal{I}}$, together with normalization, is the crucial requirement for determination of a price vector in this approach. However, in a general case it is possible to find vector of the prices unequivocally, only if the rank of a system of vectors $\{v_i\}_{\mathcal{I}}$ given by contract v is equal $l - 1$ (l is the number of goods), *i.e.*, if $\text{rank}(\{v_i\}_{\mathcal{I}}) = l - 1$. On the other hand, other contracts may coexist at current time moment, and, moreover contracts signed in the past and existing now should also take part in “determination of proportions of an exchange”. As a result, for *aggregated* proper contractual trajectory, the given approach yields the system of equations:

$$\langle p(t), x_i(t) - \mathbf{e}_i \rangle = 0, \quad \forall i \in \mathcal{I}, \quad t \geq 0. \quad (5.4.2)$$

At the same time, for a *coalition* trajectory, the similar system takes a form:

$$\langle p(t), v_i^S(t) \rangle = 0, \quad \forall i \in S, \forall S \in \mathbf{K}, \quad t \geq 0. \quad (5.4.3)$$

Moreover, in both cases it should be carried out some normalizing condition, for example, $p_1(t) = 1$.

Clearly, that it is absolutely not certain that the specified systems have a unique solution: the system can be unsolvable or, contrary, it may have infinity many solutions. However, in such case it is possible to try as the prices to take some approximating solution. Certainly, it is necessary to do in some regular way, so that a trajectory $p(t)$, $t \geq 0$ has “not bad” mathematical properties. Having this in mind, one can use known methods of a finding of approximating solution, for example, the method of least squares. In so doing so-called “generalized inverse matrix” is entered into consideration, which exists and is unique for any rectangular matrix. Let us pay attention to this method.

Let us consider a system of the linear equations with some rectangular matrix:

$$Ax = b.$$

An approximating solution of the system can be found with the help of *generalized inverse* matrix A^+ as $x = A^+b$, and, if system is overdetermined (a number of independent equations is more than unknown variables), this gives a solution by a method of the least squares. Then, if a rank of matrix is equal to the number of columns, $(A^tA)^{-1}$ exists (this is Grama matrix) and matrix A^+ can be found as

$$A^+ = (A^tA)^{-1}A^t.$$

In a general case, the matrix A^+ is set via the following four conditions:

$$AA^+A = A; \quad A^+AA^+ = A^+; \quad A^+A \text{ \& } AA^+ \text{ symmetrical.}$$

These conditions completely determine a matrix A^+ and were offered by Penrose ([Penrose, 1955](#)), which has also proved existence and uniqueness of a matrix A^+ . This problem was considered also in Moore’s works and papers of other authors, this is why one can meet in the literature the name “Moore-Penrose inverse” (see [Greene, 1993](#) and [Searle, Hausman, 1970](#)). In a general case, the matrix A^+ also has formula representation (see [Greene, 1993](#), p. 45), that in particular allows to conclude continuously differential character of the solution.

So, if to apply a generalized inverse matrix associated price process $p(t)$ to an aggregated trajectory can be determined as

$$p(t) = A^+b_1, \quad b_1 = (1, 0, \dots, 0),$$

where the matrix A of dimension $(n+1) \times l$, with rows a_k , $k = 1, \dots, n+1$ which are equal: $a_1 = b_1$ and $a_k = x_{k-1}(t) - e_{k-1}$, $k \geq 2$, accordingly.

For a coalitional trajectory associated price process $p(t)$ can be set by the same formula, in which, however, the matrix A has dimension $(\sum_{S \in \mathbf{K}} |S| + 1) \times l$, first

row $a_1 = b_1$, and all other are indexed by pairs (i, S) , $i \in S \in \mathbf{K}$ and coincide with vectors v_i^S .

For a proper contractual trajectory, it is also possible to try to determine price process as the differential equation. Really, for an aggregated trajectory, formal differentiation by time system (5.4.2) (notwithstanding that it may have not the exact solution!), with the account (5.2.8), (5.4.2) yields

$$\begin{aligned} \langle \dot{p}(t), x_i(t) - \mathbf{e}_i \rangle + p(t)\dot{x}_i &= 0 \Rightarrow \langle \dot{p}(t), x_i(t) - \mathbf{e}_i \rangle = -p(t)[\lambda^{\min}(x, v)(x_i(t) - \mathbf{e}_i) + v_i] \Rightarrow \\ \langle \dot{p}(t), \mathbf{e}_i - x_i(t) \rangle &= p(t)v_i(x), \quad \forall i \in \mathcal{I}, \quad \dot{p}_1(t) = 0. \end{aligned} \quad (5.4.4)$$

Further, applying a generalized inverse matrix, the system (5.4.4) can be approximately solved relative to $\dot{p}(t)$ and can be written in an explicit form. It is possible also to try to solve systems (5.4.2), (5.4.4) jointly and find a direct dependence of $\dot{p}(t)$ from current consumptions $x(t)$ and an exogenous trade rule $v(x)$.

The similar actions can be made for a coalitional trajectory. Differentiating system (5.4.3), by virtue of Definition 5.3.1, items (ii), (iii), and applying (5.4.3), in the case when a new contract w^S is signed by a coalition $S \in \Psi(t)$ we find

$$\begin{aligned} \langle \dot{p}(t), v_i^S \rangle + p(t)\dot{v}_i^S &= 0 \Rightarrow \langle \dot{p}(t), v_i^S \rangle = -p(t)[\lambda_S^{\min}(x, v^S, w^S)v_i^S + w_i^S] \Rightarrow \\ \langle \dot{p}(t), v_i^S(t) \rangle &= -p(t)w_i^S(x), \quad \forall i \in S, \quad \dot{p}_1(t) = 0. \end{aligned} \quad (5.4.5)$$

In all other cases (the breaking of contract or when a coalition is passive), we obtain

$$\langle \dot{p}(t), v_i^S(t) \rangle = 0, \quad \forall i \in S.$$

What preliminary conclusions and observations can be made about suggested variant of price process?

(i) The current prices are determined as an average (approximate) vector of exchange proportions overall (gross) contracts, signed and saved (*i.e.* unbroken) during contractual process up to the current time moment. It seems that it corresponds to economic intuition on how, in mass manner, the individuals can find the prices acting in the real markets. For example, a similar method is used for a finding of market price for the objects of real estate.

(ii) We need the analysis of price process to be continued, at least for some particular model examples. Moreover, it seems us that an opportunity to apply price process for a finding of suitable Lyapunov's function has to be analyzed, to establish convergence of proper contractual process, at least in limited frameworks.

(iii) Now it is not clear how in general suggested price process is related with Walrasian tâtonnement. However, it is possible to show that in an economy with two agents and two goods, the associated price process is practically equivalent to Walrasian tâtonnement. Moreover, if the trajectory converges to equilibrium allocation, then it is simple to prove that price process converges to the equilibrium prices.

(iv) It is possible to consider modifications of described price process, even essentially different price processes, with the purpose to find variant interrelated with the different points of view. For example, it seems tempting to study still non-described idea (now it is not elaborated yet): to put into correspondence to contractual process some optimal control problem, which has a contractual trajectory as an optimal solution. Then one can try to find price process from necessary conditions of optimality.

Conclusion to Chapter 5

In this chapter, contractual processes were introduced and in detail described; they are, first of all, proper contractual ones, in which partial breaking of the contracts is allowed. With this in mind in Section 5.1.2 several basic hypotheses, determining the character of contracts' breaking process, are formulated in a general kind and for major particular cases. They are the following:

- (**IB**) — instantaneous breaking of the contracts;
- (**UB**) — uniform breaking of all contracts;
- (**CUB**) — uniform breaking of gross within-coalitional contracts.

Combinations of these hypotheses result in proper contractual trajectories of a different kind of a generality. Under (**IB**) and (**UB**), contractual trajectory turns out *aggregated* (Definition 5.2.1), under (**IB**) and (**CUB**)—*coalitional-contractual* (Definition 5.3.1); formal and mathematically reasonable definitions are presented. In my opinion, the coalitional-contractual trajectory should serve the central concept in further researches. Besides, in Section 5.3.2 there was described specific contractual process, adequate a case of the pairwise bargains and to simultaneous breaking of contracts not only in active but also in passive coalitions.

At last, concept of *trade rule* was introduced (Section 5.4.1); this is a map, unequivocally determining mutually beneficial contract for the current consumption plans, and possibly having some additional specific mathematical properties. By use of a trade rule, a contractual trajectory for each mentioned kind is unequivocally determined. Mathematical speciality of the approach is that, in accordance with the Definitions of 5.2.1, 5.3.1 with incorporated trade rules, contractual trajectory is defined as the solution of ordinary differential equations with a discontinuous right-hand side: despite the fact that we are considering continuous rules!

For considered kinds of proper contractual trajectories in Section 5.4.2 a variant of parallel price process is offered. In this process, the current prices are determined as an average (in a specific sense) vector of exchange proportions under all bargains, actually carried out for the current time moment.

Chapter 6

Convergence of contractual processes

Before the beginning to describe obtained results, we consider two particular examples of economy with two individuals and two commodities. These examples are interesting because they reveal in Edgeworth box the geometrical course of contractual processes with partial breaking of contracts. One can easily observe that our process is convergent in these cases.

6.1 Contractual process in 2×2 economy

Further, an agent is called *active* at a current time moment t if he/she realizes a breaking of aggregated contract at this moment of time.

6.1.1 Two examples

Now we consider two particular examples of economy with two individuals and two commodities. These examples are interesting because they reveal in Edgeworth box the geometrical course of contractual processes with partial breaking of contracts. One can easily observe that our process is convergent in these cases.

For both examples positive orthant in 2-dimensional plane presents individual consumption sets, *i.e.*, $X_i = \mathbb{R}_+^2$, $i = 1, 2$. The examples are differentiated via agents' utilities and endowments.

Example 6.1.1 (COBB–DOUGLAS UTILITIES) Let preferences be presented by Cobb–Douglas utilities in logarithmic form as follows:

$$u_1(x_1, x_2) = \frac{1}{4} \ln x_1 + \frac{3}{4} \ln x_2, \quad u_2(y_1, y_2) = \frac{3}{4} \ln y_1 + \frac{1}{4} \ln y_2.$$

Consider also the following initial endowments:

$$\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2) = \left(\left(\frac{9}{10}, \frac{1}{10} \right), \left(\frac{1}{10}, \frac{9}{10} \right) \right), \quad \bar{\mathbf{e}} = \mathbf{e}_1 + \mathbf{e}_2 = (1, 1).$$

Then indifference curves for first and second individual going across initial endowments point \mathbf{e}_1 in 1st agent's coordinate system are described by equations:

$$x_2 = \frac{1}{10} \left(\frac{9}{10x_1} \right)^{\frac{1}{3}}, \quad x_2 = 1 - \frac{9}{10} \left(\frac{1}{10(1-x_1)} \right)^3.$$

Calculations show that Pareto boundary is a curve determined by equation

$$x_2 = \frac{9x_1}{1+8x_1}, \quad 0 \leq x_1 \leq 1.$$

Finally, a maximal surface is composed via two curves and it is the low envelope for them:

$$x_2 = \left(\frac{3x_1}{40x_1 - 9} \right), \quad x_1 > \frac{9}{40} \quad \& \quad x_2 = \left(\frac{28 - 31x_1}{37 - 40x_1} \right), \quad x_1 < \frac{28}{31}.$$

An illustration of this example in Edgeworth box is given in Figure 6.1.1.

In considered case proper contractual process is convergent to unique equilibrium $((\frac{1}{4}, \frac{3}{4}), (\frac{3}{4}, \frac{1}{4}))$. This is developed in the following way: if its trajectory is in limits of maximal surface (shaded area of Figure 6.1.1), then individuals are cooperated and signed some barter contracts giving a rise of utilities. A current consumption point is moving in such a manner as long as it starts to leave the maximal surface limits. If a new contract starts to lead the point behind maximal surface and new position is under control of 1st agent (we call him 'active' in contractual process), this is the left low part of box restricted by budget line, then this agent partially breaks aggregated contract and a current point of trajectory is projected onto maximal surface along to straight line going at initial endowments point. Analogous thing takes place for the second agent if a new contract leads the point to the area under 2nd agent control (right upper part of the box behind the budget line). As it is shown in figure, in both cases the next point of trajectory moves over maximal surface and approaches to equilibrium. Thus in the limit our trajectory achieves equilibrium allocation where 1st agent consumption is $(\frac{1}{4}, \frac{3}{4})$.

The following example is well known in literature and presents an economy with a multiplicity of equilibria.

Example 6.1.2 (EXPONENTIAL UTILITIES) Let preferences be determined by the following utilities functions:

$$u_1(x_1, x_2) = x_1 - 100e^{-\frac{x_2}{10}}, \quad u_2(y_1, y_2) = y_2 - 110e^{-\frac{y_1}{10}}.$$

Let $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2) = ((40, 0), (0, 50))$ be an initial endowments allocation. Now the indifference curves of both agents going across endowments are defined by equations

$$x_2 = -10 \ln \left(\frac{x_1 + 60}{100} \right), \quad x_2 = 110 - 110e^{-\frac{x_1 - 40}{10}}.$$

Pareto boundary is a straight line defined as:

$$x_2 = x_1 - 40 + 10 \ln 110.$$

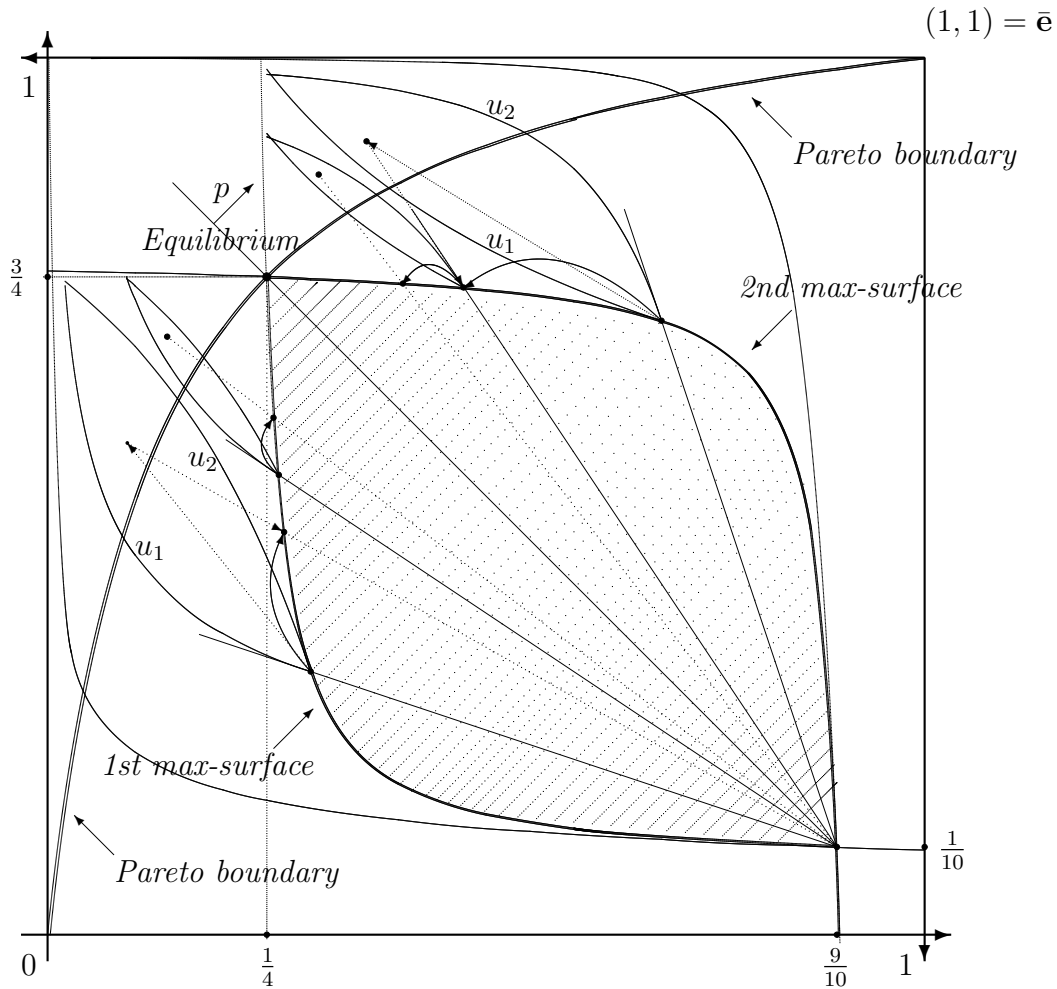


Figure 6.1.1: Contractual process in an economy with 2 agents and Cobb–Douglas utilities

Maximal surface is composed by means of two curves, relative to 1st and 2nd agents:

$$x_1 = 40 - 10x_2 e^{-\frac{x_2}{10}}, \quad x_2 = 11(40 - x_1) e^{\frac{x_1 - 40}{10}}.$$

There are three equilibrium allocations, A, B, C , where 1st agent consumption bundles are the following:

$$x^{(1)} \approx (3.2212, 10.226), \quad x^{(2)} \approx (13.17211, 20.1769), \quad x^{(3)} \approx (32.2579, 39.2627).$$

An illustration of this example in Edgeworth box is given in Figure 6.1.2. Contractual process is developed similar to that described above: in the limits of maximal surface agents sign mutually beneficial contracts as long as an allocation behind area restricted by maximal surface is reached. On the other hand, then a trajectory tends to leave maximal surface area and a current point is under control of 1st agent (he is active, this touch-dotted line in figure), the agent partially breaks aggregated contract

6.1.2 Analysis of contractual processes in 2×2 economies

The geometry of contractual trajectories in the considered examples tells us that at least in economy with two individuals and two goods the contractual process has to converge to equilibrium under rather general assumptions. Really, in addition to imposed above assumption **(D)** that model is smooth, we only need to require $\mathcal{P}_i(\mathbf{e}_i) \subset \text{int}X_i$, $i = 1, 2$, that together with **(D)** actually provides the coincidence of proper contractual allocations with equilibrium ones.

To prove this hypothesis, let us consider some contractual trajectory $x(t) = (x_1(t), x_2(t))$, $t \in [0, +\infty)$ satisfying Definition 5.2.1 and defined via some rule of trade

$$v^{\mathcal{I}} : \mathcal{A}(X) \rightarrow \mathfrak{L}^c = \{(v_1, v_2) \in (\mathbb{R}^2)^{\mathcal{I}} \mid v_1 + v_2 = 0\}.$$

As soon as economy has only two agents and $\mathcal{I} = \{1, 2\}$ a sole coalition in which an exchange of commodities may be realized, then trading rule consists of unique map $v^{\mathcal{I}}(\cdot)$, where upper index \mathcal{I} can be omitted. By definition $v_2(x) = -v_1(x)$, $\forall x \in \mathcal{A}(X)$ and it is enough to set only function $v_1 : \mathbb{R}_+^2 \rightarrow \mathbb{R}^2$, where the vector $(v_1(x_1), -v_1(x_1)) = v^{\mathcal{I}}(x_1)$ is associated with the instant contract, which is signed the members of coalition $\{1, 2\}$ at the moment $t \in [0, +\infty)$ provided that a current allocation is $x(t) = (x_1, x_2)$, $x_2 = \mathbf{e}_1 + \mathbf{e}_2 - x_1$. In addition, this function should be continuous and to define mutually beneficial contract $v = v^{\mathcal{I}}(x_1)$: $v \neq 0$ if and only if

$$\exists v \in \mathfrak{L}^c : u_i(x_i + v_i) > u_i(x_i), \quad i = 1, 2 \quad \& \quad \partial_{v_i} u_i(x_i) > 0, \quad i = 1, 2.$$

In other words, if *there is* at least one *opportunity* for a mutually beneficial exchange, then *one of variants* of such exchange *should be realized* in the form of the mutually beneficial contract; the contract can be *zero* only if *there are* no *opportunities* for a mutually beneficial exchange of commodities.

For contractual process in economy with two individuals, only two following alternatives can be realized.

- (i) There is an individual such that since some moment τ , almost everywhere on $[\tau, +\infty)$, *only he/she* can be active (probably both are passive); thus utility of this individual monotonously does not decrease along a trajectory, *i.e.*, for example, for the first agent, it has to be

$$u_1(x_1(t')) \geq u_1(x_1(t)), \quad \forall t' \geq t \geq \tau.$$

- (ii) The case described in (i) is not true, *i.e.*, there are monotonously increasing sequences $t'_{k+1} > t'_k$, $t''_{k+1} > t''_k$, $k \in \mathbb{N}$, $t'_k \rightarrow +\infty$, $t''_k \rightarrow +\infty$ when $k \rightarrow +\infty$, such that at the moments t'_k the 1st individual is active, and t''_k are the moments of 2nd agent activity, $k = 1, 2, \dots$. Thus, utilities of both individuals can oscillate, growing and decreasing, and this situation does not change when time elapses.

Analysis of alternatives (i), (ii) is realized in two subsequent lemmas. The first establishes that if alternative (i) is true, then every limit point of a contractual trajectory is Pareto optimal. The second alternative causes the greatest difficulties and second lemma states that in some sense there is the monotonicity of utilities along a trajectory but it has “piecewise” character.

Lemma 6.1.1 *Let alternative (i) be fulfilled. Then each limit point of a contractual trajectory is Pareto optimal. Hence, every **interior** limit point is equilibrium one.*

Proof of Lemma 6.1.1. So let alternative (i) be fulfilled and let starting at the moment $\tau \geq 0$, only 1st individual can be active and second is passive for all $t \geq \tau$. Further, let \tilde{x}_1 (1st agent bundle) be any limit point of trajectory. Define $\tilde{x}_2 = \mathbf{e}_1 + \mathbf{e}_2 - \tilde{x}_1$ and show that allocation $(\tilde{x}_1, \tilde{x}_2)$ is Pareto optimal. Assuming contrary due to trade rule definition we conclude $\langle \nabla u_1(\tilde{x}_1), v_1(\tilde{x}_1) \rangle > 0$ and by continuity this property has to be fulfilled in some neighborhood of the point \tilde{x}_1 , *i.e.*

$$\exists \varepsilon > 0 : \quad \langle \nabla u_1(x_1), v_1(x_1) \rangle > 0, \quad \forall x_1 \in B_{2\varepsilon}(\tilde{x}_1),$$

where $B_{2\varepsilon}(\tilde{x}_1)$ is a closed ball with the radius 2ε centered at the point \tilde{x}_1 . Since the ball is a compact set and by continuity, it is equivalent to

$$\exists \varepsilon > 0, \delta > 0 : \quad \langle \nabla u_1(x_1), v_1(x_1) \rangle > \delta, \quad \forall x_1 \in B_{2\varepsilon}(\tilde{x}_1). \quad (6.1.1)$$

We observe from this that one can come to a contradiction if one manages to show that the current point of trajectory is in the ball during infinite (by measure) time.

Really, since the 2nd agent is passive (almost everywhere) and because 1st agent utility is increasing monotonically along the trajectory (starting at the moment τ) the following estimations are fulfilled:

$$\begin{aligned} u_1(x_1(t)) - u_1(x_1(\tau)) &= \int_{\tau}^t \frac{du_1(x_1(\zeta))}{d\zeta} d\zeta = \int_{\tau}^t \langle \nabla u_1(x_1(\zeta)), \dot{x}_1(\zeta) \rangle d\zeta \geq \\ &\geq \int_{\tau}^t \langle \nabla u_1(x_1(\zeta)), v_1(x_1(\zeta)) \rangle d\zeta \geq \int_{[\tau, t] \cap \Theta} \langle \nabla u_1(x_1(\zeta)), v_1(x_1(\zeta)) \rangle d\zeta \geq \delta \cdot \mu([\tau, t] \cap \Theta). \end{aligned}$$

Here $\Theta \subset [\tau, +\infty]$ is the set of all time moments when a current point of trajectory $x_1(\zeta)$ is located in the ball $B_{2\varepsilon}(\tilde{x}_1)$ and $\mu([\tau, t] \cap \Theta)$ is Lebesgue measure of the set $[\tau, t] \cap \Theta$. If $\mu(\Theta) = +\infty$ we have $\mu([\tau, t] \cap \Theta) \rightarrow +\infty$ for $t \rightarrow +\infty$. Then due to the last estimation it has to be $u_1(x_1(t)) \rightarrow +\infty$ that is impossible since the set of all allocations is compact and utility function is continuous.

Let us show that $\mu(\Theta) = +\infty$. It is obvious if starting at some time moment $t \geq \tau$ all points of trajectory are located in the ball. In the contrary case, one can find an enumerable set of moments $t_k, t'_k, k = 1, 2, \dots$ such that $\|x_1(t_k) - \tilde{x}_1\| < \varepsilon$ and $t'_k > t_k$ is a *closest* after t_k time moment when the trajectory leaves the ball, *i.e.*

$$\|x_1(t'_k) - \tilde{x}_1\| = 2\varepsilon \quad \& \quad \|x_1(\zeta) - \tilde{x}_1\| < 2\varepsilon, \quad \forall \zeta \in [t_k, t'_k).$$

However, in this case we have an estimation:

$$\varepsilon \leq \|x_1(t'_k) - x_1(t_k)\| = \left\| \int_{t_k}^{t'_k} \dot{x}_1(\zeta) d\zeta \right\| \leq \int_{t_k}^{t'_k} \|\dot{x}_1(\zeta)\| d\zeta \leq c \int_{t_k}^{t'_k} d\zeta = c(t'_k - t_k),$$

where $c > 0$ is an upper bound for the norm of right-hand part of the law (5.2.8), *i.e.* this value satisfies

$$c \geq \|\lambda^{\min}(x, v_1)(x_1 - \mathbf{e}_1) + v_1(x_1)\|, \quad \forall x_1 \in B_{2\varepsilon}(\tilde{x}_1).$$

Due to imposed assumptions and from the compactness and continuity of objects that we need, it is easy to prove that the right-hand part of this inequality is bounded from above and, therefore, such $c > 0$ does exist. As a result, we have got the estimation

$$(t'_k - t_k) \geq \frac{\varepsilon}{c} > 0, \quad \forall k = 1, 2, \dots$$

Moreover, via construction all intervals $[t_k, t'_k]$ are pairwise non-intersected and $[t_k, t'_k] \subset \Theta$, $\forall k = 1, 2, \dots$. Therefore, $\mu(\Theta) = +\infty$. Thus we obtain a contradiction that proves Pareto optimality of the allocation under study.

To state the second part of the lemma remember that every allocation from the interior of the direct product of consumption sets which is Pareto optimal and simultaneously stable relative to the partial break of the gross contract is an equilibrium, see Theorem 1.2.2, Chapter 1, § 1.2.1. ■

Remark 6.1.1 One can easily see from the proof that this lemma is also true for an economy with any number of agents and commodities. It is important only, that alternative (i) is fulfilled. Moreover, if the economy has only two agents and alternative (i) is true then it is easy to prove that all limit points have equal utilities for both agents, that for strictly concave functions is possible only if allocations are equal (see the first part of Theorem 6.1.1 proof). ■

Remark 6.1.2 From the proof of Lemma 6.1.1 and due to Remark 6.1.1 one can also conclude convergence of contractual process without a break of contracts. However, now it will be convergence to some Pareto optimal allocation: it is obvious, that the limit point is not obliged to be equilibrium since it is not a limit point of allocations that are stable relative to partial break of gross contract. However, when utilities are strictly concave there is only one limit point, that easily follows from the fact of coincidence of utility (follows from monotonicity along a trajectory) for all individuals in all limit points. ■

Lemma 6.1.2 *Let alternative (ii) be fulfilled. Then there exist two monotonously increasing sequences of the moments of time τ_k^i , for $i = 1, 2$, such that $\tau_k^1 < \tau_k^2 < \tau_{k+1}^1$, $\langle \nabla u_i(x_i(\tau_k^i)), x_i(\tau_k^i) - \mathbf{e}_i \rangle = 0$, $i = 1, 2$, $\forall k \in \mathbb{N}$ and*

$$u_i(x_i(\tau_k^i)) < u_i(x_i(\tau)) < u_i(x_i(\tau_{k+1}^i)), \quad \forall \tau \in (\tau_k^i, \tau_{k+1}^i), \quad i = 1, 2 \quad (6.1.2)$$

holds.

The content of this lemma is illustrated in Figures 6.1.3, 6.1.4. Figure 6.1.3 demonstrates dynamics of utility for sequences of time moments τ_k^i , $i = 1, 2$, $k = 1, 2, \dots$ constructed in the proof and described in Lemma 6.1.2. Possible and impossible dynamics of proper-contractual trajectories are demonstrated in Figure 6.1.4; it is shown that the only fragment of trajectory, represented in the top part of the figure is possible—all other variants result in an impossible cycle contradicting with the choice of the moments τ_k^i .

Proof of Lemma 6.1.2. Let τ' be some time moment when the first agent is active. Now define a time moment when 1st agent is active $\tau_1^1 \geq \tau'$ and such that it is earlier

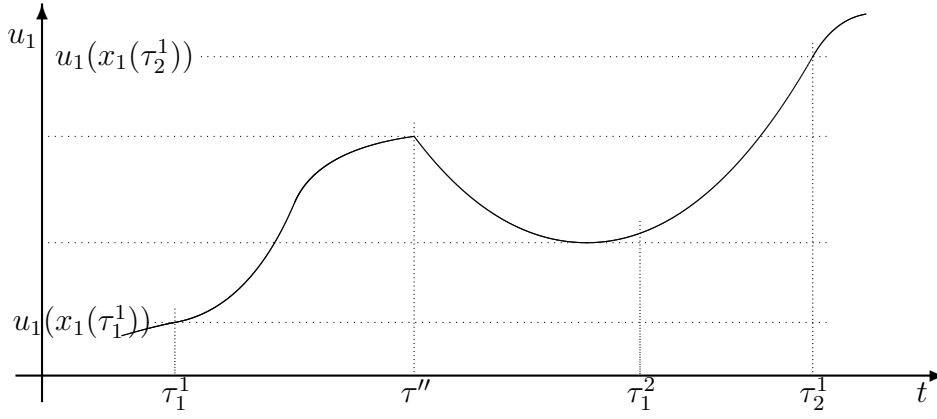


Figure 6.1.3: *Utility dynamics of contractual process in 2×2 economy, alternative (ii)*

of *first* moment $\tau'' > \tau'$ when 2nd agent is active and such that on the interval (τ_1^1, τ'') both agents are passive. Here τ_1^1 is the *latest* moment of 1st agent activity on the interval $[\tau', \tau'')$. Analogously, for 2nd agent, one can find a moment τ_1^2 as a moment of last his/her activity up to the nearest moment $\tau''' > \tau''$ when 1st agent is active. In view of compactness and continuity of objects under study, all considered time moments exist. For example, τ'' and τ_1^1 can be found by formulas

$$\tau'' = \min\{t \in [\tau', +\infty) \mid \langle \nabla u_2(x_2(t)), x_2(t) - \mathbf{e}_2 \rangle = 0\},$$

$$\tau_1^1 = \max\{t \in [\tau', \tau''] \mid \langle \nabla u_1(x_1(t)), x_1(t) - \mathbf{e}_1 \rangle = 0\}.$$

Further taking the point τ''' as “initial” (*i.e.* instead of τ') in the described above procedure, one can find the moments τ_2^1 and τ_2^2 , accordingly. Show that constructed in this way fragments of sequences that we need to be found obey the requirement (6.1.2). With this in mind first, let us better understand the geometry of moving of a trajectory and reveal some peculiarities of this moving.

By construction on intervals $[\tau_1^1, \tau'']$ and $[\tau_1^2, \tau_2^1]$ the 1st agent utility increases: it is so because only mutually beneficial contracts are signed during the contractual process and also because in our intervals 2nd agent is passive. It has to be shown that for all points t from the interval $[\tau'', \tau_1^2]$ the inequality $u_1(x_1(t)) > u_1(x_1(\tau_1^1))$ is fulfilled. Let us do it.

Now consider the moment τ'' . By construction the following relations

$$\langle \nabla u_1(x_1(\tau'')), v_1(x_1(\tau'')) \rangle > 0, \quad \langle \nabla u_2(x_2(\tau'')), v_1(x_1(\tau'')) \rangle < 0,$$

$$\langle \nabla u_2(x_2(\tau'')), x_1(\tau'') - \mathbf{e}_1 \rangle = 0$$

have to be true. Moreover if $h_2(x_2(\tau'')) = \nabla u_2(x_2(\tau'')) - \nabla^2 u_2(x_2(\tau''))(x_1(\tau'') - \mathbf{e}_1)$ satisfies¹

$$\langle h_2(x_2(\tau'')), v_1(x_1(\tau'')) \rangle < 0,$$

then the condition (5.2.5) of contracts break is violated and it means that a trajectory only “touches” with maximal surface at the point $x_1(\tau'')$ and then “leaves” it.

¹Remember that $x_2(\tau'') - \mathbf{e}_2 = -(x_1(\tau'') - \mathbf{e}_1)$ and $v_2(x_2(\tau'')) = -v_1(x_1(\tau''))$.

Therefore, in a neighborhood of the moment τ'' a break of contracts does not occur and both utilities are locally increased. A break of contracts may occur only if

$$\langle h_2(x_2(\tau'')), v_1(x_1(\tau'')) \rangle \geq 0$$

and if for small $\Delta t > 0$ at the points $\tau'' + \Delta t$ this inequality is strict. Thus after the “going through” the point $x_1(\tau'')$ a trajectory $x_1(t)$ will move some non-zero time in the framework of ε -extension of a cone with the vertex at the point $x_1(\tau'')$ which is defined by inequalities:

$$\langle h_2(x_2(\tau'')), x_1 \rangle \geq \langle h_2(x_2(\tau'')), x_1(\tau'') \rangle,$$

$$\langle \nabla u_2(x_2(\tau'')), x_1 \rangle \leq \langle \nabla u_2(x_2(\tau'')), x_1(\tau'') \rangle = \langle \nabla u_2(x_2(\tau'')), \mathbf{e}_1 \rangle.$$

More exactly, due to $\langle h_2(x_2(t)), \dot{x}_2(t) \rangle = 0$, see (5.2.2), a limit deviation of trajectory will be realized along an edge of this cone, see Figure 6.1.4.

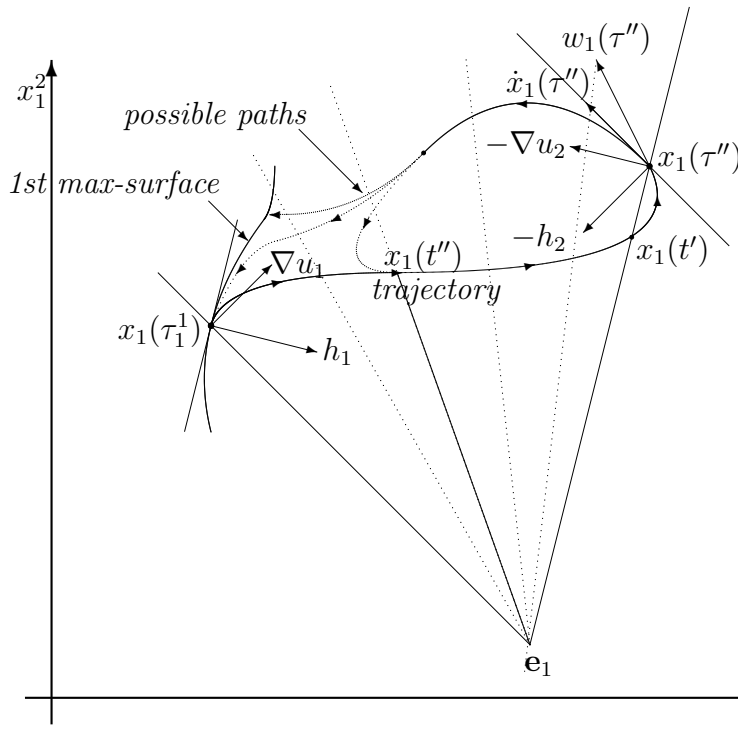


Figure 6.1.4: Possible and “impossible” moving of contractual trajectory in 2×2 economy, alternative (ii)

Further, on the interval $[\tau_1^1, \tau'']$ in the plane, a trajectory $x_1(t)$ circumscribes a continuous curve with the ends $x_1(\tau_1^1)$ and $x_1(\tau'')$ such that for $t \in (\tau_1^1, \tau'')$ the points $x_1(t)$ are located strictly “below” than a point of the maximal surface being intersected with the ray starting from \mathbf{e}_1 and going through the point $x_1(t)$, because $\langle \nabla u_i(x_i(t)), x_i(t) - \mathbf{e}_i \rangle > 0$, $i = 1, 2$. Moreover, for $t < \tau''$ and close to τ'' it has to be

$$\langle \nabla u_2(x_2(\tau'')), x_1(\tau'') \rangle < \langle \nabla u_2(x_2(\tau'')), x_1(t) \rangle \iff \langle \nabla u_2(x_2(\tau'')), \frac{x_1(t) - x_1(\tau'')}{\tau'' - t} \rangle > 0,$$

because $v_2(x_1(\tau'')) = -v_1(x_1(\tau'')) \approx \frac{x_1(t) - x_1(\tau'')}{\tau'' - t}$. Thus there exists a moment $t' < \tau''$ such that $x_1(t') - \mathbf{e}_1 = \gamma(x_1(\tau'') - \mathbf{e}_1)$ for some $0 < \gamma < 1$ and, simultaneously, all points of trajectory from interval $t \in [\tau_1^1, t']$ obey $\langle \nabla u_2(x_2(\tau'')), x_1(t) \rangle < \langle \nabla u_2(x_2(\tau'')), \mathbf{e}_1 \rangle$. A similar inequality has to be fulfilled for the points $x_1(t)$, $t \in [\tau'', \tau_1^2]$ because contracts are mutually beneficial and 1st agent is passive on this time interval.

Further, we are going to the final part of the proof. Assume that for some $t \in [\tau'', \tau_1^2]$ the inequality $u_1(x_1(t)) \leq u_1(x_1(\tau_1^1))$ is fulfilled. Now from the continuity and due to presented above reasonings, it follows that there are moments $t'' < \tau''$ and $\tau'' < t''' \leq \tau_1^2$ such that $x_1(t'') = x_1(t''')$ is true. Consider the first possible moment of this type (one needs to take minimal t''' having this property). For $t'' > \tau_1^1$ we have a contradiction since then our trajectory is cycling (due to the law of change is autonomous) and never arrives at a point on the 1st agent maximal surface but it has to be so at the moment $\tau_2^1 > t'''$. Therefore, it has to be $t'' = \tau_1^1$. However, $x_1(\tau_1^1)$ is a point on the 1st agent maximal surface where the 2nd agent is passive. Hence there is a neighborhood of $x_1(\tau_1^1)$ such that 1st agent utility strictly increases along *every* trajectory starting from any point from the neighborhood. Therefore, for all small enough $\varepsilon > 0$, it has to be $u_1(x_1(t''' - \varepsilon)) < u_1(x_1(t'''))$. Moreover, for some $\varepsilon > 0$ no point $x_1(t)$, $t \in (t''' - \varepsilon, t''')$ can be located on the 2nd agent maximal surface (otherwise at the point $x_1(t''') = x_1(\tau_1^1)$ both individuals are active that is possible only at an equilibrium which trajectory can never leave). Therefore the last moment of trajectory being on the 2nd agent maximal surface, by definition this is the moment τ_1^2 , has to be realized *earlier* the moment t''' because the point $x_1(t''') = x_1(t'') = x_1(\tau_1^1)$ is located on the 1st agent maximal surface. Thus, it has to be $t''' > \tau_1^2$, but this is impossible. The obtained contradictions finish the proof of Lemma 6.1.2. ■

The major result of the section is the following theorem on the convergence of contractual trajectories to equilibrium in an economy with two agents and two goods.

Theorem 6.1.1 *Let the economy have two agents and two commodities. Let utilities be smooth, strictly concave, and non-satiated on \mathbb{R}_+^2 . Then for any continuous trading rule the contractual trajectory by Definition 5.2.1 converges to some properly contractual allocation. Hence, in conditions when equilibrium allocations coincide with properly contractual ones, every properly contractual trajectory converges to equilibrium.*

Proof of Theorem 6.1.1. In conditions of the theorem, the considered above alternatives (i), (ii) take place. So, it is enough to show that for any alternative, the contractual process converges to a properly contractual allocation.

Let alternative (i) be true. Let's prove that $x_1(t)$ converges to \tilde{x}_1 when $t \rightarrow +\infty$. To do it, let's assume that the trajectory has two different limit points $\tilde{x}^1 = (\tilde{x}_1^1, \tilde{x}_2^1)$, $\tilde{x}^2 = (\tilde{x}_1^2, \tilde{x}_2^2)$ and $\tilde{x}^1 \neq \tilde{x}^2$. Due to Lemma 6.1.1, both of them are Pareto optimal and thus $u_1(\tilde{x}_1^1) = u_1(\tilde{x}_1^2)$. Let's assume, for example, that $u_2(\tilde{x}_2^1) \leq u_2(\tilde{x}_2^2)$. Further, consider any allocation represented as a convex combination of the limit points with strictly positive coefficients, for example, one can take $\tilde{x}' = \frac{1}{2}\tilde{x}^1 + \frac{1}{2}\tilde{x}^2$. Now, by virtue

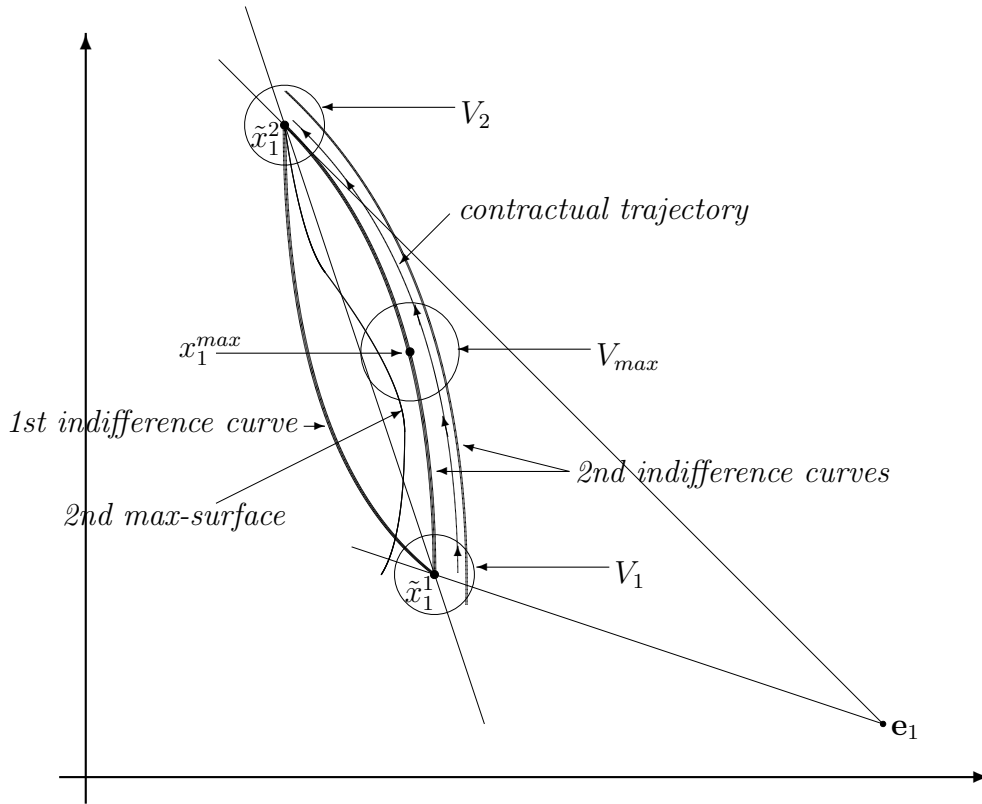


Figure 6.1.5: “The impossible” moving of contractual trajectory along a limit path under alternative (ii) and $\tilde{x}_1^1 \neq \tilde{x}_1^2$

of strict concavity of utility functions conclude

$$u(\tilde{x}') \gg u(\tilde{x}^1) \iff u_1(\tilde{x}') > u_1(\tilde{x}_1^1), u_2(\tilde{x}') > u_2(\tilde{x}_2^1),$$

that contradicts Pareto optimality of allocation \tilde{x}^1 . Thus all limit points of a trajectory coincide and, hence, the trajectory converges.

Further, we analyze alternative (ii). With this purpose, one can apply Lemma 6.1.2 and consider limit points of sequences $\{x_1(\tau_k^1)\}_{k \in \mathbb{N}}$ and $\{x_1(\tau_k^2)\}_{k \in \mathbb{N}}$. Without loss of generality, one can think that these sequences converge themselves. Determine

$$\tilde{x}_1^1 = \lim_{k \rightarrow \infty} x_1(\tau_k^1), \quad \tilde{x}_2^1 = \mathbf{e}_1 + \mathbf{e}_2 - \tilde{x}_1^1, \quad \tilde{x}_1^2 = \lim_{k \rightarrow \infty} x_1(\tau_k^2), \quad \tilde{x}_2^2 = \mathbf{e}_1 + \mathbf{e}_2 - \tilde{x}_1^2.$$

In view of (6.1.2), we have

$$u_1(\tilde{x}_1^1) = \sup_{t \geq \tau_1^1} u_1(x_1(t)) = u_1(\tilde{x}_1^2), \quad u_2(\tilde{x}_2^1) = \sup_{t \geq \tau_2^1} u_2(x_2(t)) = u_2(\tilde{x}_2^2).$$

It is clear, that point \tilde{x}_1^1 is on the maximal surface of 1st agent and \tilde{x}_2^2 is on the maximal surface of 2nd, and that for both individuals the allocations are equivalent by the utility. Further, we shall show that actually coincide not only utilities but also allocations, *i.e.* $\tilde{x}_1^1 = \tilde{x}_1^2$. For a two-goods economy this allocation will be obviously properly contractual (equilibrium) since it is on the maximal surface of every agent.²

²It is not sufficient in general, but it will be so if the allocation is Pareto optimal.

Let's assume now that $\tilde{x}_1^1 \neq \tilde{x}_1^2$. These points are on a common indifference curve of 1st agent, and, accordingly, the points $\tilde{x}_2^1 \neq \tilde{x}_2^2$ are on an indifference curve of the 2nd individual. Reasoning in Edgeworth box, for example in the coordinate system of the 1st agent, we see that two points $\tilde{x}_1^1 \neq \tilde{x}_1^2$ are connected by two continuous curves which are the pieces of the boundary of two (convex) sets of a utility level. Let's connect the specified points by a linear segment, *i.e.*, consider the set $\{\gamma\tilde{x}_1^1 + (1 - \gamma)\tilde{x}_1^2 \mid 0 < \gamma < 1\}$. By virtue of the strict concavity of utility functions, the value of utility at points of this segment is strictly more than the utility level at its ends for both individuals. This implies that for one of the agents the part of the indifference curve, going through the points $\tilde{x}_1^1, \tilde{x}_1^2$ and placed strictly between these points, cannot intersect the maximal surface of the agent, see Figure 6.1.5 (every ray going from initial endowments through one of the considered points first intersects with one of two indifference curves and then it hits at a point of the segment; therefore (via concavity) when a point moves along the ray utility increases at the point of intersection with indifference curve). Let this be a case of 2nd agent indifference curve. Further for the 2nd agent indifference curve let us find a point x_1^{max} where 1st agent utility is maximal, *i.e.*, define x_1^{max} from relation

$$u_1(x_1^{max}) = \max\{u_1(x_1) \mid u_2(\mathbf{e}_1 + \mathbf{e}_2 - x_1) = u_2(\mathbf{e}_1 + \mathbf{e}_2 - \tilde{x}_1^1)\}.$$

Obviously, in the Edgeworth box for 2nd agent indifference curve, this point is placed strictly between points $\tilde{x}_1^1, \tilde{x}_1^2$. Further find neighborhoods V_1, V_2 of points $\tilde{x}_1^1, \tilde{x}_1^2$ and a neighborhood V_{max} of point x_1^{max} satisfying the following conditions:

- 1) At every point from V_{max} the 1st agent utility *is strictly more* than his/her utility at *any* point from neighborhoods V_1, V_2 ;
- 2) For every point from V_1 if the trajectory passes through this point (*i.e.* it is starting from this point as a point of initial data in Cauchy problem) then it certainly passes through some point of neighborhood V_{max} .

Clearly that such neighborhoods can be found, since first it is possible to find neighborhoods satisfying 1) and then if necessary to reduce neighborhoods V_1, V_2 . However, now we come to the contradiction because the 1st agent utility is non-monotonically changed in the part of a trajectory where the 2nd agent is passive—it contradicts to the property that every contract defined by trading rule is mutually beneficial.³ ■

³One can yield a contradiction in a faster way: it is enough to notice that on a ray, outgoing from \mathbf{e}_1 and passing through the point \tilde{x}_1^2 , the point of 1st agent utility maximum should settle down “closer” to the point \mathbf{e}_1 , rather than a similar maximum point for the 2nd utility. It follows from the fact that points $\tilde{x}_1^1, \tilde{x}_1^2$ are located on a common 1st agent indifference curve, *i.e.*, $u_1(\tilde{x}_1^1) = u_1(\tilde{x}_1^2)$. But it means that $\tilde{x}_1^1 \neq \tilde{x}_1^2$ is impossible.

Another way to obtain a contradiction is to notice that a trajectory, starting from a neighborhood of point \tilde{x}_1^2 never can get in a neighborhood of \tilde{x}_1^1 , since for this it has “to overcome” an area of point x_1^{max} neighborhood, in which the vector field of the trading rule is oppositely directed.

6.2 Local stability in contractual processes

In this section, we consider the problem of local stability of contractual processes. But first, we recall classical definitions.

Let's consider some autonomous differential equation $\dot{x} = F(x)$ and let \bar{x} be its stationary (or critical, equilibrium) point, *i.e.* a point for which $F(\bar{x}) = 0$.⁴

A stationary point \bar{x} is called locally stable if for each neighborhood of this point it is possible to find (another) neighborhood such that the solution of the equation starting from any point of the last neighborhood (initial data in a neighborhood) will never leave the limits of first neighborhood. Formally:

$$\forall \varepsilon > 0 \exists \delta > 0 : \text{if } \|x(0) - \bar{x}\| < \delta, \text{ then } \|x(t, x(0)) - \bar{x}\| < \varepsilon \quad \forall t \geq 0,$$

where $x(t, x(0))$ is the solution of Cauchy problem with an initial point $x(0) \in \text{dom}F$. However, for our case it is not sufficient to have only ordinary stability, it is also necessary that in the limit current allocation becomes equilibrium one (because we are interested in converging processes). Formally, in addition, it is necessary to require the following:

$$\lim_{t \rightarrow +\infty} x(t, x(0)) = \bar{x}, \quad \forall x(0) : \|x(0) - \bar{x}\| < \delta.$$

Thus, speaking about local stability of contractual process, we mean local asymptotic stability. Similar definitions are given for the case of differential inclusions.

As usual, we begin our analysis from the study of 2×2 economy case. It is a rather simple case and one can obtain a characterization of locally stable equilibria already from the analysis of Diagrams 6.2.6 a), b). In these diagrams (in the style of Edgeworth box) instead of projection onto agents' maximal surfaces, it is shown the projection onto tangent hyperplane to maximal surface at equilibrium point \bar{x} that is correct for the local analysis. The only difference between cases a) and b) is actually that we *have renumbered* the tangent hyperplanes determined by vectors h_i , $i = 1, 2$ calculated at a point \bar{x} . However, this changed the case radically: locally stable equilibrium has turned into unstable! As it is shown in Figure 6.2.6 a), some current points x' , x'' are approaching to equilibrium but after a change of surfaces numeration they are moving off from equilibrium as it is shown in Figure 6.2.6 b).

Thus one can notice that in local stability the key role is played by an inter-location of vectors h_1 , h_2 and p concerning equilibrium point. Moreover, this allows us to formulate a hypothesis for the work: an equilibrium is stable if (locally) every current consumption of an *active* individual is *strictly less* preferable of his/her consumption in equilibrium. This hypothesis is fulfilled for the situations similar described in Figure 6.2.6 a) and is violated for 6.2.6 b). For 2×2 economy it is possible strictly to prove the validity of our hypothesis. However, already for the 3 goods economy, this hypothesis becomes rather doubtful.

The considerations below show that for economy with two agents and 3 commodities, our hypothesis can be true only in some degenerated cases.

⁴For differential inclusion $\dot{x} \in F(x)$ stationary condition has the form $0 \in F(\bar{x})$. However, by virtue of specific properties of inclusion $\dot{x} \in F(x)$, corresponding to contractual processes condition $0 \in F(\bar{x})$ is equivalent to $\{0\} = F(\bar{x})$.

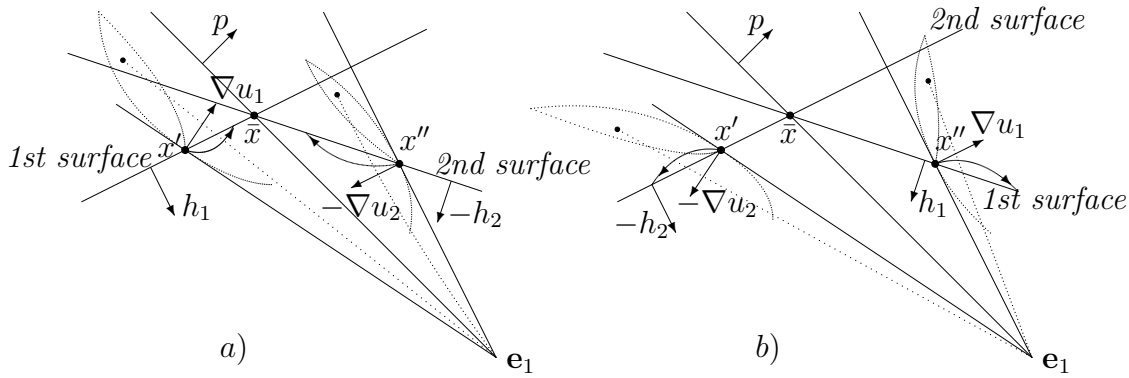


Figure 6.2.6: *Locally stable a) and unstable b) equilibrium \bar{x}*

Let's consider economy with two individuals and with commodity space of dimension $l \geq 3$. Let $x \in \mathbb{R}_+^l$ denotes the 1st individual consumption and $y \in \mathbb{R}_+^l$ is applied to denote 2nd agent consumption and let preferences be defined via Cobb–Douglas functions:

$$u_1(x) = \prod_{j=1}^l x_j^{\alpha_j}, \quad \alpha_j \gg 0, \quad \sum \alpha_j = 1, \quad u_2(y) = \prod_{j=1}^l y_j^{\beta_j}, \quad \beta_j \gg 0, \quad \sum \beta_j = 1.$$

Let $\mathbf{e}^i \gg 0$, $i = 1, 2$ be some initial endowments. It is well known that in such economy there is an unique equilibrium which we denote by (\bar{x}, \bar{y}) . Let's show that in *every* neighborhood of this equilibrium there are points (x, y) (allocations) such that an individual, say 1st, is active and the 2nd individual is passive, *i.e.* $\langle \nabla u_1(x), x - \mathbf{e}^1 \rangle = 0$ & $\langle \nabla u_2(y), y - \mathbf{e}^2 \rangle > 0$, and in addition $u_1(x) > u_1(\bar{x})$.

With this purpose let us any allocation (\tilde{x}, \tilde{y}) such that only 1st individual is active and $u_1(\tilde{x}) > u_1(\bar{x})$. Such allocations in 3 goods economy exist (computer finds them in particular examples) though they are absent in two-goods economy. Further, reasoning in Edgeworth box style we can connect points \tilde{x} and \bar{x} by linear segment and show that the points from the segment interior are strictly inside of maximal surface. It follows from the fact that for Cobb–Douglas functions the restriction $\langle \nabla u_1(x), x - \mathbf{e}^1 \rangle \geq 0$ is a restriction defined by concave function (thus the set of allocations limited by the maximal surface is convex). For simplicity, taking the logarithm of utility, and calculating a gradient, one finds

$$\langle \nabla \ln u_1(x), x - \mathbf{e}^1 \rangle = 1 - \sum \frac{\alpha_j \mathbf{e}_j^1}{x_j},$$

where in the right-hand part of equality a strict concave function is written down. It follows from this that if from the point of initial endowments \mathbf{e}^1 to let out a ray going through some point of segment $x(\gamma) = \gamma \tilde{x} + (1 - \gamma) \bar{x}$, $\gamma \in (0, 1)$ then the point of intersection of the ray and the 1st agent maximal surface is placed on the ray further than point $x(\gamma)$. Thus, the 1st agent utility at a point of intersection is greater than his/her utility at a point $x(\gamma)$, which in turn is more than his/her utility

at a point of equilibrium. So, we have shown that in all points which are located on a continuous curve on the maximal surface, it is constructed as all points of its intersection with all designed rays, the 1st agent utility is strictly more than utility at a point of equilibrium. It refutes our working hypothesis since if it is incorrect already for Cobb–Douglas utilities then in general it cannot be considered acceptable.

So, in formulated above form, our characteristic property for an equilibrium to be locally stable relative to general contractual processes is unsatisfactory. One of our opportunities to elaborate positive result is to relax the stability requirement and to require (local) convergence not for any rule of trade but for the rules having some additional and economically reasonable properties. With this in mind, we first try to understand the basic distinction between 2-goods and 3-goods economies.

In 2-goods case and for a small enough neighborhood of an equilibrium point, if the current allocation is on the maximal surface, then any mutually beneficial exchange between the agents with necessity (almost always) involves the break of gross contract, *i.e.*, contractual process goes with obligatory break of contracts. Differently, if the current point of a trajectory is placed in a neighborhood of stable equilibrium, then process further goes according to alternative (i) of previous paragraph: when time elapsed, there can be only one active individual and his/her current consumption is strictly less preferable of equilibrium one (hence, using results of the previous paragraph one can easily establish convergence of a trajectory to equilibrium). This statement is true as soon as for a small enough neighborhood of equilibrium the normalized gradients of utility functions are almost equal to the vector of equilibrium prices but the vectors h_i , $i = 1, 2$ being calculated at equilibrium are disproportionate to equilibrium prices. However, already in 3-goods economy it is not so, and in any neighborhood of equilibrium one can find an allocation on the maximal surface admitting a mutually beneficial exchange without break of contracts, *i.e.*, it is possible Pareto improvement without break! This motivates the following

Definition 6.2.1 *A rule of trade $v : \mathcal{A}(X) \rightarrow \mathcal{L}^c$ is called **benevolent**, if $v(x)$ does not attract the break of contracts in all situations when a mutually beneficial exchange without break is possible.*

*Contractual UB–process is called **benevolent**, if it is defined by a benevolent rule of trade.*

In substantial terms, Definition 6.2.1 means that before the individuals sign a new contract, they carefully investigate opportunities for a mutually beneficial exchange being aimed to find a contract without subsequent break of signed earlier contracts. The signing of a contract with subsequent break is carried out only if there is no any other opportunity to get an agreement.

Formally, for process by Definition 5.2.1 the concept of a benevolent rule of trade requires performance of the following conditions.

Let $x \in \mathcal{A}(X)$ be some allocation stable relative to the break of aggregated contract $x - \mathbf{e} = v$ and let

$$\mathcal{I}^a(x) = \{i \in \mathcal{I} \mid \langle \nabla u_i(x_i), x_i - \mathbf{e}_i \rangle = 0\} \neq \emptyset$$

be *nonempty* set of all active individuals. Let us define

$$W^{fr}(x) = \{w \in \mathcal{L}^c \mid \langle \nabla u_i(x_i), w_i \rangle > 0, \forall i \in \mathcal{I} \ \& \ \langle h_i, w_i \rangle > 0, \forall i \in \mathcal{I}^a(x)\}. \quad (6.2.1)$$

This is the set (possibly empty) of all mutually beneficial contracts that being signed do not attract the break of aggregated contract $x - \mathbf{e}$.⁵ If $W^{fr}(x) \neq \emptyset$ then

$$\langle h_i, v_i(x) \rangle \geq 0, \quad \forall i \in \mathcal{I}^a(x) \ \& \ \langle \nabla u_i(x_i), v_i(x) \rangle > 0, \quad \forall i \in \mathcal{I}. \quad (6.2.2)$$

The following statement gives some criterion of local stability in economy with two agents.

Proposition 6.2.1 *Let $\bar{x} = (\bar{x}_1, \bar{x}_2)$ be an **isolated** equilibrium allocation in the economy with two agents and let standard assumptions be satisfied. Then if for some neighborhood $V_{\bar{x}}$ in $\mathcal{A}(X)$ of point \bar{x} for every $y \in V_{\bar{x}}$ satisfying $\langle \nabla u_i(y_i), y_i - \mathbf{e}_i \rangle \geq 0$, $i = 1, 2$ the following relation*

$$[\exists j : \langle \nabla u_j(y_j), y_j - \mathbf{e}_j \rangle = 0 \ \& \ W^{fr}(y) = \emptyset] \quad \Rightarrow \quad u_j(y_j) \leq u_j(\bar{x}_j) \quad (6.2.3)$$

holds, then equilibrium is locally stable relative to (locally) benevolent contractual processes.

Note that in the left-hand part of relation (6.2.3) there is described a situation in which a mutually beneficial exchange without the subsequent break is impossible. Thus, the requirement (6.2.3) tells us that in such cases an active individual prefers equilibrium consumption to the current one; the last can also be written down as $\langle \nabla u_j(y_j), \bar{x}_j - y_j \rangle > 0$ for $\bar{x}_j \neq y_j$ (it follows from strict concavity of utility).

Proof of Proposition 6.2.1. Let $V_{\bar{x}}$ be a neighborhood of equilibrium point $\bar{x} = (\bar{x}_1, \bar{x}_2)$ which existence is postulated in Proposition 6.2.1. Consider a vector-function $f : V_{\bar{x}} \rightarrow \mathbb{R}^2$ where

$$f_i(y_i) = \min\{u_i(y_i), u_i(\bar{x}_i)\}, \quad i = 1, 2.$$

Show that this function is (non-strictly) monotonically increasing along benevolent contractual trajectory.

In fact, for $x(t) \in V_{\bar{x}}$ either $\mathcal{I}^a(x(t)) = \emptyset$ but then at the point $x(t)$ contractual process is going without break and monotonicity is obvious, or $\mathcal{I}^a(x(t)) \neq \emptyset$. In the last case if $W^{fr}(x(t)) \neq \emptyset$ then monotonicity follows from (6.2.2), in a contrary case $W^{fr}(x(t)) = \emptyset$ and one can apply (6.2.3). Further, we first note that $(u_1(x_1(t)), u_2(x_2(t))) < (u_1(\bar{x}_1), u_2(\bar{x}_2))$ is impossible because otherwise the equilibrium \bar{x} Pareto dominates $x = x(t)$ and $W^{fr}(x(t)) \neq \emptyset$. Therefore, since by (6.2.3) it has to be $u_j(x_j(t)) \leq u_j(\bar{x}_j)$, $j \in \mathcal{I}^a(x(t))$ and then $u_i(x_i(t)) > u_i(\bar{x}_i)$ for $i \neq j$. However, in this case by specification $f_i(x_i(t')) = f_i(\bar{x}_i)$ for all $t' \geq t$ close enough to t . The activity of j also implies that the function $f_j(x_j(t'))$ is locally increasing for $t' \geq t$.

⁵Here $\mathcal{L}^c = \{(v_1, \dots, v_n) \in L^{\mathcal{I}} \mid \sum_{\mathcal{I}} v_i = 0\}$ is the space of contracts. Remember that condition $\langle h_i(x), w_i \rangle \geq 0$ means that individual $i \in \mathcal{I}^a$ is not interested in the break of contract $x - \mathbf{e}$ when w is signed at a state x .

Further, due to the assumption that utility functions are strictly concave and since equilibrium allocation is Pareto optimal, it is not difficult to prove that the sets

$$V_{\bar{x}}^\varepsilon = \{y \in \mathcal{A}(X) \mid f_1(y_1) \geq f_1(\bar{x}_1) - \varepsilon \ \& \ f_2(y_2) \geq f_2(\bar{x}_2) - \varepsilon\}, \quad \varepsilon > 0$$

form a basis of neighborhoods for the point \bar{x} in $\mathcal{A}(X)$.⁶ To see this it is enough to note that $\bigcap_{\varepsilon > 0} V_{\bar{x}}^\varepsilon = \{\bar{x}\}$. However, this is true because only Pareto optimal points can be in the intersection but for strictly concave utility functions it is impossible to find two different allocations which are Pareto optimal and simultaneously have equal agents' utilities. Choosing now $\varepsilon > 0$ from condition $V_{\bar{x}}^\varepsilon \subset V_{\bar{x}}$ and taking $V_{\bar{x}}^\varepsilon$ as a neighborhood of initial data, we conclude that the first condition of local stability (see above) is valid: a trajectory being at least one time in this neighborhood can never leave it (due to the monotonicity of f along a benevolent trajectory).

Further, the monotonicity of f along a trajectory and the property $f(x(t)) \leq f(\bar{x})$ allows us to consider f as a Lyapunov vector-function, but we need to show that $\lim_{t \rightarrow \infty} x(t) = \bar{x}$. Let us do it. Let \tilde{x} be any limit point of trajectory $x(t)$. The monotonicity of f implies that $\lim_{t \rightarrow \infty} f(x(t))$ does exist and also $\lim_{t \rightarrow \infty} f(x(t)) = f(\tilde{x}) \leq (u_1(\bar{x}_1), u_2(\bar{x}_2))$. Show that this inequality can be fulfilled only as equality. To do it first of all notice that the case $f(\tilde{x}) \ll (u_1(\bar{x}_1), u_2(\bar{x}_2))$ is obviously impossible (by Lemma 6.1.1 and Remark 6.1.1). Hence, at least for one component equality is realized. Let, for example, for 1st agent and for all large enough t it is realized: $\exists \delta > 0: u_1(x_1(t)) < u_1(\bar{x}_1) - \delta$. Now (6.2.3) and benevolence imply that the utility of this agent is monotonically increased for all t large enough. Moreover, it is easy to see that if $u_2(\tilde{x}_2) > u_2(\bar{x}_2)$ then we are in a condition of alternative (i) from the previous section, § 6.1.2 (page 223) because only 1st agent can be active for t large enough. Therefore, due to Lemma 6.1.1 and Remark 6.1.1 the allocation \tilde{x} is Pareto optimal. However, in so doing, \tilde{x} has to be stable relative to partial break of gross contract $\tilde{x} - \mathbf{e}$. Hence \tilde{x} has to be an equilibrium allocation from the neighborhood $V_{\bar{x}}^\varepsilon$. Choosing now $\varepsilon > 0$ so that the neighborhood does not include equilibria different from \bar{x} and taking this neighborhood as a neighborhood for initial data, one concludes $\tilde{x} = \bar{x}$.

Thus we have proven that $(u_1(\tilde{x}_1), u_2(\tilde{x}_2)) \geq (u_1(\bar{x}_1), u_2(\bar{x}_2))$. However, once again it means that \tilde{x} is Pareto optimal and therefore it is an equilibrium. Hence $\tilde{x} = \bar{x}$. As a result: we have found a neighborhood such that any trajectory defined by benevolent contractual process which is going through some point of the neighborhood has all limits points equal to \bar{x} . This trajectory converges to \bar{x} . ■

Remark 6.2.1 Notice that for 2-goods economy an equilibrium \bar{x} is isolated if vectors h_1, h_2 calculated at the equilibrium point are non-collinear. Moreover, under assumption (D) and if equilibrium is an interior point, then it is possible to prove that any contractual process in 2×2 economy is locally benevolent. To see this, it is enough to note that for any allocation y on the maximal surface and from a neighborhood of equilibrium a cone of improvements

$$z \in \mathbb{R}^2 \mid \langle \nabla u_1(y_1), z \rangle > 0 \ \& \ \langle \nabla u_2(y_2), z \rangle < 0$$

⁶It means that all these sets are neighborhoods and that every neighborhood includes a set of this type.

does not intersect a half-plane $\langle h_1(y_1), z \rangle \leq 0$ when 1st agent is active, or does not intersect a half-plane $\langle h_2(y_2), z \rangle \geq 0$ when 2nd agent is active. It will be so since in a neighborhood of equilibrium the utility gradients are close to the equilibrium prices and in view of $h_i(y_i) \approx h_i(\bar{x}_i)$, $\langle h_i(\bar{x}_i), \mathbf{e}_i - \bar{x}_i \rangle > 0$, $i = 1, 2$.

Thus Proposition 6.2.1 corresponds with our previous reasonings and serves their formal proof. Moreover, this allows us to formulate a computable criterion of local stability for equilibrium in economy 2×2 . ■

In presented form it is not easy to extend the result of Proposition 6.2.1 to the case of any number of agents because the fact that economy has only two agents is essential in the proof. Really, if there are only two individuals and one of them is active in a current state from a suitable neighborhood of equilibrium, then one can conclude that for benevolent trajectories the utility of passive agent is strictly more than his/her utility in equilibrium. If there are more than one passive individual, then one can surely assert only that among them there is at least one individual with strictly greater utility than in equilibrium.

Despite the difficulties to study the general case of local stability, in the following section we shall continue investigation of benevolent proper-contractual processes. Under certain assumptions it will be proved that these processes generically converge to equilibrium.

6.3 Convergence of benevolent *UB*-processes

We begin the analysis from detailed research of a general case of benevolent trajectories. With this in mind, we first study situations in which a mutually beneficial exchange without break of contracts is impossible. The following lemma describes necessary conditions that such a situation takes place.

Lemma 6.3.1 *Let $x \in \text{ri}\mathcal{A}(X)$ be an allocation stable relative to the partial break of gross contract $x - \mathbf{e}$. Let standard assumptions be satisfied and let mutually beneficial exchange without the subsequent break of contracts be impossible, i.e., $W^{fr}(x) = \emptyset$. Then there is a vector $p \neq 0$ so that for each individual $i \in \mathcal{I}$ there are real numbers $\alpha_i \geq 0$, $\beta_i \geq 0$ such that*

$$p = \alpha_i \nabla u_i(x_i) + \beta_i h_i(x_i), \quad \forall i \in \mathcal{I} \quad (6.3.1)$$

and the following complementarity slackness conditions:

$$\beta_i \cdot \langle \nabla u_i(x_i), x_i - \mathbf{e}_i \rangle = 0, \quad \forall i \in \mathcal{I}$$

are fulfilled.

Complementarity slackness conditions in this lemma just serve a convenient form to describe the fact that for each passive individual (if $\langle \nabla u_i(x_i), x_i - \mathbf{e}_i \rangle > 0$) the value $\beta_i \geq 0$ should be zero while for an active agent (if $\langle \nabla u_i(x_i), x_i - \mathbf{e}_i \rangle = 0$) it can be strictly more than zero. Further, it is easy to see that up to the normalization of p

the values α_i and β_i in the formula (6.3.1) are unequivocally defined since $\nabla u_i(x_i)$ and $h_i(x_i)$ are the non-collinear couple of vectors for any active individual in the current allocation. It allows us among the active individuals to correctly identify the agents who can really influence the course of the contractual process via a break of the contracts.

So, we shall call an active individual i *really active*, if $\beta_i > 0$. Accordingly, if $\beta_i = 0$ for an active agent then he/she may be called *fictitiously active* (locally in process behavior of these individuals similar to passive ones). Also, let's name the individual *really passive* if he/she is passive *or* fictitiously active.

Proof of Lemma 6.3.1. By definition (6.2.1) of the set $W^{fr}(x)$ one can equivalently rewrite condition $W^{fr}(x) = \emptyset$ in the following way.

Let us define $B_i(x) = \{z \in L \mid \langle \nabla u_i(x_i), z \rangle > 0\}$ if the individual i is passive and let $B_i(x) = \{z \in L \mid \langle \nabla u_i(x_i), z \rangle > 0 \ \& \ \langle h_i(x_i), z \rangle > 0\}$ for the active individual. Then

$$\prod_{\mathcal{I}} B_i(x) \cap \mathfrak{L}^c = \emptyset,$$

where, remember, $\mathfrak{L}^c = \{(v_1, \dots, v_n) \in L^{\mathcal{I}} \mid \sum_{\mathcal{I}} v_i = 0\}$ is the space of contracts. Notice that $B_i(x) \neq \emptyset$ for active i since $\nabla u_i(x_i) \neq 0$, $\langle \nabla u_i(x_i), x_i - e_i \rangle = 0$ and $\langle h_i, x_i - e_i \rangle < 0$ (hence vectors are non-collinear), and of course $B_i(x) \neq \emptyset$ for passive agents. Therefore, one can apply separation theorem and find $\pi = (p_1, \dots, p_n) \neq 0$ such that

$$\langle \pi, \prod_{\mathcal{I}} B_i(x) \rangle \geq \langle \pi, \mathfrak{L}^c \rangle.$$

We see that functional π is bounded from above on subspace \mathfrak{L}^c which is possible only if $\langle \pi, \mathfrak{L}^c \rangle = 0$ and, therefore, in standard manner one can conclude that $p_i = p_j$, $\forall i \neq j$. Denote $p = p_i \neq 0$. Further, it is easy to see that $\langle \pi, \prod_{\mathcal{I}} B_i(x) \rangle \geq 0$ is possible only if (applying $\pi = (p, \dots, p)$)

$$\langle p, B_i(x) \rangle > 0, \quad \forall i \in \mathcal{I}$$

is true. Thus, for every active i the inequality $\langle p, z \rangle > 0$ is a corollary of two inequalities: $\langle \nabla u_i(x_i), z \rangle > 0$, $\langle h_i(x_i), z \rangle > 0$, and for passive agent only of first of them. Now applying Farkas lemma (or again separation theorem) we conclude the existence of $\alpha_i \geq 0$, $\beta_i \geq 0$ demanded in the statement of lemma. ■

The property of a trajectory to be benevolent is rather qualified requirement which is applied to a rule of trade. In particular, it is easy to see that the vector p which existence was proved in Lemma 6.3.1 being normalized as $\|p\| = 1$ is a continuous function of the current point of a trajectory.

Lemma 6.3.2 *Let $x(t)$, $t \geq 0$ be a benevolent UB-trajectory by Definition 5.2.1 and let standard assumptions be satisfied. Then the vectors $p = p(x(t))$, $\|p\| = 1$, existing by Lemma 6.3.1 at all points of trajectory $x(t)$ present a continuous function in its domain: an area where beneficial exchange without contracts breaking is impossible.*

It is clear that in lemma conditions, when there is the specified continuous dependence of a vector p from the current point of a trajectory, the same thing can be said about the coefficients α_i, β_i of decomposition (6.3.1): they are also continuously dependent on time. It follows from the fact (already mentioned) that vectors $\nabla u_i, h_i$ being calculated at required points, are noncollinear.

Proof of Lemma 6.3.2. It follows from the continuity of a trajectory $x(t)$ that for each individual i the set of all moments $t > 0$ where he/she is passive is open on interval $(0, +\infty)$, because the set is defined via condition $\langle \nabla u_i(x_i(t)), x_i(t) - \mathbf{e}_i \rangle > 0$. Due to Lemma 6.3.1 we have $p(x(t)) = \nabla u_i(x_i(t)) / \|\nabla u_i(x_i(t))\|$ in every moment t where individual i is passive. Therefore, since the gradient of utility continuously depends on trajectory points, one can conclude that as the function of time $p(t)$ changes continuously in a neighborhood of t . Now we need to show that if $\dot{x}(t) \neq 0$, *i.e.*, if the point $x(t)$ is not equilibrium, and if the mutually beneficial exchange without break is impossible, then a passive individual exists. Assuming that all individuals are active via (6.3.1) (the mutually beneficial exchange without break is impossible) we have

$$p(t) = \alpha_i \nabla u_i(x_i(t)) + \beta_i h_i(x_i(t)), \quad \forall i \in \mathcal{I}.$$

Further, let us multiply these equalities on vectors $x_i(t) - \mathbf{e}_i \neq 0$ and then sum the received equalities. As a result, since from the activity of the individuals we have $\langle \nabla u_i(x_i), x_i - \mathbf{e}_i \rangle = 0, \forall i \in \mathcal{I}$ and due to $\sum (x_i(t) - \mathbf{e}_i) = 0$ one obtains

$$0 = \sum \beta_i \langle h_i(x_i(t)), x_i(t) - \mathbf{e}_i \rangle.$$

Since $\beta_i \geq 0$ and $\langle h_i(x_i), x_i - \mathbf{e}_i \rangle < 0, \forall i \in \mathcal{I}$, then the last equality is possible only if $\beta_i = 0$ for all i , that is possible only in equilibrium. It is a contradiction. ■

Further, we will be interested in some specific properties of proper-contractual trajectories by Definition 5.2.1 (not necessarily benevolent!). In fact, we need to clear those situations, when at the points of a trajectory more than one active individual may exist. With this in mind, we remind that at the current point of a trajectory $x(t)$ the measure of break of the gross contract is defined as a minimum (provided that it is less than zero, otherwise a break does not occur) of some values determining desirable break for the active individuals. Desirable break of a gross contract for the agent i is defined by value

$$\lambda_i(x(t), v(x(t))) = \frac{\langle h_i(x_i(t)), v_i(x_i(t)) \rangle}{\langle h_i(x_i(t)), \mathbf{e}_i - x_i(t) \rangle},$$

see Lemma 5.2.1. However, for two individuals simultaneously define the size of break of the contracts in *nearest subsequent* after t moments of time, it is necessary that the measure of a desirable break of gross contract coincides with a general minimum and, therefore, both measures should coincide among themselves. This motivates the following definition.

Definition 6.3.1 A contractual trajectory $x : [0, +\infty) \rightarrow \mathcal{A}(X)$ (process) is called **non-degenerate** if for all non-equilibrium points $x(t)$ of its hit on a maximal surface for each couple of active individual $i, j, i \neq j$ (i.e. if $\langle \nabla u_i(x_i(t)), x_i(t) - \mathbf{e}_i \rangle = \langle \nabla u_j(x_j(t)), x_j(t) - \mathbf{e}_j \rangle = 0$) the following inequality

$$\lambda_i(x(t), v(x(t))) = \frac{\langle h_i(x_i(t)), v_i(x_i(t)) \rangle}{\langle h_i(x_i(t)), \mathbf{e}_i - x_i(t) \rangle} \neq \frac{\langle h_j(x_j(t)), v_j(x_j(t)) \rangle}{\langle h_j(x_j(t)), \mathbf{e}_j - x_j(t) \rangle} = \lambda_j(x(t), v(x(t)))$$

holds.

A trading rule is called **non-degenerate** relative to initial endowments allocation $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ if a generated trajectory with initial data $x(0) = \mathbf{e}$ is non-degenerate.

Non-degenerate contractual trajectories are easier in the analysis because in each moment of time only one individual sets a measure of break of gross contract, i.e. he/she can be considered being an active “leader” of the contractual process. However, for non-degenerate processes of a general form when time elapses, there can be a change of the leader. Below, we will see that it does not occur in the case of benevolent trajectories.

With this in mind, we first show that for *any not stabilized* trajectory the set of those moments where there are two or more active agents has a structure similar to discrete one.⁷

Lemma 6.3.3 Let $x(t)$ be a non-degenerate UB-contractual trajectory by Definition 5.2.1 and 6.3.1. Let at the moment τ the derivative $\dot{x}(\tau) \neq 0$ and an agent i be active, i.e. $\langle \nabla u_i(x_i(\tau)), x_i(\tau) - \mathbf{e}_i \rangle = 0$. Then for some $\varepsilon > 0$ and for all points from $(\tau, \tau + \varepsilon)$ only one of the following alternatives takes place:

- (i) If $\lambda_i(x(\tau), v(x(\tau))) = \lambda^{\min}(x(\tau), v(x(\tau))) < 0$ **only** individual i is active and all other agents are passive.
- (ii) If $\lambda_i(x(\tau), v(x(\tau))) = \lambda^{\min}(x(\tau), v(x(\tau))) = 0$ individual i **may be** active but it is not certainly the case however **all other** agents are certainly **passive**.
- (iii) If $\lambda_j(x(\tau), v(x(\tau))) > \lambda^{\min}(x(\tau), v(x(\tau))) = 0$ for each active individual then in interval $(\tau, \tau + \varepsilon)$ **all agents are passive**.

The statement of this lemma implies the following important

Corollary 6.3.1 If a contractual trajectory is non-degenerated and $\dot{x}(t) \neq 0, \forall t \geq 0$, then the set of all time-moments where the number of active individuals can be two or greater is no more than enumerable.

⁷Strictly speaking, this set can have limit points, however it does not influence the subsequent analysis.

Remark 6.3.1 The statement of Lemma 6.3.3 may seem almost obvious; however, it is necessary to remember that we deal with the solution of a differential equation with a discontinuous right-hand part and, therefore, this solution is not obliged to be a differentiable function. Therefore, an appropriate analysis should be realized with special carefulness.

It seems that the result of the lemma can be extended to a general case of not necessarily non-degenerate trajectories. Certainly, it has to be another edition in which possible such variant of alternative (i): from the set all active individuals satisfying this alternative condition there may be separated the group of the individuals each of them is active on some open interval of time directly contiguous to the considered time moment. However, the strict proof of this fact is not presented while... As a consequence to generalized lemma, one can hope to receive this fact: the number of the moments of time, when the set of the active individuals is reconstructed, is no more than enumerable. ■

Proof of Lemma 6.3.3. First let us choose $\varepsilon > 0$ so that if for $j \in \mathcal{I}$ inequality $\langle \nabla u_j(x_j(\tau)), x_j(\tau) - \mathbf{e}_j \rangle > 0$ is fulfilled then for all $t \in (\tau, \tau + \varepsilon)$ the similar inequality $\langle \nabla u_j(x_j(t)), x_j(t) - \mathbf{e}_j \rangle > 0$ is also fulfilled. It is possible in view of continuous dependence of a trajectory from time and since all functions participating in inequality are continuous.

Further, let $\langle \nabla u_j(x_j(\tau)), x_j(\tau) - \mathbf{e}_j \rangle = 0$ be fulfilled for some $j \in \mathcal{I}$, $j \neq i$, *i.e.*, j is another agent distinct from i which is active at the moment τ . Applying Definition 6.3.1 suppose, for example, that

$$\frac{\langle h_i(x_i(\tau)), v_i(x_i(\tau)) \rangle}{\langle h_i(x_i(\tau)), \mathbf{e}_i - x_i(\tau) \rangle} < \frac{\langle h_j(x_j(\tau)), v_j(x_j(\tau)) \rangle}{\langle h_j(x_j(\tau)), \mathbf{e}_j - x_j(\tau) \rangle}$$

holds. From a continuity of functions participating in the inequality, it is possible also to find a neighborhood of point $x(\tau)$ in $\mathcal{A}(X)$ and a neighborhood of point $v(x(\tau))$ in the space of contracts \mathfrak{L}^c such that the similar inequality is true for any point from these neighborhoods replacing $x(\tau)$ and $v(x(\tau))$, accordingly. Let $\delta > 0$ be such that

$$\frac{\langle h_i(x_i), w_i \rangle}{\langle h_i(x_i), \mathbf{e}_i - x_i \rangle} < \frac{\langle h_j(x_j), w_j \rangle}{\langle h_j(x_j), \mathbf{e}_j - x_j \rangle}, \quad \forall x \in B_\delta(x(\tau)) \cap \mathcal{A}(X), \quad \forall w \in B_\delta(v(x(\tau))) \cap \mathfrak{L}^c$$

is fulfilled, where $B_\delta(y)$ denotes a ball of radius $\delta > 0$ centered at y in an appropriate space, and vectors $h_i(x_i), h_j(x_j)$ are formally defined by formula (5.2.2) and are calculated at the designated point of space. Moreover, without loss of generality, it is possible also to think that *all contracts* from $B_\delta(v(x(\tau))) \cap \mathfrak{L}^c$ are *beneficial* at every point from $B_\delta(x(\tau)) \cap \mathcal{A}(X)$. This obviously follows from the definition of the mutually beneficial contract and from the continuity of all functions participating in required inequalities. Besides, it is possible to think that *numerator and denominator* in expressions from the last formula *do not change a sign* for all points of chosen neighborhoods and that this is true for any pair of active individuals at the moment τ . At last, reducing if necessary, $\varepsilon > 0$ can be chosen so that all points $x(t)$ for $t \in (\tau, \tau + \varepsilon)$ are in the limits of the chosen neighborhood $B_\delta(x(\tau))$ of $x(\tau)$, *i.e.* for the time not more than $\varepsilon > 0$ the trajectory does not leave this neighborhood.

Further, let us establish the validity of alternative (i). It is necessary to show that in conditions of (i) an arbitrarily chosen point of the trajectory $x(t)$, $t \in (\tau, \tau + \varepsilon)$ is located on the maximal surface of individual i , i.e., that $\langle \nabla u_i(x_i(t)), x_i(t) - \mathbf{e}_i \rangle = 0$ is fulfilled.

Assuming $\langle \nabla u_i(x_i(t)), x_i(t) - \mathbf{e}_i \rangle > 0$ find a maximum of all those moments $t' \in [\tau, t)$ where the current point of trajectory $x(t')$ is located on the maximal surface of agent i . As $x(\tau)$ is on the maximal surface of the agent i , this maximum does exist and obviously that at this moment the point of a trajectory is located on the maximal surface. Let s denote this maximum. Now we have $\langle \nabla u_i(x_i(s)), x_i(s) - \mathbf{e}_i \rangle = 0$ and $\langle \nabla u_i(x_i(\zeta)), x_i(\zeta) - \mathbf{e}_i \rangle > 0$ for all $\zeta \in (s, t)$. By definition, in the interval (s, t) the law of change of a trajectory (5.2.8) is set by the contract $v(x(\zeta))$ and by (in general discontinuous) function $\lambda^{\min}(\cdot)$ which in conditions of alternative (i) by the choice of ε and because individual i is passive for all $\zeta \in (s, t)$ has to satisfy

$$\lambda^{\min}(x(\zeta), v(x(\zeta))) > a > b > \frac{\langle h_i(x_i(\zeta)), v_i(x(\zeta)) \rangle}{\langle h_i(x_i(\zeta)), \mathbf{e}_i - x_i(\zeta) \rangle} = g_i(x(\zeta), v(x(\zeta))) \quad (6.3.2)$$

for some real a, b . Further, the vector $h_i(x_i)$ participating in the right-hand part of inequality (6.3.2) is defined by formula (5.2.2) and, therefore, it is the gradient of function $F(x_i) = \langle \nabla u_i(x_i), x_i - \mathbf{e}_i \rangle$ which defines maximal surface by equation $F(x_i) = 0$. So we have $F(x_i(s)) = 0$ and the value $F(x_i(t))$ can be found by formula

$$F(x_i(t)) = \int_s^t \frac{F(x_i(\zeta))}{d\zeta} d\zeta = \int_s^t \langle \nabla_{x_i} F(x_i(\zeta)), \dot{x}_i(\zeta) \rangle d\zeta,$$

where the function under integral is summarized (since $x(\cdot)$ is an absolutely continuous function). Substituting expression of under-integral functions ($\dot{x}_i(\zeta)$ via the law of trajectory) in view of (6.3.2) and $\langle h_i(x_i(\zeta)), x_i(\zeta) - \mathbf{e}_i \rangle < 0, \forall \zeta \in (s, t)$ we obtain the following estimation

$$\begin{aligned} F(x_i(t)) &= \int_s^t \langle h_i(x_i(\zeta)), \lambda^{\min}(\zeta)(x_i(\zeta) - \mathbf{e}_i) + v_i(x_i(\zeta)) \rangle d\zeta \leq \\ &\leq a \int_s^t \langle h_i(x_i(\zeta)), (x_i(\zeta) - \mathbf{e}_i) \rangle d\zeta + \int_s^t \langle h_i(x_i(\zeta)), v_i(x_i(\zeta)) \rangle d\zeta \leq \\ &\leq (a - b) \int_s^t \langle h_i(x_i(\zeta)), x_i(\zeta) - \mathbf{e}_i \rangle d\zeta + \int_s^t \langle h_i(x_i(\zeta)), g_i(\zeta)(x_i(\zeta) - \mathbf{e}_i) + v_i(x_i(\zeta)) \rangle d\zeta = \\ &= (a - b) \int_s^t \langle h_i(x_i(\zeta)), x_i(\zeta) - \mathbf{e}_i \rangle d\zeta. \end{aligned}$$

The last equality in the chain of estimations is true because the second integral (summand) is equal to zero: by definition of $g_i(\zeta) = g_i(x(\zeta), v(x(\zeta)))$ in (6.3.2) and due to

$$\langle h_i, \frac{\langle h_i, v_i \rangle}{\langle h_i, \mathbf{e}_i - x_i \rangle} (x_i - \mathbf{e}_i) + v_i \rangle = 0, \quad x_i \neq \mathbf{e}_i.$$

Since $\int_s^t \langle h_i(x_i(\zeta)), x_i(\zeta) - \mathbf{e}_i \rangle d\zeta < 0$ then as a result we conclude

$$F(x_i(t)) = \langle \nabla u_i(x_i(t)), x_i(t) - \mathbf{e}_i \rangle < 0,$$

that contradicts the initial assumption. Thus alternative (i) has proven.

In a part of the proof of alternatives (ii) and (iii), we only note that it can be done in accordance with the same method as stated above. The difference consists in the formulation of a requirement similar to (6.3.2) but written down concerning other parameters: only this thing is important to obtain the key estimations. For example, for the proof of alternative (ii), for the individual $j \neq i$ which is active at the moment τ for a suitable time interval one needs to apply

$$\lambda^{\min}(x(\zeta), v(x(\zeta))) < a < b < \frac{\langle h_j(x_j(\zeta)), v_j(x(\zeta)) \rangle}{\langle h_j(x_j(\zeta)), \mathbf{e}_j - x_j(\zeta) \rangle} = g_j(x(\zeta), v(x(\zeta))).$$

Lemma 6.3.3 is proven. ■

Further, we show that while equilibrium is not attained along a benevolent trajectory, there is only one individual that can be “really” active. Remember that really active is such individual, which in the relation (6.3.1) from Lemma 6.3.1 has parameter $\beta_i \neq 0$.

Theorem 6.3.1 *Let $x(t)$ be a non-degenerate benevolent trajectory and let standard assumptions be fulfilled. Then only one of the following alternatives can be true:*

- (i) *There are no (almost) time moments on the interval $[0, +\infty)$ when the breaking of gross contract $x(t) - \mathbf{e}$ is realized, i.e., for almost all time moments during the contractual process all individuals are passive.*
- (ii) *There exists a time moment $\tau > 0$ such that for almost all time moments on the interval $[0, \tau)$ all individuals are passive and the contrary at the moment τ : all individuals are active, i.e., $x(\tau)$ is an equilibrium.*
- (iii) *There exist time moments $\tau_1 > 0$ and $\tau_2 > \tau_1$ such that for almost all time moments on the interval $[0, \tau_1)$ all individuals are passive and there is the only real active individual on the interval $[\tau_1, \tau_2)$ and if $\tau_2 \neq +\infty$ then at the moment τ_2 all individuals are active, i.e., $x(\tau_2)$ is an equilibrium.*

Theorem 6.3.1 describes rather important properties of non-degenerate benevolent trajectories which allow us to conclude the convergence of this type of trajectories to an equilibrium.

Corollary 6.3.2 *Let the standard assumptions be fulfilled. Then any benevolent rule of trade generating non-degenerate contractual process defines proper-contractual UB-trajectory, for which all limit points are equilibria. Thus non-degenerate benevolent processes are quasi-globally stable.*

Proof of Corollary 6.3.2. If an alternative (ii) or (iii) when $\tau_2 < +\infty$ is realized, then the convergence of a contractual trajectory to equilibrium is obvious. If the alternative (i) or (iii) with $\tau_2 = +\infty$ is realized then we are in the condition of alternative (i) from Section 6.2 (see page 223) and now we can apply Lemma 6.1.1 and Remark 6.1.1 that proves equilibrium properties of any limit point. ■

Proof of Theorem 6.3.1. Let us determine a time moment $\tau > 0$ as the first moment when the mutually beneficial exchange without a partial break of gross contract $x(t) - \mathbf{e}$ is impossible. If there are no such moments, then alternative (i) is realized. However, if the set of all such moments is not empty, then being the closed set it always has the minimal element. Therefore, moment τ is determined correctly. Further, if $\dot{x}(\tau) = 0$ then $x(\tau)$ is an equilibrium and, therefore, the alternative (ii) is realized. Let $\dot{x}(\tau) \neq 0$. Now we are able to apply Lemma 6.3.1 and can conclude that at the moment τ there is at least one really active individual. Otherwise $\beta_i(\tau) = 0$ for all $i \in \mathcal{I}$ that is possible only in equilibrium. Let $\mathcal{I}^{ra}(\tau) \subset \mathcal{I}$ be a nonempty set of all really active individuals at the moment τ . Further, one can apply the fact that our trajectory is non-degenerate and show that alternative (i) of Lemma 6.3.3 is true. We need to prove that $\lambda_i(x(\tau), v(x(\tau))) < 0$ for some active individual $i \in \mathcal{I}$ at the moment τ . To do it we need to show $\langle h_i(x_i(\tau)), v_i(x_i(\tau)) \rangle < 0$. With this in mind, one can apply Lemma 6.3.1 and conclude the existence of a vector $p(\tau) \neq 0$ and, for each i , numbers $\alpha_i(\tau) \geq 0$, $\beta_i(\tau) \geq 0$ such that (6.3.1) is carried out:

$$p(\tau) = \alpha_i(\tau) \nabla u_i(x_i(\tau)) + \beta_i(\tau) h_i(x_i(\tau)), \quad \forall i \in \mathcal{I}.$$

Further, for each really active individual i , from $\mathcal{I}^{ra}(\tau) \neq \emptyset$ multiply the appropriate equality on vector $v_i(x_i(\tau))$ and then sum the received equalities. The obtained result can be written down as

$$\begin{aligned} \langle p(\tau), \sum_{\mathcal{I}^{ra}(\tau)} v_i(x_i(\tau)) \rangle - \sum_{\mathcal{I}^{ra}(\tau)} \alpha_i \langle \nabla u_i(x_i(\tau)), v_i(x_i(\tau)) \rangle &= \\ &= \sum_{\mathcal{I}^{ra}(\tau)} \beta_i(\tau) \langle h_i(x_i(\tau)), v_i(x_i(\tau)) \rangle. \end{aligned}$$

Since each summand in the right-hand part of this equality has a positive factor (strictly more than zero) we shall receive the required property if it will be established that the value in the left-hand part of equality is negative. But it is so because by the definition of contract $\sum_{\mathcal{I}^{ra}(\tau)} v_i(x_i(\tau)) = -\sum_{\mathcal{I} \setminus \mathcal{I}^{ra}(\tau)} v_i(x_i(\tau))$ that in view of its mutual benefit for really passive individuals gives

$$\langle p(\tau), \sum_{\mathcal{I}^{ra}(\tau)} v_i(x_i(\tau)) \rangle = -\langle p(\tau), \sum_{\mathcal{I} \setminus \mathcal{I}^{ra}(\tau)} v_i(x_i(\tau)) \rangle < 0.$$

Therefore, the left-hand part of previous equality is the summation of negative and non-positive values, and as a whole it is negative.

So, at present the moment, we have proven that alternative (i) of Lemma 6.3.3 is realized. This implies that for some $\varepsilon > 0$ on the interval $(\tau, \tau + \varepsilon)$ the contractual

process goes with a break of gross contract, and only one agent is active. Let i_0 be this individual. Only this individual (from complementarity slackness conditions from Lemma 6.3.1) can be *really* active on the interval $(\tau, \tau + \varepsilon)$ and, therefore,

$$\beta_{i_0}(t) > 0, \quad \beta_j(t) = 0, \quad \forall j \neq i, \quad \forall t \in (\tau, \tau + \varepsilon).$$

At last, applying Lemma 6.3.2 we can in these relations pass to limit for $t \rightarrow \tau + 0$ (all functions are continuous) concluding that $\beta_{i_0}(\tau) > 0$ and $\beta_j(\tau) = 0, \forall j \neq i_0$. Thus, the individual i_0 is the *sole* really active agent on the interval $[\tau, \tau + \varepsilon)$.

Below, on former assuming that $x(\tau)$ is not equilibrium, we prove the validity of alternative (iii).

With this in mind, we first show that for every $t > \tau$ the mutually beneficial exchange without a partial break of gross contract $x(t) - \mathbf{e}$ is impossible. Assuming opposite, find $t' > \tau$ as *infimum* of all moments where the exchange without break is possible. It is clear the set of all such moments forms an open set on $(\tau, +\infty)$ and t' can not belong to it. Therefore, at the moment t' the mutually beneficial exchange without a break is impossible. Besides, $x(t')$ can not be an equilibrium since the exchange goes after moment t' . Now we can apply Lemmas 6.3.1, 6.3.3 and reasoning similar to described above we can conclude the existence of $\delta > 0$ such that at *every* point of the interval $[t', t' + \delta)$ the contractual process is realized with a partial break of contracts. As a result, we come to the contradiction with a choice of the moment t' .

Further, we define $\tau_1 = \tau$ and find the moment τ_2 as infimum of all those moments of time from $(\tau_1, +\infty)$ when there is at least one another active individual distinct from i_0 . If there are no such moments, then $\tau_2 = +\infty$, and everything is proven. Let us assume $\tau_2 < +\infty$ and show that $x(\tau_2)$ is an equilibrium. First, note that at the moment τ_2 only individual i_0 can be really active. At this moment the mutually beneficial exchange without break is impossible, and on the interval (τ_1, τ_2) only i_0 is active, therefore, applying Lemma 6.3.1 we obtain

$$\beta_{i_0}(t) > 0, \quad \beta_j(t) = 0, \quad \forall j \neq i_0, \quad \forall t \in (\tau_1, \tau_2).$$

Further, in view of Lemma 6.3.2 all functions $\beta_i(\cdot), i \in \mathcal{I}$ are continuous on $[\tau_1, \tau_2]$ and, passing to limits by $t \rightarrow \tau_2 - 0$ we conclude

$$\beta_{i_0}(\tau_2) \geq 0, \quad \beta_j(\tau_2) = 0, \quad \forall j \neq i_0.$$

Further, if $\beta_{i_0}(\tau_2) \neq 0$ we are in conditions of alternative (i) from Lemma 6.3.3 and hence for some $\varepsilon > 0$ on the interval $(\tau_2, \tau_2 + \varepsilon)$ does exist only one active individual. By the choice τ_2 , this individual can not be i_0 . Therefore i_0 is passive on $(\tau_2, \tau_2 + \varepsilon)$ and once again via Lemma 6.3.1, we conclude $\beta_{i_0}(t) = 0$ on the interval $(\tau_2, \tau_2 + \varepsilon)$. However, $\beta_{i_0}(t)$ is a continuous function on $(\tau_2, \tau_2 + \varepsilon)$ by Lemma 6.3.2. Now passing to a limit by $t \rightarrow \tau_2 + 0$ we conclude $\beta_{i_0}(\tau_2) = 0$. The received contradiction proves that $\beta_{i_0}(\tau_2) = 0$. However, above it was established that $\beta_j(\tau_2) = 0, \forall j \neq i_0$. This is possible only for equilibrium point (since at the point $x(\tau_2)$ gradients of all individuals are pairwise collinear this is Pareto optimum, which is

also stable relative to the partial break of gross contract). Theorem 6.3.1 is proven. ■

One of our main results is presented in the following theorem on the convergence of non-degenerate benevolent UB -processes.

Theorem 6.3.2 *Let \mathcal{E} be a regular economy and let the standard assumptions be fulfilled. Then any non-degenerate benevolent UB -process converges to an equilibrium.*

As a corollary of this theorem, applying Thom's theorems on density and openness of transversal sections, it seems possible to prove the following result on generic convergence of benevolent contractual processes to an equilibrium.

Corollary 6.3.3 *For almost all economies of C^2 -class, every benevolent contractual UB -process converges to equilibrium.*

Proof of Theorem 6.3.2. Theorem 6.3.1 and Corollary 6.3.2 can be applied in the conditions of this theorem. Thus, each limit point of a trajectory is an equilibrium. Further, we show that in conditions of Theorem 6.3.2 every benevolent trajectory can have only one limit point.

Assume contrary and let x, y be two *different* limit points of a trajectory. Let's consider a linear segment with the ends x, y , *i.e.*, the set $\{\gamma x + (1 - \gamma)y \mid 0 \leq \gamma \leq 1\}$. Across each point $z(\gamma) = \gamma x + (1 - \gamma)y$, $\gamma \in (0, 1)$ of the segment one can conduct a hyperplane so that points x, y are *strictly* in the different half-spaces. For example, such hyperplane $H(\gamma)$ can be conducted as a hyperplane that has $y - x$ as a vector of its normal. It is clear that in such a manner we can define a family of pairwise-not-crossed hyperplanes depending on parameter $\gamma \in (0, 1)$ such that our *two different* limit points of trajectory are placed *in two different* open half-spaces defined by $H(\gamma)$. Hence, when time elapses the trajectory crosses every hyperplane an infinite number of times and any limit point of these points of crossing is also a limit point of trajectory and, therefore, this is an equilibrium. Thus, the economy has a continuum of different equilibria, since for different γ we have limit points from different parallel hyperplanes. However, each regular economy has a finite number of equilibria. This contradiction proves that there is the only limit point and, hence, the benevolent UB -contractual process converges to an equilibrium. ■

6.4 Examples of contractual processes: convergence and cycling

In this section⁸ we consider some additional examples revealing the character of contractual processes. Additionally, computer programs and results of computer modeling are described.

⁸These examples and programs were constructed in collaboration with Sergey Kolbin, IV year student of MMF NSU (during 2005–2006 years I supervised his bachelor diploma).

6.4.1 The convergence of UB -processes in a 2×3 economy

In this item we describe a computer program simulating the proper contractual process for 2×3 economy and also one numerical example of work of this program is presented.

The model of the economy with 2 agents and 3 goods is considered, in which the preferences are defined by Cobb–Douglas functions in the logarithmic form:

$$u_1(x) = a_{11} \ln(x_1) + a_{12} \ln(x_2) + a_{13} \ln(x_3), \quad x \gg 0,$$

$$u_2(y) = a_{21} \ln(y_1) + a_{22} \ln(y_2) + a_{23} \ln(y_3), \quad y \gg 0.$$

At the start of the program the parameters of utility functions and initial endowments are also determined:

$$a = ((a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23})), \quad \mathbf{e} = ((\mathbf{e}_1^1, \mathbf{e}_1^2, \mathbf{e}_1^3), (\mathbf{e}_2^1, \mathbf{e}_2^2, \mathbf{e}_2^3)).$$

In addition, specific parameters: $step > 0$ (step) and $tochn > 0$ (closeness) are specified. Applying these data program finds equilibrium and constructs a sequence of allocations $(x, y)^{(0)}, (x, y)^{(1)}, \dots, (x, y)^{(n)}$ corresponding to the rules of the proper contractual process (this one can see from the algorithm), and represents the results graphically.

The only equilibrium is found from the following system of equations:

$$\mathbf{e}_1^1 + \mathbf{e}_2^1 = \bar{x}_1 + \bar{y}_1, \quad \mathbf{e}_1^2 + \mathbf{e}_2^2 = \bar{x}_2 + \bar{y}_2, \quad \mathbf{e}_1^3 + \mathbf{e}_2^3 = \bar{x}_3 + \bar{y}_3,$$

$$\nabla u_1(\bar{x}) \cdot \bar{x} = \nabla u_1(\bar{x}) \cdot \mathbf{e}_1, \quad \nabla u_1(\bar{x}) = \alpha \cdot \nabla u_2(\bar{y}), \quad \alpha > 0.$$

Schema of program algorithm:

0. Program finds equilibrium.

1. $(x, y)^{(0)} := \mathbf{e}$.

2. Program finds a mutually beneficial contract $v^{(n)} = (v_1, v_2)$ as follows:

(*) in cube $[-0.5 \times step, 0.5 \times step]^3$ a point v_1 is randomly chosen relative to uniform distribution and $v_2 := -v_1$ is defined. If contract $v = (v_1, v_2)$ is mutually beneficial (*i.e.* $u_1(x^{(n)} + v_1) > u_1(x^{(n)})$ and $u_2(y^{(n)} + v_2) > u_2(y^{(n)})$) then **item 3**; otherwise return to (*).

3. $(x, y)^{(n+1)} := (x, y)^{(n)} + v^{(n)}$.

If after signing of this contract the trajectory does not leave the limits of the maximal surface then **item 4**. Otherwise, this point is projected onto the surface: from a linear segment $[(x, y)^{(n)} + v, \mathbf{e}]$ a point inside area limited by the maximal surface is chosen at a distance no more $0.01 \times step$ from the maximal surface. Then:

$(x, y)^{(n+1)} :=$ the projected point.

4. Point $(x, y)^{(n+1)}$ becomes visible on the monitor, and point's parameters are written into the file.

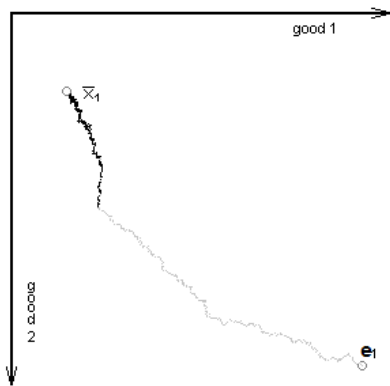


Figure 6.4.7: *Exchange dynamics between 1st and 2nd goods*

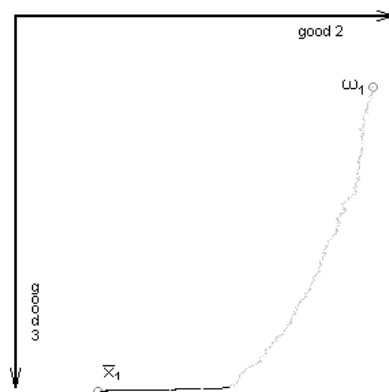


Figure 6.4.8: *Exchange dynamics between 2nd and 3rd goods*

5. If the distance from $(x, y)^{(n+1)}$ to equilibrium is more than $\text{step} \times \text{tochn}$ then **item 2**. Otherwise, we think trajectory arrives at equilibrium and program stops.

Table 6.4.1 reduces the results of the program work for the following numerical example. The preferences are defined by utility functions

$$u_1(x) = \ln(x_1) + \ln(x_2) + 9 \ln(x_3), \quad u_2(y) = 9 \ln(y_1) + 10 \ln(y_2) + \ln(y_3).$$

Initial endowments: $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2) = ((9, 9, 2), (1, 5, 8))$. Then (the only) equilibrium allocation is $(\bar{x}, \bar{y}) = ((1.761, 2.258, 9.454), (8.239, 11.742, 0.546))$.

In the program the parameters $\text{step}=0.1$, $\text{tochn}=1$ were given and one can see that a generated by the computer sequence of allocations $(x, y)^{(n)}$ “arrives” to equilibrium. Table 6.4.1 presents every 30th point of sequence $x^{(1)}, x^{(2)}, \dots, x^{(n)}$, *i.e.*, only the consumption of 1st agent is described: the amount of the first good consumed by the 1st agent at each 30th step forms the first column, second column contains analogous results for 2nd good and so on. The estimations of the contract break are specified in fourth column. These estimations are defined as a quotient of length of real progress of a trajectory to the length of signed contract, *i.e.*, written $\frac{\|(x, y)^{(n+1)} - (x, y)^{(n)}\|}{\|v^{(n)}\|}$ when the trajectory is moving by the maximal surface and it is applied “inside” if the current point is in the interior of area limited by maximal surface.

Graphically, the trajectory is presented as follows. Figure 6.4.7 represents $(x_1(t), x_2(t))$ for each step of the program work. The gray color is applied for the points inside the limits of the maximal surface, black—the moving by the maximal surface. Similarly, the dynamics of pairs $(x_2(t), x_3(t))$ and $(x_3(t), x_1(t))$ are represented in Figures 6.4.8 and 6.4.9, accordingly.

6.4.2 The absence of convergence for a 4×2 economy under assumptions CUB, IBA

Let us consider an economy with 4 individuals which utilities define a family of indifference curves as it is presented in Figure 6.4.11. At the starting time moment ($t = 0$) the current allocation coincides with the initial endowments: $x_1(0) = \mathbf{e}_1 = (9, 1)$, $x_2(0) = \mathbf{e}_2 = (3, 3)$, $x_3(0) = \mathbf{e}_3 = (3, 9)$, $x_4(0) = \mathbf{e}_4 = (3, 4)$. Further, let in a time

Table 6.4.1: *Proper contractual trajectory: computation of each 30th step for 1st agent*

1st good $x_1^{(k)}$	2nd good $x_2^{(k)}$	3rd good $x_3^{(k)}$	break degree: $\frac{\ (x,y)^{(k+1)} - (x,y)^{(k)}\ }{\ v^{(k)}\ }$
9	9	2	\mathbf{e}_1
9.002	8.956	2.048	inside
8.234	8.802	2.647	inside
7.455	8.353	3.469	inside
6.785	8.092	4.285	inside
6.044	7.681	5.148	inside
5.167	7.187	6.021	inside
4.638	6.673	6.825	inside
3.743	6.347	7.705	inside
3.261	5.454	8.669	inside
2.642	4.876	9.398	0.845
2.552	4.420	9.410	0.532
2.396	4.218	9.416	0.726
2.345	3.756	9.426	0.668
2.384	3.384	9.433	0.831
2.370	3.119	9.437	0.334
2.259	2.955	9.441	0.470
2.184	2.813	9.444	0.404
2.119	2.676	9.446	0.841
2.027	2.611	9.447	0.953
1.918	2.610	9.448	0.949
1.880	2.545	9.449	0.945
1.925	2.397	9.451	0.824
1.882	2.368	9.451	0.943
1.844	2.360	9.452	0.129
1.761	2.258	9.454	equilibrium

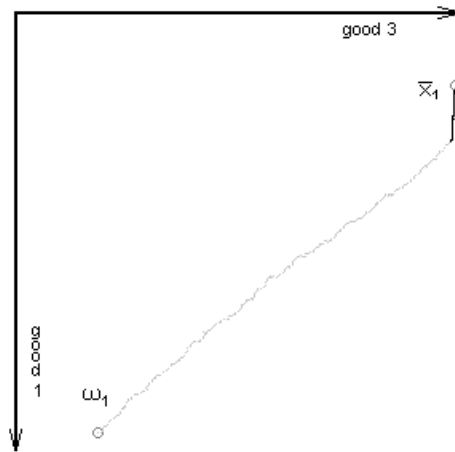


Figure 6.4.9: Exchange dynamics between 1st and 3rd goods

interval $(0, t_1)$ the coalition $\{1, 2\}$ be active, in an interval (t_1, t_2) the coalition $\{2, 3\}$ is active, $\{3, 4\}$ is active in limits (t_2, t_3) and in an interval (t_3, t_4) the coalition $\{4, 1\}$ is active. Let's assume also that in the nearest future the order of coalition activity is: $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}$ and that each time interval of activity is short enough. Suppose that for this interval of time a coalition has enough time only to break contracts signed in the past and have no time to sign a new contract. The specified modes of coalitions activity in time are visualized in Figure 6.4.10.

Let additionally the signing of intra-coalition contracts is realized in the following way. On the segment $(0, t_1)$ a contract $v^{\{1,2\}} = (\overrightarrow{e_1 A_1}, \overrightarrow{e_2 A_2})$ is signed and realized. The bundles of goods for the first and second individuals at the moment t_1 are A_1 and A_2 , accordingly. Further, on (t_1, t_2) a contract $v^{\{2,3\}} = (\overrightarrow{A_2 B_2}, \overrightarrow{e_3 A_3})$ is signed and realized by $\{2, 3\}$. Thus, the bundles of goods for 2nd and 3rd individuals at the moment t_2 are B_2 and A_3 , accordingly. Further, on an interval (t_2, t_3) a contract $v^{\{3,4\}} = (\overrightarrow{A_3 B_3}, \overrightarrow{e_4 A_4})$ is realized by $\{3, 4\}$. So, the bundles of 3rd and 4th at the moment of time t_3 are B_3 and A_4 , accordingly. At last, on an interval (t_3, t_4) a contract $v^{\{4,1\}} = (\overrightarrow{A_4 B_4}, \overrightarrow{A_1 B_1})$ is realized. In doing so the bundles of 4th and 1st at

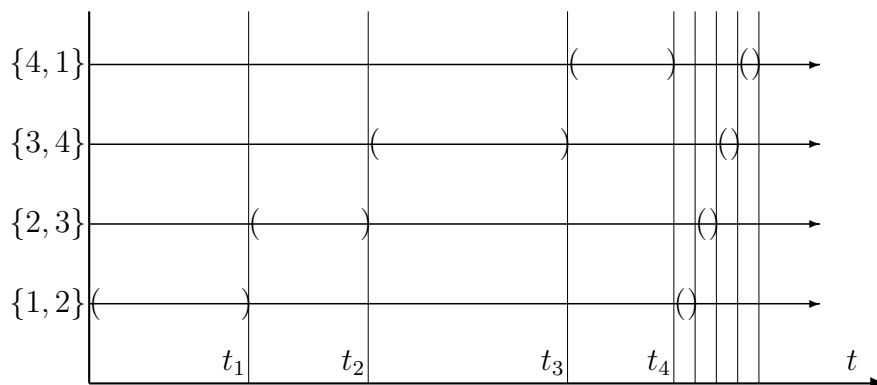


Figure 6.4.10: Activity order for bilateral coalitions in 4×2 economy

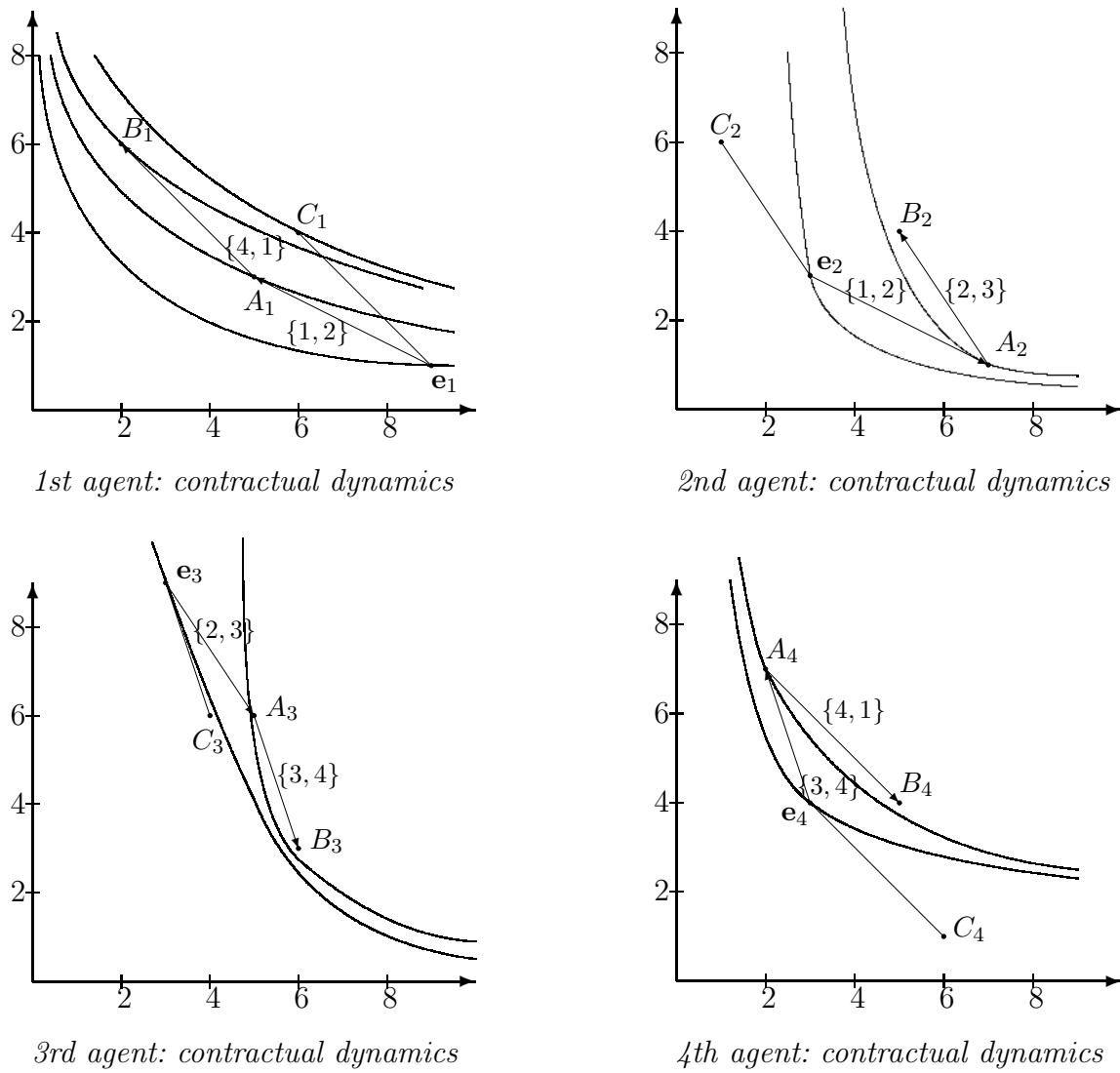


Figure 6.4.11: An example of 4×2 economy where CUB-process is cycling and does not converge to equilibrium

the moment t_4 become equal to B_4 and B_1 , accordingly. Thus, at the moment of time t_4 the commodity bundle of agent i corresponds to the point B_i in the figure, and the point A_i is the result of the first stage of barter exchange of the individual i : for each individual, it is realized on a time interval and with an appropriate partner. In each presented diagram the coalitions presented which have signed contract resulting from the specified consumption program. One can also see from the figures that all the above-stated contracts are mutually beneficial. Moreover, completing these diagrams if necessary it is possible to design appropriate indifference curves in such a manner that each signing of the new contract is carried out in accordance with principles of proper contractual process, wherein each stage of barter exchange the role of “new” endowments plays the sum of initial endowments with a flow of goods received from contracts signed him in *others* coalitions.

When all acts of described bilateral contracts are realized, at the moment t_4 the coalition $\{1, 2\}$ is active again. However, now the 1st agent finds it is favorable for him partially break contract $v^{\{1,2\}}$ with the 2nd agent and he/she completely breaks it raising his/her utility from consumption. As a result, new commodity bundles of 1st and 2nd agents are the vectors corresponding to the points C_1 and C_2 in the figure, accordingly. In so doing, since the interval of coalition activity is short, the agents have no time to sign a new contract. Next active coalition is $\{2, 3\}$. Now 2nd agent completely breaks the contract $v^{\{2,3\}}$ signed him in the past in coalition $\{2, 3\}$. Consumption bundles of 2nd and 3rd become e_2 and C_3 , accordingly. Then coalition $\{3, 4\}$ is next to be active. Now 3rd agent observes that it is favorable for him to break off the contract $v^{\{3,4\}}$. As a result, e_3 and C_4 become the consumption bundles of 3rd and 4th, accordingly. Finally, coalition $\{4, 1\}$ becomes an active one. However, once again there is an agent, now it is 4th, which desires to break off coalition contract $v^{\{4,1\}}$. The result of this is that e_4 and e_1 become 4th and 1st consumption bundles, accordingly.

Thus, one can see that the contractual trajectory has returned to the allocation of initial endowments and considered contractual process is cycled: the signing of the last contract by coalition $\{4, 1\}$ caused a breaking chain (like an avalanche) of all contracts! What can be said about this occasion? If the 1st individual were able to expect such development of events at a stage when he/she was signing a contract with 4th agent or if at once after signing this contract he/she would limit appetites and has refused to break the contract with 2nd agent, then the destruction of the contractual structure of economy did not occur. However, the behavior of such type should be clearly incorporated into the model of contractual process and this would mean essential modernization of our theoretical conceptions about proper-contractual processes.

6.4.3 Absence of convergence for 3×2 economy and *UB*-process with a piecewise-continuous trade rule

This example demonstrates the importance of assumption on continuity of a trade rule applied in the above analysis of proper-contractual *UB*-processes.

Let us consider an economy with 3 agents and 2 goods. In the model an ini-

tial resources allocation $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and Cobb–Douglas utility functions in the logarithmic form are defined:

$$\begin{aligned} u_1(x) &= a_{11} \ln(x_1) + a_{12} \ln(x_2), \quad x \gg 0, \\ u_2(y) &= a_{21} \ln(y_1) + a_{22} \ln(y_2), \quad y \gg 0, \\ u_3(z) &= a_{31} \ln(z_1) + a_{32} \ln(z_2), \quad z \gg 0. \end{aligned}$$

Being defined model parameters, one can find a sole equilibrium $\bar{\zeta} = (\bar{x}, \bar{y}, \bar{z})$ from the following system of equations:

$$\begin{aligned} \mathbf{e}_1^1 + \mathbf{e}_2^1 + \mathbf{e}_3^1 &= \bar{x}_1 + \bar{y}_1 + \bar{z}_1, \quad \mathbf{e}_1^2 + \mathbf{e}_2^2 + \mathbf{e}_3^2 = \bar{x}_2 + \bar{y}_2 + \bar{z}_2, \\ \nabla u_1(\bar{x}) &= \alpha \cdot \nabla u_2(\bar{y}) = \beta \cdot \nabla u_3(\bar{z}), \quad \alpha > 0, \quad \beta > 0, \\ \nabla u_1(\bar{x}) \cdot \bar{x} &= \nabla u_1(\bar{x}) \cdot \mathbf{e}_1, \quad \nabla u_2(\bar{y}) \cdot \bar{y} = \nabla u_2(\bar{y}) \cdot \mathbf{e}_2. \end{aligned}$$

Further, we will describe the program simulating a proper contractual process for a 3×2 economy for the specified trade rule

$$w : \mathcal{A}(X) \rightarrow \mathfrak{L}^c = \{(v_1, v_2, v_3) \in \mathbb{R}^6 \mid v_1 + v_2 + v_3 = 0\}$$

and then a numerical example of this program realization will be also considered.

At the start of the program, there are determined the parameters of utility functions, initial endowments, and an unambiguously defined trade rule. The specific parameter $step > 0$ is also determined. Applying these data program finds an equilibrium, and constructs a sequence of allocations $\zeta^{(1)}, \zeta^{(2)}, \dots, \zeta^{(n)}$ (here $\zeta = (x, y, z)$) corresponding to the rules of the proper contractual process (this follows from the algorithm), and represents the results graphically.

Schema of program algorithm:

0. The program finds equilibrium.
1. $\zeta^{(0)} := \mathbf{e}$.
2. Program finds a mutually beneficial contract $w(\zeta^{(n)}) = (w_1, w_2, w_3)$ in accordance with trade rule.
3. $\zeta^{(n+1)} := \zeta^{(n)} + w(\zeta^{(n)})$.

If after signing of this contract the trajectory does not leave the limits of the maximal surface, then **item 4**. Otherwise, the projection of this point onto the maximal surface is realized: from a linear segment $[\zeta^{(n)} + w(\zeta^{(n)}), \mathbf{e}]$ a point inside area limited by the maximal surface is chosen at a distance no more $0.01 \times step$ from the maximal surface. Then:

$$\zeta^{(n+1)} := \text{the projected point.}$$

4. The point is visualized onto the monitor and its parameters are written into the file.
5. If the distance from $\zeta^{(n+1)}$ to equilibrium is more than $0.01 \times step$ then **item 2**. Otherwise, we think that the trajectory arrives at equilibrium and the program stops.

Further, we study the numerical example. Let preferences be defined on $\text{int}\mathbb{R}_+^2$ via the following utility functions:

$$u_1(x) = 47 \ln(x_1) + 23 \ln(x_2), \quad u_2(y) = 18 \ln(y_1) + 54 \ln(y_2), \quad u_3(z) = 17 \ln(z_1) + 21 \ln(z_2).$$

Initial endowments are: $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = ((1, 4), (15, 3), (2, 5))$. Let us also determine the following *piecewise-continuous* trade rule $w : \mathcal{A}(X) \rightarrow \mathfrak{L}^c$ which is different in each of the following areas A_1, A_2 dividing the set of all feasible allocations into two parts. In each area there is a constraint only on $x \gg 0$, and variables $y \gg 0$ and $z \gg 0$ may take any values. Define

$$A_1 = \{\zeta = (x, y, z) \geq 0 \mid x_2 \leq -0.1463x_1 + 2.6968\},$$

$$A_2 = \{\zeta = (x, y, z) \geq 0 \mid x_2 > -0.1463x_1 + 2.6968\}$$

and denote $w(\zeta) = w'(\zeta)$ when $\zeta \in A_1$ and $w(\zeta) = w''(\zeta)$ for $\zeta \in A_2$.

Further, let us formally define for the area A_1 a trade rule $w' : A_1(X) \rightarrow \mathfrak{L}^c$ via the following algorithm (in A_2 the rule $w''(\zeta)$ is similarly defined).

First, for each couple of individuals $i, j, i < j$ we construct an auxiliary vector $v^{ij}(\zeta)$ specifying some mutually beneficial exchange for the current allocation ζ . For each couple of agents, this vector is constructed by some common rule; we describe it, for example, for a pair $\{1, 2\}$. Let us define

$$g(\zeta) = \nabla u_1(x) + \nabla u_2(y)$$

and consider the orthogonal complement to $g(\zeta)$ in \mathbb{R}^2 , *i.e.* a straight line defined via equation $\langle \chi, g(\zeta) \rangle = 0$, $\chi = (\chi_1, \chi_2) \in \mathbb{R}^2$. It follows from the definition $g(\zeta)$ that if gradients are *non-collinear* then for the point of the line the values $\langle \chi, \nabla u_1(x) \rangle$ and $\langle \chi, \nabla u_2(y) \rangle$ are nonzero and have a *different sign*; for collinear gradients, they are equal to zero. Further consider a directional vector of the line, *e.g.* it may be $(\frac{-g_2}{g_1+g_2}, -\frac{g_1}{g_1+g_2}) = \bar{\chi}$ (it defines the line in a parametrial form $\chi = \gamma \bar{\chi}$, $\gamma \in \mathbb{R}$). At last define $v^{12}(\zeta) = \bar{\chi}$ if $\langle \bar{\chi}, \nabla u_1(x) \rangle > 0$ and put $v^{12}(\zeta) = -\bar{\chi}$ for $\langle \bar{\chi}, \nabla u_1(x) \rangle < 0$. Clearly that in this way we correctly define a continuous map $\zeta \rightarrow v^{12}(\zeta)$ which obeys the condition that contract $(v^{12}(\zeta), -v^{12}(\zeta))$ is mutually beneficial in the non-strict form:

$$\langle v^{12}(\zeta), \nabla u_1(x) \rangle \geq 0, \quad \langle -v^{12}(\zeta), \nabla u_2(y) \rangle \geq 0.$$

Moreover, if gradients are *non-collinear*, then these inequalities are realized in *strict* form.

Further, let us define a trade rule having appropriate properties and constructed via described maps $v^{ij}(\zeta)$.

For area A_1 , the rule is defined by the formula

$$w'(x, y, z) = \beta(\zeta) \left(v^{12}(x, y) + \frac{v^{13}(x, z)}{20}, -v^{12}(x, y) + \frac{v^{23}(y, z)}{20}, \frac{-v^{13}(x, z) - v^{23}(y, z)}{20} \right),$$

and for A_2 by formula

$$w''(x, y, z) = \beta(\zeta) \left(\frac{v^{12}(x, y) + v^{13}(x, z)}{20}, v^{23}(y, z) - \frac{v^{12}(x, y)}{20}, -v^{23}(y, z) - \frac{v^{13}(x, z)}{20} \right),$$

where β is a scalar parameter chosen in an appropriate way that continuously depends on current allocation $\zeta = (x, y, z)$.

Next, let us show that vectors in brackets of the last expressions present mutually beneficial contracts. The fact that they are contracts is checked directly. They are also (non-strict) the mutually beneficial contracts because for $i < j$ by construction $v^{ij}(\zeta)$ we have

$$\langle \nabla u_i(\zeta), v^{ij}(\zeta) \rangle \geq 0, \quad \langle \nabla u_j(\zeta), -v^{ij}(\zeta) \rangle \geq 0,$$

that summing appropriate inequalities gives $\langle \nabla u_i(\zeta), w'_i(\zeta) \rangle \geq 0$, $i = 1, 2, 3$ and an analogous thing for $w''(\zeta)$ can be obtained. Moreover, if for the current allocation there is at least *one couple of individuals, whose gradients are non-collinear*, then *all* these inequalities have to be *strict*, *i.e.* they are really mutually beneficial contracts.

At last, if we shall manage to find parameter β so that to change the length of a vector specifying in our rule the “direction” of exchange (barter proportions) then we can receive the increase of individual utilities as a result of contract $w(\zeta)$ signing; *i.e.*, we need to do so that

$$u_1(x + w_1(x, y, z)) > u_1(x), \quad u_2(y + w_2(x, y, z)) > u_2(y), \quad u_3(z + w_3(x, y, z)) > u_3(z) \quad (6.4.1)$$

be true for an appropriate area of definition.

Further, we describe a method allowing us to find parameter β for $w'(\zeta)$ (for $w''(\zeta)$ it is done analogously). Define

$$\beta(\zeta) = \frac{1}{2} \min\{b_1, b_2, b_3\},$$

where once again $b_i = \min\{\text{step}, c_i\}$, $i = 1, 2, 3$ for the values c_i which are found in the following way:

$i = 1$, then c_1 is a solution of equation $u_1(x + c_1(v^{12}(x, y) + \frac{v^{13}(x, z)}{20})) = u_1(x)$ if it is solvable one; otherwise $c_1 = +\infty$.

$i = 2$, then c_2 is a solution of equation $u_2(y + c_2(-v^{12}(x, y) + \frac{v^{23}(y, z)}{20})) = u_2(y)$ if it is solvable one; otherwise $c_2 = +\infty$.

$i = 3$, then c_3 is a solution of equation $u_3(z + c_3(\frac{-v^{13}(x, z) - v^{23}(y, z)}{20})) = u_3(z)$ if it is solvable one; otherwise $c_3 = +\infty$.

It is easy to see that variables $b_i(\zeta) > 0$ are correctly determined and are the continuous functions of its argument (formally one needs to apply the implicit function theorem and use strict concavity of utilities). Hence, $\beta(\zeta) > 0$ is also a continuous function. It has to be also clear from construction that (6.4.1) is fulfilled that finishes the description of a trade rule. In conclusion, we note only that program parameter $\text{step} > 0$ appearing in construction of $b_i(\zeta)$ is used not only to define a necessary value (to do it one can take any positive number) but also to adjust the “length” of trajectory moving along a vector specifying the exchange proportions. Thus, reducing parameter $\text{step} > 0$, we approach the described discrete process to the theoretical continuous process.

For the given example, the contractual trajectory constructed by the program does not converge to equilibrium. It is visible from the following Figure 6.4.12 in which the points $\zeta^{(n)} = (x^{(n)}, y^{(n)}, z^{(n)})$ of a proper-contractual trajectory are depicted. It is

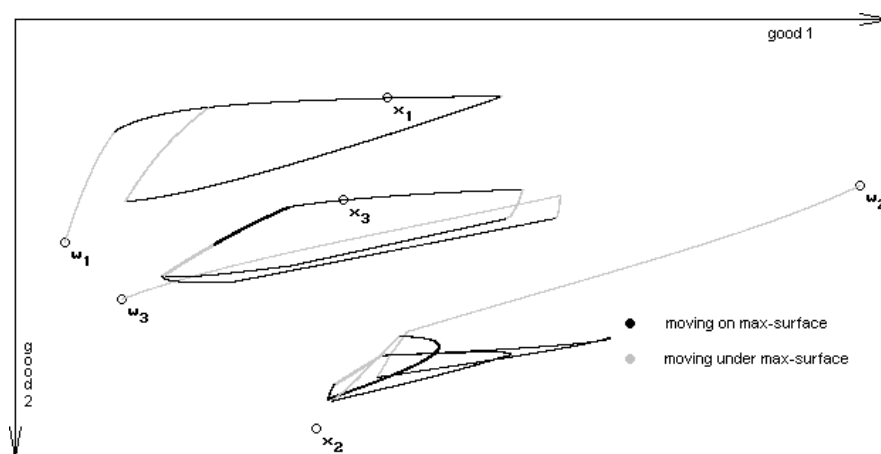


Figure 6.4.12: *Dynamics of proper contractual process with **piecewise** continuous trading rule in 3×2 economy*

curious to note, that if one applies any of rules $w'(\zeta)$, $w''(\zeta)$ as a rule for *the whole* set of feasible allocations $\mathcal{A}(X)$ then in our economy the proper-contractual UB -process is converging in the computer sense.

Conclusion to Chapter 6

Two important illustrative examples of the economy with two individuals and two products open the Chapter study. These examples demonstrate the nature and reveal the specifics of the contractual process. Further, the presented analysis of the convergence of contractual trajectories has given the following results:

- For the economies with 2 individuals and 2 commodities convergence of proper contractual processes relative to any continuous trading rule has proven under rather general assumptions (Theorem 6.1.1). Local stability of equilibria was also investigated and a reasonable criterion for this was suggested.
- The special type of *benevolent* rules of trade stood out as rules which determine a new contract allowing the break of gross barter contract only if being realized *every* new mutually beneficial contract involves contracts' breaking (Definition 6.2.1). Just for this class of benevolent processes, the basic positive results on convergence were received.
- The theorem on convergence to equilibrium of non-degenerate benevolent UB -contractual processes has been proven (Theorems 6.3.1, 6.3.2). In addition, local stability of equilibria relative to benevolent trading rules was analyzed; however appropriate theorem was proven only for 2 agents economy with an arbitrary finite commodity space (Proposition 6.2.1).
- A series of model examples are presented that demonstrate specific properties of contractual processes in different cases. There are examples for converged

processes and also two examples where the contractual process is cycling. The last has theoretical implications. First of them presents economy with 4 agents and 2 commodities and coalitional contractual process (CUB). The second example is designed for an economy with 3 agents UB -process with a *piecewise continuous* trading rule is cycling (continuity of trading rules is one of necessary assumption).

Part III
Information

Originally, model structures and solution concepts (equilibria, core *etc.*) adequately considering incompleteness and asymmetry in the individual information have been developed in the 1970's of the last century. The information is modeled as a partition of (future) states of nature that defines the space of contingent commodities—goods that are consumed in different states of the world are considered as different ones in spite of the fact that they are identical according to their physical characteristics. The economic life is going on in this space—trade and exchanges of commodities are carried out, and actual consumption of contingent goods is realized. Individual information imposes restriction on the structure of possible contracts—contracts as mappings should be measurable according to the individual information partitions of elementary states of the world. Some new specific equilibrium concepts appear: Walrasian expectations, rational expectations equilibrium, *etc.* Also, new concepts of the core were elaborated, depending on the hypotheses that are accepted for the informational interchange and for making of collective decisions. The best known among others are the coarse core, the fine core and the private core. See the next sections for more details on the description of an economy with the differential information and for the solution concepts studied in its context.

In this Part economies with differential information are studied. The potential possibility to extend the contract-based approach to the *DIE*-economies seems quite natural. Indeed, Radner, one of founders of the theory of *DIE*-economies, already used the term “contract” (in the description of the equilibrium concepts) approximately in the same manner as it is done here, see [Radner \(1982\)](#). Certainly, the requirements for the contracts’ *admissibility* for a *DIE*-economy should include their measurability relative to suitable partition of elementary states space. However, this partition differs in different core and equilibrium concepts. Moreover, the *DIE*-economies theory so far does not demonstrate appropriate conformity between the (different) core and equilibrium concepts,¹ *i.e.*, the situation differs from the case of a complete market. However, the contractual approach provides a clear-cut possibility to establish this conformity. One can reason as in the standard case: for the webs realizing core allocations, in addition to their common properties, it is enough to require their stability relative to the partial contracts’ breaking. Further, one has to suggest a price characterization for the obtained allocations. In this way some new concepts of equilibrium can be discovered.

¹Moreover, the widely known *WEE* and *REE* equilibrium concepts are criticized in literature, *e.g.* see *Preface* in [DI-economies \(2005\)](#).

Chapter 7

Information as a proper part of mathematical model

Information plays a crucial role in the economic analysis. This refers not only to the individual decision-making, but also to the functioning of the economy as a whole. The information plays an important role in resources allocation realized through the system of markets. However, in a classical economic modeling, the information and its distribution among agents were not considered in a proper way. For example, in the context of the Arrow–Debreu–McKenzie model, it is implicitly supposed that economic life proceeds in a separately taken time period and agents possess an enough amount of information on economic variables to make rational decisions, and transactions are carried out during infinitesimal time periods, *etc.*¹ (*e.g.* see [Mas-Colell et al. \(1995\)](#)). However, in the real world, individuals are forced to make decisions under uncertainty and a shortage of information and, moreover, they are asymmetrically informed on states of the world.² Unfortunately classical economic theory does not pay enough attention to the issue of information asymmetry.

Originally, model structures and solution concepts (equilibria, core *etc.*) adequately considering incompleteness and asymmetry in the individual information have been developed in the 1970s of the last century. In particular, in these models the space of contingent commodities started to be considered: goods that are consumed in different states of the world are considered as different ones in spite of the fact that they are identical according to their physical characteristics. The information is modeled as a partition of (future) states of nature and relays with the space of contingent commodities. The economic life is going on this space—trade and exchanges of commodities are carried out, and actual consumption of contingent goods is realized. Individual information imposes restriction on the structure of possible contracts—contracts as mappings should be measurable according to the individual information partitions of elementary states of the world. Some new specific equilibrium concepts appear: Walrasian expectations, rational expectations equilibrium, *etc.* Also, new concepts of the core were elaborated, depending on the hypotheses that are accepted

¹There are the expanded treatments of Arrow–Debreu model admitting asymmetrically informed agents (*e.g.* see [Radner \(1982\)](#)), however they also are not quite satisfactory.

²They are uncertainly defined factors influencing economic indicators and agents' welfare.

for the informational interchange and for making of collective decisions. The coarse, fine and private cores are the most known among others. The model of economy with the differential information and solution concepts studied in its context is described in details in the following sections.

DIE-economies are investigated in this part of monography by means of the contract based approach. The idea to extend the contract-based approach to the *DIE*-economies seems quite natural. Indeed, Radner, one of founders of the theory of *DIE*-economies, already used the term “contract” (in the description of the equilibrium concepts) approximately in the same manner as it was done below, see [Radner \(1982\)](#). Certainly, the requirements of the contracts’ *admissibility* for a *DIE*-economy should include their measurability relative to suitable partition of elementary states space. However, this partition differs in different core and equilibrium concepts. Moreover, the *DIE*-economies theory so far does not demonstrate appropriate conformity between the (different) core and equilibrium concepts,³ *i.e.* the situation differs from the case of a complete market. However, the contractual approach provides a clear-cut possibility to establish this conformity. One can reason as in the standard case (see [Marakulin \(2003, 2011\)](#)): for the webs realizing core allocations, besides their common properties, it is enough to require their stability relative to the partial contracts’ breaking. Further, one has to suggest a price characterization for the obtained allocations. This approach may be useful in discovering new concepts of equilibrium and perhaps the most correct one can be found among others.

A model of an economy with differential information was first clearly formulated in [Radner \(1968\)](#) where also an appropriate generalization of Walrasian equilibrium was introduced (Walrasian expectations equilibrium—*WEE*). Moreover, the existence theorem of this equilibrium is also proved in the paper. Further, seminal paper [Wilson \(1978\)](#) appeared, in which specific concepts of coarse and fine core are introduced, several interesting specific examples are studied, and original concepts of equilibrium are offered (they did not receive names in the modern literature, see [Glycopantis, Yannelis \(2005\)](#)). Wilson’s paper attracted the attention of economists and had a profound impact on the development of economic theory in the context of models with differentiated information. It is necessary to mention another work of Radner ([Radner, 1978](#)), which supposes that individuals are capable of extracting the information from prices distribution and introduces the important concept of rational expectations equilibrium (*REE*). This concept is specific and postulates that consumers maximize *conditional* expected utility⁴ with the account of initial and additional information provided by prices. In the subsequent research by many authors, the theory of information-differentiated economy was developed in many directions. Different core concepts were specified and studied (see section below), the existence and relationships of different equilibrium concepts were studied, important issues of incentive compatibility were analyzed⁵ and of implementation of core allocations as

³Moreover, the widely known *WEE* and *REE* equilibrium concepts are criticized in literature, *e.g.* see *Preface* in [DI-economies \(2005\)](#).

⁴This basically differs the concept from the *WEE*-equilibrium introduced in [Radner \(1968\)](#) that also is called Radner equilibrium (not to confuse with *REE*).

⁵This means the absence of the revealed motives for agents to misinform other agents about states

an equilibrium in some strategic game (specified via initial model), and also a number of other directions were considered; the most complete surveys of the literature are presented in *DI-economies* (2005), *Economic Theory 18* (2001), *Schwalbe* (1999).

In Wilson's paper, a concept of communication system (non-formalized) appears as a tool, transferring the information from one agent to another. In the subsequent works of Allen (1991a, 1994) this idea is generalized and formalized in the form of an information rule: this is a mapping which transforms the information of members of a coalition in the form that can be applied to yield a coalition allocation that can dominate current allocation. Thus for the economy with the information asymmetry Allen introduces a general method to define a core—applying different rules in a context of the same model (and, therefore, different measurability requirements for dominating allocation) one can obtain almost all concepts known in the literature (coarse, fine, private and so on). In *Schwalbe* (1999) Allen's approach is developed further and a concept of *maximal information* is introduced: the largest information that an economic agent can receive being a member of all possible coalitions he/she enters with the initial information. The measurability concerning the maximum of information is required for an allocation to be feasible in the economy as a whole. At the same time, coalitions can dominate, as Allen assumed, only through intra-coalition allocation measurable concerning the information received by a rule from the initial one. However, for both authors it is not clear what actually occurs with the information—participating in different coalitions and, extracting a new information and applying it to form a total allocation, agents forget everything and try to dominate it using transformed initial information again. There is something defective in this view on (intra-model) economic life. According to the author, this approach is not quite satisfactory. However, can one offer something constructive instead of this?

In real life every allocation is a result of exchange transactions between economic agents. Moreover, information interchange is going on also, and not every transaction (exchange) can be realized, one of the reasons is the information shortcoming. Here *contractual approach* and its views can work very fruitfully, one will see it below. For the model with asymmetric information, the admissibility should include the requirement of measurability determined by the distribution of the information among economic agents. Besides, contractual approach assumes certain dynamics of exchange processes, during which an information interchange can occur. It is natural to suppose that information distribution in the economic environment is non-uniform, non-static, and changes eventually during economic interactions between agents. However, what may result from information exchanges? In our opinion, it may be so-called *limit information* introduced in *Marakulin* (2009). The limit information is a result of an information exchange realized by a chain of economic interactions between individuals in the framework of an intra-coalition (barter) commodity exchange and the ongoing information interchange (definitions below). As examples show, generally different “chains” can lead to different distributions of the information that cannot be improved in subsequent exchange operations. It is proved in *Marakulin* (2009) that for the *monotonic rule of information sharing* the limit information is unique and this means

of the world—potentially there is such possibility, because an agent can know a state precisely (after its realization) but others are not capable to distinguish it from (some) other states.

that in this case the concept becomes correct and can be effectively applied to study *DIE*-economies.

7.1 Differential information economic model

In this section, the simplest model of an economy with differential (asymmetric) information is described. In the model framework, it is possible to consider some key issues of the economic theory: the concepts of equilibria and various definitions of the core and their implementation as well as some important existence results are provided as well.

The information can be considered from two points. On the one hand, the information has some properties of the goods (like physical) and thereby it is similar to other market goods: it can be bought, sold or exchanged on the information markets. On the other hand, the information can be individualized and considered as the characteristic of the agent similar to his/her initial endowments and preferences. Our model takes into account both aspects of the information. For the model described both points can be applied, but anyway we follow the second way: information is a constituent part of agents' characteristics.

7.1.1 Agents and their information

Let us consider an exchange economy with a finite set of agents $\mathcal{I} = \{1, 2, \dots, n\}$. The specific feature of the economy is explicitly introduced information on the states of the world (nature), that agents have. Generally, different agents possess different information and their information can vary during agents' economic activities.

The information is modeled as follows. Let us consider measurable space $(\Omega, P(\Omega))$ of the events of nature. In the general case, $P(\Omega)$ is an algebra of subsets-events¹. However, for further analysis, we really need a finite space of elementary events, this is why we suppose for simplicity that Ω is a *finite* set (P is derived from "partition") and will apply the term "partition" instead of "algebra" or "field" (that now is equivalent among each to other). The elements $a, b, c \in \Omega$ are called states of nature (world) or elementary events and $\mathbf{w} \in \Omega$ denotes a typical state of nature. Here $P(\Omega)$ is a partition² of Ω . Partition P_i is associated with the i th agent information. If the information element of partition consists of several states of nature then it means that the agent *is not able to distinguish* these states. This is a way to describe the ability of an agent to distinguish events.

Let $P_i(\mathbf{w})$ denote an element of P_i that includes the state $\mathbf{w} \in \Omega$. It is said that the information P is *finer than (better of)* information P' if each element P is a subset of some element of P' , *i.e.*, $P(\mathbf{w}) \subseteq P'(\mathbf{w}), \forall \mathbf{w} \in \Omega$. So, one information is finer than another one if it is capable to better distinguish elementary events of nature. On

¹That usually is denoted as \mathcal{A} ; in other sources one also speaks about the *field of events*, denoting it as \mathcal{F} . This terminology is usually applied for an infinite set of Ω .

²The *partition* of a set is a set of pairwise disjoint nonempty subsets such that their union gives the whole set.

the set of all partitions of Ω , the relation “*finer*” defines a partial ordering and it is denoted as

$$P \succeq P' \iff P \text{ finer than } P' \iff P' \text{ coarser than } P.$$

On the set of all partitions, \succeq defines the lattice structure, *i.e.*, any finite set of partitions has supremum and infimum.

The ordered set (train) of the individualized partitions $\mathbb{P} = (P_i)_{i \in \mathcal{I}}$ is called the **information structure of the economy**.

The relation \succeq for information partitions induces a partial ordering on a set of all information structures that is defined by a rule:

$$\mathbb{P} \succeq \mathbb{P}' \iff P_i \succeq P'_i, \quad \forall i \in \mathcal{I}: \quad \mathbb{P} = (P_i) \quad \& \quad \mathbb{P}' = (P'_i).$$

The relation \succeq is applied also for the coalition information structures—for comparison of different information provisions of some coalition.

*Information is called **perfect** or **complete** if it includes only one-element subsets of Ω . An information structure of the economy is called **perfect** if the information of each agent is perfect.*

*An information structure $\mathbb{P} = (P_i)_{i \in \mathcal{I}}$ is called **asymmetric**, if $P_i \neq P_j$ for some $i, j \in \mathcal{I}, i \neq j$.*

In further analysis, we will assume without loss of generality that

$$\bigcap_{i \in \mathcal{I}} P_i(\mathbf{w}) = \{\mathbf{w}\} \quad \forall \mathbf{w} \in \Omega \quad \iff \quad \bigvee_{i \in \mathcal{I}} P_i = \Omega^*, \quad (7.1.1)$$

i.e., the supremum of individual information forms a perfect information Ω^* (otherwise one can identify indistinguishable states), setting an appropriate equivalence relation, and then passing to the consideration of equivalence classes as a new set of elementary events.

• **Commodities and consumption plans.** Let there be l physically different commodities (goods) in the economy. Thus the space of the physical commodities is l -dimensional Euclidian space \mathbb{R}^l . Individuals can potentially consume different bundles of physical (contingent) commodities in different events of the world. The space of contingent commodities is $L = (\mathbb{R}^l)^\Omega = \mathfrak{M}(\Omega, \mathbb{R}^l)$, this is the set of all maps from the space of elementary events Ω in \mathbb{R}^l .

Let P be some partition of Ω . Function f with domain Ω is called *P -measurable*, if it is a constant³ on the elements of P . For a partition P specify the set

$$\mathfrak{M}_P(\Omega, \mathbb{R}^l) := \{f : \Omega \rightarrow \mathbb{R}^l \mid f|_{P(\mathbf{w})} = \text{const}\},$$

³For a finite algebra of events this definition is equivalent to the standard one.

this is a subspace of $\mathfrak{M}(\Omega, \mathbb{R}^l)$ of all P -measurable functions. Space $\mathfrak{M}_P(\Omega, \mathbb{R}^l)$ for $P = P_i$ is the set of *information admissible* consumption bundles of the individual i and this is a basic feature of an economy with differentiated information.

Information may be considered as a component of the consumption bundle. Thus uncertainty is modeled in the following way: the goods, besides a place and time of availability, are specified also by a state of the world. As soon as the physical goods are considered in the interrelation with the states of the world, it is natural to postulate that space of the admissible consumption bundles depends on the information. Thereupon *the generalized space of the goods*

$$\mathfrak{M}(\Omega, \mathbb{R}^l) \times \mathfrak{P}$$

arises (\mathfrak{P} denotes the set of all partitions). In this space, the goods are presented ordered pairs, consisting of state contingent commodities and information. Thus the information becomes a part of goods definition. To illustrate this idea, let us consider the following example.

Example 7.1.1 Let there be two states of nature: a —agent i is infected and b —agent i is not infected. There are two variants of the information: $P_i = \{\{a\}, \{b\}\}$ and $P'_i = \{\{a, b\}\}$; in the first case, the agent distinguishes states, and in the second one does not. Assume that there is a medicine that can cure the illness of the agent. It seems obvious that the medicine with the information P_i is not the same commodity as for the information P'_i . ■

So, initial endowments $(\mathbf{e}_i, P_i^0) \in \mathfrak{M}(\Omega, \mathbb{R}^l) \times \mathfrak{P}$ for each individual $i \in \mathcal{I}$ consist of the endowments of goods $\mathbf{e}_i \in \mathfrak{M}(\Omega, \mathbb{R}^l)$ and his/her initial stock of the information P_i^0 (partition of Ω). For simplicity, let's assume that individuals are able to consume only non-negative quantities of physical goods and that $\mathbf{e}_i \geq 0, \forall i \in \mathcal{I}$.

The consumption set of the agent $i \in \mathcal{I}$ is defined by the following:

$$\mathbb{X}_i := \{(x, P) \in \mathfrak{M}(\Omega, \mathbb{R}_+^l) \times \mathfrak{P} \mid x - \mathbf{e}_i \in \mathfrak{M}_P(\Omega, \mathbb{R}^l)\}.$$

Obviously that $(\mathbf{e}_i, P_i^0) \in \mathbb{X}_i$ since the contract $\mathbf{e}_i - \mathbf{e}_i = 0$ is compatible with any information. Simplifying notations, we specify:

$$X_i := \mathfrak{M}(\Omega, \mathbb{R}_+^l) \quad i \in \mathcal{I}.$$

Thus consumption plan of the agent i is represented by the ordered couple (x_i, P_i) , where the first component $x_i \in \mathfrak{M}(\Omega, \mathbb{R}_+^l)$ is a bundle of contingent commodities and the second one is an information of the agent. The consumption set of the agent consists of all consumption plans that are realized by a web of contracts compatible with the information P_i . Once again notice that agents cannot consume the goods independently of their information. That is if an agent does not distinguish two states of the world and if his/her consumption plan realized by a contract is such that it is different in these states of nature, then it is not placed in the agent's consumption set.

A train consisting of consumption plans of agents $((x_i, P_i))_{i \in \mathcal{I}}$ where $(x_i, P_i) \in \mathbb{X}_i$ is called an *allocation*.

7.1.2 Information sharing rule. Limit information

Nonempty subsets of \mathcal{I} , *i.e.* groups of agents, are called *coalitions*. To simplify reasonings, let us presume that all coalitions are permissible⁴ and let $C := 2^{\mathcal{I}} \setminus \{\emptyset\}$. It is supposed that members of a coalition are able to exchange their own experience, that is, to change their individual information. Redistributions of agents' initial endowments are carried out on the basis of this modified information. Let's consider the process of *informational exchange* among the agents who entered into a coalition.

In the most general form, an informational interchange is described by the information rule that is one of the model parameters. For every coalition and its set of (private) information, the information rule puts into correspondence a new set of information, formally this is defined as follows:

Information rule for a coalition $S \subseteq \mathcal{I}$ is a mapping $k_S : \mathfrak{P}^S \rightarrow \mathfrak{P}^S$ for each one-element set $S = \{i\}$ satisfying $k_{\{i\}}(P) = P$, $\forall i \in \mathcal{I}$, $P \in \mathfrak{P}$.

Information rule for an economy is $(2^{|\mathcal{I}|} - 1)$ -tuple of mappings $k = (k_S)_{S \in C}$ where to each coalition S there corresponds the proper rule k_S .

Substantially it is possible to express in such a manner. A member of a coalition $i \in S$ with the information P_i after joining to the coalition S has an access to the information $P'_i = k_S^i((P_j)_{j \in S})$, this is the i th component (projection) of mapping k_S calculated at the "point" $P_S = (P_i)_{i \in S}$. Exactly this information the individual i can use within the limits of coalition activity. For singleton coalitions, information change does not occur.

So, the result of an information rule application is new information that agents can use for the redistributions of initial endowments within the limits of the coalition.

A map $k = (k_S)_{S \in C}$ is called an **information-sharing rule** if $k_S((P_i)_{i \in S}) \succeq (P_i)_{i \in S}$ for all $(P_i)_{i \in \mathcal{I}}$, $\forall S \in C$.

Now let us consider some known in the literature examples that actually explain the concept of an information rule and also they are applied in the different core concepts:

Example 7.1.2 Coarse information rule $k^c := (k_S^c)_{S \in C}$ is defined as follows: $k_S^c((P_i)_{i \in S}) = (P_S^c)_{i \in S}$ where $P_S^c := \bigwedge_S P_i$ for any coalition S . Here $\bigwedge_S P_i$ denotes the finest partition received from the bundle of the information $(P_i)_{i \in S}$ such that for any P_i each element of P_i is a subset of some element from partition $\bigwedge_S P_i$. Thus, $\bigwedge_S P_i$ is infimum of partitions P_i , $i \in S$. ■

This rule describes a situation where members of a coalition *are unable to communicate* at all, and this rule is not a *sharing information* rule.

In this case, each agent should understand the exchange transaction as a whole, *i.e.*, all agents should distinguish significant events for the transaction as for themselves and so for partners—such rules are. One can notice that this does not mean

⁴Without specific troubles all obtained results can be transited to the case of a bounded set of permissible coalitions.

that agents have to forget their initial information to join a coalition. However, it can happen so that agents are unable to use more of their initial information at the transaction conclusion. Notice that the coarse information rule is symmetric, but is neither dense nor bounded.⁵

Example 7.1.3 *Fine* information rule $k^f := (k_S^f)_{S \in C}$ is defined as follows: $k_S^f((P_i)_{i \in S}) = (P_S^f)_{i \in S}$, where $P_S^f := \bigvee_{i \in S} P_i$ for any coalition S and $\bigvee_S P_i$ is the supremum of the information partitions: $(\bigvee_S P_i)(\mathbf{w}) := \bigcap_{i \in S} P_i(\mathbf{w}) \forall \mathbf{w} \in \Omega$. ■

This is a rule such that members of a coalition can *exchange completely* the information that they have. This information is obtained as follows: take the information of each member of a coalition and construct the coarsest partition that is finer than each individual one. In this case, information that coalition members are applied for a redistribution of resources is not worse than its individual information.

Certainly, particular cases of an information rule are the rule where information interchange does not occur at all (this is the *private* rule) and zero rule where irrespective of the initial information the information $P_i = \{\Omega\}$ is attributed to each member of a coalition.

Finally, one may note that information exchange can certainly depend on not only individual information of coalition members but also a current allocation, *i.e.* current bundles of consumed contingent commodities. Moreover, it seems natural to think that the only incentive to share information is the possibility (hope) to sign a new mutually beneficial contract for new information endowments. To formally describe the extended treatment of informational rule it is enough to change the *domain* for mappings k_S , $S \in C$ and thinks they are defined on the sets $\mathbb{X}_S = \prod_{i \in S} \mathbb{X}_i$ instead of $\mathfrak{P}_S = \mathfrak{P}^S$, *i.e.* one has to assume

$$k_S : \mathbb{X}_S \rightarrow \mathfrak{P}^S, \quad \forall S \in C.$$

This extended treatment of the informational rule will be applied in further analysis.

- **Information produced by the rule.** Initial informational structure of an economy jointly with transforming information rule both capable to produce secondary information structures which play an important role in a subsequent analysis. Thereby, applying information rules and probably several times coarse, fine, or also maximal and limit informational structures are produced. Let us further consider the notion of maximal information ([Schwalbe, 1999](#)).

Definition 7.1.1 (SCHWALBE 1999) *An information $P_i^{max} := \bigvee_{S \in C} k_S^i((P_j^0)_{j \in S})$ is called **maximal information** of agent $i \in \mathcal{I}$ relatively an informational rule $k = (k_S)_{S \in C}$.*

⁵In accordance with current terminology, a rule of information sharing is called **dense**, if $k_T^i((P_j)_{j \in T}) \succeq k_S^i((P_j)_{j \in S})$ for each $i \in \mathcal{I}$ and all $S, T \subseteq \mathcal{I}$ such that $i \in S \subset T$. This means that if a coalition increases in size, then the information of each member can only be improved. Information rule is called **bounded** if $k_T^i((P_j)_{j \in T}) \succeq k_S^i((P_j)_{j \in S})$ for each $i \in \mathcal{I}$ and all $S \subseteq T$.

Thus, the maximal information is such information the agent would receive that if he/she could join all coalitions simultaneously. However, each individual enters a coalition activity with the initial information. The case realized looks paradoxical; however, probably maximal information has to be interpreted as potentially and initially possible amount of information that can be collected from informational endowments.

The author suggested another point of view in (Marakulin, 2009) where the concept of limit information was introduced.

Definition 7.1.2 (MARAKULIN 2007) *An information is called **limit information** if a sequence of coalitions exists such that the information formed at the last stage of information exchange cannot further be changed (unimprovable for each agent).*

Note that the concept of limit information implies that, in each act of information sharing, agents can use the information that they have received earlier from preceding exchanges. Thus, information can be accumulated, and agents can learn.⁶

Formally, limit information $\mathbb{P}^{lim} = (P_i^{lim})_{i \in \mathcal{I}}$ is defined by a (finite) chain of coalitions $S_1, S_2, \dots, S_m \subseteq \mathcal{I}$, so that:

- (i) $P_i^\xi = k_{S_\xi}^i((P_j^{\xi-1})_{j \in S_\xi})$, $i \in S_\xi$, $P_i^\xi = P_i^{\xi-1}$, $i \in \mathcal{I} \setminus S_\xi$, $\xi = 1, 2, \dots, m$;
- (ii) $\forall S \subseteq \mathcal{I}$, $\forall i \in S$, $k_S^i((P_j^m)_{j \in S}) = P_i^m = P_i^{lim}$.

In general, limit information can depend on its implementing coalition chain. However, for a *monotonic information sharing* rule, it is uniquely defined (Marakulin, 2009). This notion is of a high theoretical interest, which is why we discuss it now in more detail.

One can get an illusion that maximal and limit information are differently defined but the same object. However, Example 7.1.4 described below illustrates it is not true and uniqueness not always takes place. Besides, Example 7.1.4, along with Example 7.1.5, shows that limit information may be finer and may be coarser than maximal information.⁷

Example 7.1.4 Consider 4 states of nature $\Omega = \{a, b, c, d\}$ and 3 agents with their initial information:

$$P_1^0 = \{a, b, c, d\}, \quad P_2^0 = \{\{a, b\}, \{c, d\}\}, \quad P_3^0 = \{\{a, c\}, \{b, d\}\}.$$

Numbers of coalitions: $S_1 = \{1, 2\}$, $S_2 = \{1, 3\}$, $S_3 = \{2, 3\}$, $S_4 = \{1, 2, 3\}$.

The rule of sharing information for a coalition $S \subseteq \mathcal{I}$: *In the coalition S the act of sharing information occurs if S contains an agent and there is an element $E \subset \Omega$ of his/her informational partition such that **any other** member of the coalition has an element of his/her partition that includes the **whole** event E . Then*

⁶This is rather “eductive learning” and “rational learning” than “adaptive learning”.

⁷The rule of information sharing and examples have been constructed by Anna Kadyrova in her diploma in 2003.

all agents of coalition S become able to distinguish the event E and, if possible, the next act of information sharing occurs (within the iteration), etc. If these conditions are not satisfied, then information is not exchanged.

Note that if an individual is able to distinguish events $E \subset \Omega$, $E' \subset \Omega$, and, in addition, $E \subset E'$, then this individual is also able to distinguish the event $E' \setminus E$.

Informally, the described rule implies the following. Suppose that there exists a member of a coalition S that realizes that he/she is able to distinguish an event E while the rest of members S are able to distinguish broader events than E . Then this agent initiates information sharing. The agent transmits the ability to distinguish these events to the rest of the agents. Transmitting his/her knowledge to the others, this agent expects to receive additional information from them, immediately or, possibly, in the subsequent iterations of information sharing, being a member of other coalitions.

Let us consider how 1st agent information can be transformed. Suppose that at first stage agent 1 entered the coalition $S_1 = \{1, 2\}$. Then his/her information P_1^1 (the superscript is the iterations number) equals $\{\{a, b\}, \{c, d\}\}$. Obviously, this information cannot be improved in the course of any subsequent intra-coalition exchange and, hence, it is the limit.

Consider another possible sequence of coalitions. Suppose that at first agent 1 joined the coalition $S_2 = \{1, 3\}$. Then his/her information P_1^1 is assigned the value $\{\{a, c\}, \{b, d\}\}$. Again, this information is limit.

Finally, note that $P_1^{max} = \{\{a\}, \{c\}, \{b\}, \{d\}\}$, i.e., in this case the maximal information of agent 1 is finer than any version of its limit information. ■

The following example shows that limit information may be finer than maximal information.

Example 7.1.5 Consider 8 states of nature $\Omega = \{a, b, c, d, e, f, g, h\}$ and 3 agents with their initial information:

$$\begin{aligned} P_1^0 &= \{\{b, c\}, \{a, d, f\}, \{e, g, h\}\}, \\ P_2^0 &= \{\{a, b\}, \{c, d, e, f\}, \{g, h\}\}, \\ P_3^0 &= \{\{a, b, c, h\}, \{d, e\}, \{f, g\}\}. \end{aligned}$$

Suppose that the rule of sharing information is the same as in the previous example and coalitions are denoted as earlier.

Then consider maximal information \mathbb{P}^{max} . By definition, for agent 1 we have:

$$P_1^{max} = \bigvee_{S \ni 1} k_S^1(P^0) = k_{S_1}^1(P_1^0, P_2^0) \vee k_{S_2}^1(P_1^0, P_3^0) \vee k_{S_4}^1(P_1^0, P_2^0, P_3^0),$$

where

$$\begin{aligned} k_{S_1}^1(P_1^0, P_2^0) &= \{\{b, c\}, \{a, d, f\}, \{e\}, \{g, h\}\}, \\ k_{S_2}^1(P_1^0, P_3^0) &= \{\{b, c\}, \{a, d, f\}, \{e, g, h\}\}, \\ k_{S_4}^1(P_1^0, P_2^0, P_3^0) &= \{\{b, c\}, \{a, d, f\}, \{e, g, h\}\}. \end{aligned}$$

As a result, we have:

$$P_1^{max} = \{\{b, c\}, \{a, d, f\}, \{e\}, \{g, h\}\}.$$

Next we find out a possible version of limit information. Suppose that at first the coalition $S_1 = \{1, 2\}$ is “active”. One obtains:

$$\begin{aligned} P_1^1 &= k_{S_1}^1(P_1^0, P_2^0) = \{\{b, c\}, \{a, d, f\}, \{e\}, \{g, h\}\}, \\ P_2^1 &= k_{S_1}^2(P_1^0, P_2^0) = \{\{a, b\}, \{c, d, f\}, \{e\}, \{g, h\}\}, \\ P_3^1 &= P_3^0 = \{\{a, b, c, h\}, \{d, e\}, \{f, g\}\}. \end{aligned}$$

Suppose that next the coalition $S_2 = \{1, 3\}$ is “active”. We have:

$$\begin{aligned} P_1^2 &= k_{S_2}^1(P_1^1, P_3^1) = \{\{a, f\}, \{b, c\}, \{d\}, \{e\}, \{g, h\}\}, \\ P_2^2 &= P_2^1 = \{\{a, b\}, \{c, d, f\}, \{e\}, \{g, h\}\}, \\ P_3^2 &= k_{S_2}^3(P_1^1, P_3^1) = \{\{a, h\}, \{b, c\}, \{d\}, \{e\}, \{f, g\}\}. \end{aligned}$$

Suppose that the coalition $S_1 = \{1, 2\}$ is “active” again. We have:

$$\begin{aligned} P_1^3 &= k_{S_1}^1(P_1^2, P_2^2) = \{\{a, f\}, \{b, c\}, \{d\}, \{e\}, \{g, h\}\}, \\ P_2^3 &= k_{S_1}^2(P_1^2, P_2^2) = \{\{a, b\}, \{c, f\}, \{d\}, \{e\}, \{g, h\}\}, \\ P_3^3 &= P_3^2 = \{\{a, h\}, \{b, c\}, \{d\}, \{e\}, \{f, g\}\}. \end{aligned}$$

It is easily seen that any further process of intra-coalition information sharing cannot continue. As a result, for agent 1 we have:

$$P_1^{lim} = \{\{b, c\}, \{a, f\}, \{d\}, \{e\}, \{g, h\}\}.$$

Thus, the limit information P_1^{lim} of agent 1 is finer than his/her maximal information P_1^{max} . Therefore, in this example limit information is coarser than the finest information (*i.e.*, complete information) and is finer than maximal information, *i.e.*, lies between them. ■

In order to apply the concept of limit information in a well-defined way, we need to reveal conditions of uniqueness, *i.e.*, the conditions under which it does not depend on how coalitions in the sequence of information transactions are ordered. Below, we formulate a sufficient condition for the uniqueness of limit information. It turns out that, for each *monotone* rule of information sharing, limit information is unique.

Definition 7.1.3 An information rule $k = (k_S)_{S \in C}$ is **monotone** if, for each coalition $S \in C$ and any informational structures, from $\mathbb{P} \succeq \mathbb{P}'$ it follows that $k_S(\mathbb{P}) \succeq k_S(\mathbb{P}')$.

In other words, an information rule is a monotone if, for each coalition, it preserves the natural *partial* order relation on the set of all informational structures. Here it is appropriate to note that a non-monotone rule needs not to have a property that, for some $\mathbb{P} \succ \mathbb{P}'$ and some coalition S , $k_S(\mathbb{P}) \prec k_S(\mathbb{P}')$ is true. The relation \succeq is only a *partial* ordering on the set of informational structures, *i.e.*, there are non-comparable structures. If a rule is non-monotone for some coalition S and a pair of informational structures, $\mathbb{P} \succ \mathbb{P}'$ & $k_S(\mathbb{P}) \not\preceq k_S(\mathbb{P}')$ is true. This is not equivalent to the previous statement. This situation is illustrated by the rule from Example 7.1.4, where $\mathbb{P}' = k_{S_1}(\mathbb{P}^0) \succ \mathbb{P}^0$ but both $k_{S_2}(\mathbb{P}') \not\preceq k_{S_2}(\mathbb{P}^0)$ and $k_{S_2}(\mathbb{P}') \not\preceq k_{S_2}(\mathbb{P}^0)$ hold, *i.e.*, the structures that are obtained by the rule of information sharing are non-comparable.

Theorem 7.1.1 (ON UNIQUENESS) If a rule of **information sharing** $k \in \mathfrak{R}$ is **monotone**, then the limit informational structure \mathbb{P}^{lim} obtained by the rule k is a unique one.

Proof of Theorem 7.1.1. Let $\alpha = \{S_1, S_2, \dots, S_r\}$ and $\beta = \{T_1, T_2, \dots, T_q\}$ be two sequences of coalitions, which generate, in accordance with Definition 7.1.2, two versions of limit information, \mathbb{P}_α^{lim} and \mathbb{P}_β^{lim} . We will show that $\mathbb{P}_\alpha^{lim} = \mathbb{P}_\beta^{lim}$.

Indeed, by Definition 7.1.2, for any rule of information sharing, we have

$$\mathbb{P}^0 \preceq k_{S_1}(\mathbb{P}^0) = \mathbb{P}_\alpha^1 \preceq k_{S_2}(\mathbb{P}_\alpha^1) = \mathbb{P}_\alpha^2 \preceq \dots \preceq k_{S_r}(\mathbb{P}_\alpha^{r-1}) = \mathbb{P}_\alpha^r = \mathbb{P}_\alpha^{lim} \Rightarrow \mathbb{P}^0 \preceq \mathbb{P}_\alpha^{lim}.$$

We sequentially apply the monotonicity of the rule and the definition of limit information. By doing so, we conclude that

$$\mathbb{P}_\beta^1 = k_{T_1}(\mathbb{P}^0) \preceq k_{T_1}(\mathbb{P}_\alpha^{lim}) = \mathbb{P}_\alpha^{lim} \Rightarrow \mathbb{P}_\beta^2 = k_{T_2}(\mathbb{P}_\beta^1) \preceq k_{T_2}(\mathbb{P}_\alpha^{lim}) = \mathbb{P}_\alpha^{lim} \dots \Rightarrow$$

$$\mathbb{P}_\beta^{q-1} = k_{T_{q-1}}(\mathbb{P}_\beta^{q-2}) \preceq k_{T_{q-1}}(\mathbb{P}_\alpha^{lim}) = \mathbb{P}_\alpha^{lim} \Rightarrow \mathbb{P}_\beta^{lim} \preceq \mathbb{P}_\alpha^{lim}.$$

By using similar reasoning, first with respect to the sequence β and then with respect to α , we find out that $\mathbb{P}_\beta^{lim} \succeq \mathbb{P}_\alpha^{lim}$. Hence, $\mathbb{P}_\beta^{lim} = \mathbb{P}_\alpha^{lim}$. Theorem 7.1.1 is proved. ■

Corollary 7.1.1 *If a rule of information sharing $k \in \mathfrak{R}$ is monotone, then the limit informational structure \mathbb{P}^{lim} is finer than the maximal one, i.e., $\mathbb{P}^{lim} \succeq \mathbb{P}^{max}$.*

Proof of Corollary 7.1.1. It is sufficient to show that, for any coalition $S \subseteq \mathcal{I}$ and any $i \in \mathcal{I}$, $k_S^i((P_j^0)_{j \in S}) \preceq \mathbb{P}_i^{lim}$ holds. However, this is the case, since if the coalition S is added to the beginning of any sequence of coalitions implementing limit information, then the new chain also implements limit information. At the same time, as the rule of information sharing is repeatedly applied, information can become only finer at each stage and it automatically becomes limit after the last stage. Further, in order to complete this proof, we need to use Definition 7.1.1:

$$P_i^{max} := \bigvee_{S \in \mathcal{C}} k_S^i((P_j^0)_{j \in S}) \preceq \mathbb{P}_i^{lim}.$$

■

Remark 7.1.1 Presented in this section results can be generalized in at least two important ways:

First, the measurable space (an algebra of events) can be infinite: the corresponding result can be found in [Marakulin \(2009\)](#).

Second, the result of Theorem 7.1.1 can be strengthened assuming that in the chain of (monotone) coalition rules realizing a limit information the rules that themselves are also changing are applied, but again in a monotone way. Here it is monotonicity with respect to the partial ordering specified for the information sharing rules. For this case Theorem 7.1.1, might serve as a corollary of a more general result. ■

7.1.3 Preferences

For a general and detailed Arrow–Debreu model, the preferences of individuals should be defined for their consumption plans. However, for the setting considered, information is part of the consumption, which requires a correct interpretation. Indeed,

being a part of the consumption bundle, information performs the admissibility: information itself is unable to generate utility, but also nobody can consume information incompatible commodity bundle. In the general preference is presented as a point-to-set mapping $\mathcal{P}_i : \mathbb{X}_i \Rightarrow \mathbb{X}_i$, whereas above $\mathcal{P}_i(x, P)$ is the set of all plans (x', P') strictly preferred to the plan (x, P) , that habitually is written as

$$(x', P') \in \mathcal{P}_i(x, P) \iff (x', P') \succ_i (x, P),$$

where functions $x'(\cdot) - \mathbf{e}_i(\cdot)$ and $x(\cdot) - \mathbf{e}_i(\cdot)$ are (required) measurable with respect to partitions P' and P respectively. Note that the measurability restrictions are imposed for the deviations of consumption bundles (contract), in accordance with the definition of consumption sets.

One can assume further for simplicity, that preferences are defined via utility functions $u_i : \mathbb{X}_i \rightarrow \mathbb{R}$, $i \in \mathcal{I}$, where consumption sets \mathbb{X}_i have been specified above. It is supposed usually that utilities are invariant relative to information:

$$u_i(x, P) = u_i(x, P'), \quad \forall (x, P), (x, P') \in \mathbb{X}_i,$$

that can be interpreted as the fact that the information itself does not generate any utility.⁸ This assumption means that there is a utility $u_i(x) = u_i(x, \Omega^*)$, defined on $(\mathbb{R}_+^l)^\Omega$ such that

$$u_i(x, P) = u_i(x, \Omega^*), \quad \forall (x, P) \in \mathbb{X}_i.$$

Thus if a partition P is *fixed* then the set of all P -measurable functions forms a subspace in $(\mathbb{R}^l)^\Omega$ and formally utility function $u_i(\cdot, P)$ can be defined as a reduction of known function $u_i(\cdot, \Omega^*)$. In this sense, the utility does not depend on information. This allows us to consider a point-to-set mapping $\mathcal{P}_i(\cdot)$, $i \in \mathcal{I}$ as being defined on $X_i = \mathfrak{M}(\Omega, \mathbb{R}_+^l)$, and moreover further we will use the same notation.

In the known literature there are widely used utilities defined via so-called *randomized* a posteriori utility functions $u_i : \Omega \times \mathbb{R}_+^l \rightarrow \mathbb{R}$, where $u_i(\mathbf{w}, x_i(\mathbf{w}))$ is a “value” of i 's utility in a state ‘ \mathbf{w} ’ for the consumption plan $x_i(\mathbf{w})$. In that case, the expected utility is used (ex-ante expected utility), calculated by the formula⁹

$$u_i(x_i) = \sum_{\mathbf{w} \in \Omega} u_i(\mathbf{w}, x_i(\mathbf{w})) q_i(\mathbf{w}),$$

where $q_i(\mathbf{w})$ is defined in the model an (individualized) prior,¹⁰ *i.e.* a priori defined probability of a state \mathbf{w} realization. It traditionally happened because of the need to apply conditional expectation of utility: for the introduction and studies of such concepts as an equilibrium in rational expectations (*REE*) and incentive compatibility—stability relative to potentially misreported states of the world. On the other hand, it is clear that one possible manage with this functional form of utility without its (expected) probabilistic specification. Moreover, this is no accident: as we will see this is the only correct way to model preferences for differentiated information.

⁸Again, it is important to remember that the flow $x - \mathbf{e}_i$ has to be measurable relative to both partitions P and P' .

⁹In current literature, they are also referred to as “Bayesian utilities”.

¹⁰In general $q_i(\cdot)$ is the density of a priori probability distribution (prior).

Indeed, the point is that for a realized (or not) event E preferences should not be directly connected with preferences for $\mathbf{w} \notin E$. Here I speak about preferences as an ordinal category, instead of utility, which of course will depend on consumption in $\mathbf{w} \notin E$. This means that preferences of the individual can be defined, for example, by means of a generalized-separable function having the following functional form:

$$u_i(x_i(\cdot)) = \psi_i(\varphi_1^i(x_i(\mathbf{w}_1)), \varphi_2^i(x_i(\mathbf{w}_2)), \dots, \varphi_k^i(x_i(\mathbf{w}_k))), \quad (7.1.2)$$

where $\Omega = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$. For a correct presentation it is necessary to assume that function $\psi_i(\cdot)$ is continuous and strictly monotonically increasing by each of arguments, and $\varphi_{\mathbf{w}}^i(\cdot)$ are ordinary quasi-concave functions setting preferences. It is obvious, that for any fixed consumption outside of any (only one!) elementary event $\mathbf{w} \in \Omega$ (or for an event $E \in P_i$) any function of the presented form induces identical preferences on the space of contingent commodities reduced to the state \mathbf{w} . This corresponds to the notion of *weakly* separable utility.

Strongly separable utility has the following functional form:

$$u_i(x_i(\cdot)) = \psi_i(\varphi_1^i(x_i(\mathbf{w}_1)) + \varphi_2^i(x_i(\mathbf{w}_2)) + \dots + \varphi_k^i(x_i(\mathbf{w}_k))), \quad (7.1.3)$$

$$\Omega = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}.$$

The essential difference is that for every set $E \subset \Omega$ of elementary events for contingent commodities, related with states from E , there is inducted a preference relation that does not depend on current consumption in external for E states $\mathbf{w} \in \Omega \setminus E$. For the proof of this result see [Barten, Bohm \(1982\)](#), [Debreu \(1960\)](#). Moreover, the requirements for a general case are clear also: it is necessary that consumption in events exterior (distinct from) to any set of elementary events do not influence preferences within the limits of the given event. Formally in the general case of strong separability, it should be

$$\begin{aligned} \Pr_E \left[\mathcal{P}(x_{|E}, y_{|\Omega \setminus E}) \cap ((x_{|E}, y_{|\Omega \setminus E}) + \mathcal{L}^E) \right] &= \Pr_E \left[\mathcal{P}(x_{|E}, z_{|\Omega \setminus E}) \cap ((x_{|E}, z_{|\Omega \setminus E}) + \mathcal{L}^E) \right] \\ &\forall \text{ measurable } y_{|\Omega \setminus E}, z_{|\Omega \setminus E} : \Omega \setminus E \rightarrow \mathbb{R}_+^l, \end{aligned} \quad (7.1.4)$$

where $\Pr_E[\cdot]$ is a projecting operator onto the subspace of functions defined on $E \subset \Omega$. This can be defined as a (pointwise) multiplication of a function from L on characteristic function χ^E of set $E \subseteq \Omega$.¹¹ Probably the latter property is easier to perceive in terms of binary relations:

$$\begin{aligned} \forall E \subset \Omega, \quad \forall \text{ measurable } x_{|E}, \xi_{|E} : E \rightarrow \mathbb{R}_+^l \\ (x_{|E}, y_{|\Omega \setminus E}) \succ (\xi_{|E}, y_{|\Omega \setminus E}) \iff (x_{|E}, z_{|\Omega \setminus E}) \succ (\xi_{|E}, z_{|\Omega \setminus E}) \\ \forall \text{ measurable } y_{|\Omega \setminus E}, z_{|\Omega \setminus E} : \Omega \setminus E \rightarrow \mathbb{R}_+^l. \end{aligned} \quad (7.1.5)$$

Of course, the separable utilities is a very strong requirement (especially for the strong separability), and it would be too simplistic and incorrect to assume it in

¹¹It is defined as $\chi^E(\mathbf{w}) = 1$ for $\mathbf{w} \in E$ and zero otherwise.

many other theoretical constructs. However, we are dealing with a model where a comparative analysis is often carried out at the level of individual events, and their combination (ie information) can dynamically be changed over time and hence the set-event may be changed. Moreover, in this context, the separability over the events of nature is appropriate and does not cause substantial model criticism. Simultaneously, the analysis provides a specific mathematical comfort and simplifies the presentation of meaningful concepts.

In this sequel, it is postulated that the preferences are strongly separable (in terms of utility functions one assumes form (7.1.2)).

(Sep) For each $i \in \mathcal{I}$ preferences $\mathcal{P}_i : X_i \Rightarrow X_i$ are **strongly separable**, i.e. it obeys (7.1.5) and for every $x_i \in X_i$ there is an open convex $G_i \subset L$ such that $\mathcal{P}_i(x_i) = G_i \cap X_i$ and $x_i \in \overline{\mathcal{P}_i(x_i)} \setminus \mathcal{P}_i(x_i)$.

Assume additionally

(NS) Preferences have extremely non-satiated consumption bundles for every possible elementary market, i.e., for every allocation $x = (x_i)_{\mathcal{I}} \in X = L_+^{\mathcal{I}}$ and every $E \subseteq \Omega$ does exist a bundle $z^E = z^E(x) \in \mathbb{R}^l$, such that

$$x_i + z^E \chi^E \in \mathcal{P}_i(x_i) \quad \forall i \in \mathcal{I}.$$

These assumptions can be compared with similar requirements of Part I: The assumption **(A)** of Chapters 1, 2 pages 19, 67 and the assumption **(S)** of Chapter 3 on page 125. Notice that in view of **(Sep)** assumption **(NS)** implies the existence of $z^E \in \text{ri}\mathcal{L}_E^+$ such that $x_i + \lambda z^E \chi^E \in \mathcal{P}_i(x_i) \forall i \in \mathcal{I}$ for some real $\varepsilon > 0$ and every $0 < \lambda < \varepsilon$. Clearly, that for monotonic preferences **(NS)** is always true. It should be emphasized that **(NS)** is consistent with the processes of information sharing since it can be applied to any kind of allocations that may appear in the contractual process with a simultaneously going sharing of information. In terms of utilities (7.1.2), assumption **(NS)** in particular means (not only!), that functions $\varphi_{\mathbf{w}}^i : \mathbb{R}_+^l \rightarrow \mathbb{R}$ are quasi-concave and locally non-satiated on \mathbb{R}_+^l , $\forall \mathbf{w} \in \Omega$, $\forall i \in \mathcal{I}$.

If preferences can be presented by differentiable utility functions, then as usual we are talking about the smooth model: economy \mathcal{E}^{di} is called **smooth** if for each $i \in \mathcal{I}$

$$\mathcal{P}_i(x_i) = \{y \in X_i \mid u_i(y) > u_i(x_i)\} \quad \forall x_i \in X_i$$

for a differentiable function u_i , defined on an open neighborhood of X_i , and such that $\nabla u_i(x_i) \neq 0 \forall x_i \in X_i$ holds.

Finishing this section, let us turn again to the randomized utility functions. Due to Debreu's theorem (Debreu, 1960) every complete continuous preorder on $(\mathbb{R}_+^l)^\Omega$, which obeys **(Sep)**, can be defined via summation of utility functions determined for the "multipliers" \mathbb{R}_+^l . These functions should be taken as randomized utility $u_i : \Omega \times \mathbb{R}_+^l \rightarrow \mathbb{R}$, where $u_i(\mathbf{w}, x_i(\mathbf{w}))$ is the "value" of utility in a state \mathbf{w} for the plan $x_i(\mathbf{w})$. Obtained resulting total utility can always be interpreted as the (ex-ante)

expected utility. Renormalizing the result one can also write a formula form

$$u_i(x_i) = \sum_{\mathbf{w} \in \Omega} u_i(\mathbf{w}, x_i(\mathbf{w})) q_i(\mathbf{w}). \quad (7.1.6)$$

Here $q_i(\mathbf{w})$ is (as in many papers) given initially in the model (individualized) a priori probability of state $\mathbf{w} \in \Omega$ ¹². It is correct to assume

$$\forall i \in \mathcal{I} \quad \sum_{\mathbf{w} \in E} q_i(\mathbf{w}) > 0, \quad \forall E \in \mathcal{P}_i$$

(or even postulate $q_i(\mathbf{w}) > 0, \forall \mathbf{w} \in \Omega$ and for each i), that corresponds agents' preferences are non-satiated for every distinctive event of the world. Of course, the resulting probability $q_i(\cdot)$ is a matter of interpretation and a tribute to tradition. However, it is important that this way does not lose in generality.

Thus, the subject of this study is the model of exchange economy with differential information, which can be briefly expressed as follows:

$$\mathcal{E}^{di} = \langle \mathcal{I}, \mathbb{R}^l, \Omega, (k_S(\cdot))_{S \in C}, (X_i, \mathcal{P}_i, P_i, \mathbf{e}_i)_{i \in \mathcal{I}} \rangle.$$

7.2 Allocations, core and equilibria

Definition of allocation has a great value for the consideration of the core of an economy with differentiated information. Certainly, feasible allocations of an economy should be at least physically admissible. However, if the economy with asymmetric information is considered then the admissibility definition should take into account also the agents' initial information and the information rule that operates in the model.

In the literature, there are two well-known definitions of feasible allocation. In the first one (applied by Yannelis et al.), an allocation is called feasible if it is compatible with the initial information of individuals. Thus, by Yannelis, the ability of information exchange between agents is not accepted into account. The second one appeared in the works of Allen (see, *e.g.*, Allen (1991a,b)) where the ability of an information exchange is taken into consideration. By Allen allocations measurable relative to the information accessible to all members of the economy are considered as feasible, *i.e.*, it is the measurability relative to coarsest information. These definitions are not free from shortcomings. Yannelis' definition is poorly motivated by a substantial party: it is intuitively clear that feasibility of allocation should somehow depend on the information rule, the informational world is not static. Definition by Allen is excessively rigid, that can entail emptiness of the core and also it can happen that agents cannot use any information at all and only initial endowments become (single) feasible allocation. Being motivated by these reasons, Schwalbe ([Schwalbe, 1999](#)) introduces the concept of *maximal information* that is applied to him further to define feasible allocation (see Section 7.1.2, Definition 7.1.1). Thus, the maximal

¹²In general $q_i(\cdot)$ is the density of the a priori probability distribution (prior).

information is such information that would be received by the agent if he/she would be able to join to all possible coalitions simultaneously. However, each individual enters a coalition activity with the initial information. Another point of view was suggested in Marakulin (2009) (see Section 7.1.2, Definition 7.1.2), where the concept of *limit information* was introduced.

7.2.1 Allocations

Information induces a particular type of allocation used in the model to specify a relevant core concept (equilibrium). In the most general form, we arrive at the following definition.

Definition 7.2.1 An allocation $((x_i, P_i))_{i \in \mathcal{I}}$ is called **feasible** if the maps $x_i : \Omega \rightarrow \mathbb{R}_+^l$ and the partitions P_i , $i \in \mathcal{I}$ obey:

$$(i) \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i,$$

$$(ii) (x_i - \mathbf{e}_i) : \Omega \rightarrow \mathbb{R}^l \text{ is measurable relative to } P_i^{eco} \forall i \in \mathcal{I}.$$

Here the first condition reflects physical feasibility of the allocation, while the second one—its informational feasibility since it requires the measurability of gross contract (net trade) realizing the allocation. Notice that it should be specially stipulated about what information P_i^{eco} in item (ii) is applied. In particular, P_i^{eco} can coincide with P_i^{max} (Schwalbe), P_i^0 (Yannelis), $\bigwedge_{j \in \mathcal{I}} P_j$ (Allen) or even P_i^{lim} (Marakulin).

7.2.2 Core

A core concept in an economy with differentiable information has own its specificity and, as opposed to the complete market, is generally not unequivocally defined. This is so because, besides the physical feasibility of an allocation it is necessary to consider its information feasibility, *i.e.*, its measurability concerning the information partition(s). Thus, it is necessary to define what kind of measurability is applied for a final allocation in an economy as a whole and what measurability requirements are applied for the intra-coalition allocations to dominate current resources allocation. Available in the literature (not all!) and other possible variants of these requirements are presented in Table 7.2.1: Here the first column presents requirements to allocation in an economy and the first row to intra-coalition allocations. The intersection of a column and a row presents the name of the core concept, where in the last row two new author concepts are presented. Resuming this table, one can notice that the approach using an information rule is the most general: commonly it is enough to specify a rule.

The non-emptiness of the core of any type, and thus, the validness of the proposed solutions is considered below and is solved in the theorem which generalizes the known Scarf's theorem. From the information point of view, the answer is the following: allocations under consideration would be measurable with respect to information structure finer than the information used to coalitional domination. Let us pass now to consider equilibrium concepts.

econ. \ coal.	$\wedge_S k_S^i(\mathbb{P})$	$\wedge_S P_i$	P_i	$\{k_S^i(\mathbb{P})\}$	P_i^{lim}	$\vee_S P_i$	$\vee_S k_S^i(\mathbb{P})$	author
$\wedge_{\mathcal{I}} P_i$		strong coarse	α			β		Yannelis
P_i		coarse	private			fine		Yannelis
$\vee_{\mathcal{I}} P_i$		γ	δ			weak fine		Yannelis
$\forall f$ -allocations		coarse				fine		Wilson
$\wedge_{\mathcal{I}} k_{\mathcal{I}}^i(\mathbb{P})$	coarse							Allen
$\{k_{\mathcal{I}}^i(\mathbb{P})\}$							fine	Allen
$\{P_i^{max}\}$				k-core				Schwalbe
$\{P_i^{lim}\}$				k-limit	limit			Marakulin

Table 7.2.1: *Known in literature variety of cores for DI-economies*

7.2.3 Equilibria

Consider first the historically introduced concept of Walrasian equilibrium presented in terms of expected utility. This is the so-called *WEE*-equilibrium (Walrasian Expectation Equilibrium), introduced in [Radner \(1968\)](#).

For the model, it is assumed in addition, the existence of a specific partition \mathcal{F} ¹³ of Ω such that $\mathcal{F} \succeq \bigvee_{i \in \mathcal{I}} P_i$, i.e. this partition is finer than any individual information partition. Further, the \mathcal{F} -measurable *system of the prices* $p : \Omega \rightarrow \mathbb{R}_+^l$ is considered that is used to define budget constraints. In view of the assumptions, all values can be defined and calculated: the initial endowments $\mathbf{e}_i(\cdot)$ are also assumed to be measurable with respect to the initial agent's information P_i .

Definition 7.2.2 (WEE-EQUILIBRIUM) *A couple (p, x) such that $x = (x_i)_{\mathcal{I}}$, $x_i \in X_i$, $i \in \mathcal{I}$ is called **Walrasian expectations or Radner equilibrium** if the following conditions are satisfied:*

- (i) $x_i : \Omega \rightarrow \mathbb{R}_+^l$ are P_i -measurable $\forall i \in \mathcal{I}$ and $p : \Omega \rightarrow \mathbb{R}_+^l$ is \mathcal{F} -measurable;
- (ii) x_i maximizes expected utility under informational and budget constraints:

$$[y_i : \Omega \rightarrow \mathbb{R}_+^l \text{ is } P_i \text{ - measurable}] \ \& \ \sum_{\mathbf{w} \in \Omega} y_i(\mathbf{w})p(\mathbf{w}) \leq \sum_{\mathbf{w} \in \Omega} \mathbf{e}_i(\mathbf{w})p(\mathbf{w}),$$

i.e. a value $\sum_{\mathbf{w} \in \Omega} u_i(\mathbf{w}, y_i(\mathbf{w}))q_i(\mathbf{w})$ is maximized for $y_i = x_i$, $i \in \mathcal{I}$;

- (iii) $\sum_{i \in \mathcal{I}} x_i \leq \sum_{i \in \mathcal{I}} \mathbf{e}_i$ & $\sum_{\mathbf{w} \in \Omega} p(\mathbf{w}) \sum_{i \in \mathcal{I}} x_i(\mathbf{w}) = \sum_{\mathbf{w} \in \Omega} p(\mathbf{w}) \sum_{i \in \mathcal{I}} \mathbf{e}_i(\mathbf{w})$.

Notice that this is a free disposal equilibrium concept formed on stage before the realization of any state of nature and in the absence of any kind of additional information about the realized event.

It is easy to see that this is an ordinary Walrasian equilibrium, considered relative to preferences specifically defined and with respect to the material balance in the form of inequality. If we exclude this (old-fashioned) opportunity, then we arrive at the following definition. Recall the notation:

$$\mathfrak{M}_P(\Omega, \mathbb{R}^l) := \{f : \Omega \rightarrow \mathbb{R}^l \mid f|_{P(\mathbf{w})} = \text{const}\} = \mathcal{L}_P,$$

¹³In general settings, this is σ -algebra of events.

where for $P = P_i = P_i^0$ as initial information we apply notations

$$\mathcal{L}_i = \mathcal{L}_{P_i}, \quad i \in \mathcal{I}.$$

Definition 7.2.3 (PRIVATE EQUILIBRIUM) A couple (x, p) , $x = (x_i)_{\mathcal{I}} \in X$, $p : \Omega \rightarrow \mathbb{R}^l$, $p \neq 0$ is called (ex ante) private **quasi-equilibrium**, if it obeys:

- (i) $(x_i - \mathbf{e}_i) : \Omega \rightarrow \mathbb{R}_+^l$ is P_i -measurable for all $i \in \mathcal{I}$,
- (ii) $0 \not\equiv \langle p, (\mathcal{P}_i(x_i) - \mathbf{e}_i) \cap \mathcal{L}_i \rangle \geq 0$, $i \in \mathcal{I}$,
- (iii) $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$.

If all inequalities in (ii) are **strict**, then couple (x, p) is called an **equilibrium**.

A direct comparison of definitions shows that the concept of private equilibrium coincides with the *WEE*-equilibrium up to the material balance in the form of equity, as well as a more general point of view on preferences. In addition, this definition is spared from the requirement of measurability of the initial endowments, and a price map is regarded as a functional from L' , the space of (algebraic) dual to $L = \mathfrak{M}(\Omega, \mathbb{R}^l) = L'$, that, in general, simplifies the subject description. The concept of private equilibrium appears in the works of Yannelis (see [Yannelis \(1991\)](#)) and several other authors. Further, we consider the construction of equilibria in rational expectations.

As well as for *WEE*-equilibrium, there is a partition \mathcal{F} and \mathcal{F} -measurable *system of the prices* $p : \Omega \rightarrow \mathbb{R}_+^l$. It is supposed that having received price signals, agents are able to extract from them information, and then they use it to make their own rational decisions. Formally, function $p(\cdot)$ generates a partition $\sigma(p)$ of Ω , this is the coarsest partition concerning which $p(\cdot)$ is measurable. Having learned $\sigma(p)$, the agent i has the access to the information $\sigma(p) \vee P_i = \mathcal{G}_i$ and applies it for buying and selling.

Definition 7.2.4 (REE-EQUILIBRIUM) A couple (p, x) such that $x = (x_i)_{\mathcal{I}} \in X$, $p : \Omega \rightarrow \mathbb{R}_+^l$, $\|p(\mathbf{w})\| = 1$, $\forall \mathbf{w} \in \Omega$ is called **rational expectations equilibrium**, if the following conditions are satisfied:

- (i) $\forall i \in \mathcal{I}$ the plan $x_i : \Omega \rightarrow \mathbb{R}_+^l$ is \mathcal{G}_i -measurable;
- (ii) $\forall i \in \mathcal{I}$ and $\forall \mathbf{w} \in \Omega$ the plan x_i maximizes the conditional expected utility (interim expected utility) $\sum_{\mathbf{w}' \in \Omega} u_i(\mathbf{w}', x_i(\mathbf{w}')) q_i(\mathbf{w}' | \mathcal{G}_i)(\mathbf{w})$ ¹⁴ under budget constraint¹⁵ $p(\mathbf{w})x_i(\mathbf{w}) \leq p(\mathbf{w})\mathbf{e}_i(\mathbf{w})$, $(x_i' - \mathbf{e}_i)$ is \mathcal{G}_i -measurable;
- (iii) $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$.

¹⁴Here $q_i(\mathbf{w}' | \mathcal{G}_i)(\mathbf{w})$ is conditional probability distribution when the element $\mathcal{G}_i(\mathbf{w})$ of partition \mathcal{G}_i is realized, $\mathbf{w} \in \mathcal{G}_i(\mathbf{w}) \subset \Omega$.

¹⁵It is useful to remember that in view of \mathcal{G}_i -measurability of all functions from the inequality and therefore that all of them are constant on $\mathcal{G}_i(\mathbf{w})$, because of homogeneity of budget inequality, this inequality is true on the whole $\mathcal{G}_i(\mathbf{w})$ (according to content) and not only for the state \mathbf{w} .

The notion of *REE*-equilibrium appeared in the works of [Kreps \(1977\)](#) and [Radner \(1978\)](#). Notice, that this concept is an intermediate one since all individual decisions are made not before but after a price signal. Another difference with *WEE*-equilibrium consists of a multiplicity of budget constraints: there is one for each event that the agent is able to understand.

Conditional expectation applied in item (ii) requires an explanation and, on the other hand, creates a (false) impression that it is absolutely necessary to use functional type (7.1.6) in the utility presentation as an expected one. Certainly, the latter one is untrue and this functional type is used only because of the convenience of exposition and interpretation rather than a matter of fact the concept of *REE*-equilibrium. The matter is that once an individual i understood that a (new) event $\mathbf{w} \in \Omega$ is implemented: more precisely, the individual understood that the event $\mathcal{G}_i(\mathbf{w})$ as a *whole* is realized (he/she is able to understand only $\mathcal{G}_i(\mathbf{w})$ and not \mathbf{w} —there is no such information), then the individual begins to qualify consumption and contracts, not in the category of initial preferences, but as reduced to the event $\mathcal{G}_i(\mathbf{w})$, *i.e.* in terms of conditional expected utility.

Due to (7.1.5) on the space $\mathcal{L}_+^E = \mathfrak{M}(E, \mathbb{R}_+^l)$, $E = \mathcal{G}_i(\mathbf{w})$ there is correctly specified preference relation $\mathcal{P}_i^E(\cdot)$ which employs the individual. However, note that by the admissibility of information he/she will evaluate only the functions (or rather their changes) that are *constant everywhere* on $E = \mathcal{G}_i(\mathbf{w})$, *i.e.* they are bundles from \mathbb{R}_+^l . As a result, condition (ii) of the definition can be rewritten in the following (equivalent) form, which will later be used in the definition of the concept of equilibrium with differentiated agents:

$$(ii) \langle p^E, (\mathcal{P}_i^E(x_i^E) - \mathbf{e}_i^E) \rangle > 0, \quad \langle p^E, x_i^E - \mathbf{e}_i^E \rangle = 0, \quad \forall E \in \mathcal{G}_i.$$

Of course, the reader has not forgotten that all the objects used in the latter condition are l -dimensional—based on the fact of information accessibility and by construction: $p(\cdot)$ is constant on $\mathcal{G}_i(\mathbf{w})$.

Thus, the concept of *REE*-equilibrium is defined by a set of budget constraints, which have a chance to be realized at some intermediate time moment when “tomorrow” has already occurred and the random variable (an elementary event) was realized, but the uncertainty is still not fully resolved.

7.3 Incentive compatibility or profitable lie

When information is asymmetrically distributed, we meet an important issue such as the possibility of false informing of partners about the realized state of nature. This kind of deception can be beneficial to the agent in the case that its partners are unable to distinguish false and true states and if the realized deception agent has real benefits (implements a more preferred consumption). Thus, the informational validity of an allocation or a contract should also take into account the factor of honest interaction between the partners, which can be reliably implemented only when the participants in economic transactions have no incentives for information cheating. Further we first

consider several illustrative examples borrowed from *DI-economies* (2005; Preface) and then present precise definitions.

7.3.1 Definitions and examples

Example 7.3.1 (WHEN A PERSON CAN LIE PROFITABLY) Consider a two-person economy with 3 equal probable states of the world $\{a, b, c\} = \Omega$, and in each state one good $x \geq 0$ can be consumed and it gives for an agent utility \sqrt{x} . Thus preferences are defined for consumption sets $X_i = \mathbb{R}_+^3$ via utility functions (for a current state of the world) as follows:

$$u_i(\mathbf{w}, x_i(\mathbf{w})) = \sqrt{x_i(\mathbf{w})}, \quad \mathbf{w} \in \{a, b, c\}, \quad i = 1, 2.$$

Let¹⁶

$$\begin{aligned} \mathbf{e}_1(a, b, c) &= (10, 10, 0), & P_1 &= \{\{a\}, \{b\}, \{c\}\}, \\ \mathbf{e}_2(a, b, c) &= (10, 0, 10), & P_2 &= \{\{a, c\}, \{b\}\} \end{aligned}$$

be initial endowments and individual information. Now consider the following allocation

$$x_1(a, b, c) = (10, 5, 5), \quad x_2(a, b, c) = (10, 5, 5),$$

which is measurable with respect to the full information $P_1 \vee P_2 = \{\{a\}, \{b\}, \{c\}\}$ —this information is accessible as a result of the mutual information exchange. Obviously, this allocation is Pareto optimal with respect to full information (belongs to “weak fine core”, see below). However, what happens if state $\mathbf{w} = a$ is realized?

If state ‘a’ is realized, then only 1st agent knows with certainty which state is implemented, 2nd is not able to distinguish it from ‘c’. This creates the potential possibility for 1st agent to send a false message and declare ‘c’—in order to affect the implementation of the contract according to this event. The temptation becomes real when the agent is able to improve his/her (current a posteriori) utility as a result of the substitution. In our case contract is a vector $v_1 = (v_1(a), v_1(b), v_1(c)) = (0, -5, +5) = -v_2$ and, if the 2nd agent believes to the 1st, then he/she delivers +5 units for 1st agent and they have (x'_1, x'_2) :

$$u_1(x'_1(a)) = u_1(\mathbf{e}_1(a) + v_1(c)) = \sqrt{15} > \sqrt{10} = u_1(x_1(a)),$$

$$u_2(x'_2(a)) = u_2(\mathbf{e}_2(a) + v_2(c)) = \sqrt{5} < \sqrt{10} = u_2(x_2(a)).$$

As we can see, the temptation for the first individual is a real threat of fraud for 2nd. What is the reason for this situation? The main reason is that the principle of ***incentive compatibility*** is violated, and the contract implementing final allocation *is not* measurable with respect to personal information: in the elementary events ‘a’ and ‘c’ 2nd agent must act differently, despite the fact that he does not distinguish the states. ■

¹⁶I’ve changed a little bit the example in a part of P_1 : originally 1st agent does not distinguish states ‘a’ and ‘b’, but then it is not clear how he/she can know what is the state realized: he/she has to report ‘c’ only if ‘a’ and not ‘b’ is realized.

The following example shows that potentially profitably lie cannot affect only individuals, but also coalitions.

Example 7.3.2 (WHEN COALITIONS CAN CHEAT PROFITABLY) Let us consider a three-person differential information economy with two goods denoted by $x \geq 0$, $y \geq 0$ and the three states $\{a, b, c\} = \Omega$, which are assumed to be equally probable. Utility functions, randomized initial endowments, and individualized information are presented below.

Agents' preferences are defined for the consumption sets $X_i = \mathbb{R}_+^2$, and in the current state of the world are represented by the following utilities:

$$u_i(\mathbf{w}, x_i(\mathbf{w}), y_i(\mathbf{w})) = \sqrt{x_i(\mathbf{w})y_i(\mathbf{w})}, \quad \mathbf{w} \in \{a, b, c\}, \quad i = 1, 2, 3.$$

Initial endowments and private information are as follows:

$$\begin{aligned} \mathbf{e}_1(a, b, c) &= ((20, 0), (20, 0), (20, 0)), & P_1 &= \{\{a, b, c\}\}, \\ \mathbf{e}_2(a, b, c) &= ((0, 10), (0, 10), (0, 5)), & P_2 &= \{\{a, b\}, \{c\}\}, \\ \mathbf{e}_3(a, b, c) &= ((10, 10), (10, 10), (20, 30)), & P_3 &= \{\{a\}, \{b\}, \{c\}\} \end{aligned}$$

Consider an allocation

$$\begin{aligned} (x_1, y_1)(a, b, c) &= ((10, 5), (10, 5), (12.5, 7.5)), \\ (x_2, y_2)(a, b, c) &= ((10, 5), (10, 5), (2.5, 2.5)), \\ (x_3, y_3)(a, b, c) &= ((10, 10), (10, 10), (25, 25)) \end{aligned}$$

and argue that this allocation is incentive compatible *individually, but not coalitional*.

The allocation is implemented by contract $v = (v_1, v_2, v_3) \in \mathbb{R}^{18}$, where

$$\begin{aligned} v_1(a, b, c) &= ((-10, 5), (-10, 5), (-7.5, 7.5)), \\ v_2(a, b, c) &= ((10, -5), (10, -5), (2.5, -2.5)), \\ v_3(a, b, c) &= ((0.0, 0.0), (0.0, 0.0), (5.0, -5.0)). \end{aligned}$$

For the presented informational distribution, cheating is potentially possible only by a third agent; no one can cheat 3rd agent (and an agent which joins him) because he/she has full information. On the other hand, the third may declare 'a' when 'b' is realized and vice versa, but this does not give him additional profit. Thus, this allocation is individually incentive compatible, but is it also a coalitional one?

Agents 2 and 3 can form a coalition and if 'c' is realized they can announce 'b' for 1st agent. Then jointly they have

$$v_2(b) + v_3(b) = (10, -5) \gg (7.5, -7.5) = v_2(c) + v_3(c)$$

and, of course, are able to mutually benefit redistribute the obtained surplus (2.5, 2.5). Thus, this allocation *is not* (transfer) coalitional incentive compatible.

Notice that in this example we again have that contract implementing the allocation is non-measurable relative to the individual informational partitions. Here v_1 is not measurable relative to P_1 . ■

The above examples illustrate the idea of incentive compatibility and demonstrate that measurability of contracts regarding individual information is sufficient for the contract and the final allocation is incentive compatible and, therefore, no one would have desired cheating with information. It seems that coalitional incentive compatibility should be recognized as the most correct requirement of this type. The precise definition is given below, but first I want to give one more example from *DI-economies* (2005; Preface, example 0.4). This example demonstrates the idea that information can bring real benefits to individuals, especially for specific coalition-contractual terms.

Example 7.3.3 (INFORMATIONAL ADVANTAGE) Similarly to example 7.3.1 let us consider an economy with three equally probable states of nature $\{a, b, c\} = \Omega$, where one good denoted by $x \geq 0$ can be consumed by each of three agents and again is estimated as utility \sqrt{x} . So current utility functions (relative to a state of the world) are:

$$u_i(\mathbf{w}, x_i(\mathbf{w})) = \sqrt{x_i(\mathbf{w})}, \quad \mathbf{w} \in \{a, b, c\}, \quad i = 1, 2, 3.$$

Initial endowments and individual information are the following:

$$\begin{aligned} \mathbf{e}_1(a, b, c) &= (5, 5, 0), & P_1 &= \{\{a, b\}, \{c\}\}, \\ \mathbf{e}_2(a, b, c) &= (5, 0, 5), & P_2 &= \{\{a, c\}, \{b\}\}, \\ \mathbf{e}_3(a, b, c) &= (0, 0, 0), & P_3 &= \{\{a\}, \{b\}, \{c\}\}. \end{aligned}$$

Now consider an allocation

$$\begin{aligned} x_1(a, b, c) &= (4, 4, 1), \\ x_2(a, b, c) &= (4, 1, 4), \\ x_3(a, b, c) &= (2, 0, 0). \end{aligned}$$

Further, we first note that this allocation is incentive compatible since only the third individual has the possibility to misreport agents 1 or 2 being joined with one of them (together they always can detect lie). However, this can not lead them to a profitable result because a barter contract is measurable regarding personal information and, thus, misinformation does not affect the fulfillment of contractual obligations for a cheated person. Thus, this is an allocation from the private core, see Definition 8.1.1 of Section 8.1. Note that if the 3rd agent would not have any information, *i.e.* for $P_3 = \{\{a, b, c\}\}$, then the contract to implement the allocation ceases to be measurable. Obviously in this case, for any elements of private core 3rd agent gains '0'. Thus, for 3rd individual information is taken into account and it is his/her advantage.

Note also that each of the classical equilibrium concepts known in modern literature—Walrasian and Rational in expectations—are not able to identify this phenomenon, because in all versions the value of 3rd individual initial endowments is '0', and he/she is not able to buy anything. ■

Now let us consider the basic concept of incentive compatibility applied in the model of an economy with asymmetric information.

Definition 7.3.1 An allocation $x = (x_i)_{i \in \mathcal{I}} \in \mathcal{A}(X)$ realized by a contract $x - \mathbf{e} = v = (v_i)_{i \in \mathcal{I}}$ is called **coalitional incentive compatible (CIC)**, if for each coalition $S \subset \mathcal{I}$, $S \neq \emptyset$ there is no state $a \in \Omega$ such that

$$(i) \{a\} = \bigcap_{i \in S} P_i(a),$$

$$(ii) \exists b \in \bigcap_{j \in \mathcal{I} \setminus S} P_j(a), b \neq a \mid$$

$$(a) \forall i \in S \quad u_i(a, \mathbf{e}_i(a) + v_i(b)) > u_i(a, x_i(a));$$

$$(b) \exists j \notin S: \quad u_j(a, \mathbf{e}_j(a) + v_j(b)) < u_j(a, x_j(a)).$$

Notice that items (i) and (ii) are formulated here in accordance with the assumption (7.1.1) which implies

$$\left[\bigcap_{i \in S} P_i(\mathbf{w}) \subseteq \bigcap_{j \in \mathcal{I} \setminus S} P_j(\mathbf{w}) \right] \Rightarrow \bigcap_{i \in S} P_i(\mathbf{w}) = \{\mathbf{w}\}.$$

So, according to the definition an allocation $x = (x_i)_{i \in \mathcal{I}} \in \mathcal{A}(X)$ is called coalitional incentive compatible if there is not a group of individuals (coalition), which due to (i) is able to identify actually realized elementary state of nature $a \in \Omega$ and due to (ii) is able to misreport members of complementary coalition announcing a false state $b \neq a$ and so that each member of the coalition is better off. Furthermore, by (ii, b) at least one member of the complementary coalition $\mathcal{I} \setminus S$ is worse off, *i.e.*, his/her utility in true state ‘ a ’ is lower as a result of contract implementation for the false state ‘ b ’. In other words, if an allocation is coalitional incentive compatible then *no group of individuals is able to gain from a lie* to the detriment of other economic agents.

The concept of incentive compatibility in the context of a model with differentiated information appeared in *Glycopantis et al. (2001)*, but the requirement (ii, b) was not applied there. However, its addition is of theoretical significance: if the efficiency is treated in the limited framework of measurable contracts, then there is no conflict with incentives—each member of the complementing coalition is not able to distinguish a false signal from the true one, but it does not affect the performance of the contract and the final result—after final authorization of uncertainty the result for the non-coalition member will be the same. Note that the contractual measurability is sufficient but unnecessary for the property of incentives compatibility. Thus, *contractual* concepts of different types described below (*private* contractual core and equilibrium, solutions for a model with differentiated agents, and interim concepts of Chapter 9) transfer coalitional incentive compatibility (it is not so according to definitions presented in the literature).

The adding of the item (ii, b) separates the effect of information sharing from the effect of fraud—exchanging information and goods, coalition members are able to improve a posteriori utility without damage for the complementary coalition. Now

two effects—fraud with information and effective information sharing are split and analyzed, respectively. The following example confirms this.

Briefly, an idea is the following one. Let there be three agents and three states of the world $\Omega = \{a, b, c\}$. Assume the first agent knows everything, the second one is able to distinguish ‘a’ the third agent is able to distinguish nothing. Suppose endowments and a posteriori utilities of 1st and 2nd individual in the states ‘a’ and ‘b’ are equal and the same holds for the third agent. Suppose that preferences and endowments in the state ‘c’ are such that the Pareto optimal contracts in the event $\{b, c\}$ between 1st and 2nd agents differ from their Pareto border in the state ‘a’. Then if the state ‘b’ is implemented and after the false signal ‘a’ from the coalition $S = \{1, 2\}$, members of S can win by definition (i) – (ii, a) —without damage to the 3rd agent ((ii, b) is violated). It can happen, for example, in a private equilibrium.

The following is a numerical example in which the participation of a third agent is not specified (formally one can always add an agent so that his endowments coincide with the consumption in private equilibrium). Moreover, the effect of the “information sharing” is observed here not only in equilibrium allocation but also at any Pareto optimal point.

Example 7.3.4 (INFORMATION SHORTAGE AND SHARING) There are $\Omega = \{a, b, c\}$, two products consumed in non-negative quantities, and two agents with the following endowments and information:

$$\begin{aligned} \mathbf{e}_1(a, b, c) &= ((1, 2)_a, (1, 2)_b, (1, 2)_c), & P_1 &= \{\{a\}, \{b\}, \{c\}\}, \\ \mathbf{e}_2(a, b, c) &= ((2, 1)_a, (2, 1)_b, (2, 1)_c), & P_2 &= \{\{a\}, \{b, c\}\}. \end{aligned}$$

Let a posteriori utilities be presented as

$$\begin{aligned} u_1(a, x) &= u_1(b, x) = \ln(x_1) + \ln(x_2), & u_1(c, x) &= F \ln(x_1) + G \ln(x_2), & F > 0, & G > 0, \\ u_2(a, y) &= u_2(b, y) = \ln(y_1) + \ln(y_2), & u_2(c, y) &= \ln(y_1) + \ln(y_2), \\ u_i(\xi) &= \sum_{\mathbf{w} \in \Omega} u_i(\mathbf{w}, \xi(\mathbf{w})), & i &= 1, 2. \end{aligned}$$

Here it is important that endowments and a posteriori utilities of 1st and 2nd individuals coincide in the states ‘a’ and ‘b’, but the 1st utility in state ‘c’ is different from other states.

Assume $(1+F)(1+G) = 4$. Then for every Pareto optimal (and rational) allocation its partial allocation in the state ‘a’ dominates by a *posteriori utility* the allocation of the state ‘b’. In a particular case $F = 2, G = \frac{1}{3}$

$$\begin{aligned} x(a, b, c) &\approx ((1.4, 1.4)_a, (1.7, 1.1)_b, (1.7, 1.1)_c), \\ y(a, b, c) &\approx ((1.6, 1.6)_a, (1.3, 1.9)_b, (1.3, 1.9)_c) \end{aligned}$$

is a private equilibrium and now straightforward calculation shows that there is “information sharing effect”. The detailed arguments are presented below.

It is obvious that every private core allocation $(x, y) \in L_+^2$ is an interior point of consumption sets. This allows to apply the first-order conditions and simultaneously to normalize prices setting

$$p(a, b, c) = \nabla u_1(a, b, c, x) = \left(\left(\frac{1}{x_1(a)}, \frac{1}{x_2(a)} \right), \left(\frac{1}{x_1(b)}, \frac{1}{x_2(b)} \right), \left(\frac{F}{x_1(c)}, \frac{G}{x_2(c)} \right) \right).$$

Analogously from $y(b) = y(c)$ and second agent consumption problem, one has

$$(p(a), p(b) + p(c)) = \alpha(\nabla u_2(y(a)), \nabla u_2(y(b)) + \nabla u_2(y(c))).$$

As a result, due to the first-order conditions, one concludes

$$x(a, b, c) = \left(\left(\frac{1}{p_1(a)}, \frac{1}{p_2(a)} \right), \left(\frac{1}{p_1(b)}, \frac{1}{p_2(b)} \right), \left(\frac{F}{p_1(c)}, \frac{G}{p_2(c)} \right) \right),$$

$$y(a) = \alpha \left(\frac{1}{p_1(a)}, \frac{1}{p_2(a)} \right), \quad y(b) = y(c) = \alpha \left(\frac{2}{p_1(b) + p_1(c)}, \frac{2}{p_2(b) + p_2(c)} \right).$$

Next, consider the commodity balance relations. From $x(b) + y(b) = (3, 3) = x(c) + y(c)$ one concludes

$$x(b) = x(c) \Rightarrow p_1(c) = Fp_1(b), \quad p_2(c) = Gp_2(b).$$

Now $x_1(b) + y_1(b) = 3 = x_2(b) + y_2(b)$ implies

$$p_1(b) = \frac{1 + F + 2\alpha}{3(1 + F)}, \quad p_2(b) = \frac{1 + G + 2\alpha}{3(1 + G)}.$$

Analogously $x_1(a) + y_1(a) = 3 = x_2(a) + y_2(a)$ yields

$$p_1(a) = p_2(a) = \frac{1 + \alpha}{3}.$$

Calculations show that

$$u_1(x(a)) > u_1(x(b)) \iff x_1(a)x_2(a) > x_1(b)x_2(b) \iff p_1(b)p_2(b) > p_1(a)p_2(a),$$

$$u_2(y(a)) > u_2(y(b)) \iff (1 + F)(1 + G)p_1(b)p_2(b) > 4p_1(a)p_2(a).$$

Now choosing F and G satisfying $(1 + F)(1 + G) = 4$, we provide an increase or decrease of utility *simultaneously for both individuals* and, thereby, the allocation $(x(a), y(a))$ of 'a' being Pareto optimal, will dominate (or be equivalent to) the allocation $(x(b), y(b))$. To be sure, let us carry out again the calculations. We have

$$p_1(b)p_2(b) = \frac{(1 + F + 2\alpha)(1 + G + 2\alpha)}{3(1 + F)3(1 + G)} = \frac{(\bar{F} + \alpha)(\bar{G} + \alpha)}{9},$$

where $\bar{F} = \frac{(1+F)}{2}$, $\bar{G} = \frac{(1+G)}{2}$ and $\bar{F}\bar{G} = 1$. This value has to be compared with

$$p_1(a)p_2(a) = \frac{(1 + \alpha)^2}{9}.$$

However, $\bar{F} > 0$, $\bar{G} > 0$ and $\bar{F}\bar{G} = 1$ always imply $\bar{F} + \bar{G} \geq 2$ and so that equality is possible only if $\bar{F} = \bar{G} = 1$. So, if F and G are not simultaneously equal to 1, then

$$(\bar{F} + \alpha)(\bar{G} + \alpha) = \alpha^2 + (\bar{F} + \bar{G})\alpha + 1 > (1 + \alpha)^2,$$

that proves inequality we needed.

Note that real $\alpha > 0$ parameterizes the Pareto frontier. Thus, in this example, every Pareto optimal allocation (from the interior) is such that the allocation in the state 'a' dominates the allocation of 'b' (via a posteriori utility) and the reason for this is (implicit) information sharing.

Now let us find the value α corresponding to the private equilibrium; for this one applies the budget equality of the first agent. Calculating gradients and inner products, one finds

$$F + G + 4 = (p_1(a) + p_1(b) + p_1(c)) + 2(p_2(a) + p_2(b) + p_2(c)) = \frac{1}{3}(6 + F + 2G + 9\alpha) \Rightarrow$$

$$\alpha = \frac{2F + G + 6}{9}.$$

For $F = 2$ and $G = \frac{1}{3}$ we have $(1 + F)(1 + G) = 4$ and $\alpha = \frac{31}{27} = 1\frac{4}{27}$. Now equilibrium allocation is calculated via presented above formulas:

$$x(a, b, c) = \left(\left(\frac{81}{58}, \frac{81}{58} \right)_a, \left(\frac{243}{143}, \frac{54}{49} \right)_b, \left(\frac{243}{143}, \frac{54}{49} \right)_c \right).$$

■

Note that Definition 7.3.1 does not address the measurability of the allocation or the contract that implements it. Moreover, in addition to the model's substantial content, the measurability of contracts relative to individual information is also a basic sufficient condition for the final allocation to be coalitional incentive compatible. (See Krasa, Yannelis (1994) and Hahn, Yannelis (1997) for related concepts.)

Besides the coalitional incentive compatibility, the modern theory also addresses its modifications: individual and transfer-coalitional ones.

- An allocation is called **individually incentive compatible (IIC)** if it obeys Definition 7.3.1 for the case of singleton coalitions.¹⁷

- An allocation is called **transfer coalitional incentive compatible (TCIC)** if it obeys Definition 7.3.1, wherein addition coalition members are able to redistribute resources among themselves, i.e., item (ii) is rewritten as follows:

$$\left\{ \begin{array}{l} \exists b \in \bigcap_{j \in \mathcal{I} \setminus S} P_j(a), b \neq a \ \& \ \exists w = (w_i)_{i \in S} \in L^S, \sum_S w_i = 0 \mid \\ (a) \ \forall i \in S \ u_i(a, \mathbf{e}_i(a) + w_i(a) + v_i(b)) > u_i(a, x_i(a)); \\ (b) \ \exists j \notin S: u_j(a, \mathbf{e}_j(a) + v_j(b)) < u_j(a, x_j(a)). \end{array} \right.$$

Notice that the definition does not suppose TCIC allocation be implemented via measurable contract (so as for CIC), but it is so also for a transfer barter contract w . From the definitions, the transfer coalitional form of incentive compatibility is the strongest one. Here we have:

$$\text{TCIC} \Rightarrow \text{CIC} \Rightarrow \text{IIC}.$$

¹⁷Note that now a requirement (ii, b) of Definition 7.3.1 is not demanded, because here there is no way for information exchange.

7.3.2 Maximin expected utilities

Sometimes a coalition incentive compatibility can be provided in the absence of the measurability of implementing contracts. However, anyway it should be measurability but now it is in the context of individual perception of final consumption, expressed in terms of preference. This can be achieved, for example, for *maximin* preferences. Preferences of this type were introduced and axiomatized in [Gilboa, Schmeidler \(1989\)](#), and then they were reformulated ([de Castro, Yannelis, 2010](#)) in the form of (7.3.7) below.

• *Let the agents of DI-economy have a posteriori utility functions $u_i : \Omega \times \mathbb{R}^l \rightarrow \mathbb{R}$ and let individual probability distributions $\mu_i(\cdot)$ be associated with the information partitions P_i , $i \in \mathcal{I}$.*

Maximin expected preferences are defined as follows:

$$U_i^{min}(x_i) = \sum_{E \in P_i} \left(\min_{\mathbf{w} \in E} u_i(\mathbf{w}, x_i(\mathbf{w})) \right) \mu_i(E), \quad x_i : \Omega \rightarrow \mathbb{R}_+^l. \quad (7.3.7)$$

In substantial terms, maximin preferences correspond to the cautious behavior of economic agents, *i.e.*, for a pessimistic perception of future events that is valued as a guaranteed minimum: it seems to be a plausible scenario.

From the modeling point of view, a most relevant option of maximin preferences is a special case of a posteriori preferences when they are identical at agent's indistinguishable states of the world. This is a case where a posteriori utility function $u_i(\cdot, y)$ is measurable regarding the agent's information partition for any fixed consumption $y \in \mathbb{R}_+^l$, *i.e.*, when

$$\forall i \in \mathcal{I}, \forall y \in \mathbb{R}_+^l \quad u_i(\mathbf{w}, y) = u_i(\mathbf{w}', y) \quad \forall \mathbf{w}, \mathbf{w}' \in P_i(\mathbf{w}) \quad (7.3.8)$$

holds. Indeed, the minimum in (7.3.7) can be correctly interpreted only in the case of comparable values and, for example, it can be money. However, even if it is so, it is not clear why the same consumption is indistinguishable for the individual states one attaches the different (utility) value—no reason for this, for beginning an agent needs at least to understand what the difference is.

Note that the maximin preferences do not satisfy hypothesis **(Sep)**, or rather they are *not* separable ones. The main advantage of these preferences and a subject of theoretical interest is presented in the Proposition below.

Definition 7.3.2 *Core allocations of economy \mathcal{E}^{di} , in which all agents are equipped with maximin preferences, form **maximin core** $\mathcal{C}^{min}(\mathcal{E}^{di})$; these allocations are not assumed to be measurable.*

Before we formulate and prove the main positive feature of maximin core elements—they are coalitional incentive compatible—we consider an important example of [Liu, Yannelis \(2013\)](#), which clarifies the issue.

Example 7.3.5 Let us consider the economy from Example 7.3.1 (2 agents and 3 equally probable states of nature, one commodity), which differs only in that now the

1st agent is not able to distinguish between ‘a’ and ‘b’. Also, let as above preferences of agents be defined on $X_i = \mathbb{R}_+^3$ and generated by the following ex-post utilities:

$$u_i(\mathbf{w}, x_i(\mathbf{w})) = \sqrt{x_i(\mathbf{w})}, \quad \mathbf{w} \in \{a, b, c\}, \quad i = 1, 2.$$

Initial endowments and information of individuals:

$$\begin{aligned} \mathbf{e}_1(a, b, c) &= (10, 10, 0), & P_1 &= \{\{a, b\}, \{c\}\}, \\ \mathbf{e}_2(a, b, c) &= (10, 0, 10), & P_2 &= \{\{a, c\}, \{b\}\} \end{aligned}$$

Now consider an allocation

$$\bar{x}_1(a, b, c) = (10, 8, 2), \quad \bar{x}_2(a, b, c) = (10, 2, 8)$$

and show that it is an element of maximin core. In an economy with 2 agents, it is sufficient to show that the allocation is a feasible allocation that is both individually rational and Pareto optimal. Now a direct calculation of utilities for $i = 1, 2$ gives

$$U_i^{min}(\bar{x}_i) = \frac{2}{3}\sqrt{\min\{10, 8\}} + \frac{1}{3}\sqrt{2} = \frac{\sqrt{50}}{9} > \frac{\sqrt{40}}{9} = \frac{2}{3}\sqrt{\min\{10, 10\}} + \frac{1}{3}\sqrt{0} = U_i^{min}(\mathbf{e}_i),$$

that proves the first condition. To check the Pareto optimality, one can verify that the allocation $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of a maximization problem

$$\max_{x_1, x_2} (U_1^{min}(x_1) + U_2^{min}(x_2))$$

subject to $x_1(a, b, c) + x_2(a, b, c) = (10, 5, 5)$, $x_1(a, b, c) \geq 0$ and $x_2(a, b, c) \geq 0$. So it proves that $\bar{x} = (\bar{x}_1, \bar{x}_2)$ belongs to the maximin core.

However, this allocation fails to be an element of the private core or to correspond to any of the equilibrium concepts either their generalizations discussed below. The reason is that $x = (x_1, x_2)$ fails to be measurable regarding informational partitions of economic agents. ■

Theorem 7.3.1 *Let a posteriori utilities $u_i : \Omega \times \mathbb{R}_+^l \rightarrow \mathbb{R}$ of individuals obey*

- (i) $\forall \mathbf{w} \in \Omega$ $u_i(\mathbf{w}, \cdot)$ is strictly monotonic for each commodity,
- (ii) $\forall y \in \mathbb{R}_+^l$ $u_i(\cdot, y)$ and initial endowments $\mathbf{e}_i(\cdot)$ are measurable subject to individual information.

*Then every allocation of maximin core $\mathcal{C}^{min}(\mathcal{E}^{di})$ is a **transfer coalitional incentive compatible**.*

Remark 7.3.1 From the proof, one can easily see that Theorem 7.3.1 is executed on a broader class of *upper contractual* allocations. In addition, item (ii, b) of Definition 7.3.1 is not demanded. This occurs because an increase of a posterior utility in a state of the world still does not imply the growth of utility from the element of information partition, but this evaluates the allocation individuals reached.

Effective communication does not occur, because the agents are not able to use it since utility is not determined for a new partition.

Notice that Theorem 7.3.1 assumptions are so to provide the maximin utility $U^{min}(\cdot)$ generates indifference relation such that every equivalence class contains a measurable consumption plan. Probably it is a crucial property (of course not the only one) for a solution concept to be incentive compatible for possible nonmeasurable allocations. One can easily suggest a lot of methods construct preferences having this property and different with maximin. For example, one can consider ‘*max*’ instead of ‘*min*’ in (7.3.7), or apply *convex* combinations of $(x_i(\mathbf{w}))_{\mathbf{w} \in E}$ which coefficients can vary only over $E \in P_i$, etc. Now it is only a hypothesis that is a subject for future studies. \blacksquare

Proof of Theorem 7.3.1. We argue by contradiction. By Definition 7.3.1, one finds a coalition $S \subset \mathcal{I}$, $S \neq \emptyset$, a state $a \in \Omega$ and contract $w = (w_i)_{i \in S} \in (\mathbb{R}^l)^S$, $\sum_S w_i = 0$ such that for $v = x - \mathbf{e}$ the following is true:

$$(i) \{a\} = \bigcap_{i \in S} P_i(a),$$

$$(ii) \exists b \in \bigcap_{j \in \mathcal{I} \setminus S} P_j(a), b \neq a: \forall i \in S$$

$$U_i^{min}(x_i(\Omega \setminus \{a\}), \mathbf{e}_i(a) + w_i(a) + v_i(b)) > U_i^{min}(x_i).$$

By the definition of maximin utility (7.3.7), the latter for $z_i(a) = \mathbf{e}_i(a) + w_i(a) + v_i(b)$ and $\forall i \in S$ can be written in a form

$$\begin{aligned} & \left(\min_{\mathbf{w} \in P_i(a)} u_i(\mathbf{w}, x_i(\Omega \setminus \{a\}), z_i(a)) \right) \mu_i(P_i(a)) + \sum_{E \in P_i, E \neq P_i(a)} \left(\min_{\mathbf{w} \in E} u_i(\mathbf{w}, x_i(\mathbf{w})) \right) \mu_i(E) > \\ & > \left(\min_{\mathbf{w} \in P_i(a)} u_i(\mathbf{w}, x_i(\mathbf{w})) \right) \mu_i(P_i(a)) + \sum_{E \in P_i, E \neq P_i(a)} \left(\min_{\mathbf{w} \in E} u_i(\mathbf{w}, x_i(\mathbf{w})) \right) \mu_i(E). \end{aligned}$$

Omitting second identical summand for both sides and $\mu_i(P_i(a)) > 0$ one finds

$$u_i(a, \mathbf{e}_i(a) + w_i(a) + v_i(b)) \bigwedge_{\mathbf{w} \in P_i(a), \mathbf{w} \neq a} \min u_i(\mathbf{w}, x_i(\mathbf{w})) > \min_{\mathbf{w} \in P_i(a)} u_i(\mathbf{w}, x_i(\mathbf{w}))$$

that implies

$$\forall i \in S \quad u_i(a, x_i(a)) < \min_{\mathbf{w} \in P_i(a), \mathbf{w} \neq a} u_i(\mathbf{w}, x_i(\mathbf{w})).$$

At the same time, if there were found $j \in \mathcal{I} \setminus S$ such that

$$u_j(a, x_j(a)) > \min_{\mathbf{w} \in P_j(a), \mathbf{w} \neq a} u_j(\mathbf{w}, x_j(\mathbf{w})),$$

then agent j would be able to transfer some of his/her resources to the members of the coalition S irreducibly for his/her maximin utility, *i.e.*, for some $z \in \mathbb{R}_+^l$, $z \neq 0$ we could have $x_j(a) - z \geq 0$ with

$$u_j(a, x_j(a) - z) > \min_{\mathbf{w} \in P_j(a), \mathbf{w} \neq a} u_j(\mathbf{w}, x_j(\mathbf{w})).$$

Now uniformly distributing $z > 0$ among agents of S , due to preferences monotonicity one finds $u_i(a, x_i(a) + \frac{1}{n-1}z) > u_i(a, x_i(a))$, that together with the previous inequality means *strict* increasing of maximin utility for the members of coalition S , and non-decreasing of it for all other individuals. This contradicts the allocation x ¹⁸ belongs to the core. So one has

$$\forall j \notin S \quad u_j(a, x_j(a)) = \min_{\mathbf{w} \in P_j(a)} u_j(\mathbf{w}, x_j(\mathbf{w})) \Rightarrow u_j(a, x_j(a)) \leq u_j(b, x_j(b)).$$

Finally, note that the measurability of endowments and utilities yields: $j \notin S$ $\mathbf{e}_j(a) = \mathbf{e}_j(b)$ implies $x_j(b) = \mathbf{e}_j(a) + v_j(b) \Rightarrow u_j(a, \mathbf{e}_j(a) + v_j(b)) \geq u_j(a, x_j(a))$. Together with (ii) it means that for the state 'a' one finds a contract $w' \in (\mathbb{R}^l)^{\mathcal{I}}$, defined as $w'_i = w_i + v_i(b)$, $i \in S$ & $w'_j = v_j(b)$, $j \in \mathcal{I} \setminus S$, which improves the maximin utility of all individuals, and for the members of the coalition $S \neq \emptyset$ strictly. For the elements of maximin core, it is impossible. ■

Conclusion to Chapter 7

In this chapter, a detailed model of an economy with differential (asymmetric) information in the most general form was described. Information is individualized and is described as a partition of possible states of nature. The model is also equipped with a rule of information sharing, which transforms information as a result of the intra-coalition contractual activity. There are two derived distributions of information obtained by a rule: the concept of maximal information by Schwalbe and limit information by Marakulin.

A series of concepts of core and equilibria known in the literature were considered. Among others, variants of fine and coarse cores, k -core by Schwalbe, and versions of the core with limit information are presented. The most significant and well known in the literature concepts of equilibrium are described: Walrasian expectations equilibrium (*WEE*), private equilibrium, and rational expectations equilibrium (*REE*).

An important theoretical notion of incentive compatibility was described and its important modification was suggested: under the new definition, all notions applying measurable contracts are incentive compatible. In this context, the notion of maximin expected utilities was presented—these utilities generate a core with unmeasurable contracts which is always incentive compatible.

¹⁸This is a weak Pareto domination, that in our context is equivalent to the strong one.

Chapter 8

Private core and equilibrium in a contractual economy

8.1 Private core. Lemma on beneficial contract

Before we formulate and prove different mathematical statements, let us attempt to understand that the contractual approach can add to our understanding of the functioning of the economy with asymmetrically informed agents.

Let's imagine that there are "today" and "tomorrow" and that *today* we need to plan our tomorrow consumption. We don't have exact information about what will happen tomorrow, but at least for ourselves, we know that exactly we *will be able* to understand, *i.e.* when this "tomorrow" will start to be realized. In the model the events which we will be able to understand tomorrow form a *partition* of the set Ω of all elementary events of tomorrow. For different agents, these partitions are different ones, and this is the informational asymmetry exhibited.

Planning for tomorrow's consumption, the individual should agree with other agents on what commodity changes we will make tomorrow. However, the satisfaction of the individual from the consumption of tomorrow commodities essentially depends on what events will happen tomorrow. For example, if I plan tomorrow a trip to the forest, but if there will be rain, I will get wet and by the evening my temperature will rise and then it will be required febrifugal and other medicines. If the rain doesn't happen, then for me the value of an umbrella and medical products will be insignificant. However, already today I should provide a possibility of tomorrow's rain and conclude agreements on the change of some commodities (money?) on an umbrella and medicines. Moreover, the planning on an umbrella will be desirable for making only when it will be precisely known, that the rain will happen, and concerning medicines—when the temperature has already risen (or it is precisely known, that will rise). Here it is important, that the individual first of all *should understand the event* which specifies the exchange. It, of course, concerns each side in the exchange agreement that we intend to conclude already today; the contract concluded in this agreement should start from the text: "If there will be a rain, then..." and/or from the text: "If there will be a rain and my temperature will rise..." The mathematical design of these situations can be the following.

A (barter) contract $v = (v_i)_{\mathcal{I}}$ is an n -tuple of maps $v_i : \Omega \rightarrow \mathbb{R}^l$, $i \in \mathcal{I}$, satisfying the measurability

$$\forall i \in \mathcal{I}, \quad v_i(\cdot) \text{ measurable subject to } P_i, \quad (8.1.1)$$

and also a standard balancing assumption:

$$\forall \mathbf{w} \in \Omega, \quad \sum_{i \in \mathcal{I}} v_i(\mathbf{w}) = 0. \quad (8.1.2)$$

The first of these requirements says that if an individual is not able to distinguish one elementary event from another one, they have to belong to the same element of partition. For example, if a and b from Ω are indistinguishable for i , it means that a and b have placed in a common element of an informational partition, and then obligations of an individual under the contract should be identical, *i.e.*, it should be $v_i(a) = v_i(b)$ and the function as a whole should be constant on each of elements of an informational partition. In this context, condition (8.1.1) restricts the area \mathcal{W} of permissible contracts.

If there are no other restrictions imposed and regarding contracts breaking one supposes only a possibility of their full break, then as a contractual allocation, we will receive in accuracy allocations from the private core which definition is the following one.

Definition 8.1.1 *Private core $\mathcal{C}^{pr}(\mathcal{E}^{di})$ for an economy \mathcal{E}^{di} with asymmetrically informed agents consists of allocations $x = (x_i)_{\mathcal{I}} \in X$ such that*

- (i) $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$,
- (ii) $(x_i - \mathbf{e}_i) : \Omega \rightarrow \mathbb{R}^l$ is P_i -measurable for all $i \in \mathcal{I}$,
- (iii) $\nexists S \subseteq \mathcal{I} : \exists y^S = (y_i)_S \mid \forall i \in S, y_i \in X_i$ is such that $(y_i - \mathbf{e}_i)$ is P_i -measurable, $y_i \succ_i x_i$ & $\sum_S (y_i - \mathbf{e}_i) = 0$.

Here condition (iii) presents usual coalitions non-domination requirement considered with the account of measurability of dominating allocation relative to the private information. Clearly one can take $V = \{x - \mathbf{e}\}$ as a web of contracts implementing allocation x from the core $\mathcal{C}^{pr}(\mathcal{E}^{di})$. The coincidence of concepts (initial and contractual) is checked directly: just one needs to take into account the differences in notation and terminology. However, the concept of contractual allocation correctly corresponding to the notion of the private core is still a bit stronger than presented in Definition 8.1.1.

Recall that the web of contracts V is called upper stable, if the allocation implemented by this web $x = \mathbf{e} + \sum_{v \in V} v$, there is no coalition that has a mutually beneficial contract, *i.e.*, coalitions are unable to improve consumption of their members by the signing of a new contract and *without breaking of old contracts*:

$$\nexists v = (v_i)_{\mathcal{I}} \in \mathcal{W} : \quad x_i + v_i \succ_i x_i \quad \forall i \in \text{supp}(v).$$

To correctly define contractual private core one has to replenish (i)–(iii) with the requirement that the web $\{x - \mathbf{e}\}$ is upper stable (the break of $v = \{x - \mathbf{e}\}$ is presumed for domination in (iii)). This motivates the following definition.

• Let $\mathcal{PB}^{pr}(\mathcal{E}^{di}) \subset \mathcal{A}(\mathcal{E})^1$ denote the set of all upper stable contractual allocations in a differential information economy by Definition 1.1.2.

• Private contractual core is the set of all contractual upper stable elements of the private core, i.e. it is

$$\mathcal{C}^{pr}(\mathcal{E}^{di}) \cap \mathcal{PB}^{pr}(\mathcal{E}^{di}).$$

This set can also be described as the set of all allocations $x = (x_i)_{\mathcal{I}} \in X$, which are the elements of the private core where the consumption allocation is considered also as an initial endowments allocation, i.e. for $\mathbf{e} = x$. The easiest way to identify a private contractual core is to modify item (iii) of Definition 8.1.1, writing it in the form

$$(iii)^* \nexists S \subseteq \mathcal{I} : \exists y^S = (y_i)_S \mid \forall i \in S, y_i \in X_i \text{ is such that } (y_i - \mathbf{e}_i) \text{ is } P_i\text{-measurable, } y_i \succ_i x_i \ \& \ \sum_S (y_i - \mathbf{e}_i) = 0 \ \text{or} \ \sum_S (y_i - x_i) = 0.$$

Below we shall specify an equilibrium concept that is well corresponding to the private core, but first, we turn to consider an important property of the private core elements, this is incentive compatibility that in substantial terms means stability relative to informational cheating. As it was noted above (see Definition 7.3.1, page 284 and the comments on it), all solution concepts applying measurable contracts (information admissibility) generate an incentive compatible type of solution by Definition 7.3.1 (including the item (ii, b)), wherein for upper stable contractual allocations this is always the strongest coalitional transfer compatibility. However, for the sake of completeness and a better understanding of the problem, it seems also useful to consider another approach known in the literature, where there is no analog of (ii, b). However, the presented case does not need this, since an exchange of information is not demanded.

Below we apply a specific definition named in *Koutsougeras, Yannelis (1993)* as **weak coalition incentive compatibility**.

Proposition 8.1.1 *Let $x \in \mathcal{PB}^{pr}(\mathcal{E}^{di})$, i.e. contract $v = x - \mathbf{e}$ obeys (8.1.1), (8.1.2) (informational measurability) and is upper (contractual) stable: no coalition can sign a new mutually beneficial contract without the break of v . Then the allocation is **incentive compatible** in the following sense:*

There do not exist a coalition $S \subset \mathcal{I}$, $S \neq \emptyset$ and a state $a \in \Omega$ such that

$$(i) \ \{a\} = P_i(a), \ i \in S;$$

$$(ii) \ \exists b \in \bigcap_{j \in \mathcal{I} \setminus S} P_j(a), \ b \neq a: \quad u_i(a, \mathbf{e}_i(a) + v_i(b)) > u_i(a, x_i(a)), \ \forall i \in S.$$

¹Applied here the superscript *pr* appeals to the private information of individuals that is taken into account in Pareto frontier specification.

Notice that now usually imposed assumption $P_i(a) \in \bigwedge_S P_j$ (e.g. see [Koutsougeras, Yannelis, 1993](#)) is almost equivalent to (i). Indeed, since $\bigwedge_S P_j$ is a partition one concludes $\bigcap_S P_j(a) \in \bigwedge_S P_j$ that together with $\bigcup_S P_j(a) \subseteq (\bigwedge_S P_j)(a)$ implies $P_i(a) = \bigcap_S P_j(a) = \bigcup_S P_j(a) = P_j(a)$, $j \in S$.

Proof of Proposition 8.1.1. We argue by contradiction. Find a coalition $S \subset \mathcal{I}$ and a state $a \in \Omega$ such that conditions (i), (ii) are fulfilled. The measurability of v_j relative to P_j for undistinguished states $a, b \in \Omega$ realizes $v_j(b) = v_j(a)$, $\forall j \in \mathcal{I} \setminus S$. Now by contract definition, one concludes

$$\sum_S v_i(b) = - \sum_{\mathcal{I} \setminus S} v_j(b) = - \sum_{\mathcal{I} \setminus S} v_j(a) = \sum_S v_i(a).$$

Therefore, coalition S is able to conclude a new contract $w(a) = v(b) - v(a)$ for $\mathbf{w} = a$ and $w(\mathbf{w}) = 0$, $\forall \mathbf{w} \neq a$, $\mathbf{w} \in \Omega$. Notice that by construction and due to (i) contract w obeys (8.1.1), (8.1.2). However, due to (ii) after contract w is signed all members of coalition S become better off that contradicts to upper stability of v . ■

Remark 8.1.1 Note that by virtue of the assumption (i) of the Proposition for a coalition of possible cheats, an exchange of information does not lead to an improvement of their individual information about the state ‘ a ’. Therefore, the members of the coalition can be better off only by deceiving members of the complementary coalition. However, this is only a special case of information distribution when item (ii) of Proposition 8.1.1 becomes an analog of the requirement (ii) of Definition 7.3.1. Of course, Proposition 8.1.1 does not describe the case in its entirety with the information fraud for allocations from the private contractual core; moreover, it can not be generalized to coalition compatibility without (ii, b), it’s just not true, see example 7.3.4, p. 285 above. ■

Now let us turn to equilibrium notion. According to the general methodology of the contractual approach, an allocation implemented by a web of admissible contracts **cannot** be an equilibrium if the web is **upper unstable** or it is **unstable** relative to **partial breakings** of contracts. It may not be enough to suggest a correct definition of equilibrium in the most general setting, but these properties should be fulfilled with necessity. So, to give a possible equilibrium treatment one needs at least to characterize upper stable webs of contracts.

Lemma 8.1.1 (ABOUT THE MUTUALLY BENEFICIAL CONTRACT) *Let $S \subseteq \mathcal{I}$, $S \neq \emptyset$ be a coalition and $A \subseteq \Omega$ be an event understandable by every coalition S member.² Then for $x \in \mathcal{A}(x)$ if **there is no** mutually beneficial exchange of contingent commodities for coalition S members then there do exist a vector $p \in (\mathbb{R}^l)^A$, $p \neq 0$, and vectors $q_i \in (\mathbb{R}^l)^A$, $i \in S$, such that*

$$\forall i \in S \quad \forall E \in P_i, \quad E \subseteq A \quad \sum_{\mathbf{w} \in E} q_i(\mathbf{w}) = 0 \quad (8.1.3)$$

²That is $\forall i \in S$ set A is P_i -measurable $\iff A \in \bigwedge_S P_i$.

and

$$\forall i \in S \quad p + q_i \neq 0 \quad \& \quad \langle \mathcal{P}_i(x_i), p + q_i \rangle \geq \langle x_i, p + q_i \rangle \quad (8.1.4)$$

holds.

Inverse: let there be vectors satisfying (8.1.3), (8.1.4) and in (8.1.4) at least one inequality is strict.³ Then for the coalition S , **there is no** mutually beneficial contract.

The lemma implies two important for further considerations corollaries.

Corollary 8.1.1 Let in Lemma 8.1.1 conditions preferences be described via differentiable utility functions and let $x = (x_i)_{\mathcal{I}}$ be an interior relative to A and S allocation.⁴ Then for the coalition S **there is no** mutually beneficial contract for event A if and only if there exists a vector $p \in (\mathbb{R}^l)^A$, $p \neq 0$ and $\lambda_i > 0$, $i \in S$, such that

$$\forall E \in P_i, \quad E \subseteq A \quad \lambda_i \sum_{\mathbf{w} \in E} \nabla_{\mathbf{w}} u_i(x_i) = \sum_{\mathbf{w} \in E} p(\mathbf{w}) \quad \forall i \in S.$$

The requirements in the last relations can be considered as a (linear) system of equations subject to $\lambda_i > 0$, $i \in \mathcal{I}$ and $p(\mathbf{w}) \in \mathbb{R}^l$, $\mathbf{w} \in A$. If a solution does exist then the contract is impossible. The following corollary presents the most simply verified criterium of mutually beneficial contract existence.

Corollary 8.1.2 Let in Lemma 8.1.1 and its Corollary 8.1.1 conditions an event $E \in P_i$ for each $i \in S$. Then a mutually beneficial contract for E does exist if and only if $h_i = \sum_{\mathbf{w} \in E} \nabla_{\mathbf{w}} u_i(x_i(\cdot))$, $i \in S$ form a non-collinear system of vectors.

So as soon as by definition of upper stable properly contactual allocation in the case $A = \Omega$ and coalition $S = \mathcal{I}$ mutually beneficial contract is impossible, then one can apply Lemma 8.1.1 conclusion. This implies the existence of map $p : \Omega \rightarrow \mathbb{R}^l$, that satisfies (8.1.4) and it can be taken as a *price* map.

Further, to derive an equilibrium notion, one has also to reveal stability relative to partial contract breakings. For differentiable utilities, this is equivalent to requirement

$$\langle \nabla u_i(x_i), x_i - \mathbf{e}_i \rangle \geq 0, \quad \forall i \in \mathcal{I},$$

that for interior points yields

$$\langle p + q_i, x_i - \mathbf{e}_i \rangle \geq 0, \quad \forall i \in \mathcal{I}.$$

Clear that all inequalities can be realized here only in the form of strict equalities. Notice also that one has considered only the case of the single-element web of contracts $V = \{x - \mathbf{e}\}$. Further, in view of (8.1.1), (8.1.3) for every contract $v = (v_i)_{\mathcal{I}} \in \mathcal{W}$ one has $\langle q_i, v_i \rangle = 0$, $\forall i \in \mathcal{I}$, that under presented assumptions means that previous relations are equivalent to $\langle p, (\mathcal{P}_i(x_i) - \mathbf{e}_i) \cap \mathcal{L}_i \rangle > 0$, $i \in \mathcal{I}$, where

$$\mathcal{L}_i = \mathfrak{M}_{P_i}(\Omega, \mathbb{R}^l) = \{f \mid f : \Omega \rightarrow \mathbb{R}^l \text{ is } P_i\text{-measurable}\}, \quad i \in \mathcal{I}.$$

³That is $\exists i : \langle \mathcal{P}_i(x_i), p + q_i \rangle > \langle x_i, p + q_i \rangle \iff \langle y, p + q_i \rangle > \langle x_i, p + q_i \rangle \forall y \in \mathcal{P}_i(x_i)$.

⁴Therefore $x_i(\mathbf{w}) \in \text{int}\mathbb{R}_+^l$, $\mathbf{w} \in A$, $i \in S$.

As a result, we are going to the concept of *private equilibrium* by Definition 7.2.3 that according to the contractual point correctly corresponds to the notion of the private core. We now turn to the proof of key Lemma 8.1.1 and its corollaries.

Further, at first, let us consider some notations. I start from the affine space $\mathcal{A}(\mathbb{P})$ of all allocations measurable relative to an information structure $\mathbb{P} = \{P_i\}_{\mathcal{I}}$, where $L = (\mathbb{R}^l)^\Omega$ is a space of contingent commodities

$$\mathcal{A}(\mathbb{P}) = \{(x_i)_{i \in \mathcal{I}} \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i \ \& \ \forall i \in \mathcal{I} \ x_i : \Omega \rightarrow \mathbb{R}^l \text{ is } P_i\text{-measurable}\}.$$

Let us write the requirement of measurability of maps $x_i(\cdot)$ in detailed form $\iff x_i(\cdot)$ is a constant on the elements of partition $P_i \iff x_i(\cdot) \in \mathfrak{M}_{P_i}(\Omega, \mathbb{R}^l) \iff$

$$\forall \mathbf{w} \in \Omega, \quad x_i(\mathbf{w}') = x_i(\mathbf{w}) \quad \forall \mathbf{w}' \in P_i(\mathbf{w}). \quad (8.1.5)$$

Lemma 8.1.2 *Let P be a partition of Ω . Then orthogonal space to the subspace $\mathfrak{M}_P(\Omega, \mathbb{R}^l)$ of all P -measurable, with values in \mathbb{R}^l functions is described as:*

$$[\mathfrak{M}_P(\Omega, \mathbb{R}^l)]^\perp = \{q : \Omega \rightarrow \mathbb{R}^l \mid \forall \mathbf{w} \in \Omega \sum_{\mathbf{w}' \in P(\mathbf{w})} q(\mathbf{w}') = 0\}. \quad (8.1.6)$$

The lemma states that functional $q_i(\cdot)$ vanishes on the maps satisfying (8.1.5) if and only if for every element of partition P_i the sum of $q_i(\mathbf{w}')$ taking over $\mathbf{w}' \in P_i(\mathbf{w}) \in P_i$ is equal to zero.

Proof of Lemma 8.1.2. Let $z \in \mathfrak{M}_P(\Omega, \mathbb{R}^l)$ and q be from a dual space. Let E_1, E_2, \dots, E_k be the elements of partition P . Calculate the inner product:

$$\begin{aligned} \langle z, q \rangle &= \sum_{\mathbf{w} \in E_1} \langle z(\mathbf{w}), q(\mathbf{w}) \rangle + \sum_{\mathbf{w} \in E_2} \langle z(\mathbf{w}), q(\mathbf{w}) \rangle + \dots + \sum_{\mathbf{w} \in E_k} \langle z(\mathbf{w}), q(\mathbf{w}) \rangle = \\ &= \langle z(\mathbf{w}_1), \sum_{\mathbf{w} \in E_1} q(\mathbf{w}) \rangle + \langle z(\mathbf{w}_2), \sum_{\mathbf{w} \in E_2} q(\mathbf{w}) \rangle + \dots + \langle z(\mathbf{w}_k), \sum_{\mathbf{w} \in E_k} q(\mathbf{w}) \rangle, \end{aligned}$$

where $\mathbf{w}_j \in E_j$, $j = 1, 2, \dots, k$ are chosen in an arbitrary way. Now since the vectors $z(\mathbf{w}_j) \in \mathbb{R}^l$ can be arbitrary ones then it is necessary and sufficient for the last summation to be zero (for every z) that $\sum_{\mathbf{w} \in E_j} q(\mathbf{w}) = 0$, $j = 1, 2, \dots, k$. The proof is complete. \blacksquare

Proof of Lemma 8.1.1 (about the mutually beneficial contract). Consider the case when there is no mutually beneficial contract. This can be written in the following way. Define

$$\mathcal{V}_S = \{v = (v_i)_S \in L^S \mid v_i : \Omega \rightarrow \mathbb{R}^l \text{ is } P_i\text{-measurable},$$

$$v_i(\mathbf{w}) = 0, \mathbf{w} \notin A, i \in S, \sum_S v_i = 0\},$$

a space of all possible contracts for coalition S and such that an exchange is realized only in the limits of event A . Then

$$\nexists v \in \mathcal{V}_S : (x_i + v_i)_S \in \prod_S \mathcal{P}_i(x_i) \iff \prod_S (\mathcal{P}_i(x_i) - \{x_i\}) \cap \mathcal{V}_S = \emptyset.$$

In the latter formula, two convex sets are intersected and the first of them has a nonempty interior. Therefore, the separation theorem can be applied and one can conclude the existence of a vector (functional) which does not vanish on the subspace of coalitional allocations defined for the event $A \subseteq \Omega$ ⁵ $f = (f_i)_{i \in S} \in (L^S)' = L^S$, $f|_A \neq 0$, such that

$$\langle f, \prod_S (\mathcal{P}_i(x_i) - \{x_i\}) \rangle \geq \langle f, \mathcal{V}_S \rangle. \quad (8.1.7)$$

As soon as \mathcal{V}_S is a subspace then $\langle f, \mathcal{V}_S \rangle = 0$ and moreover via

$$\mathcal{V}_S = \prod_{i \in S} \mathcal{V}_A^i \cap \{v = (v_i)_{i \in S} \in L^S \mid \sum_S v_i = 0\},$$

where

$$\mathcal{V}_A^i = \{v_i \in L \mid v_i : \Omega \rightarrow \mathbb{R}^l \text{ is } P_i \text{-measurable, } v_i(\mathbf{w}) = 0 \forall \mathbf{w} \notin A\},$$

then f is decomposed into a sum of two functionals $f = q + p$ such that $q = (q_i)_S$ is zero on the first of intersected sets and $p = (p_i)_S$ on the second one. Now applying Lemma 8.1.2 one can conclude:

$$\begin{aligned} \langle q, \prod_{i \in S} \mathcal{V}_A^i \rangle = 0 &\Rightarrow \langle q_i, \mathcal{V}_A^i \rangle = 0 \forall i \in S \Rightarrow \\ \forall E \in P_i, E \subseteq A &\sum_{\mathbf{w} \in E} q_i(\mathbf{w}) = 0 \quad \forall i \in S; \end{aligned} \quad (8.1.8)$$

the second part of previous conclusion yields

$$\langle p, \{v = (v_i)_{i \in S} \in L^S \mid \sum_S v_i = 0\} \rangle = 0 \Rightarrow p_i = p_j = p, \forall i \neq j, i, j \in S.$$

As a result, one has:

$$\exists p \in (\mathbb{R}^l)^\Omega : \forall i \in S \exists q_i \in (\mathbb{R}^l)^\Omega \mid f_i = p + q_i \ \& \ \forall E \in P_i, E \subseteq A \sum_{\mathbf{w} \in E} q_i(\mathbf{w}) = 0.$$

On the other hand, the last one together with (8.1.7) implies $\langle f, \prod_S (\mathcal{P}_i(x_i) - \{x_i\}) \rangle \geq 0$, that in view of $0 \in \text{cl}(\mathcal{P}_i(x_i) - \{x_i\})$, $i \in \mathcal{I}$ (local non-satiation of preferences (**NS**)) leads to conclude

$$\langle f_i, \mathcal{P}_i(x_i) \rangle \geq \langle f_i, x_i \rangle, \quad i \in S.$$

⁵First one can find a *nonzero* functional $f^A = (f_i^A)_S$ for the subspace defined by A , and then one extends it on the whole space with a saving separation property.

Up to the moment, we have $\exists i \in \mathcal{I} (p + q_i)_{|A} \neq 0$. Further let us show that $p_{|A} \neq 0$ and $(p + q_i)_{|A} \neq 0$ for all i .

For beginning one can consider monotonic preferences. Now increasing consumption of each individual for a unit of every commodity in every state of the world from A , one can construct bundles strictly preferred by each agent and from the (relative) interior of $\prod_S[(L_A + x_i) \cap \mathcal{P}_i(x_i)]$. This implies that a value of f on this allocation has to be strictly more of its value on $x = (x_i)_S$, since otherwise, inequality (8.1.7) will imply that for a neighborhood of origin functional $f_{|A}$ vanishes and therefore it is equal to zero, that contradicts to the choice of functional via separation theorem. However, in view of (8.1.8) the difference in functional values is equal to $|S| \sum_{\mathbf{w} \in A} (p(\mathbf{w}) \cdot \mathbf{1}) \neq 0$,⁶ which in particular proves that $p_{|A} \neq 0$. For the general case, one has to apply the assumption (NS) (page 275) and for $i \in S$ find a vector⁷ that defines $z_i = z^A \in \mathcal{L}_i \cap L_A$ (subspace of L inducted by A and a partition P_i) such that the bundle $x_i + z_i$ is strictly preferred to x_i and moreover a train $(x_i + z_i)_S$ belongs to the interior mentioned above. Further, due to (8.1.8) and z_i is P_i measurable one concludes $\langle q_i, z_i \rangle = 0$, that implies $\langle p, \sum_S z_i \rangle = |S| \langle p, z^A \rangle \neq 0$, that gives the first result.

Further, assume $p + q_i = 0$ for some i . Let preferences be monotonic. One has $\sum_{\mathbf{w} \in A} (p(\mathbf{w}) + q_i(\mathbf{w})) \cdot \mathbf{1} = 0$, that in view of (8.1.8) implies $\sum_{\mathbf{w} \in A} (p(\mathbf{w}) \cdot \mathbf{1}) = -\sum_{\mathbf{w} \in A} (q_i(\mathbf{w}) \cdot \mathbf{1}) = 0$ that due to the previous one is impossible. General case is analyzing similarly where from $0 < \langle f, z \rangle = \langle p, \sum_S z_j \rangle = |S| \langle p + q_i, z^A \rangle$ one comes to a necessary conclusion.

The second part of Lemma's conclusion can be proven in a standard way. Assume that in lemma conditions there exists a mutually beneficial contract $v = (v_i)_S$, i.e. there is a vector having properties: $x_i + v_i \in \mathcal{P}_i(x_i) \forall i \in S$ & $\sum_S v_i = 0$. Applying (8.1.4) and summing inequalities one finds $\sum_S p(x_i + v_i) > \sum_S p x_i$ that is impossible. ■

Proof of Corollary 8.1.1 (to Lemma 8.1.1). In view of (8.1.4), for differentiated preferences in an interior point vector $f_i = p + q_i \neq 0$ has to be proportional to the gradient of the utility function. Moreover, since $f_i \neq 0$ for each $i \in S$, then the proportionality coefficient has to be *strictly* more than zero. Therefore, one can conclude:

$$\forall i \in S \quad \exists \lambda_i > 0 : \lambda_i \nabla u_i(x_i) = p + q_i,$$

that being summed over $\mathbf{w} \in E \in P_i$, $E \subseteq A$ due to (8.1.3) yields

$$\forall E \in P_i, E \subseteq A \quad \lambda_i \sum_{\mathbf{w} \in E} \nabla_{\mathbf{w}} u_i(x_i) = \sum_{\mathbf{w} \in E} p(\mathbf{w}) \quad \forall i \in S,$$

as we wanted to prove. Easy to see that the presented arguments are reversible ones. ■

⁶Here $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^l$.

⁷Notice that due to (NS) one can find a common vector for all individuals. However, the proof of this part can be provided for individualized vectors also: for the next step, to state $p + q_i \neq 0$, $i \in S$, it is problematic.

8.1.1 Scarf's theorem with fractional coalitions

In an economic model, one can simultaneously review and compare the different game-theoretic concepts, strategic games are met here with the cooperative ones. Moreover, I hold the point that the cooperative approach is more productive than the strategic one. For us, cooperation means a contractual approach. However, the classical view is the construction of a cooperative game via an economic model and then studies its properties. One of the key concepts in the cooperative theory is the core whose existence for games with non-transferable utility has been established in the famous Scarf's theorem (Scarf, 1967). The result of this theorem—nonempty core for balanced games—then with amazing efficiency is used for modeling the perfect competition conditions through replicas of the economy, and then (in particular) it allows to prove the existence of equilibrium. In this section, we consider a generalization of Scarf's theorem, generalized to games with fractional coalitions. At the same time, the proof presented here appears to be the most effective of all known in the literature. This proof extends (and updates) to the fractional case, the proof of Scarf's theorem proposed in Danilov (1999). The results of this section are of our own game-theoretic interest.

For a model of the differential information economy, one can put into correspondence some cooperative game with non-transferable utility (to be short, an NTU-game).

Cooperative NTU-game (a game with nontransferable utility, for details, see e.g. Moulin (1988)) is a couple $(\mathcal{I}, (V(S))_{S \subseteq \mathcal{I}})$, described by the set of players (agents) $\mathcal{I} = \{1, \dots, n\}$, ($n \geq 2$) and the sets of permissible vector-payoffs $V(S) \subseteq \mathbb{R}^S$ for every (nonempty) coalition $S \subset \mathcal{I}$, which have to satisfy the following properties: $\forall S \subseteq \mathcal{I}$

- $V(S)$ is the nonempty closed subset in \mathbb{R}^S ;
- $V(S)$ is comprehensive from below, i.e., $x \in V(S)$ and $y \leq x$ imply $y \in V(S)$;
- every singleton coalition has nonempty and bounded from the above possibilities, i.e., $V(\{i\}) \neq \emptyset$ and $V(\{i\}) < +\infty$, $\forall i \in \mathcal{I}$;
- the set of all individual-rational vector-payoffs from $V(S)$, this is by definition the set

$$Q(S) := \{v \in V(S) \mid v_i \geq V(\{i\}) \forall i \in S\}, \quad (8.1.9)$$

which is bounded from above in \mathbb{R}^S .

A set $\mathcal{C}(V) \subset V(\mathcal{I})$ of all vector-payoffs dominated by no coalition is said to be the *core* of the game $(\mathcal{I}, (V(S))_{S \subseteq \mathcal{I}})$, i.e., $x \in \mathcal{C}(V) \iff$

$$x \in V(\mathcal{I}) \ \& \ \nexists S \subseteq \mathcal{I}, \ S \neq \emptyset : \exists y = (y_i)_S \in V(S) \text{ such that } y_i > x_i \ \forall i \in S.$$

Theorem 8.1.1 (SCARF 1967) *Core of balanced NTU-game is nonempty.*

Next, we consider a generalization of Scarf's theorem, extending the result to games with fractional coalitions. It also introduces the concept of balanceness, which can be used in the special case of integer coalitions (membership in the coalition is given as 0 or 1), this balanceness is applied in the classical theorem of Scarf: this definition is reproduced in the next section. Games with fractional coalitions have appeared in the papers of Allouch and Florenzano, *e.g.* see ([Allouch, Florenzano, 2004](#)).

Further, let us consider a cooperative game with fractional coalitions. For natural $r \in \mathbb{N}$, a fractional coalition is defined by a vector

$$d = (d_1, d_2, \dots, d_n) : \quad r d_i \in \{0, 1, \dots, r\}, \quad i \in \mathcal{I}$$

and by a set of feasible payoff-vectors for agents non-trivially entering in the coalitions:

$$V(d) \subset \mathbb{R}^{\text{supp}(d)}, \quad \text{supp}(d) = \{i \in \mathcal{I} \mid d_i > 0\}.$$

Let D denote a set of all non-zero fractional coalitions. Like ordinary ones, fractional coalitions can be applied to dominate a current payoff vector; doing so we are going to the notion of the fractional core, this is a set

$$\mathcal{C}_q(V) = \{x \in V(\mathcal{I}) \mid \nexists d \in D : \exists y \in V(d) \text{ such that } y_i > x_i \forall i \in \text{supp}(d)\}.$$

Further we consider a generalization of Scarf's theorem to the context of the fractional coalitions. To do this, we need to extend the notion of a balanced family of coalitions.

The family of *fractional* coalitions $\mathcal{B} \subseteq D$ is called *balanced* if for each $d \in \mathcal{B}$ there is real $\lambda_d \geq 0$ such that

$$\sum_{d \in \mathcal{B}} \lambda_d d_i = 1 \quad \forall i \in \mathcal{I}$$

holds, or in an equivalent form

$$\sum_{d \in \mathcal{B}} \lambda_d d = \mathbf{1}_{\mathcal{I}}. \quad (8.1.10)$$

A game (D, V) is called *balanced*, if for every balanced family of coalitions \mathcal{B}

$$\bigcap_{d \in \mathcal{B}} \text{pr}_{|_{S(d)}}^{-1}(V(d)) \subseteq V(\mathcal{I}).$$

Here $S(d) = \text{supp}(d)$ and $\text{pr}_{|_{S(d)}}(\cdot)$ is a projecting map onto $\mathbb{R}^{S(d)}$.

Theorem 8.1.2 *Let (D, V) be a cooperative game with fractional coalitions and let ordinary requirements are satisfied for fractional coalitions (closeness, comprehensiveness from below, (8.1.9) and etc.). Then $\mathcal{C}_q(V) \neq \emptyset$ if (D, V) is balanced.*

Proof of Theorem 8.1.2. The proof is based on the application of Kakutani's fixed point theorem to a point-to-set mapping constructed in an appropriate way. This construction is presented below.

One can think without loss of generality that one-element coalitions are able to earn zero and not more, *i.e.* $V(\mathbf{1}_{\{i\}}) = (-\infty, 0] \forall i \in \mathcal{I}$. Further, consider sets

$$\tilde{V}(d) = V(d) \cap \mathbb{R}_+^{\text{supp}(d)}$$

if the intersection is nonempty and define $\tilde{V}(d) = \{0\}$ for empty intersection, $d \in D$. By assumption (8.1.9) all these sets are nonempty compacts. Therefore, there exists a real $c > 0$ such that a cube G with the side $2c$ centered in origin includes in its interior each of these sets, *i.e.*

$$\tilde{V}(d) \subset (-c, c)^{\text{supp}(d)}, \quad \forall d \in D,$$

$$G = \{z \in \mathbb{R}^{\mathcal{I}} \mid -c(1, 1, \dots, 1) \leq z \leq c(1, 1, \dots, 1)\}$$

holds.

On the cube G we define the following point-to-set mappings:⁸

$$\chi_d(z) = \begin{cases} \{2r\}, & \text{if } z_d \in \text{int}(\tilde{V}(d) - \mathbb{R}_+^{\text{supp}(d)}), \\ \{0\}, & \text{if } z_d \notin (\tilde{V}(d) - \mathbb{R}_+^{\text{supp}(d)}), \\ [0, 2r], & \text{otherwise.} \end{cases}$$

So, by definition $\chi_d(z)$ takes value $\{2r\}$ if $z_d = (z_i)_{i \in \text{supp}(d)}$ is in the interior of “corrected” set of coalition abilities $\tilde{V}(d) - \mathbb{R}_+^{\text{supp}(d)}$, is equal to the segment $[0, 2r]$ on its boundary and coincides with $\{0\}$ behind its limits. So, if a coalition employs an agent, she is ready to pay up to $2r$ units of wealth and 0 otherwise.

Now on the cube $[0, 2r]^D$ define a mapping $\varphi(\cdot)$ with values in G by formula:

$$\varphi(\Lambda) = \underset{z \in G}{\text{argmax}} \langle z, \sum_{d \in D} \lambda_d d - \mathbf{1}_{\mathcal{I}} \rangle, \quad \Lambda = (\lambda_d)_{d \in D} \in [0, 2r]^D. \quad (8.1.11)$$

This looks like the maximization of labor excess demand value. Finally, define a map of compact $G \times [0, 2r]^D$ into itself by formula

$$\psi(z, \Lambda) = \varphi(\Lambda) \times \prod_D \chi_d(z) \subset G \times [0, 2r]^D, \quad (z, \Lambda) \in G \times [0, 2r]^D.$$

One can easily conclude via construction that the map ψ has a closed graph and nonempty convex values. Thus Kakutani's fixed point theorem conditions are satisfied and one concludes the existence of a couple $(\bar{z}, \bar{\Lambda}) \in G \times [0, 2r]^D$, such that

$$(\bar{z}, \bar{\Lambda}) \in \psi(\bar{z}, \bar{\Lambda}).$$

Further, let us show that $\bar{z} \in \mathcal{C}_q(V)$. To do it first show that in the fixed point

$$\sum_{d \in D} \bar{\lambda}_d d = \mathbf{1}_{\mathcal{I}}, \quad \bar{\lambda}_d \in \chi_d(\bar{z}), \quad d \in D \quad (8.1.12)$$

⁸They are constructed by the closing of characteristic set function graph and via the taking of convex hull of their images.

holds, *i.e.*, $\bar{\Lambda}$ is a balanced family. Assuming contrary one has:

$$\sum_{d \in D} \bar{\lambda}_d d - \mathbf{1}_{\mathcal{I}} \neq 0 \Rightarrow \exists i \in \mathcal{I} : \sum_{d \in D} \bar{\lambda}_d d_i - 1 \neq 0.$$

Assume $\sum_{d \in D} \bar{\lambda}_d d_i - 1 > 0$ for an agent i . Now by (8.1.11) one has $\bar{z}_i \in \varphi_i(\bar{\Lambda}) = \{c\}$, that due to the choice of c and construction gives $\chi_d(\bar{z}) = \{0\} \forall d \in D : d_i \neq 0$ that implies $\sum_{d \in D} \bar{\lambda}_d d_i = 0$. This contradicts the assumption.

Now assume $\sum_{d \in D} \bar{\lambda}_d d_i - 1 < 0$ for some i . Now by (8.1.11) one has $\bar{z}_i \in \varphi_i(\bar{\Lambda}) = \{-c\} < 0$, that due to construction gives $\bar{\lambda}_{\mathbf{1}_{\{i\}}} = 2r \Rightarrow \sum_{d \in D} \bar{\lambda}_d d_i \geq 2r$. One has again a contradiction. So (8.1.12) is true because other possibilities cannot be realized.

Condition (8.1.12) means that the bundle $\bar{\Lambda} = (\bar{\lambda}_d)_{d \in D}$ is balanced and by $\chi_d(z)$ construction for $\bar{\lambda}_d > 0$, $\bar{\lambda}_d \in \chi_d(\bar{z})$ it has to be $\bar{z}_d \in V(d)$, that due to the game is balanced implies $\bar{z} \in V(\mathcal{I})$. Finally, if it would occur, a coalition $d \in D$ dominates \bar{z} , then it would mean that $\bar{\lambda}_d = 2r \in \chi_d(\bar{z})$ is true for d , this again contradicts to (8.1.12). So one has found a non-dominated via coalitions payoff vector \bar{z} from $V(\mathcal{I})$ and therefore $\mathcal{C}_q(V) \neq \emptyset$. \blacksquare

8.1.2 Existence of core in economy with differentiated information

In the previous section notions of contractual allocation and contractual private equilibrium were considered, they are closely related with the known in literature notions of private core and *WEE*-equilibrium. However, now we first consider (briefly) the issue of non-emptiness of core \mathcal{E}^{di} . Here we consider various core concepts, some of those described in Section 7.2.1. Further, we first recall existing definitions adapting them to our context. First, we study the existence of a core for model \mathcal{E}^{di} .

For a model of differential information economy one can put into correspondence some cooperative game with non-transferable utility (to be short, an NTU-game). Now let us consider a game construction corresponding to the concept of ex ante private core in an economic model and where informational partitions P_i^S , $i \in S$, $S \subseteq \mathcal{I}$ are applied in an appropriate way. Partition P_i^S describes information that individual $i \in S$ can apply in the finding of a dominating allocation via coalition S . In this case the set of all permissible vector-payoffs for coalition S is determined by formula

$$V(S) = \{(\vartheta_i)_{i \in S} \leq (u_i(x_i))_{i \in S} \mid (x_i)_{i \in S} \in \mathcal{A}(S) = \mathcal{A}(\mathbb{P}^S)\},$$

where

$$\mathcal{A}(\mathbb{P}^S) = \{y^S = (y_i)_S \mid \sum_S y_i = \sum_S \mathbf{e}_i$$

$$\& (y_i - \mathbf{e}_i)(\cdot) \text{ is } P_i^S \text{ - measurable, } y_i \in (\mathbb{R}_+^l)^\Omega, i \in S\}.$$

Clearly that the sets $V(S)$ satisfy all necessary conditions, it can be checked easily due to the compactness of set of feasible allocations and via the continuity of utilities in the initial economic model.

Recall that the family \mathcal{B} of subsets in \mathcal{I} is said to be *balanced*, if for every $S \in \mathcal{B}$ there is a real $\lambda_S \geq 0$, such that

$$\sum_{S \in \mathcal{B}: i \in S} \lambda_S = 1 \quad \forall i \in \mathcal{I} \iff \sum_{S \in \mathcal{B}} \lambda_S \mathbf{1}_S = \mathbf{1}_{\mathcal{I}}$$

takes place where, by definition, $\mathbf{1}_S \in \mathbb{R}^{\mathcal{I}}$ is such a vector that $(\mathbf{1}_S)_i = 1$ for $i \in S$ and $(\mathbf{1}_S)_i = 0$ if $i \notin S$, *i.e.*, this is the indicator-function of the set S .

A game $(\mathcal{I}, (V(S))_{S \subseteq \mathcal{I}})$ is said to be *balanced* if for every balanced family \mathcal{B} of coalitions

$$\bigcap_{S \in \mathcal{B}} \text{pr}_{|_S}^{-1}(V(S)) \subseteq V(\mathcal{I}).$$

Here $\text{pr}_{|_S}(\cdot)$ is the projection map of space $\mathbb{R}^{\mathcal{I}}$ onto \mathbb{R}^S .

In view of Scarf's theorem (see previous section) the core of a balanced game $(\mathcal{I}, (V(S))_{S \subseteq \mathcal{I}})$ is nonempty. Applying this theorem and using standard arguments, one can prove the following

Proposition 8.1.2 *Let $P_i^S \preceq P_i^{\mathcal{I}}$ for every (admissible) coalition $S \subseteq \mathcal{I}$ and each $i \in S$ and let agents' preferences be defined via concave continuous utility functions.⁹ Then $\mathcal{C}(\mathcal{E}^{di}) \neq \emptyset$.*

Proof of Proposition 8.1.2. The only thing that really needs to be accurately considered is ensuring a game constructed via a model of economy is balanced one. Let \mathcal{B} be any balanced family and let $(\lambda_S)_{S \in \mathcal{B}}$ be a corresponding family of balancing coefficients. Let $v = (v_i)_{\mathcal{I}} \in \mathbb{R}^{\mathcal{I}}$ be a vector such that $\forall S \in \mathcal{B}$

$$\exists y^S = (y_i^S)_S \in \mathcal{A}(S) \mid v_i \leq u_i(y_i^S), \quad \forall i \in S$$

is true. Now by definition of balanced family for any i we have $\lambda_S v_i \leq \lambda_S u_i(y_i^S)$, that being summed by coalitions from \mathcal{B} including i , due to $u_i(\cdot)$ is a concave function, yields

$$\vartheta_i = \sum_{S \in \mathcal{B}: i \in S} \lambda_S v_i \leq \sum_{S \in \mathcal{B}: i \in S} \lambda_S u_i(y_i^S) \leq u_i \left(\sum_{S \in \mathcal{B}: i \in S} \lambda_S y_i^S \right).$$

Moreover,

$$(y_i^S - \mathbf{e}_i) \text{ is } P_i^S \text{ - measurable, } \forall S \in \mathcal{B} : i \in S \Rightarrow \\ \sum_{S \in \mathcal{B}: i \in S} \lambda_S (y_i^S - \mathbf{e}_i) \text{ is } \bigvee_{S \in \mathcal{B}} P_i^S \text{ - measurable,}$$

and in view of assumption $\bigvee_{S \in \mathcal{B}} P_i^S \preceq P_i^{\mathcal{I}}$, that for $x_i^{\mathcal{B}} = \sum_{S \in \mathcal{B}: i \in S} \lambda_S y_i^S$ implies

$$(x_i^{\mathcal{B}} - \mathbf{e}_i) \text{ is } P_i^{\mathcal{I}} \text{ - measurable, } \forall i \in \mathcal{I} \Rightarrow (x_i^{\mathcal{B}})_{\mathcal{I}} \in \mathcal{A}(\mathcal{I}).$$

As soon as we had $\vartheta_i \leq u_i(x_i^{\mathcal{B}})$, $i \in \mathcal{I}$, then now by game's definition we conclude $\vartheta \in V(\mathcal{I})$ as wanted to prove. ■

⁹It is enough to have the compactness of the set $\mathcal{A}(\mathbb{P}^{\mathcal{I}})$ of all feasible for grand coalition allocations. Now we have it by a definition of the model.

As a corollary to the proposition one can conclude core is nonempty in an economy such that for grand coalition allocations measurable relative to limit information are considered and for a coalitional domination—relative to information achieved via any fixed sequence of coalitions information sharing, the sequence is fixed for the definition. In this context we have got the definitions:

Definition 8.1.2 Let $\alpha = \{S_1, S_2, \dots, S_m\}$ be an arbitrary (possibly empty) sequence of coalitions and let $\mathbb{P} = \{P_i\}_{i \in \mathcal{I}}$ be an information structure. Define $\mathbb{P}^\alpha = \{P_i^\alpha\}_{i \in \mathcal{I}}$, where $P_i^\alpha = k_{S_m}^i(k_{S_{m-1}}^i(\dots k_{S_1}^i(\mathbb{P})))$, $i \in \mathcal{I}$.

Limit core $\mathcal{C}_{k\alpha}^{lim}(\mathcal{E}^{di})$ of **$k\alpha$ -type** consists of allocations $x = (x_i)_{\mathcal{I}} \in X$ such that:

$$(i) \quad \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i,$$

$$(ii) \quad (x_i - \mathbf{e}_i) : \Omega \rightarrow \mathbb{R}^l \text{ is } P_i^{lim}\text{-measurable for all } i \in \mathcal{I}.$$

$$(iii) \quad \nexists S \subseteq \mathcal{I} : \exists y^S = (y_i)_S \mid \forall i \in S, y_i \in X_i \text{ is such that } (y_i - \mathbf{e}_i) \text{ is } k_S^i(\mathbb{P}^\alpha)\text{-measurable, } y_i \succ_i x_i \text{ \& } \sum_S (y_i - \mathbf{e}_i) = 0.$$

Limit core of $k\alpha$ -type for $\alpha = \emptyset$ is called **limit k-core**.

Limit core $\mathcal{C}^{lim}(\mathcal{E}^{di})$ consists of allocations, which belong to every $k\alpha$ -core for all sequences α , i.e.

$$\mathcal{C}^{lim}(\mathcal{E}^{di}) = \bigcap_{\alpha} \mathcal{C}_{k\alpha}^{lim}(\mathcal{E}^{di}).$$

In core definition of $k\alpha$ type the sequence $\alpha = \{S_1, S_2, \dots, S_m\}$ presents a background (prehistory) of informational exchange acts accumulated to the time of domination. With this information, members of the coalition $S \subseteq \mathcal{I}$ apply again the rule k_S of information sharing and, if possible, dominate the current allocation. Accordingly, limit core only includes such allocations, which the coalition is not able to dominate in any prehistory of information exchanges.

Applying Proposition 8.1.2 and due to the limit information is unique (Theorem 7.1.1 of Section 7.1.2) one concludes

Corollary 8.1.3 Let consumers' preferences be defined via concave continuous utility functions and information rule $k = (k_S)_{S \in C}$ be a monotonous rule of information sharing. Then limit core and limit k-core are nonempty.

Proof of Corollary 8.1.3. What has happened with limit core? Note that the family $\mathcal{C}_{k\alpha}^{lim}(\mathcal{E}^{di})$ is filtered (centered) on α : for any finite family $\alpha_1, \alpha_2, \dots, \alpha_t$ there is a sequence $\bar{\alpha}$ such that

$$\mathcal{C}_{k\bar{\alpha}}^{lim}(\mathcal{E}^{di}) \subseteq \mathcal{C}_{k\alpha_j}^{lim}(\mathcal{E}^{di})$$

for all $j = 1, 2, \dots, t$. Hence, since the set of all allocations is compact, the intersection of the cores of type $k\alpha$ is not empty. ■

8.1.3 Private core and equilibria under perfect competition

However, the relationship between private equilibrium and its implementing contractual allocation was not clearly identified: it is necessary to clarify perfect competition case when correctly defined core and equilibrium have to coincide. While we are looking for the answer to this quest, we also clarify conditions under which ex ante private equilibria do exist.

Below standard analytic methods are applied, they are based on the study of replicated economies. Further, first recall known definitions adapting them to our context.

Let us turn now to replicated models. A differential information economy replica of volume $r \in \mathbb{N}$ is called the economy \mathcal{E}_r^{di} , in which r exact copies of each consumer from initial model \mathcal{E}^{di} are put into correspondence in \mathcal{E}_r^{di} . The agents from \mathcal{E}_r^{di} are numbered by double index (i, m) , $i \in \mathcal{I}$, $m = 1, \dots, r$, and it is put $X_{im} = X_i$, $\mathbf{e}_{im} = \mathbf{e}_i$. Agents' preferences are defined and take values in X_{im} due to identification $\mathcal{P}_{im} = \mathcal{P}_i$. An information that agents have in a replica exactly repeats the initial one and is the same for agents of the same types, *i.e.*, $P_{im} = P_i, \forall i, m$. To an initial economy \mathcal{E}^{di} allocation $x = (x_i)_{\mathcal{I}}$, we can put into correspondence the replicated economy allocation $x^r = (x_{im}^r)$ by the rule $x_{im} = x_i, \forall i, m$.

Definition 8.1.3 *An allocation $x = (x_i)_{\mathcal{I}}$ is called ex ante private Edgeworth equilibrium for model \mathcal{E}^{di} if $x^r \in \mathcal{C}(\mathcal{E}_r^{di})$ for every natural $r = 1, 2, \dots$. The set of all ex ante Edgeworth equilibria is denoted as $\mathcal{C}^e(\mathcal{E}^{di})$.*

Now let us consider the most characteristic properties of Edgeworth equilibria: their existence and relationships with ex ante private equilibria.

Theorem 8.1.3 *Let every agent's preference be determined via continuous and concave utility function on $(\mathbb{R}_+^l)^\Omega$. Then ex ante private Edgeworth equilibria exist.*

Remark 8.1.2 Probably it is not a strongest possible existence result,¹⁰ however this is still important and non-trivial result that is simply formulated and clearly proven.

■

Proof of Theorem 8.1.3. The result will be stated if one shows that symmetric part of core

$$\{x = (x_{im})_{m=1, i \in \mathcal{I}}^{m=r} \in \mathcal{C}(\mathcal{E}_r^{di}) \mid x_{im} = x_{im'} \forall m, m' = 1, \dots, r, \forall i \in \mathcal{I}\} = \mathbb{S}(\mathcal{C}(\mathcal{E}_r^{di}))$$

is nonempty for each r -replica of a studied model. To do it let us put into correspondence to economy \mathcal{E}_r^{di} a cooperative game of n -persons with the fractional coalitions. For a coalition $d = (d_1, d_2, \dots, d_n)$, $rd_i \in \{0, 1, \dots, r\} \forall i \in \mathcal{I}$ define the set

$$\mathcal{A}(d) = \{y^d = (y_i)_{\mathcal{I}} \in L_+^{\mathcal{I}} \mid$$

¹⁰One needs only the continuity and convexity of preferences (ie convex upper level sets) but the existence of utility functions does not play a special role although it is applied in presented proof. It is also important that the set of all feasible allocations is compact but the fact that commodity space is finite dimensional is not an essential factor.

$$\sum_{\mathcal{I}} d_i y_i = \sum_{\mathcal{I}} d_i \mathbf{e}_i \quad \& \quad (y_i - \mathbf{e}_i)(\cdot) \text{ is } P_i^0 \text{ - measurable, } y_i \in (\mathbb{R}_+^l)^\Omega, \quad i \in \mathcal{I} \}.$$

This set differs from the set for ordinary coalitions in the part of inter-coalitional allocations that correspond to allocation for a coalition of replica economy. Further, define

$$V(d) = \{(\vartheta_i)_{i \in \text{supp}(d)} \leq (u_i(x_i^d))_{i \in \text{supp}(d)} \mid x^d \in \mathcal{A}(d)\}.$$

Now to apply Theorem 8.1.2 it will enough to check the constructed game (D, V) is balanced one. Let us do it.

Let $\mathcal{B} \subseteq D$ be a *balanced* family of fractional coalitions and let $\lambda_d \geq 0$, $d \in \mathcal{B}$ be scalar coefficients system satisfying (8.1.10). Let $v \in \mathbb{R}^{\mathcal{I}}$ be such that $\forall d \in D$

$$(\vartheta_i)_{i \in \text{supp}(d)} \in V(d) \iff \exists x^d \in \mathcal{A}(d) : (\vartheta_i)_{i \in \text{supp}(d)} \leq (u_i(x_i^d))_{i \in \text{supp}(d)}.$$

Consider an allocation for grand coalition defined by formulas:

$$z_i = \sum_{d \in \mathcal{B}} \lambda_d d_i (x_i^d - \mathbf{e}_i), \quad i \in \mathcal{I}.$$

By construction $z_i(\cdot)$ is P_i -measurable and

$$\sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \left[\sum_{d \in \mathcal{B}} \lambda_d d_i (x_i^d - \mathbf{e}_i) \right] = \sum_{d \in \mathcal{B}} \lambda_d \sum_{i \in \mathcal{I}} d_i (x_i^d - \mathbf{e}_i) = 0$$

holds. Therefore, $z = (z_i)_{i \in \mathcal{I}}$ is a permissible contract and in view of $\sum_{d \in \mathcal{B}} \lambda_d d_i = 1$

$$z_i + \mathbf{e}_i = \sum_{d \in \mathcal{B}} \lambda_d d_i x_i^d \in (\mathbb{R}_+^l)^\Omega, \quad i \in \mathcal{I}$$

takes place. Further multiply inequality $\vartheta_i \leq u_i(x_i^d)$ on $\lambda_d d_i \geq 0$ and sum the products by $d \in \mathcal{B}$ to obtain

$$\vartheta_i = \left(\sum_{d \in \mathcal{B}} \lambda_d d_i \right) \vartheta_i \leq \sum_{d \in \mathcal{B}} \lambda_d d_i u_i(x_i^d) \leq u_i \left(\sum_{d \in \mathcal{B}} \lambda_d d_i x_i^d \right) = u_i(z_i + \mathbf{e}_i), \quad \forall i \in \mathcal{I}.$$

By definition of $V(\mathcal{I})$ this means $\vartheta \in V(\mathcal{I})$ and therefore the game is balanced. Now by Theorem 8.1.2 there is some $\bar{\vartheta} \in \mathcal{C}_q(V)$ such that $\exists x \in \mathcal{A}(\mathcal{I})$:

$$\bar{\vartheta} = (u_i(x_i))_{\mathcal{I}} \quad \& \quad \nexists d \in D \mid \exists y^d \in \mathcal{A}(d) : u_i(x_i) < u_i(y_i^d), \quad i \in \text{supp}(d). \quad (8.1.13)$$

Now let us show that $x^r = (x, x, \dots, x) \in \mathbb{S}(\mathcal{C}(\mathcal{E}_r^{di}))$. Assume contrary and let $T \subseteq \mathcal{I} \times \{1, \dots, r\}$ be a dominating coalition, *i.e.*

$$\exists z^T = (z_{im})_{(i,m) \in T} : u_i(z_{im}) > u_i(x_i) \quad \forall (i, m) \in T.$$

Further, define $T(i) = \{m \in \{1, \dots, r\} \mid (i, m) \in T\}$ where $|T(i)| = \text{card}(T(i))$ is a number of its elements and let $d_i = \frac{|T(i)|}{r}$, $i \in \mathcal{I}$ define a fractional coalition $d = (d_1, d_2, \dots, d_n)$. Further, define allocation $y^d \in \mathcal{A}(d)$ putting

$$y_i^d = \frac{1}{|T(i)|} \sum_{m \in T(i)} z_{im}, \quad i \in \text{supp}(d).$$

Now in view of the convexity of preferences (concave utilities) and since T dominates x^r conclude

$$y_i^d \succ_i x_i \quad \forall i \in \text{supp}(d),$$

that contradicts to (8.1.13). This finishes the proof that the symmetric part of every replicated economy is nonempty. Projecting this set into the space of allocations one concludes that for every $r \in \mathbb{N}$ sets

$$\{x \in \mathcal{A}(\mathcal{I}) \mid x^r \in \mathcal{C}(\mathcal{E}_r^{di})\}$$

are nonempty and compacts. Now since these sets are included one to another decreasing when r is risen, then by the lemma on included compacts their intersection is nonempty: as we wanted to prove. ■

The next stage of analysis is to put into correspondence for Edgeworth equilibria the elements of fuzzy core and then to characterize them in value units. It turns out that the elements of fuzzy core coincide with Edgeworth equilibria and they are private quasi-equilibria in fact. Clearly we have to realize this program taking into account non-symmetrically distributed information.

Recall that any vector

$$t = (t_1, \dots, t_n) \neq 0, \quad 0 \leq t_i \leq 1 \quad \forall i \in \mathcal{I}$$

may be identified with a fuzzy coalition, where the real number t_i being interpreted as the measure of agent i in the coalition. A coalition t is said to dominate (block) an allocation $x \in \mathcal{A}(X)$ if there exists $y^t \in \prod_{\mathcal{I}} X_i$ such that

$$y_i^t - \mathbf{e}_i \in \mathcal{L}_i, \quad \forall i \in \mathcal{I} \quad \& \quad \sum_{i \in \mathcal{I}} t_i y_i^t = \sum_{i \in \mathcal{I}} t_i \mathbf{e}_i \iff \sum_{i \in \mathcal{I}} t_i (y_i^t - \mathbf{e}_i) = 0 \quad (8.1.14)$$

and

$$y_i^t \succ_i x_i \quad \forall i \in \text{supp}(t) = \{i \in \mathcal{I} \mid t_i > 0\}. \quad (8.1.15)$$

For non-satiated on \mathcal{L}_i preferences, conditions (8.1.14), (8.1.15) can be equivalently rewritten in the form¹¹

$$0 \in \sum_{i \in \mathcal{I}} t_i [(\mathcal{P}_i(x_i) - \mathbf{e}_i) \cap \mathcal{L}_i].$$

The set of all feasible allocations which cannot be dominated by fuzzy coalitions is denoted by $\mathcal{C}^f(\mathcal{E}^{di})$ and is called the *fuzzy core*. If preferences are convex and non-satiated, then it can be characterized in the following way:

$$x \in \mathcal{C}^f(\mathcal{E}^{di}) \iff 0 \notin \text{co} \bigcup_{i \in \mathcal{I}} [(\mathcal{P}_i(x_i) - \mathbf{e}_i) \cap \mathcal{L}_i]. \quad (8.1.16)$$

¹¹Admitting inaccuracy here and below we will sometimes identify a vector with a single element set, including it.

Theorem 8.1.4 *Let $\mathcal{P}_i(x_i)$ be convex and open in X_i for each agent $i \in \mathcal{I}$ and every $x = (x_i)_{\mathcal{I}} \in \mathcal{A}(X)$. Then fuzzy core coincides with the set of all ex ante private Edgeworth equilibria, i.e.*

$$\mathcal{C}^e(\mathcal{E}^{di}) = \mathcal{C}^f(\mathcal{E}^{di}).$$

Proof of Theorem 8.1.4. It is enough to show that $\mathcal{C}^e(\mathcal{E}^{di}) \subseteq \mathcal{C}^f(\mathcal{E}^{di})$ to state the theorem. Further argumentation has owned specifics, but in general it is rather standard, e.g. compare with Theorem 3.2.5 proof, p. 163.

Assume contrary and find $x \in \mathcal{C}^e(\mathcal{E}^{di})$, which is dominated by a fuzzy coalition $t \neq 0$. Due to definition there is $y^t \in \prod_{\mathcal{I}} X_i$, such that (8.1.14), (8.1.15) hold. Further, for $t_i > 0$ define

$$z_i = (t_i/s_i)y_i^t + (1 - t_i/s_i)\mathbf{e}_i \iff s_i(z_i - \mathbf{e}_i) = t_i(y_i^t - \mathbf{e}_i),$$

where *rational* s_i are such that $t_i \leq s_i \leq 1$ is true and for $t_i = 0$ take $s_i = 0$. By construction, $z_s = (z_i)_{\mathcal{I}} \in \prod_{\mathcal{I}} X_i$ and for each individual the map $z_i - \mathbf{e}_i$ is measurable relative to the same partition (algebra) as is measurable $y_i^t - \mathbf{e}_i$ and

$$\sum_{i \in \mathcal{I}} s_i(z_i - \mathbf{e}_i) = 0$$

holds. Since $\mathcal{P}_i(x)$ are convex, and relatively open by assumptions then numbers s_i can be chosen so that $z_i \in \mathcal{P}_i(x)$ is true for all i satisfying $s_i > 0$. However, this contradicts to the choice of $x \in \mathcal{C}^e(\mathcal{E}^{di})$. Theorem 8.1.4 is proven. \blacksquare

As a supplement to (8.1.16), we also need an additional characterization of fuzzy core. Let us consider the sets

$$\Upsilon_i(x_i) = \text{co}(\mathcal{P}_i(x_i) \cup \{\mathbf{e}_i\}), \quad i \in \mathcal{I}.$$

In Section 1.2.3 above, these sets were applied to give a characterization of fuzzy core for a model with symmetrically distributed information. Being adopted to an asymmetrical case, it gives:

Proposition 8.1.3 *An allocation $x \in \mathcal{A}(X)$ is the element of fuzzy core if and only if for $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ relation*

$$\prod_{\mathcal{I}} \Upsilon_i(x_i) \cap \{(z_1, \dots, z_n) \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\} \cap \left(\prod_{\mathcal{I}} \mathcal{L}_i + (\mathbf{e}_1, \dots, \mathbf{e}_n) \right) = \{\mathbf{e}\}^{12} \quad (8.1.17)$$

is true.

Applying (8.1.17) one can prove the following corollary convenient in various applications. This will allow us to state the main result of this section. Also notice that content and arguments in this corollary are similar to applying in the proof of Lemma 1.2.2 from the first chapter of the monograph.

¹²Notice that characterization (8.1.17) is also valid for satiated preferences.

Corollary 8.1.4 *Let $x \in \mathcal{A}(X)$ and $\mathcal{P}_i(x_i) \neq \emptyset$ for all $i \in \mathcal{I}$. Then $x \in \mathcal{C}^f(\mathcal{E}^{di})$ implies:*

$$\prod_{\mathcal{I}} (\mathcal{P}_i(x) + \text{co}\{0, \mathbf{e}_i - x_i\}) \cap \bigcap \{(z_1, \dots, z_n) \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\} \cap (\prod_{\mathcal{I}} \mathcal{L}_i + (\mathbf{e}_1, \dots, \mathbf{e}_n)) = \emptyset. \quad (8.1.18)$$

Proof of Corollary 8.1.4. The argument in the proving of this result is based on Propositions 8.1.3 and relation (8.1.17), which characterizes fuzzy core elements. We need to show that (8.1.17) implies (8.1.18).

Assume that x satisfies (8.1.17) and suppose that (8.1.18) is false. This implies that there is a vector $t = (t_1, \dots, t_n)$, $0 \leq t_i \leq 1$ and bundles $z_i \succ_i x_i$, $i \in \mathcal{I}$ such that

$$\sum_{\mathcal{I}} z_i + \sum_{\mathcal{I}} t_i(\mathbf{e}_i - x_i) = \sum_{\mathcal{I}} \mathbf{e}_i, \quad z_i - \mathbf{e}_i \in \mathcal{L}_i, \quad i \in \mathcal{I}, \quad (8.1.19)$$

hold. Now for a real $0 < \beta \leq \frac{1}{2}$ consider the vector $y = y(\beta) = (y_i)_{i \in \mathcal{I}}$, where

$$y_i(\beta) = \beta[z_i + t_i(\mathbf{e}_i - x_i)] + (1 - \beta)x_i, \quad i \in \mathcal{I}.$$

In view of (8.1.19), and $x \in \mathcal{A}(X)$ we have $\sum_{\mathcal{I}} y_i(\beta) = \sum_{\mathcal{I}} \mathbf{e}_i$ and $y_i(\beta) - \mathbf{e}_i \in \mathcal{L}_i$, $i \in \mathcal{I}$ for every β . Now vectors $y_i(\beta)$ can be presented in the form

$$y_i(\beta) = (1 - \beta t_i)x_i + \beta t_i \mathbf{e}_i + (1 - \beta t_i) \frac{\beta}{1 - \beta t_i} (z_i - x_i), \quad i \in \mathcal{I},$$

whereby the choice of β we have $\mu_i = \frac{\beta}{1 - \beta t_i} \leq 1$. This, due to preferences assumptions for $i \in \mathcal{I}$, implies

$$\mu_i(z_i - x_i) \in \mathcal{P}_i(x) - x_i \Rightarrow \exists \eta_i \in \mathcal{P}_i(x) : \mu_i(z_i - x_i) = \eta_i - x_i.$$

Therefore, the previous formula gives

$$y_i = (1 - \beta t_i)\eta_i + \beta t_i \mathbf{e}_i,$$

that implies $y_i \in \Upsilon_i(x_i)$, $i \in \mathcal{I}$. This allows us to apply relation (8.1.17), concluding $y = y(\beta) = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ for all real $0 < \beta \leq \frac{1}{2}$. Write this equality componentwise and due to $y_i(\beta)$ specification find

$$\beta[z_i + t_i(\mathbf{e}_i - x_i)] + (1 - \beta)x_i = \mathbf{e}_i \Rightarrow z_i + t_i(\mathbf{e}_i - x_i) = x_i + \frac{\mathbf{e}_i - x_i}{\beta}$$

that has to be true for all $i \in \mathcal{I}$ and all $0 < \beta \leq \frac{1}{2}$. However, these equalities (consider different β) can be true only if $x_i = \mathbf{e}_i = z_i$, $i \in \mathcal{I}$, that due to the choice of z_i implies $x_i \succ_i x_i$ and contradicts to assumptions on preferences. Proof is completed. ■

Theorem 8.1.5 *Every ex ante private Edgeworth equilibrium is a quasi-equilibrium.*

This theorem together with Theorem 8.1.3 presents the main result of the section and says that ex ante private core shrinks to equilibria of the same type and defined by Definition 7.2.3. However, for this statement to have a perfect mathematical formalization it is necessary for the theorem conditions to complete with requirements that guarantee every quasi-equilibrium is an equilibrium in fact. Notice also that an attempt to prove this theorem starting from characterization (8.1.16) encounters with serious mathematical difficulties because the sets $(\mathcal{P}_i(x_i) - \mathbf{e}_i) \cap \mathcal{L}_i$ have *empty* interior and therefore one cannot guarantee that functional separating a set from the right-hand side of (8.1.16) and point '0' is not zero on \mathcal{L}_i , that contradicts to condition (ii) by quasi-equilibrium Definition 7.2.3.

Proof of Theorem 8.1.5. Take $x \in \mathcal{C}^e(\mathcal{E}^{di})$. Apply Corollary 8.1.4 and formula (8.1.18). Now separation theorem can be applied to find a functional (vector) $f = (f_1, f_2, \dots, f_n) \in L^{\mathcal{I}}$, $f \neq 0$ separating the left-hand side set in (8.1.18) from the intersection of others. One has:

$$\begin{aligned} & \langle f, \prod_{\mathcal{I}} (\mathcal{P}_i(x) + \text{co}\{0, \mathbf{e}_i - x_i\}) \rangle \geq \\ & \geq \langle f, \{(z_1, \dots, z_n) \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\} \cap (\prod_{\mathcal{I}} \mathcal{L}_i + (\mathbf{e}_1, \dots, \mathbf{e}_n)) \rangle. \end{aligned} \quad (8.1.20)$$

As soon as in the right-hand side of the inequality there are functional values presented on non-trivial affine subspace, then the functional $f(\cdot)$ has to be constant on the subspace that is equivalent to

$$\langle f, \prod_{\mathcal{I}} \mathcal{L}_i \cap \{(z_1, \dots, z_n) \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = 0\} \rangle = 0.$$

This implies that f is decomposed into a sum of two functionals $f = q + p$ such that $q = (q_i)_{\mathcal{I}}$ is zero on the first of intersected in last formula sets and $p = (p_i)_{\mathcal{I}}$ is zero on the second one. This in view of Lemma 8.1.2 gives:

$$\begin{aligned} \langle q, \prod_{i \in \mathcal{I}} \mathcal{L}_i \rangle = 0 & \Rightarrow \langle q_i, \mathcal{L}_i \rangle = 0 \quad \forall i \in \mathcal{I} \Rightarrow \\ \forall E \in P_i, \sum_{\mathbf{w} \in E} q_i(\mathbf{w}) & = 0 \quad \forall i \in \mathcal{I}; \end{aligned} \quad (8.1.21)$$

the second part of conclusion yields

$$\langle p, \{v = (v_i)_{i \in \mathcal{I}} \in L^{\mathcal{I}} \mid \sum_{\mathcal{I}} v_i = 0\} \rangle = 0 \Rightarrow p_i = p_j = p, \quad \forall i, j \in \mathcal{I}.$$

As a result, one has:

$$\exists p \in (\mathbb{R}^l)^\Omega : \quad \forall i \in \mathcal{I} \quad \exists q_i \in (\mathbb{R}^l)^\Omega \mid f_i = p + q_i \quad \& \quad \forall E \in P_i, \quad \sum_{\mathbf{w} \in E} q_i(\mathbf{w}) = 0.$$

Further one can realize standard argumentation as earlier in Lemma 8.1.1 (about mutually beneficial contract) and state an analog of formula (8.1.4). Computing the right-hand part in (8.1.20) accounting for the obtained conclusions one finds $\sum_{j \in \mathcal{I}} \langle p + q_j, \mathbf{e}_j \rangle$. On the other hand, take $\mathbf{e}_j = x_j + (\mathbf{e}_j - x_j) \in \text{cl}(\mathcal{P}_j(x) + \text{co}\{0, \mathbf{e}_j - x_j\})$ for $j \neq i$ and $z_i + (\mathbf{e}_i - x_i) \in (\mathcal{P}_i(x) + \text{co}\{0, \mathbf{e}_i - x_i\})$ and substitute this to the left-hand side of (8.1.20), one obtains $\langle p + q_i, z_i - x_i \rangle + \sum_{j \in \mathcal{I}} \langle p + q_j, \mathbf{e}_j \rangle$. Comparing further the value with the functional value of the right-hand side on $\mathbf{e} = (\mathbf{e}_j)_{\mathcal{I}}$ and then reducing identical terms one can conclude, taking into account that the choice of $z_i \in \mathcal{P}_i(x)$ is arbitrary,

$$\langle p + q_i, \mathcal{P}_i(x_i) \rangle \geq \langle p + q_i, x_i \rangle \quad \& \quad p + q_i \neq 0 \quad \forall i \in \mathcal{I}.$$

Here the latter inequality and $p \neq 0$ can be proven as it was done in Lemma 8.1.1. Finally, to prove budget equalities in the last argumentation for given $i \in \mathcal{I}$ instead of $(\mathbf{e}_i - x_i) \in \text{co}\{0, \mathbf{e}_i - x_i\}$ take $0 \in \text{co}\{0, \mathbf{e}_i - x_i\}$ concluding then

$$\langle p + q_i, \mathcal{P}_i(x_i) \rangle \geq \langle p + q_i, \mathbf{e}_i \rangle \quad \forall i \in \mathcal{I}.$$

Now since $x_i \in \text{cl}\mathcal{P}_i(x)$ then $\langle p + q_i, x_i \rangle \geq \langle p + q_i, \mathbf{e}_i \rangle \Rightarrow \langle p + q_i, x_i - \mathbf{e}_i \rangle \geq 0$ that in view of (8.1.21) and $x_i - \mathbf{e}_i \in \mathcal{L}_i$ yields $\langle p, x_i - \mathbf{e}_i \rangle \geq 0$, $i \in \mathcal{I}$. However due to allocation is feasible the last one is possible only then $\langle p, x_i \rangle = \langle p, \mathbf{e}_i \rangle$, $\forall i \in \mathcal{I}$. Theorem 3.2.6 is proven. \blacksquare

There is another and purely contractual presentation of private equilibrium and, therefore, *WEE*-equilibrium. This presentation is based on the notion of fuzzy contractual allocation, see Definition 1.1.7 on page 31. I firmly believe that in general this is a meaningful, substantial notion that is more natural than Edgeworth equilibrium. Moreover, for an economy with asymmetrically distributed information the concept of fuzzy contractual allocation is so efficient as for symmetrical case: the only difference with Definition 1.1.7 is an additional measurement requirement for consumption flows received from a contract, this is measurement relative to individual (private) information. For the model \mathcal{E}^{di} , this way produced notion has the following characteristic property.

Proposition 8.1.4 *A feasible allocation $x = v + \mathbf{e} \in X$, $v \in \prod_{\mathcal{I}} \mathcal{L}_i$ is fuzzy contractual if and only if*

$$\text{co}\{x_i, \mathbf{e}_i\} \cap \mathcal{P}_i(x_i) = \emptyset, \quad \forall i \in \mathcal{I}, \quad (8.1.22)$$

$$\begin{aligned} & \prod_{\mathcal{I}} [(\mathcal{P}_i(x_i) + \text{co}\{0, \mathbf{e}_i - x_i\}) \cup \{\mathbf{e}_i\}] \cap \{(z_1, \dots, z_n) \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\} \cap \\ & \quad \cap \left(\prod_{\mathcal{I}} \mathcal{L}_i + (\mathbf{e}_1, \dots, \mathbf{e}_n) \right) = \{(\mathbf{e}_1, \dots, \mathbf{e}_n)\}. \end{aligned} \quad (8.1.23)$$

One sees the only difference with symmetric case characterization: in formula (8.1.23) a new element is appeared in intersection, that defines aggregated contract $z - \mathbf{e}$ to be

measurable. The proof of Proposition 8.1.4 follows the arguments of Proposition 1.2.2. From

$$\Upsilon_i(x_i) \subset (\mathcal{P}_i(x_i) + \text{co}\{0, \mathbf{e}_i - x_i\}) \cup \{\mathbf{e}_i\}, \quad \forall i \in \mathcal{I}$$

and due to Proposition 8.1.3 one can immediately conclude that fuzzy contractual allocations are the elements of fuzzy core. On the other hand, Corollary 1.2.2 shows that (compare (8.1.18) with (8.1.23)) the difference is negligible.

Conclusion to Chapter 8

The main focus of the analysis was the private core and equilibria. It was shown that according to a contractual point of view these concepts are well match each other. Moreover, this conclusion was confirmed by the standard analysis: via the modeling of perfect competition through replicas and analyzing of Edgeworth equilibria for a model with asymmetrically distributed information. It was shown that Edgeworth equilibria are private quasi-equilibria that is one of the most important results of the section. In addition, we studied limit core and its associated concepts. A theorem on existence (non-emptiness) of the core was proven. Mathematical results on the topic:

- Lemma 8.1.1 on the existence of beneficial contract.
- Theorem 8.1.2 on the non-emptiness of cooperative game core with fractional coalitions and the theorem on core (Corollary 8.1.3 to Proposition 8.1.2) of the economy with allocations measurable relative to the limit information and for blocking via allocations measurable relative to the information of the finite level (there is a specific finite chain of coalitions of a fixed length): balanceness.
- Edgeworth equilibria (Theorem 8.1.3 on existence), fuzzy core (characterization in Proposition 8.1.3) and a limit theorem on the coincidence of the core and equilibrium (Theorem 8.1.5, all Edgeworth equilibria are ex ante private equilibria).

Chapter 9

Contractual interim core and equilibrium

In the subsequent analysis, it will be useful to have the following temporary submissions on various stages of contractual processes in an economy with the differentiated information.

Timing of contractual processes

<i>F</i>		<i>A</i>		\dots		<i>B</i>		<i>F</i>
Today afternoon: Consumption of contracts	\Rightarrow	Today after afternoon: New contracting	\Rightarrow	Night	\Rightarrow	Tomorrow before afternoon: Recontracting	\Rightarrow	Tomorrow afternoon: Consumption of contracts

It can be seen from the diagram that the consumption of products under contractual deliveries is made each day in the afternoon. After the afternoon, a new contractual process is opened, and it ends at the end of the day. Pass the night and next morning the contractual process may be reopened: the situation has changed, and the uncertainty is solved (partially), but not every individual is able to understand the result in every detail, there is still not enough information. In the afternoon all uncertainties are finally cleared, but the contractual process has already been completed: there are deliveries and consumption of products under agreements previously concluded.

Of course, the fairly conventional cycle of the contractual process in its temporal extension was presented above. However, these concepts help to better understand the contractual interaction that correctly presents the implementation of rational expectations, equilibrium, and an appropriate concept of the core.

9.1 *REE*-equilibrium: preliminary analysis

Before passing on the description of formal constructions, let's try to clarify the *REE*-equilibrium concept (in rational expectations). What is going on?

Somebody (the Lord, God) in some intermediate moment between today and tomorrow (on the diagram at night) presents for agents reviewing the prices or exacter

a function of the prices $p : \Omega \rightarrow \mathbb{R}^l$, on which tomorrow trade will be carried out. Having realized function $p(\cdot)$, agents extract from it the information as follows. They know that when tomorrow will occur, and they will see (on screen monitors, in shops) the realized price \bar{p} and one can conclude that (from morning until the afternoon) an elementary event from the set

$$p^{-1}(\bar{p}) = \{\mathbf{w} \in \Omega \mid p(\mathbf{w}) = \bar{p}\}$$

is realized. The agent i also has his own information P_i (a partition of Ω) on events of the next day. This is why he/she knows today that tomorrow he/she will be able to distinguish what event from $p^{-1}(\bar{p}) \cap P_i(\mathbf{w}) \neq \emptyset$, $\mathbf{w} \in \Omega$ is realized. However, it means that having received information about function $p : \Omega \rightarrow \mathbb{R}^l$ individual i is able to form plans of purchases and selling, *i.e.* that for every $E = E(\bar{p}, \mathbf{w}) = p^{-1}(\bar{p}) \cap P_i(\mathbf{w}) \neq \emptyset$ he/she has to find a vector $v_i(E) \in \mathbb{R}^l$ maximizing agent's utility under condition an event E is realized. In this case, the individual consumption bundle is $\mathbf{e}_i(\mathbf{w}) + v_i(E)$, $\mathbf{w} \in E$. Moreover, an equilibrium situation is, so that there is a balance of demand and supply (of purchases and selling) for every elementary event that can be realized tomorrow.

Let's imagine that God has transmitted the prices that explicitly distinguish all world states. Then in an intermediate stage when price signal is obtained, each individual will receive a possibility to understand (distinguish) every state of the world and, therefore, the own market for every of the future elementary events should be developed for which it is possible to consider competitive equilibrium and the core as in routine Arrow–Debreu model. Certainly, those prices that were transmitted by God should be the prices of equilibrium for each of the markets determined by states of the world. All said can be easily checked from the definition of equilibrium in rational expectations. However, what about core? Undoubtedly in this case one has to take the Cartesian product of cores—in this way all known theoretical results can be obtained and generalized. Thus

$$\sigma(p) = \Omega^* \Rightarrow \mathcal{C}(\mathcal{E}^{di}) = \prod_{\mathbf{w} \in \Omega} \mathcal{C}(\mathcal{E}^{di}(\mathbf{w})).$$

Before the consideration of the general case and the finding of an adequate contractual presentation for core and equilibrium in the rational expectations let us consider the following modeling example from *Kreps (1977)*, demonstrating that *REE*-equilibrium may not exist in banal situations.

Example 9.1.1 (*REE-EQUILIBRIUM MAY NOT EXIST, KREPS 1977*) Let us consider an economy with 2 agents and 2 states of nature $\{a, b\} = \Omega$, which are equally probable and in each of these states agents are able to consume two products $(z_1, z_2) \geq 0$. Let $x = (x(a), x(b)) \in X_1 = \mathbb{R}_+^4$ denote 1st agent consumption vector and let $y = (y(a), y(b)) \in X_2 = \mathbb{R}_+^4$ be applied for the second one. Initial endowments, information, and utilities (relative to the current state of the world) are the following.

$$\mathbf{e}_i = (\mathbf{e}_i(a), \mathbf{e}_i(b)) = \frac{3}{2}((1, 1)_a, (1, 1)_b), \quad i = 1, 2.$$

$$u_1(a, z) = \ln z_1 + z_2, \quad u_1(b, z) = 2 \ln z_1 + z_2 \quad \& \quad P_1 = \{\{a\}, \{b\}\}$$

$$u_2(a, z) = 2 \ln z_1 + z_2, \quad u_2(b, z) = \ln z_1 + z_2 \quad \& \quad P_1 = \{\{a, b\}\}$$

There are two possibilities to implement the *REE*-equilibrium: the new information is recognized through the price channel or not.

Consider the first possibility. Then in each of two states of nature, equilibrium should be realized, combining them, we have to find *REE*. Standardly using the first-order conditions for consumer problem subject to the normalized price of the market 'a', we have

$$(p_1, p_2) = \nabla u_1(a, x) = \left(\frac{1}{x_1}, 1 \right) \Rightarrow p_2 = 1, \quad x_1 = \frac{1}{p_1},$$

and with this in mind,

$$\exists \alpha > 0 \mid (p_1, 1) = \alpha \nabla u_2(a, y) = \alpha \left(\frac{2}{y_1}, 1 \right) \Rightarrow \alpha = 1, \quad y_1 = \frac{2}{p_1}.$$

Now from the balance of the first commodity and via budget constraints one obtains

$$3 = x_1 + y_1 = \frac{3}{p_1} \Rightarrow p_1 = 1 \Rightarrow x_1 = 1, \quad y_1 = 2;$$

$$\langle (1, 1), (1, x_2) \rangle = p e_1(a) = 3 \Rightarrow x_2 = 2, \quad y_2 = 1.$$

As a result, we find

$$p(a) = (1, 1) \quad \& \quad x(a) = (1, 2), \quad y(a) = (2, 1).$$

Arguing in this way, we find an equilibrium for the state 'b'

$$p(b) = (1, 1) \quad \& \quad x(b) = (2, 1), \quad y(b) = (1, 2).$$

As we can see, the prices in both states of the world are the same and, therefore, the second agent is still unable to distinguish the states of nature. For this case his/her consumption in indistinguishable states should be equal, but we have $y(a) \neq y(b)$, which is a contradiction.

We next consider the second possibility, and assume that the division of information has not occurred, *i.e.* $p(a) = p(b) = (p_1, p_2)$. Similarly to that presented above we find

$$\exists \alpha > 0 \mid (p_1, p_2) = \nabla u_1(a, x) = \left(\frac{1}{x_1(a)}, 1 \right) = \alpha \nabla u_1(b, x) = \alpha \left(\frac{2}{x_1(b)}, 1 \right) \Rightarrow$$

$$\alpha = 1, \quad p_2 = 1, \quad x_1(b) = \frac{2}{p_1} = 2x_1(a).$$

For the second agent we have $y(a) = y(b) = y$ and the utility is $u_2(y) = 3 \ln y_1 + 2y_2$, that gives the following solution of consumer maximization problem

$$\exists \beta > 0 \mid (p_1, 1) = \beta \nabla u_2(y) = \beta \left(\frac{3}{y_1}, 2 \right) \Rightarrow \beta = \frac{1}{2} \quad \& \quad y_1 = \frac{3}{2p_1}.$$

However, in view of the above relations and the first product balance in the two states of nature, we conclude

$$3 = x_1(a) + y_1 = \frac{1}{p_1} + \frac{3}{2p_1} = \frac{5}{2p_1} \neq 3 = x_1(b) + y_1 = \frac{2}{p_1} + \frac{3}{2p_1} = \frac{7}{2p_1}.$$

This contradiction completes the analysis. ■

So, *REE*-equilibrium may not exist, but it exists almost everywhere (generically), see [Allen \(1981\)](#). Continuing our analysis let us assume $\Omega = \{a, b, c\}$ and there are three agents $\mathcal{I} = \{1, 2, 3\}$ with initial information

$$P_1 = \{\{a\}, \{b, c\}\}, \quad P_2 = P_3 = \{\{a, b\}, \{c\}\}.$$

Let's assume that the nature has transmitted a price signal such that $p' = p(a) = p(b) \neq p(c) = p''$. What has happened? The agents have been informed not only on different variants of tomorrow prices but also they have got an ability to distinguish the future events $E' = \{a, b\}$ and $E'' = \{c\}$. Certainly, in the example, it is useful only for the first individual who has now the perfect information. However, the partition $\{E', E''\}$ is the information which each individual possesses now and everybody knows, that other ones also know (the common knowledge). Thus, already today the markets of the future purchases and selling, specified by events E' or E'' can be developed and start to function. Here for E' every deal starts with a preamble: "if E' has happened then..."; similar for E'' . And what does happen according to *REE*-equilibrium concept? The following treatment is possible:

For $E' = \{a, b\}$:

$i = 1$, $\mathbf{w} = a$, bundle $x_1(a)$ solves the problem

$$u_1(\varphi_a^1(y_1(a)), *, *) \rightarrow \max, \quad s.t. \quad y_1(a) \geq 0 \quad \& \quad \langle p', y_1(a) - \mathbf{w}_1(a) \rangle \leq 0;$$

$i = 1$, $\mathbf{w} = b$, bundle $x_1(b)$ solves the problem

$$u_1(*, \varphi_b^1(y_1(b)), *) \rightarrow \max, \quad s.t. \quad y_1(b) \geq 0 \quad \& \quad \langle p', y_1(b) - \mathbf{w}_1(b) \rangle \leq 0;$$

$i = 2$, $E = \{a, b\}$, bundle $x_2 = x_2(a, b)$ solves the problem (accounting $\mathbf{e}_2(a) = \mathbf{e}_2(b)$)

$$u_2(\varphi_a^2(y_2(a)), \varphi_b^2(y_2(b)), *) \rightarrow \max, \quad s.t. \quad y_2(a) = y_2(b) \geq 0 \quad \& \quad \langle p', y_2 - \mathbf{e}_2(a) \rangle \leq 0.$$

$i = 3$, $E = \{a, b\}$, bundle $x_3 = x_3(a, b)$ solves the problem (accounting $\mathbf{e}_3(a) = \mathbf{e}_3(b)$)

$$u_3(\varphi_a^3(y_3(a)), \varphi_b^3(y_3(b)), *) \rightarrow \max, \quad s.t. \quad y_3(a) = y_3(b) \geq 0 \quad \& \quad \langle p', y_3 - \mathbf{e}_3(a) \rangle \leq 0;$$

Balance: $x_1(a) + x_2(a) + x_3(a) = \mathbf{e}_1(a) + \mathbf{e}_2(a) + \mathbf{e}_3(a)$ &

$$x_1(b) + x_2(b) + x_3(b) = \mathbf{e}_1(b) + \mathbf{e}_2(b) + \mathbf{e}_3(b).$$

For $E'' = \{c\}$:

$i = 1$, $\mathbf{w} = c$, bundle $x_1(c)$ solves the problem

$$u_1(*, *, \varphi_c^1(y_1(c))) \rightarrow \max, \quad s.t. \quad y_1(c) \geq 0 \quad \& \quad \langle p'', y_1(c) - \mathbf{e}_1(c) \rangle \leq 0;$$

$i = 2$, $\mathbf{w} = c$, bundle $x_2(c)$ solves the problem

$$u_2(*, *, \varphi_c^2(y_2(c))) \rightarrow \max, \text{ s.t. } y_2(c) \geq 0 \ \& \ \langle p'', y_2(c) - \mathbf{e}_2(c) \rangle \leq 0;$$

$i = 3$, $\mathbf{w} = c$, bundle $x_3(c)$ solves the problem

$$u_3(*, *, \varphi_c^3(y_3(c))) \rightarrow \max, \text{ s.t. } y_3(c) \geq 0 \ \& \ \langle p'', y_3(c) - \mathbf{e}_3(c) \rangle \leq 0;$$

Balance: $x_1(c) + x_2(c) + x_3(c) = \mathbf{e}_1(c) + \mathbf{e}_2(c) + \mathbf{e}_3(c)$.

Presented relations show that events E' and E'' in the model context generate rather independent economic structures, submodels, for which reduced in the event REE -allocation has specific equilibrium properties. The model relative to $E' = \{a, b\}$ has two characteristic features:

- (i) **prices** for elementary events a and b **coincide**;
- (ii) **1st individual**, which is able to distinguish events a and b , in the submodel is **presented by two** different consumer **problems** (only prices p' are a common thing) with different utilities: $\varphi_a^1(\cdot)$ and $\varphi_b^1(\cdot)$. One may state that the **individual is “duplicated”**: one his face works on the market defined by a , another one by b .

One important relationship between items (i) and (ii) has to be noted: if (i) were broken, *i.e.* if prices in states a and b were different, then all agents were “duplicated” and the market was disintegrated in two parts. Namely, due to (i) and that some individuals cannot distinguish events, their common market is developed. Further, we analyze the case in more detail. Once again contractual approach will be applied for the analysis.

9.2 Contract-based approach and differentiated agents

According to general contractual settings, to apply the approach one has to define a set \mathcal{W} of all permissible contracts correctly reflecting model context. As it was postulated above a contract is the tuple of maps $v_i : \Omega \rightarrow \mathbb{R}^l$, $i \in \mathcal{I}$, which obeys measurability (8.1.1) and balances (8.1.2) requirements; recall them:

$$\forall i \in \mathcal{I}, \ v_i(\cdot) \text{ is } P_i - \text{measurable}, \ \forall \mathbf{w} \in \Omega \quad \sum_{i \in \mathcal{I}} v_i(\mathbf{w}) = 0. \quad (9.2.1)$$

However, does it well corresponds to REE -concept in the above example context? The main point is how the multiplicity of extremal problems is reflected in the contract and contractual interaction of individuals, *i.e.* the possibility of agent decomposition in a pair, as in an example considered above. In general, each agent has to be decomposed (split) into several artificial agents, each of them corresponds to the element of informational partition. So we introduce a new set of individuals where several agents defined via agent information partition present an initial individual.

We form a new economic model which can be studied by applying familiar methods; then the obtained results are transferred to the initial model and it helps us in its investigation.

9.2.1 Differentiated agents

Let us consider a new set of individuals, which are indexed by couples (i, E) where a first component is a number (name) of the individual, and $E \in P_i$ is an element of his/her informational partitions. Here (i, E) can be treated as the individual in one of the possible variants of *tomorrow's* implementations of agent i and it corresponds to a knowledge of this agent. So, suppose

$$\mathfrak{S} = \{(i, E) \mid i \in \mathcal{I}, E \in P_i\}.$$

Further one will specify contracts which 'duplicated agent' is able to conclude. According to construction such an agent can live and function only if the event E is realized, therefore for $(i, E) \in \mathfrak{S}$ one specifies

$$\begin{aligned} v_i^E : \Omega \rightarrow \mathbb{R}^l : v_i^E(\mathbf{w}) = v_i^E(\mathbf{w}'), \forall \mathbf{w}, \mathbf{w}' \in E \ \& \ v_i^E(\mathbf{w}'') = 0, \forall \mathbf{w}'' \in \Omega \setminus E \iff \\ v_i^E(\cdot) \text{ is } P_i\text{-measurable} \ \& \ \text{supp}(v_i^E) = E \in P_i. \end{aligned} \quad (9.2.2)$$

Certainly a tuple $(v_i^E)_{\mathfrak{S}}$ can be considered as a contract only if the balance restriction

$$\sum_{\mathfrak{S}} v_i^E = 0 \quad (9.2.3)$$

is satisfied. For the individual $(i, E) \in \mathfrak{S}$ to function as a real economic agent, one has to endow him/her with an initial commodity bundle and preferences. Put

$$\mathbf{e}_i^E = \mathbf{e}_i(\cdot) \cdot \chi^E, \quad E \in P_i \ \& \ (i, E) \in \mathfrak{S},$$

where $\chi^E(\cdot)$ is a characteristic function of the set $E \subseteq \Omega$ and let

$$\mathcal{L}_i^E = \{y \cdot \chi^E(\cdot) \in L \mid y \in \mathbb{R}^l\}$$

be a subspace corresponded to individual $(i, E) \in \mathfrak{S}$ (its dimension is l) in the space of contingent commodities $L = (\mathbb{R}^l)^\Omega$. Further, let us define preferences and consumption sets. Put

$$X_i^E = (\mathcal{L}_i^E + \mathbf{e}_i^E) \cap X_i$$

and define on X_i^E relation \succ_i^E by formula

$$y_i^E \succ_i^E x_i^E \iff \exists z_i^{\Omega \setminus E} : \Omega \setminus E \rightarrow \mathbb{R}^l \mid (y_i^E, z_i^{\Omega \setminus E}) \succ_i (x_i^E, z_i^{\Omega \setminus E}),$$

that can also be rewritten in an equivalent form as

$$\mathcal{P}_i^E(x_i^E) = [\mathcal{P}_i(x_i) \cap (x_i + \mathcal{L}_i^E)] \cdot \chi^E, \quad \text{for } x_i^E = x_i \cdot \chi^E.$$

¹Notice that any map $v_i(\cdot)$ applied in the definition of contract (8.1.2) is measurable by P_i and therefore it can be decomposed into a direct sum of maps v_i^E , $E \in P_i$.

Due to assumptions (7.1.2)–(7.1.4), this is a correct way to present preferences for an economy with differentiated information. Notice that

$$\sum_{E \in P_i} \mathcal{P}_i^E(x_i^E) \subset \mathcal{P}_i(x_i)$$

is always true, but the reverse inclusion is false. As a result of presented constructions, one comes to an economic model $\mathcal{E}^{\mathfrak{S}}$.

So, there are two possibilities to apply a contractual approach in economies with asymmetrically informed individuals:

- (i) The set of agents is the same as in the initial model of economy and contracts are completely determined by conditions (9.2.1).
- (ii) The set of agents varies on \mathfrak{S} to which contract specifications (9.2.2)–(9.2.3) are applied.

These possibilities lead to different concepts of the contractual allocations and corresponding concepts of the core and an equilibrium. In the first case, this is a priori private core and equilibrium (analogue *WEE*), studied in Chapter 8. In the second one, *new* concepts of core and equilibrium are introduced; these notions were named as core and equilibrium with *differentiated agents*.² Further, we consider the second variant, which eventually deduces us to *REE*-equilibrium.

First, let's consider the simplest possibility to introduce a core by analogue with a standard case, but now already in described above model with duplicated agents; we named it as the *D*-core.

Definition 9.2.1 *D*-core $\mathcal{C}^d(\mathcal{E}^{di})$ of economy \mathcal{E}^{di} with asymmetrically informed agents consists of allocations $x = (x_i)_{\mathcal{I}} \in X$ such that:

- (i) $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$,
- (ii) $(x_i - \mathbf{e}_i) : \Omega \rightarrow \mathbb{R}^l$ is P_i -measurable for all $i \in \mathcal{I}$.
- (iii) $\nexists S \subseteq \mathfrak{S} : \exists y^S = (y_i^E)_{(i,E) \in S} \mid \forall (i, E) \in S \ y_i^E \in X_i^E$ is such that $y_i^E \succ_i^E x_i^E$ & $\sum_{(i,E) \in S} (y_i^E - \mathbf{e}_i^E) = 0$ or $\sum_{(i,E) \in S} (y_i^E - x_i^E) = 0$.

One can see that the only but basic difference of the *D*-core from the private one consists in that dominating coalitions are formed from duplicated agents of the economy; notice that (iii) also takes into account the requirement of the web's stability from above. In so doing for the core of this type the allocations are implemented by stable systems of contracts, stable in an intermediate stage of uncertainty implementation, *i.e.* at a moment when each agent is able to understand every state of the nature $\mathbf{w} \in \Omega$ (it still is not realized!) while only in a form $P_i(\mathbf{w})$. Here coalitions of

²Differentiated information about the future induces in the present differentiation and “reproduction” of agents in the future: depending on the information, different agents “are mirrored” in different ways.

duplicated agents can be formed because uncertainty is still not definitively resolved. Later, probably, individuals learn more about the state \mathbf{w} but when it will happen all contracts will already be realized. Moreover, it should be clear that there are no possibilities for deceit here, since if the more informed agent or a coalition of agents will report a false state— \mathbf{w}' instead of true \mathbf{w} —when it cannot bring damage to the less informed individuals because in both cases in a gross contract $(v_i)_{\mathcal{I}}$ these individuals have the same vector of mutual deliveries, because by (8.1.1) one has $v_i(\mathbf{w}') = v_i(\mathbf{w})$ for $\mathbf{w}', \mathbf{w} \in P_i(\mathbf{w})$. In other words, allocations of D -core are incentive compatible.

Regarding the existence of D -core, it is enough to note that Theorem 8.1.2 is applicable here with more or less simple modifications be conditioned by the reproduction of agents—a game of economy is still balanced that is necessary for core existence.

Further, let us turn to an equilibrium concept that correctly corresponds to the D -core and contractual approach point of view. We start the analysis by deriving conditions that characterize mutually beneficial exchange for the economy with differentiated agents: this is once more lemma about the mutually beneficial contract.

Lemma 9.2.1 (ABOUT A BENEFICIAL CONTRACT FOR D -AGENTS) *Let $S \subseteq \mathcal{I}$, $S \neq \emptyset$ be a coalition and $A \subseteq \Omega$ be an event understandable by every coalition S member and let*

$$S(\mathfrak{S}) = \{(i, E) \mid i \in S, E \in P_i, E \subseteq A\}.$$

*Then if **there is no** mutually beneficial exchange of contingent commodities for coalition $S(\mathfrak{S})$ members then there does exist a vector $p \in (\mathbb{R}^l)^A$, $p \neq 0$, and vectors $q_i \in (\mathbb{R}^l)^A$, $i \in S$, such that*

$$\forall i \in S \forall E \in P_i, E \subseteq A \sum_{\mathbf{w} \in E} q_i(\mathbf{w}) = 0 \quad (9.2.4)$$

and

$$\forall i \in S \quad p + q_i \neq 0 \quad \& \quad \langle \mathcal{P}_i^E(x_i^E), p + q_i \rangle \geq \langle x_i^E, p + q_i \rangle \quad \forall E \in P_i, E \subseteq A \quad (9.2.5)$$

holds.

*Inverse: let there be vectors satisfying (9.2.4), (9.2.5). Then for a coalition S **there is no** mutually beneficial contract in which an agent $(i, E) \in S(\mathfrak{S})$ is non-trivially involved for which inequality (9.2.5) has a strict form.³*

The proof of Lemma 9.2.1 is realized similarly to Lemma 8.1.1 proof. The only modification is due to D -agents specification: in an appropriate intersection instead of $\mathcal{P}_i(x_i)$ one has to take the set $\sum_{E \in P_i, E \subseteq A} \mathcal{P}_i^E(x_i^E)$; let us omit other details. Notice also the coincidence between (9.2.4) with an analogue requirement (8.1.3) applied in Lemma 8.1.1.

Similarly to Lemma 8.1.1, Lemma 9.2.1 has an important for below analysis corollary, that is formulated to present full argumentation and to better specify contractual approach for differentiated agents.

³See footnote above.

Corollary 9.2.1 *Let in Lemma 9.2.1 conditions preferences be described via differentiable utility functions and let $x = (x_i)_{\mathcal{I}}$ be an interior allocation relative to A and S .⁴ Then **there is no** mutually beneficial contract for the coalition $S(\mathfrak{S})$ if and only if there exists a vector $p \in (\mathbb{R}^l)^A$, $p \neq 0$ and $\lambda_{i,E} \geq 0$, $(i, E) \in S(\mathfrak{S})$ non-zero for some $E \in P_i \forall i \in S$, such that*

$$\forall E \in P_i, E \subseteq A \quad \lambda_{i,E} \sum_{\mathbf{w} \in E} \nabla_{\mathbf{w}} u_i(x_i) = \sum_{\mathbf{w} \in E} p(\mathbf{w}) \quad \forall i \in S.$$

Notice some distinctions with Corollary 8.1.1: factors $\lambda_{i,E}$ have doubled indexes and, in general, some of them can be zeros.

Now for the concept of D -equilibrium to be introduced, it is enough to add budget constraints to the latter lemma conclusion.

Definition 9.2.2 *A couple (x, p) , $x = (x_i)_{\mathcal{I}} \in X$, $p : \Omega \rightarrow \mathbb{R}^l$, is called **private D -quasi-equilibrium** if it satisfies:*

- (i) $(x_i - \mathbf{e}_i) : \Omega \rightarrow \mathbb{R}^l$ is P_i -measurable for all $i \in \mathcal{I}$,
- (ii) $0 \neq \langle p^E, (\mathcal{P}_i^E(x_i^E) - x_i^E) \rangle \geq 0$, $\langle p^E, x_i^E - \mathbf{e}_i^E \rangle = 0$, $\forall (i, E) \in \mathfrak{S}$,
- (iii) $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$.

If in item (ii) all inequalities are **strict**, then pair (x, p) is called **D -equilibrium**.

Notice that due to the definition $p(\cdot) \neq 0$ there is also possible $p^E = p \cdot \chi^E \equiv 0$ for some $E \in P_i, i \in \mathcal{I}$. Let us see also an alternative form of condition (ii) presentation: $\forall (i, E) \in \mathfrak{S}$,

$$\sum_{\mathbf{w} \in E} p(\mathbf{w}) x_i(\mathbf{w}) = \sum_{\mathbf{w} \in E} p(\mathbf{w}) \mathbf{e}_i(\mathbf{w}) \quad \&$$

$$\langle \sum_{\mathbf{w} \in E} p(\mathbf{w}), y \rangle \geq 0 \quad \forall y \in \mathbb{R}^l : (y \cdot \chi_E(\cdot) + x_i) \in \mathcal{P}_i(x_i).$$

Thus individual (i, E) takes prices in aggregated form, aggregated relative to states from E , that can be treated as a form of expected prices and profit under condition of event E . Notice that if one adds in the definition the requirement

$$p(\mathbf{w}) = p(\mathbf{w}'), \quad \forall \mathbf{w}, \mathbf{w}' \in E \in P_i, \quad \forall i \in \mathcal{I},$$

then one yields the notion generalizing REE -equilibrium to the case of general preferences.

The existence of D -quasi-equilibrium can be proven applying the same methods that were used above for ex ante private equilibrium: due to replicate models and passing to fuzzy core and, for its elements, price characterization in the form of quasi-equilibria can be given.

Finishing the section let us consider the concepts of D -core and D -equilibrium, and also of private core and equilibrium in an example of DIE -economy, that admits a demonstration on Edgeworth box (in spite of the fact the space of contingent commodities is 4-dimensional one).

⁴See Corollary 8.1.1.

Example 9.2.1 (AN ECONOMY $2 \times 2 \times 2$) Let us consider a standard pure exchange economy with asymmetrically informed agents. Let there be 2 types of physically different goods, 2 agents, 2 states of the nature and assume that 2nd agent is able to distinguish them but 1st one is not. Thus we have:

$$i = 1, 2, \quad \Omega = \{a, b\}, \quad P_1 = \{\Omega\}, \quad P_2 = \{\{a\}, \{b\}\},$$

$L = \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ is the space of contingent commodities. Let $X_1 = X_2 = \mathbb{R}_+^4$ and consider the following utility functions and initial endowments:

$$\begin{aligned} u_x &= u_1(x(a), x(b)) = \ln(x_1(a)) + \ln(x_2(a)) + \ln(x_1(b)) + \ln(x_2(b)), \\ \mathbf{e}_x &= \mathbf{e}_1 = ((3\frac{1}{2}, \frac{1}{2})_a, (3\frac{1}{2}, \frac{1}{2})_b); \\ u_y &= u_2(y(a), y(b)) = 2\ln(y_1(a)) + \ln(y_2(a)) + \ln(y_1(b)) + 2\ln(y_2(b)), \\ \mathbf{e}_y &= \mathbf{e}_2 = ((1\frac{1}{2}, 2\frac{1}{2})_a, (\frac{1}{2}, 3\frac{1}{2})_b). \end{aligned}$$

Thus total endowments in the economy are presented by the vector

$$\bar{\mathbf{e}} = (\bar{\mathbf{e}}(a), \bar{\mathbf{e}}(b)), \quad \bar{\mathbf{e}}(a) = (5, 3), \quad \bar{\mathbf{e}}(b) = (4, 4).$$

Now assume that there is **no informational exchange** between agents. Now the measurability of contract $(v, -v)$, $v = (v(a), v(b)) \in L$ implies that $v(a) = v(b) = w \in \mathbb{R}^2$ and, therefore,

$$x(a) = \mathbf{e}_x(a) + v(a) = \mathbf{e}_x(b) + v(b) = x(b) = (3\frac{1}{2}, \frac{1}{2}) + (w_1, w_2).$$

In general for the states of future 2nd individual may have different consumption plans but now they also can be expressed in the terms of 2-dimension contract (w_1, w_2) :

$$y(a) = \mathbf{e}_y(a) - v(a) = (1\frac{1}{2}, 2\frac{1}{2}) - (w_1, w_2), \quad y(b) = \mathbf{e}_y(b) - v(b) = (\frac{1}{2}, 3\frac{1}{2}) - (w_1, w_2).$$

As soon as 2nd agent is not able to consume negative quantities, then

$$y(a) = (1\frac{1}{2}, 2\frac{1}{2}) - (w_1, w_2) \geq 0, \quad y(b) = (\frac{1}{2}, 3\frac{1}{2}) - (w_1, w_2) \geq 0 \quad \Rightarrow \quad (w_1, w_2) \leq (\frac{1}{2}, 2\frac{1}{2})$$

and we are coming to 2-dimensional Edgeworth box presented in 1st agent consumption plans:

$$EB = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq (x_1, x_2) \leq (4, 3)\}.$$

Now Pareto boundary can be found applying (for example) 2nd Welfare Theorem that allows to conclude that gradients have to be collinear ones. So, this boundary is described by the following system of equations:

$$\begin{cases} \frac{2}{x_1} = \lambda(\frac{2}{5-x_1} + \frac{1}{4-x_1}), \\ \frac{2}{x_2} = \lambda(\frac{2}{4-x_2} + \frac{1}{3-x_2}), \quad \lambda > 0. \end{cases}$$

Eliminating $\lambda > 0$ from the system, one obtains 3rd power equation that determines Pareto boundary in the interior points of the box. Sufficiently good its approximation is a curve defined by explicit linear equation $x_2 = \frac{3}{4}x_1$, *i.e.* this is the diagonal of the box. Further, if one adds 1st agent budget constraint (equality) to the equation of Pareto boundary, then one comes to the system of equations that describes private

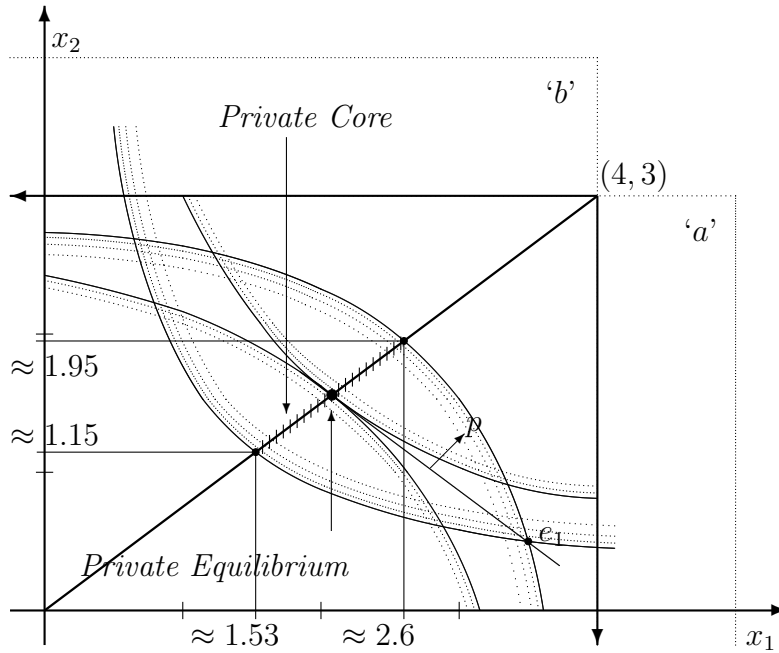


Figure 9.2.1: Private core and equilibrium

equilibrium (2nd budget constraint is fulfilled automatically). As soon as for interior point first-order conditions imply prices are collinear to the gradients of utilities, one finds

$$\langle \nabla u_1(x), x \rangle = \langle \nabla u_1(x), \mathbf{e}_1(a) \rangle \Rightarrow 4 = \frac{7}{x_1} + \frac{1}{x_2} \Rightarrow x_1 \approx \frac{25}{12} \approx 2.08, \quad x_2 \approx \frac{25}{16} \approx 1.56.$$

The results of above analysis are graphically presented in Figure 9.2.1.

Now let us consider D -core and D -equilibrium. With this in mind, let us find Pareto boundaries in the models reduced to the states a, b . For ‘ a ’ one obtains the following system of equations:

$$\begin{cases} \frac{2}{x_1} = \lambda \frac{2}{5-x_1}, \\ \frac{2}{x_2} = \lambda \frac{1}{3-x_2}, \quad \lambda > 0 \end{cases} \iff x_2 = \frac{6x_1}{x_1 + 5}. \quad (9.2.6)$$

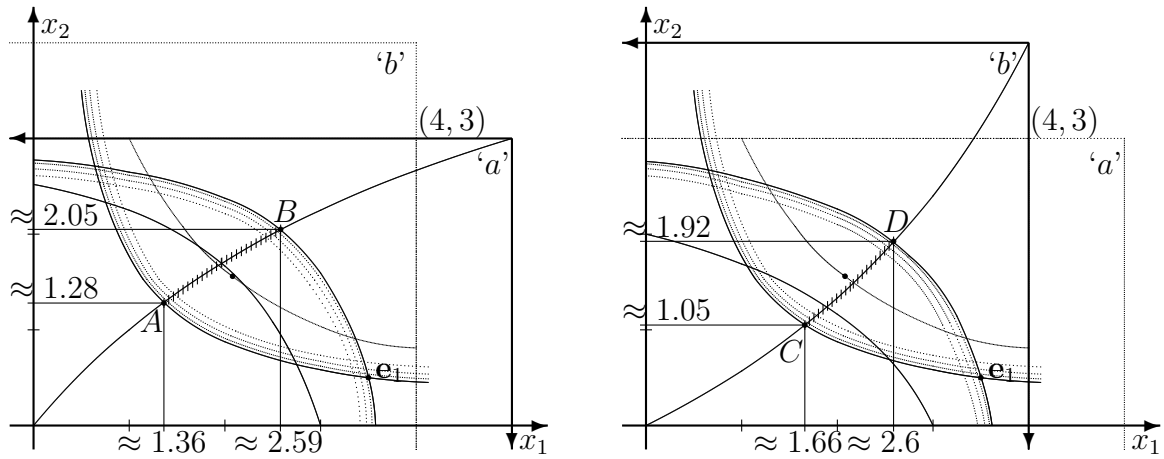
Here the “core” is presented by a curved segment on the Pareto curve, placed between two indifference curves passed through the initial endowments point; here the endpoints of the segment are $A \approx (1.36, 1.28)$ and $B \approx (2.59, 2.05)$.

For ‘ b ’ the similar system is the following:

$$\begin{cases} \frac{2}{x_1} = \lambda \frac{1}{4-x_1}, \\ \frac{2}{x_2} = \lambda \frac{2}{4-x_2}, \quad \lambda > 0 \end{cases} \iff x_2 = \frac{4x_1}{8 - x_1}. \quad (9.2.7)$$

A “core” corresponds with a curved segment of Pareto curve that is placed between points $C \approx (1.66, 1.05)$ and $D \approx (2.6, 1.92)$. Graphical construction for the Edgeworth box is provided in Figure 9.2.2.

Further, let us reduce provided constructions into joint picture presenting D -core.

Figure 9.2.2: D -core: analysis for states 'a' and 'b'

For this example, one presents only a coalition of all players that is efficient enough, because other coalitions can sign only zero-contract. This is why the elements of D -core are completely described as being Pareto optimal and individually rational ones (*i.e.*, they cannot be dominated by one-element coalitions). In our context Pareto optimality of an allocation $(\bar{x}, \bar{x}, \bar{e}(a) - \bar{x}, \bar{e}(b) - \bar{x}) \in \mathbb{R}_+^8$ means the intersection of the following three sets is empty:

$$\begin{aligned} & \{(x_1, x_2) \in \mathbb{R}^2 \mid u_1(x(a), x(b)) > u_1(\bar{x}, \bar{x}), x(a) = x(b) = (x_1, x_2)\} \cap \\ & \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid u_2^a(\bar{e}(a) - (x_1, x_2)) > u_2^a(\bar{e}(a) - (\bar{x}_1, \bar{x}_2))\} \cap \\ & \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid u_2^b(\bar{e}(b) - (x_1, x_2)) > u_2^b(\bar{e}(b) - (\bar{x}_1, \bar{x}_2))\} = \emptyset. \end{aligned}$$

Here the first one is the set of all strictly preferred consumption bundles for the 1st agent, that is written according to his/her impossibility to differentiate elementary events a and b . Two other sets are the sets of all strictly preferred consumption bundles for the duplicated agents for the 2nd individual: his/her possible implementations in the future states a and b . All sets are written in variables associated with the consumption of the 1st individual. Graphical and numerical construction for the example is provided in Figure 9.2.3, where a shaded set of points $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is represented as a curved area $ACDE$ which boundaries are the fragments of cores for events 'a' and 'b', and also fragments of indifference curves of 1st agent and the duplicate $(2, b)$. Notice that here a part of core for 'a', that includes endpoint B , is intercepted by the indifference curve of $(2, b)$ and as a result curved segment AE is considered where $E \approx (2.47, 2.5)$.

Further, in the context of example, let us consider the notion of D -equilibrium. Briefly, D -quasi-equilibrium are the allocations in which equilibria in elementary events 'a' and 'b' are implemented. In so doing for a "**complementary**" state, **prices are equal to zero**; for example, for D -equilibrium induced by event 'b' **prices for event 'a' are zeros**. Calculate these equilibria. According to analysis provided above it can be done adding to Pareto boundary equation the budget equation (equality) for the 1st agent being calculated with respect to prices

$p = (p_1, p_2) = \nabla_x u_1(x, x) = \left(\frac{2}{x_1}, \frac{2}{x_2}\right) \Rightarrow \langle p, x \rangle = 4 = \frac{7}{x_1} + \frac{1}{x_2} = \langle p, \mathbf{e}_1(a) \rangle$. As a result for 'a' one obtains the following system of equations which solution gives equilibrium consumption bundle for the 1st agent and equilibrium prices:

$$\begin{cases} x_2 = \frac{6x_1}{x_1+5} \\ \frac{7}{x_1} + \frac{1}{x_2} = 4 \end{cases} \Rightarrow 6(4x_1 - 7) = x_1 + 5 \Rightarrow x_1 = \frac{47}{23}, x_2 = \frac{47}{27} \Rightarrow p = (23, 27).$$

Similarly, one can find equilibrium induced by the event 'b'. One obtains the following system of equations for the finding of the 1st agent equilibrium consumption bundle:

$$\begin{cases} x_2 = \frac{4x_1}{8-x_1} \\ \frac{7}{x_1} + \frac{1}{x_2} = 4 \end{cases} \Rightarrow 4(4x_1 - 7) = 8 - x_1 \Rightarrow x_1 = \frac{36}{17}, x_2 = \frac{36}{25} \Rightarrow p = (17, 25).$$

So one finds the following D -equilibria:

$$p_a = (23, 27), p_b = 0, x(a) = x(b) = \left(\frac{47}{23}, \frac{47}{27}\right), y(a) = \left(\frac{68}{23}, \frac{34}{27}\right), y(b) = \left(\frac{45}{23}, \frac{61}{27}\right),$$

$$p_a = 0, p_b = (17, 25), x(a) = x(b) = \left(\frac{36}{17}, \frac{36}{25}\right), y(a) = \left(\frac{49}{17}, \frac{39}{25}\right), y(b) = \left(\frac{32}{17}, \frac{64}{25}\right).$$

Here in both cases vector y was found by formulas $y(a) = \bar{\mathbf{e}}(a) - x(a)$ and $y(b) = \bar{\mathbf{e}}(b) - x(b)$.

Now we would raise an interested and important for contractual approach question: can all presented (quasi)equilibrium allocations be implemented in real economic life and that of them have better chances for a long-run existence? An answer is presented due to contractual approach: long-run living allocation has to be at least lower stable, *i.e.* stable relative to partial breaking of contracts implementing the allocation. However, an equilibrium induced by the state 'a' does not obey this criterium. In order to assure this one needs to calculate directional derivative in direction to initial endowments: all of them have to be non-positive. However, non-zero derivative can have only duplicated agent (2, b). Being provided the calculations one finds:

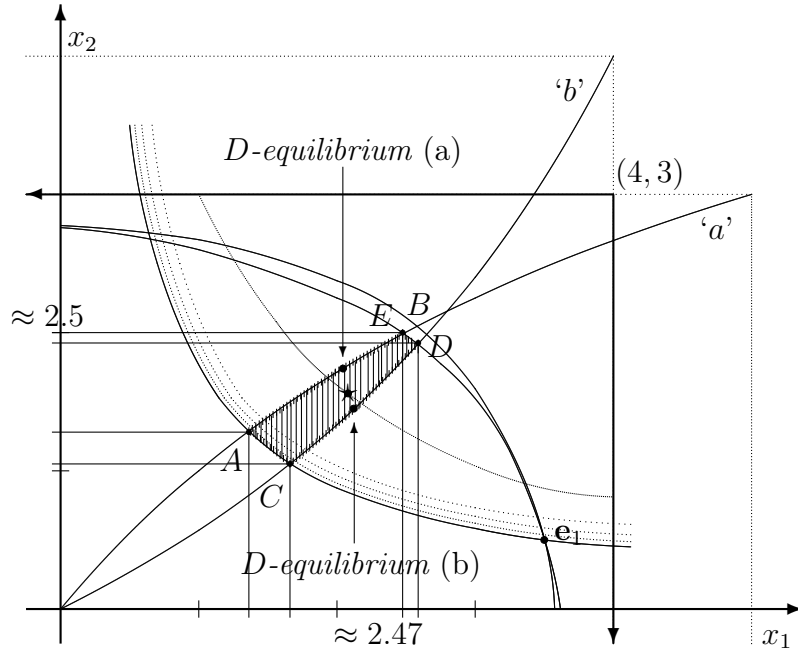
$$\nabla_y u_2^b(y) = \left(\frac{1}{y_1}, \frac{2}{y_2}\right), y(b) = \left(\frac{45}{23}, \frac{61}{27}\right) \Rightarrow \nabla_y u_2^b(y(b)) = \left(\frac{23}{45}, \frac{54}{61}\right) \Rightarrow$$

$$\mathbf{e}_2(b) - y(b) = \left(\frac{1}{2}, \frac{7}{2}\right) - \left(\frac{45}{23}, \frac{61}{27}\right) = \left(\frac{-67}{23 \cdot 2}, \frac{67}{54}\right) \Rightarrow$$

$$\langle \nabla_y u_2^b(y), \mathbf{e}_2(b) - y(b) \rangle = \left\langle \left(\frac{23}{45}, \frac{54}{61}\right), \left(\frac{-67}{23 \cdot 2}, \frac{67}{54}\right) \right\rangle = \frac{67}{61} - \frac{67}{90} > 0.$$

Thus directional derivative in the direction $\mathbf{e}_2(b) - y(b)$ is strictly more zero that implies agent-duplicate (2, b) will partially break contracts... Another quasi-equilibrium is induced by the state 'b' and is lower stable in fact that can be checked from the calculation:

$$\nabla_y u_2^a(y(a)) = \left(\frac{2}{y_1}, \frac{1}{y_2}\right) = \left(\frac{34}{49}, \frac{25}{39}\right) \Rightarrow$$

Figure 9.2.3: D -core: shaded curved area $ACDE$

$$\langle \nabla_y u_2^a(y), \mathbf{e}_2(a) - y(a) \rangle = \left\langle \left(\frac{34}{49}, \frac{25}{39} \right), \left(\frac{-47}{34}, \frac{47}{50} \right) \right\rangle = \frac{47}{78} - \frac{47}{49} < 0.$$

Thus this quasi-equilibrium has more chances to exist than the previous one and therefore the state 'b' which specifies viable quasi-equilibrium has in contractual process an advantage in comparison with the state 'a'.

In conclusion, let us show that there are no other D -(quasi)equilibria in the example, and therefore there are no real equilibria at all. It can be understood coming to a contradiction. Suppose $p_a \neq 0$ and $p_b \neq 0$ are equilibrium prices. Then according to item (ii) of Definition 9.2.2 and necessary conditions for an extremum one will have: $\exists \alpha > 0, \beta > 0, \gamma > 0$ such that $\alpha \nabla_x u_1(x, x) = p_a + p_b, \beta \nabla_y u_2(y) = p_a, \gamma \nabla_z u_2(z) = p_b$ for $y = \bar{\mathbf{e}}(a) - x, z = \bar{\mathbf{e}}(b) - x$. However, for the current train of consumption bundles it has to be true budget equalities in addition:

$$(p_a + p_b)x = (p_a + p_b)\mathbf{e}_1(a) \Rightarrow$$

$$(p_a + p_b) \perp (x - \mathbf{e}_1(a)) = 0, \quad w = (x - \mathbf{e}_1(a)) = (x - \mathbf{e}_1(b));$$

$$p_a y = p_a \mathbf{e}_2(a) \Rightarrow p_a \perp (y - \mathbf{e}_2(a)) = 0, \quad -w = (y - \mathbf{e}_2(a));$$

$$p_b z = p_b \mathbf{e}_2(b) \Rightarrow p_b \perp (z - \mathbf{e}_2(b)) = 0, \quad -w = (z - \mathbf{e}_2(b)).$$

This implies that each vector from p_a, p_b and $p_a + p_b$ is orthogonal to the vector $w \neq 0$. Therefore, because the space is two-dimensional, all three gradients utilities are *pairwise collinear* one to another. Hence couple (x, y) has to be Pareto optimal for 'a', but the couple (x, z) is optimal for 'b'. Thus the vector $x = (x_1, x_2)$ has to satisfy each of two obtained above equations of Pareto boundaries, but there are no such solutions. ■

9.2.2 Ambiguity and maximin expected preferences

The term “ambiguity” means that not only a state of the world which will happen tomorrow is unknown for individuals, but also they do not know the probability of its implementation: they can only estimate the probability of events which are the elements of agent’s informational partition. Thus, this is an uncertainty of a higher level and now even formally one can not apply the Bayesian (expected) utilities which one can find in most of current literature. The application of MEU-utilities (maximin expected utilities) solves to this problem, these utilities were already discussed in Section 7.3.2. Below, we consider the contractual dynamics under these preferences.

We begin the analysis from the establishing of a curious property of *maximin* core allocations (Definition 7.3.2), consisting in the fact that, despite the absence of measurability requirements for consumption plans, they still have a specific (weak) measurability with respect to individual information, this is the measurability of utility. The following statement holds.

Proposition 9.2.1 *Let for all individuals a posteriori utilities $u_i : \Omega \times \mathbb{R}_+^l \rightarrow \mathbb{R}$ obey:*
 $\forall \mathbf{w} \in \Omega, \forall y \in \mathbb{R}_+^l$

- (i) $u_i(\mathbf{w}, \cdot)$ is strictly monotonic by each commodity,
- (ii) $u_i(\cdot, y)$ & $\mathbf{e}_i(\cdot)$ are measurable with respect to individual information.

Assume also that $X_i = L_+$, $i \in \mathcal{I}$ and for every state of nature there exists its distinguishing individual, i.e.,

$$\forall \mathbf{w} \in \Omega \exists j \in \mathcal{I} \mid P_j(\mathbf{w}) = \{\mathbf{w}\}. \quad (9.2.8)$$

Then for $x = (x_i)_{i \in \mathcal{I}} \in \mathcal{C}^{min}(\mathcal{E}^{di})$, $\forall i \in \mathcal{I}$ a real valued function $u_i(\mathbf{w}, x_i(\mathbf{w}))$ of $\mathbf{w} \in \Omega$ is measurable relative to P_i , i.e.

$$u_i(\mathbf{w}, x_i(\mathbf{w})) = u_i(\mathbf{w}', x_i(\mathbf{w}')), \quad \mathbf{w}' \in P_i(\mathbf{w}), \quad \forall \mathbf{w}, \mathbf{w}' \in \Omega.$$

Note that the first part of the statement assumptions coincides with the conditions of the Theorem 7.3.1. The second part has a purely informational nature and allows to accumulate a surplus of resources for a part of individuals in the hands that are interested in. In above Example 7.3.5, these requirements are violated, which implies the existence of core allocations without specific (weak) measurability property.

Proof of Proposition 9.2.1. Assume the contrary and show that in this case there is a coalition that can enter into a new mutually beneficial contract.

Let $\mathbf{w} \in \Omega$ and $i \in \mathcal{I}$ be such that for an allocation $x = (x_i)_{i \in \mathcal{I}}$ of maximin core

$$u_i(\mathbf{w}, x_i(\mathbf{w})) > \min\{u_i(\mathbf{w}', x_i(\mathbf{w}')) \mid \mathbf{w}' \in P_i(\mathbf{w})\}$$

is true. In particular, it implies $x_i(\mathbf{w}) > 0$. By (9.2.8) there is $j \in \mathcal{I}$ such that $P_j(\mathbf{w}) = \{\mathbf{w}\}$. Now individual i can transmit to j a part of resources from $x_i(\mathbf{w})$, for

example $\varepsilon x_i(\mathbf{w})$ for some $0 < \varepsilon < 1$, without violating the last inequality. However, now one obtains

$$u_j(\mathbf{w}, x_j(\mathbf{w}) + \varepsilon x_i(\mathbf{w})) > u_j(\mathbf{w}, x_j(\mathbf{w})),$$

that implies

$$U_j^{min}(x'_j) > U_j^{min}(x_j), \quad x'_j(\mathbf{w}) = x_j(\mathbf{w}) + \varepsilon x_i(\mathbf{w}), \quad x'_j(\mathbf{w}') = x_j(\mathbf{w}') \quad \forall \mathbf{w}' \neq \mathbf{w}.$$

Now individual j may transmit $\delta x'_j(\mathbf{w}')$, $\mathbf{w}' \in \Omega$ to all other members of the economy, choosing $0 < \delta < 1$ so that

$$U_j^{min}((1 - \delta)x'_j) > U_j^{min}(x_j)$$

be true. As a result, we have found an allocation $x'' = (x''_k)_{k \in \mathcal{I}}$,

$$\begin{aligned} x''_j &= x'_j - (\delta - \frac{1}{n})x'_j, \\ x''_k &= x_k + \frac{1}{n}\delta x'_j, \quad k \neq i, j, \quad k \in \mathcal{I}, \\ x''_i(\mathbf{w}) &= (1 - \varepsilon)x_i(\mathbf{w}) + \frac{1}{n}\delta x'_j(\mathbf{w}) \quad \& \quad x''_i(\mathbf{w}') = x_i(\mathbf{w}') + \frac{1}{n}\delta x'_j(\mathbf{w}'), \quad \forall \mathbf{w}' \neq \mathbf{w}, \quad \mathbf{w}' \in \Omega. \end{aligned}$$

By construction $U_k^{min}(x''_k) > U_k^{min}(x_k)$, $\forall k \in \mathcal{I}$ that contradicts to the optimality of $x = (x_k)_{k \in \mathcal{I}}$. ■

Remark 9.2.1 Obviously, this statement will be also executed under weaker assumptions. For example, instead of the core, one can talk about individually rational Pareto optimal allocations. Requirement (ii) of preferences and endowments measurability also can be relaxed, and so on. I did not do this just to simplify formulation and avoid the clutter of the fact with unessential details which obscure the essence of the problem. ■

How can the contractual process be developed at maximin preferences and what is its specificity? This is the central question of the nearest considerations. To this end, we turn again to Example 9.2.1; now Proposition 9.2.1 allows a two-dimensional presentation of maximin core allocations, see Figure 9.2.3.

Example 9.2.2 (CONTINUING OF EXAMPLE 9.2.1) First note, that for an allocation from the maximin core in each state of the world its partial allocation should belong to the Pareto border, otherwise there is a mutually beneficial contract (proved by arguments similar to the proof of the last statement). Moreover, unlike the measurable case for first agent *indistinguishable* states allocations may be *different*. Therefore maximin core elements must be represented by a pair of points from the partial Pareto boundaries which correspond to states ‘ a ’ and ‘ b ’, and according to Proposition 9.2.1 they have to be situated at their intersection with the indifference curve of the first agent. Of course, we need to remember about singleton coalitions—this couple of points should be located within the area bounded by the indifference curve of the first agent and the indifference *surface* of the second, presented by the equation $u_2(y) = u_2(a, y(a)) + u_2(b, y(b)) = u_2(a, \mathbf{e}_2(a)) + u_2(b, \mathbf{e}_2(b))$. The relevant constructions are presented in Figure 9.2.4.

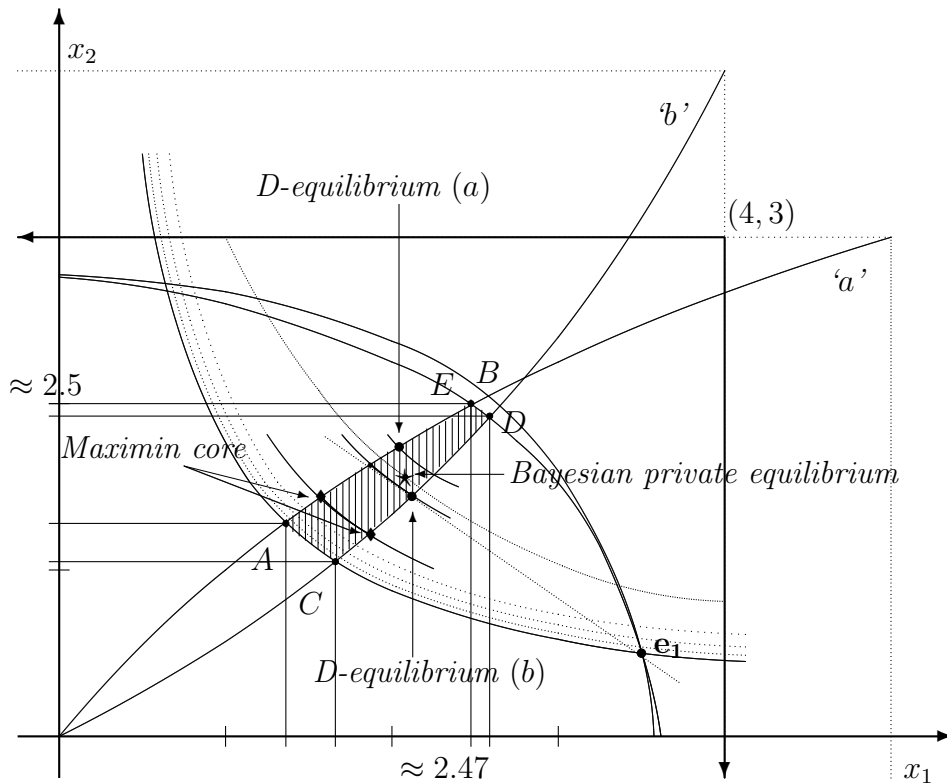


Figure 9.2.4: Maximin core and equilibria

What is going on with *private* equilibrium? Let us first consider its contractual counterpart, presented as a proper contractual allocation. It should be a core element which is stable with respect to partial breaking of contracts. For the first agent, it means that it is impossible to find $0 < \varepsilon < 1$ such that

$$u_1(x(a) + \varepsilon(\mathbf{e}_1 - x(a))) > u_1(x(a)) \quad \& \quad u_1(x(b) + \varepsilon(\mathbf{e}_1 - x(b))) > u_1(x(b))$$

holds. For the second we have $\nexists 0 < \delta < 1$

$$u_2(y(a) + \delta(\mathbf{e}_2(a) - y(a))) + u_2(y(b) + \delta(\mathbf{e}_2(b) - y(b))) > u_2(y(a)) + u_2(y(b)).$$

It is clear from geometrical considerations that the first condition identifies core allocation (pair points) located “above” of D-quasi-equilibrium of type (b), $x(b) = (\frac{36}{17}, \frac{36}{25})$. Now to find $x(a)$ one needs to solve system

$$\begin{cases} x_2(a) = \frac{6x_1(a)}{x_1(a)+5}, \\ x_1(a)x_2(a) = x_1(b)x_2(b) = \frac{36 \cdot 36}{17 \cdot 25} \end{cases} \Rightarrow (x_1(a), x_2(a)) \approx (1.87, 1.63).$$

The second condition is a non-trivial one, for further analysis we rewrite it in the form

$$\langle \nabla u_2(y(a)), \mathbf{e}_2(a) - y(a) \rangle + \langle \nabla u_2(y(b)), \mathbf{e}_2(b) - y(b) \rangle \leq 0.$$

In equilibrium this inequality turns to equality

$$\frac{3}{5 - x_1(a)} + \frac{5}{3 - x_2(a)} + \frac{1}{2(4 - x_1(b))} + \frac{7}{4 - x_2(b)} = 6,$$

written in variables corresponding to the first agent consumption. By Proposition 9.2.1 and taking into account the requirements to belong to the Pareto frontier (9.2.6), (9.2.7), this equation must be supplemented by a system

$$\begin{cases} x_1(a)x_2(a) + 5x_2(a) = 6x_1(a), \\ x_1(b)x_2(b) + 5x_2(b) = 4x_1(b), \\ x_1(a)x_2(a) = x_1(b)x_2(b). \end{cases}$$

The finding of a complete solution of the resulting system of equations is not an easy task and we omit it. Note, however, that it follows from geometrical considerations in Edgeworth box that the solution will be located between D-quasi-equilibria of (*a*) and (*b*) types and will be represented by a pair of points of the first agent indifference curve intersected with the Pareto boundaries of '*a*' and '*b*' states.

What has happened in the case of maximin preferences with the D-core and D-equilibrium? If for the notion of D-core one applies weak domination instead of a strong one⁵ then the statement of Proposition 9.2.1 still is true. Subsequent analysis is largely retained, but the behavior of 2nd individual (informed, which is split into two new agents) is modified. The elements of the core will again be presented by a pair of points formed by the intersection of the first agent indifference curve with the Pareto boundaries for states '*a*' and '*b*'. D-equilibrium is now described as above and is presented as a pair of allocations corresponding to D-(quasi)equilibrium type (*b*). All the above objects are represented in Figure 9.2.4. ■

Continuing the analysis let us raise a question: how can contractual process proceed with the maximin preferences if an information sharing is potentially allowed? Are there any incentives for informed agents to share information? As a result of this sharing, how can newly transformed preferences can be described?

I start with the last question on the transformation of the utility. Earlier in Section 7.1.3, p. 274 we have postulated independence of preferences from information and, moreover, their separability (7.1.5). Maximin preferences obviously do not satisfy this condition, but a reasonable theoretical treatment has to be conducted.

What will happen with the maximin utility of individuals after informational sharing? For instance, suppose there are three states of nature $\Omega = \{a, b, c\}$, which an individual is unable to differentiate. Now his/her maximin utility takes the form

$$U^{min}(x(a), x(b), x(c)) = \min\{u(a, x(a)), u(b, x(b)), u(c, x(c))\}.$$

Suppose now that the agent has learned to distinguish '*a*': what has happened with his/her utility, is it changed or not? For Bayesian utility nothing happens, but it is not so for maximin. Now one can not just write the amount of utility in the state '*a*' plus utility minimum for two other states: under such decomposition the utility is unreasonable changed (increased?). No objections raise the representation

$$u(a, x(a))\mu(a) + \min\{u(b, x(b)), u(c, x(c))\}\mu(\{b, c\}),$$

⁵Recall that the domination by coalition is strong, if each member of the coalition is strictly better off, with a weak domination at least one member of the coalition is so—others may be indifferent.

where $\mu(E)$ presents a probability for a new information partition. After one more information sharing utility is transformed to the form

$$u(a, x(a))\mu'(a) + u(b, x(b))\mu'(b) + u(c, x(c))\mu'(c),$$

where $\mu'(\cdot)$ is a new probability distribution such that $\mu'(b) + \mu'(c) = \mu(\{b, c\})$ and $\mu'(a) = \mu(a)$. In fact, we assume that simultaneously with the learning to distinguish events during the information sharing the agent has also learned the probability of their realization so that the probability of initial event is split into the sum of new known to him.

Thus during the information sharing process maximin utility is transformed into Bayesian. One can also assume that individuals are endowed with shadow inactivated Bayesian utility, which now is partially activated. Moreover, by Proposition 9.2.1 proposed method has no effect the utility value of core allocations, *i.e.* when ordinary methods of contractual process have been exhausted. In other words, the separability is performed in the limited sense—for the class of efficient allocations. However, what are the incentives and a resulting goal for this process? It is easier to present further considerations in the context of described above Example 9.2.1.

Indeed, it seems that the contractual process without information sharing will lead economy to one of two allocations: the maximin private equilibrium or D-(quasi)equilibrium of the type (b). These allocations are presented in the following Figure 9.2.5 (similar to 9.2.4), which explains our problem in the context of Example 9.2.1. Mentioned points are located at Pareto borders as soon as nobody is interested in new contracts. However, what can be concluded about the partial breaking of contracts?

Consider, for example, *maximin* D-(quasi)equilibrium of *type* (b), presented by a couple of points at the figure. Possible result of partial breaking by one-quarter of contracts is indicated by arrows emanating from the equilibrium pair along straight lines connecting the allocation with the initial endowments. It can be seen that the second (informed) agent is not at all interested in the breaking of contracts, as in each of the states of the world the new allocation is outside of better ones. The first agent also will not break the contract until his/her utility is defined as maximin—here it corresponds to a posteriori utility u_1 , calculated in the bottom after breaking point (represented by asterisk), as a posteriori utility is decreased in 'b' and increased in 'a'. However, after the information sharing utility is transformed into a Bayesian one and a weighted sum of the utilities in two states of the world will increase. Thus the first individual is interested in the sharing of information for a further specification of contracts (partially breaking).

In *maximin private* equilibrium for the second agent after infinitesimal partial breaking of contracts, the growth and decreasing of utility in the states of the world are balanced: 'a' gives rise, 'b' decreasing, and the aggregate marginal change is zero. The first agent is also not interested in a break by the reasons described above. However, he/she may be interested in after the information sharing, it will depend on the numerical values of new probabilities which combine the effects of growth and decreasing. The greater relative weight of state 'a' implies a greater chance for breaking.

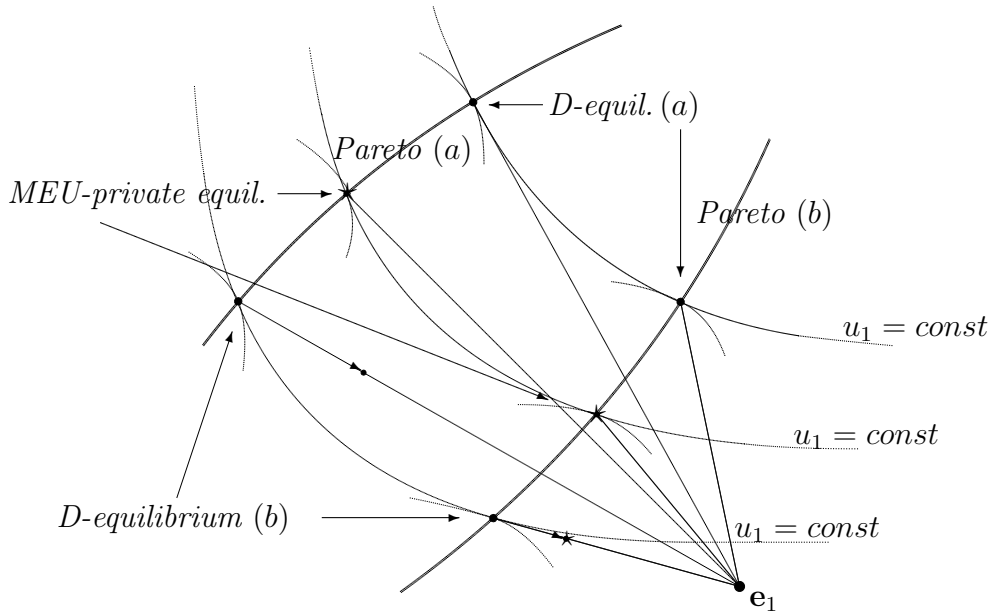


Figure 9.2.5: Contractual dynamics under maximin utilities

Contractual structure in the *maximin* D-(quasi)equilibrium of *type (a)* is unstable as the second individual is interested in the partial breaking of contracts, and it is not related with the sharing of information. For the first agent, it is unprofitable to break contracts.

From the above analysis about contractual process and implemented allocations, we conclude the following ones.

(i) Whatever the stable allocation contractual process is finished, it can be proper contractual or equilibrium allocation, the informed agent has *no incentive* to share information. Indeed, in each of the possible states of nature implemented allocation is already at the Pareto boundary and, therefore, the signing of a new contract can not be mutually beneficial. At the same time, the agent is not interested in the partial breaking of contracts.

(ii) Interesting in the exchange of information, party can only be less-informed individual, his/her interest is to obtain the possibility of *partial breaking of contracts*. However, it is important how his/her preferences are transformed after information sharing (new probabilities). In the case of D-(quasi)equilibrium of *type (b)* the first individual is solely interested in the sharing of information, in difference with the *type (a)* when there is no such interest at all. There are also intermediate solutions.

9.3 Interim core and REE-equilibria

Let us consider the following example of an asymmetric information additionally discuss the concept of the *D*-core and an equilibrium introduced above.

Let there be 2 agents, 1st know $\{\{a\}, \{b\}\}$, second— $\{\{a, b\}\}$. What will happen in case of *D*-core? The specificity is that in coalitions $\{(1, a), (2, ab)\}$ and $\{(1, b), (2, ab)\}$

by informational reasons the exchange is impossible and therefore their possibilities to dominate will be equal to one-element coalitions (even less). The only efficient coalition will be the coalition of all duplicated players (notice that it is not the same as in the case of private core). However, does it ensure the necessary stability?

If, for example, elementary event ‘a’ is realized “tomorrow morning”, then 1st agent can suggest to 2nd one to break old contract $v = (v_1^a, v_1^b, v_2^{ab})$, $v_1^a = v_1^b = -v_2^{ab}$ and to sign a new w , if there is such (w_1^a, w_1^b, w_2^{ab}) , $w_1^a = w_1^b = -w_2^{ab}$ that

$$\varphi_a^1(w_1^a + \mathbf{e}_1^a) > \varphi_a^1(v_1^a + \mathbf{e}_1^a) \quad \& \quad \psi^2(w_2^{ab} + \mathbf{e}_2^a, w_2^{ab} + \mathbf{e}_2^b) > \psi^2(v_2^{ab} + \mathbf{e}_2^a, v_2^{ab} + \mathbf{e}_2^b).$$

It is important that for the first agent a significant role plays only duplicate ‘a’, and second duplicate ‘b’ is a dummy: it is so because state ‘a’ is already realized and 1st agent knows it for certain. The second agent is not able to receive new information, but to conclude a good deal, the 1st individual accompanies him, promising identical deliveries in case of possible states. Clearly, when state ‘b’ is realized the similar things are happened only function $\varphi_a^1(\cdot)$ is replaced on $\varphi_b^1(\cdot)$.

The main question for the example: whether the contract from D -core is stable relative to described threats? There are no obvious foundations to state it. For this reason I suggest one more concept of the core which form contracts stable relative to attempts to open a new recontracting process in the time “tomorrow morning”.

Definition 9.3.1 *Interim-core $\mathcal{C}^{int}(\mathcal{E}^{di})$ for the economy \mathcal{E}^{di} with asymmetrically informed agents consist of allocations $x = (x_i)_{\mathcal{I}} \in X$ such that:*

- (i) $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$,
- (ii) $(x_i - \mathbf{e}_i) : \Omega \rightarrow \mathbb{R}^l$ is P_i -measurable for all $i \in \mathcal{I}$,
- (iii) $\forall \mathbf{w} \in \Omega \nexists S(\mathbf{w}) = S \subseteq \mathcal{I} : \exists y^S = (y_i^E)_{(i,E) \in \mathfrak{S}: i \in S}, y_i^E \in \mathcal{L}^E, (i, E) \in \mathfrak{S} \mid$
 - (a) $\forall i \in S$ for $F = P_i(\mathbf{w})$ $y_i^F \in X_i^F$ & $y_i^F \succ_i^F x_i^F$ and
 - (b) $\sum_{(i,E) \in \mathfrak{S}: i \in S} (y_i^E - \mathbf{e}_i^E) = 0$ or $\sum_{(i,E) \in \mathfrak{S}: i \in S} (y_i^E - x_i^E) = 0$.

One can notice again that due to item (iii)(a) preference in consumption is required only for duplicates $(i, F) \in \mathfrak{S}$ such that $F = P_i(\mathbf{w})$ under supposition the nature has realized state \mathbf{w} . Comparing definitions one concludes that interim core is a subset of D -core, *i.e.*

$$\mathcal{C}^{int}(\mathcal{E}^{di}) \subseteq \mathcal{C}^d(\mathcal{E}^{di})$$

is always true.

A remarkable property of the interim core is that all its elements are coalitional incentive compatible, *i.e.*, they are **CIC**-allocations by Definition 7.3.1 on page 284. Intuitively, this means that the allocation and the contract that implements it are stable relative to the informational fraud of the groups of individuals.

Proposition 9.3.1 *Let for each agent of \mathcal{E}^{di} his/her preferences be identical in undistinguished states of nature, i.e., in terms of utilities they are represented as*

$$u_i(x_i) = \sum_{\mathbf{w} \in \Omega} u_i(\mathbf{w}, x_i(\mathbf{w}))q_i(\mathbf{w}), \quad x_i \in L_+,$$

where for every fixed $y \in \mathbb{R}_+^l$ function⁶ $u_i(\cdot, y)$ is P_i measurable.⁷ Let also endowments $\mathbf{e}_i : \Omega \rightarrow \mathbb{R}_+^l$ be measurable relative to individual information P_i , $i \in \mathcal{I}$.

Then every allocation $x \in \mathcal{C}^{int}(\mathcal{E}^{di})$ is **transfer coalitional incentive compatible**.

Proof of Proposition 9.3.1. We argue by contradiction. By Definition 7.3.1 find a coalition $S \subset \mathcal{I}$, $S \neq \emptyset$ and a state $a \in \Omega$ such that for $v = x - \mathbf{e}$

$$(i) \{a\} = \bigcap_{i \in S} P_i(a),$$

$$(ii) \exists b \in \bigcap_{j \in \mathcal{I} \setminus S} P_j(a), b \neq a \ \& \ \exists w = (w_i)_{\mathcal{I}} \in L^S, \sum_S w_i = 0:$$

$$u_i(a, \mathbf{e}_i(a) + v_i(b) + w_i(a)) > u_i(a, x_i(a)) \quad \forall i \in S$$

holds. Further, the measurability of v_j relative to P_j for undistinguished states $a, b \in \Omega$ realizes $v_j(b) = v_j(a)$, $\forall j \in \mathcal{I} \setminus S$. Now by contract definition one concludes

$$\sum_S v_i(b) = - \sum_{\mathcal{I} \setminus S} v_j(b) = - \sum_{\mathcal{I} \setminus S} v_j(a) = \sum_S v_i(a).$$

Consequently, the coalition can sign a new contract $h = (h_i)_{\mathcal{I}}$ defined as follows: $h_i(\mathbf{w}) = v_i(b) - v_i(a) + w_i(a)$, $\forall \mathbf{w} \in \Omega$ for $i \in S$ and $h_j = 0$, $j \in \mathcal{I} \setminus S$. This is a contract in view of the latter formula and $\sum_S w_i = 0$. Moreover, the agreement does not depend on the states of nature, therefore it is measurable. Simultaneously, the measurability of endowments implies: for all $i \in S$ and $y_i = x_i + w_i$, $\forall \mathbf{w} \in P_i(a)$

$$y_i(\mathbf{w}) = \mathbf{e}_i(\mathbf{w}) + v_i(\mathbf{w}) + h_i(\mathbf{w}) = \mathbf{e}_i(a) + v_i(a) + h_i(a) = \mathbf{e}_i(a) + v_i(b) + w_i(a) \geq 0.$$

Also, the measurability of $u_i(\cdot, y)$ implies $u_i(\mathbf{w}, y) = u_i(\mathbf{w}', y) \quad \forall \mathbf{w}, \mathbf{w}' \in P_i(a), \forall y \in \mathbb{R}_+^l$ that via (ii) and previous equality yields:

$$\forall i \in S, \forall \mathbf{w} \in P_i(a) \quad u_i(\mathbf{w}, y_i^S(\mathbf{w})) > u_i(\mathbf{w}, x_i(\mathbf{w})) \implies$$

$$\forall i \in S \text{ for } F = P_i(a) \quad \sum_{\mathbf{w} \in F} u_i(\mathbf{w}, y_i^S(\mathbf{w}))q_i(\mathbf{w}) > \sum_{\mathbf{w} \in F} u_i(\mathbf{w}, x_i(\mathbf{w}))q_i(\mathbf{w}) \iff$$

$$\forall i \in S, \text{ for } F = P_i(a) \quad y_i^F \in X_i^F \ \& \ y_i^F \succ_i^F x_i^F.$$

Thus, the state $\mathbf{w} = a$ and coalition S are found so that item (ii) of Definition 9.3.1 is violated. This contradicts $x \in \mathcal{C}^{int}(\mathcal{E}^{di})$. ■

A negative (but expected) property of the interim core is that it can be empty: analyzed above Example 9.2.1 demonstrates this and, moreover, it is a typical case for asymmetric information. Our analysis is continued by

⁶See comments to formula (7.1.6).

⁷The P_i -measurability of the ex post utility functions is often assumed in modern studies of games with incomplete information.

Example 9.3.1 Consider a pure exchange economy with asymmetrically informed agents from the previous section, Example 9.2.1. One can note from item (iii) of interim core definition that consumption bundles corresponding to a current elementary event $\mathbf{w} \in \Omega$ have to form an allocation from the core of economy reduced to the state \mathbf{w} . Moreover, it has to be true for *every* elementary event. These cores corresponding to the states ‘a’ and ‘b’ were specified during Example 9.2.1 analysis, they are represented in Figure 9.2.2, and Figure 9.2.3 presents a summarized picture. It shows that for the area bounded by indifference curves, Pareto boundaries being separately specified for these events are not intersected, *i.e.* presented **requirements are inconsistent**. Moreover, the situation cannot be changed in general for any small variation of utilities and will become even worse if one will try to increase commodity space dimension: this is so because dimension of Pareto boundaries will be the same (number of agents minus 1), *i.e.* it will be a one-dimensional curve in 3 and more dimensional space and (it possible to say) the probability of these curves have nonempty intersection is zero. Notice that asymmetry of informational distribution plays the key role in this conclusion.

One can conclude from example presented that in order to have a nonempty interim core one has to require the coincidence of individual preferences, associated with different states of the world and induced to commodity space (space of contracts!). In other words to obtain nonempty intersection of cores and, therefore, nonempty interim core one needs to require each person that is able to distinguish events has equal preferences (at least locally) for the states of the world that cannot be distinguished by another agent. However namely when interim core is empty for a current allocation realized via a web of contracts, the agents have incentives to open information: as a result of their informational sharing they discover new possibilities to conclude new mutually beneficial contract.

It is obvious that there are informational structures, when the core is not empty under rather weak other assumptions. However, it is possible to show formally that generically it can be only for the symmetric information and for asymmetrically informed agents interim core generally is empty. ■

The following statement says that if in D -equilibrium according to Definition 9.2.2 one requires, in addition, that if, for any two states of the world that are indiscernible at least for one individual, relative prices are identical (have equal proportions) in these states, then this allocation belongs to interim core.

Proposition 9.3.2 *Let (x, p) be a D -equilibrium by Definition 9.2.2 such that*

$$\forall i \in \mathcal{I} \quad \forall \mathbf{w}, \mathbf{w}' \in E \in P_i, \quad \exists \lambda_{\mathbf{w}'} \geq 0 : \lambda_{\mathbf{w}'} p(\mathbf{w}) = p(\mathbf{w}'). \quad (9.3.1)$$

*Then $x \in \mathcal{C}^{int}(\mathcal{E}^{di})$, *i.e.* this is an interim-core allocation.*

Note that D -equilibrium considered in the statement is such that being proper renormalized price mapping can be represented as a map measurable with respect to individual information for each individual, *i.e.*, relative to $\bigwedge_{\mathcal{I}} P_i$ that is “common knowledge”.

Proof of Proposition 9.3.2. By contradiction: assume there is $\mathbf{w} \in \Omega$ such that a coalition S dominates allocation x :

$$\exists y^S = (y_i^E)_{(i,E) \in \mathfrak{S}: i \in S} \mid \sum_{(i,E) \in \mathfrak{S}: i \in S} (y_i^E - \mathbf{e}_i^E) = 0 \quad \& \quad [y_i^F \succ_i^F x_i^F, F = P_i(\mathbf{w})] \quad \forall i \in S.$$

By (ii) from Definition 9.2.2, conclude

$$\langle p^F, y_i^F \rangle > \langle p^F, x_i^F \rangle = \langle p^F, \mathbf{e}_i^F \rangle, \quad F = P_i(\mathbf{w}) \quad \forall i \in S,$$

that by $y_i(\mathbf{w}') - \mathbf{e}_i(\mathbf{w}') = y_i(\mathbf{w}) - \mathbf{e}_i(\mathbf{w})$ for all $\mathbf{w}' \in P_i(\mathbf{w})$, $i \in S$ and (9.3.1) yields

$$0 < \langle p^F, y_i^F - \mathbf{e}_i^F \rangle = \langle p(\mathbf{w}), y_i(\mathbf{w}) - \mathbf{e}_i(\mathbf{w}) \rangle \sum_{\mathbf{w}' \in F} \lambda_{\mathbf{w}'}, \quad \forall i \in S \Rightarrow$$

$$\langle p(\mathbf{w}), \sum_S (y_i(\mathbf{w}) - \mathbf{e}_i(\mathbf{w})) \rangle > 0,$$

that implies $\sum_S (y_i(\mathbf{w}) - \mathbf{e}_i(\mathbf{w})) \neq 0$ and therefore contradicts to (iii)(b) of Definition 9.3.1. \blacksquare

Proposition 9.3.2 characterizes D -equilibria that are the elements of the interim core, however it would be necessary to present or clarify their existence conditions. The following proposition is important for understanding this problem: there are analyzed D -equilibrium allocations, realized as fuzzy contractual ones, such that there are no individuals having incentives to share information, because it is impossible to sign new beneficial contract.

Proposition 9.3.3 *Let $x = (x_i)_{i \in \mathcal{I}}$ be a fuzzy contractual allocation with differentiated agents. Then there exist prices $p : \Omega \rightarrow \mathbb{R}^l$, $p \neq 0$ for each $i \in \mathcal{I}$ and $E \in P_i$ satisfying*

$$\sum_{\mathbf{w} \in E} p(\mathbf{w}) x_i(\mathbf{w}) = \sum_{\mathbf{w} \in E} p(\mathbf{w}) \mathbf{e}_i(\mathbf{w})$$

and

$$\left\langle \sum_{\mathbf{w} \in E} p(\mathbf{w}), z \right\rangle \geq 0, \quad \forall z \in \mathbb{R}^l : \quad x_i + z \cdot \chi^E \in \mathcal{P}_i(x_i). \quad (9.3.2)$$

Thus fuzzy contractual allocation is D -quasi-equilibrium .

Let, in addition, the allocation $x = (x_i)_{i \in \mathcal{I}}$ be such that there is **no** possibility to sign a new mutually beneficial **contract for any kind of informational sharing**. Then (9.3.2) is replaced by a more qualified requirement

$$\sum_{\mathbf{w} \in E} p(\mathbf{w}) z(\mathbf{w}) \geq 0, \quad \forall z : \Omega \rightarrow \mathbb{R}^l : \quad x_i + z \cdot \chi^E \in \mathcal{P}_i(x_i). \quad (9.3.3)$$

It can be specially noted the difference in formulas (9.3.2) and (9.3.3): a *vector* $z \in \mathbb{R}^l$ is applied in the first case, but a *function* $z : \Omega \rightarrow \mathbb{R}^l$ in the second one.

For contractual process the meaning of the proposition is that if (9.3.3) is invalid for fuzzy contractual allocation with D -agents, then informational sharing may happen that will imply new acts in contractual process.

Proof of Proposition 9.3.3. State the first part of proposition. First, its conditions have to be presented in the convenient for mathematical analysis form. To do it let us reformulate relation (8.1.23) for the case of differentiated agents. Define

$$\hat{\mathcal{P}}_i^E(x_i) = \{x_i^E + z \mid z : \Omega \rightarrow \mathbb{R}^l, z(\mathbf{w}) = 0, \mathbf{w} \notin E \ \& \ x_i + z \in \mathcal{P}_i(x_i)\}.$$

Then for fuzzy contractual allocation with fixed information one has

$$\begin{aligned} & \prod_{i \in \mathcal{I}} \sum_{E \in P_i} [(\hat{\mathcal{P}}_i^E(x_i) + \text{co}\{0, \mathbf{e}_i^E - x_i^E\}) \cup \{\mathbf{e}_i^E\}] \cap \\ & \bigcap \{z = (z_i)_{\mathcal{I}} \mid z_i = \sum_{E \in P_i} z_i^E, \forall i \in \mathcal{I} \ \& \ \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i \ \& \ (z_i^E - \mathbf{e}_i^E)_{|_E} = \text{const}, \ \& \\ & (z_i^E - \mathbf{e}_i^E)_{|\Omega \setminus E} = 0, \forall (i, E) \in \mathfrak{S}\} = \{(\mathbf{e}_1, \dots, \mathbf{e}_n)\}. \end{aligned} \quad (9.3.4)$$

Notice that it differs with (8.1.23): there is no necessity to include a third element in the latter intersection; this element was applied in (8.1.23) in order to specify contracts measurable relative to individual information. Moreover, here sets $\hat{\mathcal{P}}_i^E(x_i)$ are applied instead of $\mathcal{P}_i^E(x_i)$ in the first set under intersection.

Further, in order to present price characteristics for the allocation, let us apply to intersection (9.3.4) separation theorem and consider a liner functional (vector) $\pi = (p_i)_{i \in \mathcal{I}} \neq 0$ separating these sets. This functional has the following specific structure, that can be checked arguing in similar manner to the proof of Lemma 8.1.1 (about mutually beneficial contract, see page 298):

$$\exists p : \Omega \rightarrow \mathbb{R}^l \ \& \ q_i : \Omega \rightarrow \mathbb{R}^l, p + q_i \neq 0, \forall i \in \mathcal{I}$$

such that $\forall i \in \mathcal{I} \ \forall E \in P_i$

$$\begin{aligned} & \sum_{\mathbf{w} \in E} q_i(\mathbf{w}) = 0 \ \& \\ & \langle \hat{\mathcal{P}}_i^E(x_i), p^E + q_i^E \rangle \geq \langle x_i^E, p^E + q_i^E \rangle = \langle \mathbf{e}_i^E, p^E + q_i^E \rangle. \end{aligned} \quad (9.3.5)$$

Notice that here functional $p^E + q_i^E$ supports set $\hat{\mathcal{P}}_i^E(x_i)$, where in difference with $\mathcal{P}_i^E(x_i)$, for different states of nature from E different commodity bundles can be consumed, *i.e.* $(z_i - \mathbf{e}_i)_{|_E} \neq \text{const}$ is possible for $z_i \in \hat{\mathcal{P}}_i^E(x_i)$. Moreover, notice that now one cannot guaranty that $p^E + q_i^E \neq 0$ for all $E \in P_i$; it known only that at least one of them is not zero (since $p + q_i \neq 0$). Further, due to $\langle \chi^E, q_i^E \rangle = 0$ one can avoid q_i from relations (9.3.5) and for $x_i + z \cdot \chi^E \in \mathcal{P}_i(x_i)$, $z \in \mathbb{R}^l$ rewrite it in an equivalent form: $\forall (i, E) \in \mathfrak{S}$,

$$\left\langle \sum_{\mathbf{w} \in E} p(\mathbf{w}), x_i^E - \mathbf{e}_i^E \right\rangle = 0 \ \& \ \left\langle \sum_{\mathbf{w} \in E} p(\mathbf{w}), z \right\rangle \geq 0, \forall z \in \mathbb{R}^l : x_i + z \cdot \chi^E \in \mathcal{P}_i(x_i),$$

that proves (9.3.2) and the first part of lemma. Notice that this exactly thing required by definition of D -quasiequilibrium.

Further, the fact that after every possible informational sharing there are no mutually beneficial contracts is equivalent to that there are no beneficial contracts relative to complete information. However, one has to remember that current allocation is still realized via a web of contracts obtained for differentiated agents and only these contracts can be broken: partially, fuzzy or as a whole. This means that in order for all conditions of the second part to be satisfied, it is sufficient that the second set from intersection be replaced by the similar one but without measurement requirements (one needs to study the case when all distinguished sets have only one element, *i.e.* when $E = \{\mathbf{w}\}$, $\mathbf{w} \in \Omega$). As a result, for $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ we have

$$\prod_{i \in \mathcal{I}} \sum_{E \in \mathcal{P}_i} [(\hat{\mathcal{P}}_i^E(x_i) + \text{co}\{0, \mathbf{e}_i^E - x_i^E\}) \cup \{\mathbf{e}_i^E\}] \cap \{z = (z_i)_{i \in \mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\} = \{\mathbf{e}\}.$$

Further, standardly applying separation theorem one concludes the existence of “price” map $p : \Omega \rightarrow \mathbb{R}^l$ such that $\forall (i, E) \in \mathfrak{S}$,

$$\begin{aligned} \sum_{\mathbf{w} \in E} p(\mathbf{w})x_i(\mathbf{w}) &= \sum_{\mathbf{w} \in E} p(\mathbf{w})\mathbf{e}_i(\mathbf{w}) \ \& \\ \sum_{\mathbf{w} \in E} p(\mathbf{w})z(\mathbf{w}) &\geq 0, \ \forall z : \Omega \rightarrow \mathbb{R}^l : x_i + z \cdot \chi^E \in \mathcal{P}_i(x_i) - \end{aligned}$$

as we wanted to prove. ■

The last proposition reveals some specific features of equilibrium in the case when agents have no incentives to open information. However, the rising question is to estimate how typical is this case and can one qualify it as a generical one for D -equilibria to obey (9.3.3) in addition. With this in mind, let us comparatively analyze this for D -equilibria. Now compare the number of independent constraints and the number of applied variables in the model situation considered above where 1st agent is able to distinguish events a and b , but 2nd *is not able* to do it. Calculate the balance between variables and constraints (variables minus constraints). So for D -equilibria one has:

Direct variables applied for allocation:

$$l \text{ (1st for 'a')} + l \text{ (1st for 'b')} + l \text{ (2nd for 'ab')} - 2l \text{ (number of balance equations)} = l.$$

Price variables and budget constraints⁸ yield:

$$l - 1 \text{ (} p(a) \text{ normed)} + l \text{ (for } p(b)) - 2 \text{ (number of independent budget equations)} = 2l - 3.$$

First-order conditions (collinearity of prices and gradients):

$$1 - l \text{ (1st for 'a')} + 1 - l \text{ (1st for 'b')} + 1 - l \text{ (2nd for 'ab')} = 3 - 3l.$$

⁸Standardly: if material balance is true, then one (any) of budget equalities followed by other ones. This is why a number of independent constraints is a unit less of total quantity.

As a result, summing right-hand parts, one finds

$$l + 2l - 3 + 3 - 3l = 0,$$

i.e. in a generic situation for twice differentiable utility functions one can expect a *finite* number of D -equilibria.

What are the changes in the calculations for allocation satisfying (9.3.3)? Only the final addend is changed, namely in first-order conditions for 2nd individual l additional constraints⁹ are appeared, *i.e.*, the total balance will be negative one and one can expect that generically there are no these allocations at all... For more detailed analysis one can turn to the consideration of Example 9.2.1.

However, what happens in general case and what concept of an equilibrium does correspond to the interim core? To answer the question correctly, we need the following technical result.

Lemma 9.3.1 *Let \mathcal{E}^{di} be a smooth economy and $x = (x_i)_{\mathcal{I}} \in \mathcal{C}^{int}(\mathcal{E}^{di})$ be an **interior** allocation from interim core. Then there are prices $p : \Omega \rightarrow \mathbb{R}^l$, $p(\mathbf{w}) \neq 0 \forall \mathbf{w} \in \Omega$ and real $\lambda_{i,E} > 0$, $(i, E) \in \mathfrak{S}$ such that*

$$\lambda_{i,E} \sum_{\mathbf{w} \in E} \nabla_{\mathbf{w}} u_i(x_i) = p(\mathbf{w}') = p(\mathbf{w}'') \quad \forall \mathbf{w}', \mathbf{w}'' \in E, \quad \forall (i, E) \in \mathfrak{S}. \quad (9.3.6)$$

Therefore, for these prices one also has

$$\left\langle \sum_{\mathbf{w} \in E} p(\mathbf{w}), z \right\rangle > 0, \quad \forall z \in \mathbb{R}^l : \quad x_i + z \cdot \chi^E \in \mathcal{P}_i(x_i). \quad (9.3.7)$$

Condition (9.3.6) means the measurability of the price mapping with respect to infimum of individual informations. In view of (9.3.7), this mapping plays the similar role as the usual prices in the classical Welfare Theorem, but here it is with respect to differentiated agents. Notice the relationship of (9.3.7) with relation (9.3.2), that characterizes fuzzy contractual allocation with D -agents and that corresponds to (ii) of D -(quasi)equilibrium definition (Definition 9.2.2): a difference is only the strict form of inequality.

Proof of Lemma 9.3.1. Analyzing Definition 9.3.1 one finds that for every $\mathbf{w} \in \Omega$ allocation $x(\mathbf{w}) = (x_i(\mathbf{w}))_{\mathcal{I}}$ is Pareto optimal in the model reduced to the event \mathbf{w} , where agents' utilities are (locally) defined via $\vartheta_i(y_i) = u_i(z_i(\cdot))$ as follows

$$z_i(\mathbf{w}') = \begin{cases} y_i + \mathbf{e}_i(\mathbf{w}'), & \mathbf{w}' \in P_i(\mathbf{w}), \\ x_i(\mathbf{w}'), & \mathbf{w}' \notin P_i(\mathbf{w}). \end{cases}$$

This standardly implies the existence of vector $p(\mathbf{w}) \in \mathbb{R}^l$, $p(\mathbf{w}) \neq 0$ and real $\mu_{i,\mathbf{w}} > 0$ such that $\mu_{i,\mathbf{w}} \nabla \vartheta_i([x_i - \mathbf{e}_i](\mathbf{w})) = p(\mathbf{w})$. Thus for normalized $p(\mathbf{w})$ one can conclude (because $p(\mathbf{w}) = \mu_{i,\mathbf{w}} \nabla \vartheta_i([x_i - \mathbf{e}_i](\mathbf{w})) = p(\mathbf{w}')$), that

$$p(\mathbf{w}) = p(\mathbf{w}') \quad \text{for } \mathbf{w}, \mathbf{w}' \in E, \quad (i, E) \in \mathfrak{S}.$$

⁹Instead of $\nabla_a u_2(x_2^a, x_2^b) + \nabla_b u_2(x_2^a, x_2^b) = \lambda(p^a + p^b)$ for $x_2^a - \mathbf{e}_2^a = x_2^b - \mathbf{e}_2^b$ one has to use $\nabla_{ab} u_2(x_2^a, x_2^b) = \lambda(p^a, p^b)$.

This proves (9.3.6) in view of $\sum_{\mathbf{w} \in E} \nabla_{\mathbf{w}} u_i(x_i) = \nabla \vartheta_i([x_i - \mathbf{e}_i](\mathbf{w}))$ for initial utilities. Now relation (9.3.7) follows directly. Lemma is proven. ■

Now one can formulate an equilibrium notion well corresponding to interim core concept. This definition is based on Lemma 9.3.1 result and the only that we need now is to add to relations (9.3.6), (9.3.7) a budget feasibility (inequality realized as equality).

Definition 9.3.2 *A couple (x, p) is called interim equilibrium if it is a D -equilibrium according to Definition 9.2.2 and in addition:*

$$\forall i \in \mathcal{I}, \quad \forall \mathbf{w} \in \Omega \quad p(\mathbf{w}) = p(\mathbf{w}') \neq 0 \quad \forall \mathbf{w}' \in P_i(\mathbf{w}).$$

holds. Thus, the price mapping must be measurable with respect to each individual information, i.e., relative to $\bigwedge_{\mathcal{I}} P_i$.

Notice that this definition almost exactly corresponds to the notion of *REE*-equilibrium, see page 280. The only difference is the specific probability context applied for *REE* and also the possibility to consider finer informational structures, enriched via price informational channel. Notice also that interim equilibrium being an element of interim core (due to Proposition 9.3.2) may not exist; the existence of it can be provided for a specific informational structure, but to achieve it informational exchange has to be realized.

What is the dynamics of a contractual process that drives economy to interim equilibrium? Clearly this process has to include an endogenous process of informational sharing. I shall continue the analysis of considered above example, in which $\Omega = \{a, b\}$, $P_1 = \{\{a\}, \{b\}\}$, $P_2 = \{\Omega\}$.

What will be in an equilibrium? In the variant that corresponds to private equilibrium, the 1st agent should reach a maximum of utility under constraint

$$p(a)x(a) + p(b)x(b) \leq p(a)\mathbf{e}_1(a) + p(b)\mathbf{e}_1(b).$$

On the other hand, in the variant corresponding to D -equilibrium, this agent has to solve two problems:

$$\varphi_a^1(x(a)) \rightarrow \max \quad s.t. \quad p(a)x(a) \leq p(a)\mathbf{e}_1(a),$$

$$\varphi_b^1(x(b)) \rightarrow \max \quad s.t. \quad p(b)x(b) \leq p(b)\mathbf{e}_1(b).$$

Second agent in both cases solves a problem under an additional constraint:

$$[p(a) + p(b)]y(a, b) \leq p(a)\mathbf{e}_2(a) + p(b)\mathbf{e}_2(b), \quad y(a, b) = y(a) = y(b).$$

Clearly solutions of these problems have to be feasible in equilibrium.

The question is raised: will contractual process be completed in this (D -equilibrium) allocation or can it continue developing? Probably the first agent will wish to share the information with 2nd and to transmit him the ability to distinguish a and b ? He/she can do it only if he/she will receive real advantage (profit). This

advantage can be realized as a new favorable contract—having learned a difference between states, 2nd agent will agree to sign contract beneficial for the first agent. Moreover, there is not another way—without the sharing of information—to reach an agreement.

Let us apply Lemma 9.2.1 result and write the conditions of this agreement for differentiated agents. Lemma and its Corollary 9.2.1 imply the existence of vectors of (equilibrium) prices $p(a), p(b), q \in \mathbb{R}^l$:

$$\exists (v(a), v(b)) : \langle p(a), v(a) \rangle > 0, \quad \langle p(b), v(b) \rangle > 0 \quad -$$

these are payoffs of 1st in every future state and

$$\langle p(a) + q, -v(a) \rangle > 0, \quad \langle p(b) - q, -v(b) \rangle > 0$$

are payoffs of second agent (possibly not all > 0 , one zero can be); here q is a vector which exists via condition that *there is no* mutually beneficial contract under *fixed* information (Pareto optimality), see Lemma 9.2.1. There are two possibilities: $q \neq 0$ and $q = 0$.

As soon as for $0 \neq q \neq \lambda p(a) \forall \lambda$ vectors $p(a)$ and $-[p(a) + q]$ are noncollinear, then there exists $v(a)$ satisfying

$$\langle p(a), v(a) \rangle > 0 \quad \& \quad \langle p(a) + q, -v(a) \rangle > 0.$$

Thus there is the following: 1st says to 2nd: “I can teach you to distinguish ‘ a ’ and ‘ b ’, and you will teach me to distinguish event $\{a, b\}$ (from 2nd partition) and then we will be able to find mutually beneficial exchange...”

When this cannot happen in general (let the allocation be interior)? Lemmas 8.1.1, 9.2.1, their Corollaries 8.1.1, 8.1.2, 9.2.1 and Proposition 9.3.3 give the answers. It will not happen if and only if for event $a \in \Omega$

$$\forall z \in \mathbb{R}^l \quad \langle p(a), z \rangle > 0 \Rightarrow \langle p(a) + q_i(a), z \rangle > 0 \quad \forall i \in \mathcal{I},$$

is true and similar requirement for the event $b \in \Omega$:

$$\forall z \in \mathbb{R}^l \quad \langle p(b), z \rangle > 0 \Rightarrow \langle p(b) + q_i(b), z \rangle > 0 \quad \forall i \in \mathcal{I}.$$

In view of Farkas’s lemma, 1st of these conditions is equivalent to the vectors $p(a)$ and $p(a) + q_i(a)$ are collinear for all $i \in \mathcal{I}$; the same thing for a second condition but now for event b . Certainly, speaking about prices $p(a)$ and $p(a) + q_i(a)$ we mean gradients of utilities related with prices via conditions $\exists \lambda_{1a} > 0: \nabla_a u_1(x_1) = \lambda_{1a} p(a)$ and $\exists \lambda_{2ab} > 0: \nabla_a u_2(x_2) = \lambda_{2ab} (p(a) + q_2)$ $\nabla_b u_2(x_2) = \lambda_{2ab} (p(b) - q_2)$ and similar relations for other individuals and for the state ‘ b ’. However, the collinearity of $p(a)$ and $p(a) + q_i(a)$ implies collinearity of $p(a)$ and $q_i(a) \neq 0$ and analogously the collinearity of $p(b)$ and $q_i(b) = -q_i(a)$ that implies $p(a)$ and $p(b)$ are collinear vectors and therefore (due to Proposition 9.3.2) this allocation belongs to Interim core. All this is true for the case $q \neq 0$ and then collinearity means the implementation of Interim equilibrium as a special case of the D -equilibrium.

Thus if informed individual $i = 1$ has noncollinear gradients $\nabla_a u_1(x_1)$ and $\nabla_b u_1(x_1)$ then mutually beneficial **contract** with individual $i = 2$ will **not be found** only if

$$(\nabla_a u_2(x_2), \nabla_b u_2(x_2)) = (\alpha \nabla_a u_1(x_1), \beta \nabla_b u_1(x_1))$$

for some real $\alpha > 0$ and $\beta > 0$, and this is the case¹⁰ $q = 0$. Moreover, due to Lemma 9.3.1 for noncollinear gradients individual $i = 1$ understands clearly that if one of states ‘ a ’ or ‘ b ’ will be realized ‘tomorrow morning’ then he/she or other agents will renew contractual process, *i.e.* current allocation is not stable in fact. Both instances motivate $i = 1$ to open (share) today information to $i = 2$ and then contractual process can be proceeding under a new informational allocation. This will proceed till a new contractual allocation corresponding to D -equilibrium with proportional prices (equal being renormalized!) at indistinguishable states of the world, *i.e.* then an interim equilibrium is realized.

Further, one describes a possible procedure of mutually beneficial contract searching during contractual process ongoing with sharing of information. Let for an event $a \in \Omega$ vectors $p(a) + q_i(a)$ and $p(a) + q_j(a)$ are non-collinear. Now consider the event

$$E_{ij}(a) = P_i(a) \cap P_j(a).$$

Individuals i, j can be taught to distinguish this event if they share information about $P_i(a)$ and $P_j(a)$. Further, if vectors

$$\sum_{\mathbf{w} \in E_{ij}(a)} (p(\mathbf{w}) + q_i(\mathbf{w})) \quad \& \quad \sum_{\mathbf{w} \in E_{ij}(a)} (p(\mathbf{w}) + q_j(\mathbf{w})) \quad (9.3.8)$$

are noncollinear then due to Corollaries 8.1.1, 9.2.1 to lemmas on mutually beneficial contract these individuals will be able to find a mutually beneficial exchange for the event $E_{ij}(a)$; otherwise if vectors (9.3.8) are collinear then a third participant can be invited to joint to coalition, let his/her number be k , and such that

$$E_{ijk}(a) = P_i(a) \cap P_j(a) \cap P_k(a) \neq E_{ij}(a),$$

i.e. k -th individual is able to improve information about state ‘ a ’. Now if vectors

$$\sum_{\mathbf{w} \in E_{ijk}(a)} (p(\mathbf{w}) + q_i(\mathbf{w})) \quad \& \quad \sum_{\mathbf{w} \in E_{ijk}(a)} (p(\mathbf{w}) + q_j(\mathbf{w})) \quad \& \quad \sum_{\mathbf{w} \in E_{ijk}(a)} (p(\mathbf{w}) + q_k(\mathbf{w}))$$

are noncollinear then coalition $\{i, j, k\}$ is able to find a new mutually beneficial contract; otherwise one needs to involve in a coalition one more individual and so on. This process has to finish at some time because by (7.1.1)

$$\bigcap_{\mathcal{I}} P_i(a) = \{a\},$$

¹⁰Notice that in this case there is no necessity to share information at all, and if nevertheless 1st will teach 2nd agent to distinguish ‘ a ’ and ‘ b ’, then nothing will be happened and current allocation will not change. However, after informational sharing the allocation will obtain an additional stability.

and among vectors $p(a) + q_i(a)$, $i \in \mathcal{I}$ there exists at least one non-collinear pair.

If the moment is appropriate for a mutually beneficial exchange to occur, then information interchange is realized and ordinary contractual process is going up to approach to a new dead-end situation: there is no possibility for an exchange without new information sharing.

So, what is the result? The system as a whole can come (if it is not cycled) to allocation such that even after the information sharing a mutually beneficial exchange is impossible. Besides, it should be stable contractual allocation. However, what is the case?

(i) This method and also applying other ways that can be outside the model framework (for example via a price channel) leads us to a new information distribution that is finer than the previous one but in general is coarse than supreme information: $\tilde{\mathbb{P}} = (\tilde{P}_i)_{\mathcal{I}} \succeq (P_i)_{\mathcal{I}} = \mathbb{P}$, $\tilde{P}_i \preceq \bigvee_{j \in \mathcal{I}} P_j$, $i \in \mathcal{I}$.

(ii) For the attained level of informational distribution a new mutually beneficial contract is impossible even after additional information sharing (via Lemma 9.2.1), formally: $\exists p : \Omega \rightarrow \mathbb{R}^l$, $p \neq 0$ and $\lambda_{i,E} > 0$, $(i, E) \in \mathfrak{S}$, such that

$$\forall E \in \tilde{P}_i, \quad \lambda_{i,E} \sum_{\mathbf{w} \in E} \nabla_{\mathbf{w}} u_i(x_i) = \sum_{\mathbf{w} \in E} p(\mathbf{w}) \quad \forall i \in \mathcal{I}. \quad (9.3.9)$$

Moreover, for the stability of allocation at the moment of ‘tomorrow morning’, it has to be the true condition (9.3.1) of prices are proportional at all states of the world that are indistinguishable for an individual; this can be written as for every $\mathbf{w} \in \Omega$ the family of vectors $\{\sum_{\mathbf{w}' \in P_i(\mathbf{w})} \nabla_{\mathbf{w}'} u_i(x_i)_{i \in \mathcal{I}}\}$ is collinear one. At the same time the requirement for a system of gradients $\{\nabla_{\mathbf{w}} u_i(x_i)\}_{i \in \mathcal{I}}$ be collinear for $\forall \mathbf{w} \in \Omega$, that formally implies impossibility to sign a new mutually beneficial contract after informational sharing, seems to be excessive one. This is so because the informed agents are not motivated enough to share information and because the ability to distinguish or not distinguish something is a private property of an individual and in general this is unknown for a contractual partner. Here the disproportion in gradients (prices, exchange proportions) can be considered as a signal to share information: there is no exchange without this signal. Thus one has described a contractual process delivering a fixed-point allocation that corresponds to interim equilibrium, but it is happened now for enriched informational structure.

And what is about core? It has to be a core in the model with agents’ ‘duplicates’ constructed for an information distribution $\tilde{\mathbb{P}} = (\tilde{P}_i)_{\mathcal{I}}$, realized at the last stage of information sharing:

$$\mathcal{C}(\mathcal{E}^{di}, \mathfrak{S}, \tilde{\mathbb{P}}), \quad \mathfrak{S}(\tilde{\mathbb{P}}) = \{(i, E) \mid i \in \mathcal{I}, E \in \tilde{P}_i\}.$$

For a further convenient and short usage, let us call this core a *D-interim-core*. Now one always has by construction

$$\forall i \in \mathcal{I} \quad \tilde{P}_i \preceq \bigvee_{j \in \mathcal{I}} P_j.$$

Moreover, there are reasons to think that in a generic case (for almost all economies) the described process will lead to supreme information, *i.e.* it will be

$$\forall i \in \mathcal{I} \quad \tilde{P}_i = \bigvee_{j \in \mathcal{I}} P_j = P_V.$$

For this case one will have

$$\mathcal{C}(\mathcal{E}^{di}, \mathfrak{S}, \tilde{\mathbb{P}}) = \mathcal{C}(\mathcal{E}^{di}, \mathfrak{S}, P_V) = \prod_{E \in P_V} \mathcal{C}(\mathcal{E}^{di}(E)),$$

where $\mathcal{C}(\mathcal{E}^{di}(E))$ is a core in the model $\mathcal{E}^{di}(E)$ with agents as in initial model, but commodity space is changed to

$$\{z : E \rightarrow \mathbb{R}^l \mid z(\mathbf{w}) = z(\mathbf{w}'), \forall \mathbf{w}, \mathbf{w}' \in E\}$$

and with endowments preferences of the initial model are inducted on a positive cone of this space (this is applied as a consumption set). In particular, if (7.1.1) is true, then

$$P_V = \Omega^* \Rightarrow \mathcal{C}(\mathcal{E}^{di}) = \prod_{\mathbf{w} \in \Omega} \mathcal{C}(\mathcal{E}^{di}(\mathbf{w})).$$

The final remark. The presented core can be considered also as a core that generically corresponds to the concept of *REE*-equilibrium. In general case one needs to take the core of an economy with differentiated agents: in this model the individual information is specified so as it is presented in *REE*-concept: supremum of the initial information and an information received via the price channel. Here one can feel inconformity between the concept of core (stability on the basis of the barter trade) and the presence of prices and trade under these prices... However, *REE*-equilibrium concept was not introduced by me.

9.3.1 Universal equilibrium and core notions in *DI*-economies

In the previous sections, different possibilities to apply a contractual approach to the model with differentiated information were proposed and analyzed. I think that contractual approach turned out as an efficient tool that allows us to investigate not only private core and equilibrium but also to introduce a sequence of new concepts (*D*-core and equilibrium, interim concepts *etc*), driving us to a contractual analogue of *REE*-equilibrium notion. In this section I want to propose one more equilibrium notion based on a complex-contractual treatment: under some circumstances this implements private equilibrium, under other ones *REE*-equilibrium, but often and often it will present something intermediate one.

Now let us think about that is the fundamental difference between two equilibrium concepts, private equilibrium (*WEE*) and equilibrium in rational expectations (*REE*), between private equilibrium in contractual sense and interim equilibrium (when nobody wants to reveal information). I believe that the main difference consists in:

- (i) For the first (private) case contracts concluded today at the end of negotiation stage will be *certainly realized tomorrow*, nevertheless that tomorrow some individuals will wish somehow to change contracts, *i.e.* the fact that concluded contracts are implemented confirming the agent's reputation as a reliable partner is more important than some temporary gains.
- (ii) For the second case, this is the case of interim equilibrium and core, all contracts are initially less stable and, according to the game rules, at the moment "tomorrow morning" *contractual process can be renewed*, individuals will break existing and sign new contracts; they will also reveal information.

Further, I would like to note first that if one will not take into account possibilities of information exchange then statistically (in expectations) the first noted variant is preferable for individuals since it leads to higher utility values. The weak point in this approach is stability and implementation. Is it possible for presented approaches to be combined in a form of united concept? Certainly it is if one applies a contractual point of view. Namely, one can specify for every coalition a subspace of strongly stable contracts, contracts of this kind will certainly be implemented tomorrow; this is a subspace of space of contracts where the distribution of information is already taken into account. Yet, it is not forbidden for individuals to conclude all other contracts but their tomorrow implementations are not guaranteed: tomorrow morning a new contractual process can be initiated. Agreements realized among a narrow group of responsible individuals (reliable gentlemen, they are always better than their word) can serve as the example of strong kind contracts. Another example presents the trade of future contracts: the deal being concluded today is realized tomorrow (a month or half year later *etc.*). Contracts of the second kind look like preliminary agreements implementing a preliminary stage of final contract search that in view of informational constraints can be found only "tomorrow morning" when some information will appear on a true state of the world.

In general, how does the contractual process to go? There are two stages, during the first one only high-stable contracts are concluded, and all other contracts are signed in second stage, where in difference with the first one informational exchange can also be going being an integrated part of a contractual process that drives economy to an interim equilibrium. Further, one considers a formalized mathematical presentation (unfortunately rather cumbersome ones).

Let us consider model \mathcal{E}^{di} of pure exchange economy with differentiated information described on page 276. Let us add to the model a new element $\mathcal{W} = \bigcup_{S \subseteq \mathcal{I}} \mathcal{W}^S$, where $\mathcal{W}^S \subseteq L^{\mathcal{I}}$ is a space of strong stable contracts that can be signed by coalition $S \subseteq \mathcal{I}$ members, they are contracts that after finish of contractual stage "today" will certainly implemented tomorrow¹¹. Let \mathcal{W}_i be a **projection** \mathcal{W} on i th actor in the space of allocations $L^{\mathcal{I}}$. Further, let us consider two notions, a core and equilibrium, that well correspond to contractual point of view and differential information.

Definition 9.3.3 *An allocation $x \in \prod_{\mathcal{I}} X_i$ is an element of contractual core if there exists a web of contracts $V = \{v^S\}_{S \subseteq \mathcal{I}}$, $v^S \in W^S$, $S \subseteq \mathcal{I}$ such that $y = \mathbf{e} + \sum_{v^S \in V} v^S \in$*

¹¹Here $v = (v_i)_{\mathcal{I}} \in \mathcal{W}^S \Rightarrow v_i = 0$ for $i \notin S$.

$\prod_{\mathcal{I}} X_i$ and the allocations x and y obey conditions: $y = (y_i)_{\mathcal{I}}$ is an element of a private core with respect to \mathcal{W} , i.e.

$$(i) \sum_{i \in \mathcal{I}} v_i = 0,$$

$$(ii) v_i : \Omega \rightarrow \mathbb{R}^l \text{ is } P_i\text{-measurable for all } i \in \mathcal{I},$$

$$(iii) \nexists S \subseteq \mathcal{I} : \exists z^S = (z_i)_{i \in S}, \quad (z^S - \mathbf{e}^S) \in \mathcal{W}^S \mid \forall i \in S, z_i \in X_i \text{ is such that} \\ (z_i - \mathbf{e}_i) \text{ is } P_i\text{-measurable, } z_i \succ_i y_i \text{ \& } \sum_{i \in S} (z_i - \mathbf{e}_i) = 0;$$

and the allocation x is an element of interim core relative to endowments $y = (y_i)_{\mathcal{I}}$ and (some) informational structure $\tilde{\mathbb{P}} = (\tilde{P}_i)_{i \in \mathcal{I}}$, obtained as a result of a contractual process of informational sharing, i.e. x obeys in addition

$$(iv) \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i,$$

$$(v) (x_i - y_i) : \Omega \rightarrow \mathbb{R}^l \text{ is } \tilde{P}_i\text{-measurable for all } i \in \mathcal{I},$$

$$(vi) \forall \mathbf{w} \in \Omega \nexists S(\mathbf{w}) = S \subseteq \mathcal{I} : \exists z^S = (z_i)_{i \in S} \in (\mathbb{R}_+^l)^S, \sum_{i \in S} (z_i - y_i(\mathbf{w})) = 0 \text{ \& } \\ z_i \cdot \chi_F \succ_i^F x_i^F, \forall i \in S \text{ for } F = \tilde{P}_i(\mathbf{w})$$

Notice a small technical difference in the presentation of (vi) from the latter definition and (iii) from Definition 9.3.1. On the existence of contractual core, one can note that the existence of a private core with respect to \mathcal{W} can be stated without problems applying technique presented in the proof of Theorem 8.1.2. Further, one can take any allocation from the core and activate a contractual process that leads us to a fixed point from the D -core (the existence of this can be stated directly similarly as in Theorem 8.1.2). Now if this point is not in interim core, then an informational sharing is realized and a new contractual process is initiated that again leads to a point from D -core, but now already relative to enriched informational structure, and so on.

Further, let us consider a two stages contractual concept of equilibrium.

Definition 9.3.4 A couple of allocations $(y, x) \in X \times X$ and a couple of price mappings (q, p) , $q, p : \Omega \rightarrow \mathbb{R}^l$ is called contractual equilibrium if the couple (y, q) is a private equilibrium with respect to \mathcal{W} , i.e., first the following is true:

$$(i) (y_i - \mathbf{e}_i) : \Omega \rightarrow \mathbb{R}^l \text{ is } P_i\text{-measurable for all } i \in \mathcal{I},$$

$$(ii) 0 \not\equiv \langle q, (\mathcal{P}_i(y_i) - \mathbf{e}_i) \cap \mathcal{L}_i \cap \mathcal{W}_i \rangle \geq 0^{12}, i \in \mathcal{I},$$

$$(iii) \sum_{i \in \mathcal{I}} y_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i.$$

¹²Recall that \mathcal{L}_i is, in the space of contingent commodities, a space of maps measurable with respect to i^{th} agent informational partition.

Second requirement is that pair (x, p) is an interim equilibrium with respect to endowments $y = (y_i)_{\mathcal{I}}$ and an informational structure $\tilde{P}_i = P_i \vee \sigma(p)$, where $\sigma(p)$ is the algebra of events (field), generated by the price map $p(\cdot)$ ¹³:

(iv) $(x_i - y_i) : \Omega \rightarrow \mathbb{R}^l$ is \tilde{P}_i -measurable for all $i \in \mathcal{I}$,

(v) $\langle p^E, (\mathcal{P}_i^E(x_i^E) - y_i^E) \rangle > 0$, $\langle p^E, x_i^E - y_i^E \rangle = 0$, $\forall i, E = \tilde{P}_i(\mathbf{w}), \forall \mathbf{w} \in \Omega$,

(vi) $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i = \sum_{i \in \mathcal{I}} y_i$.

On the existence of equilibria defined via the above conception, one can only note that everything that is known for the core one needs to apply to find fuzzy contractual elements that correspond to equilibria. Notice also that an attempt to consider an equilibrium with common prices for both stages instead of two-stages $q(\cdot), p(\cdot)$ leads us to the notion whose existence is unclear. Moreover, it is also unclear how it can be interpreted, since in this case contractual process has to be treated as going simultaneously for strong and weak stable contracts.

Remark 9.3.1 Introduced notions can be illustrated by the conceptual example. Assume that there is a group responsible for the own word agents, these people always keep the promises given by them even if it happens to be unprofitable at present and they can have damage in comparison with a case of a break of concluded earlier (yesterday) contracts. This contract was mutually beneficial within the limits of information known yesterday and according to presented uncertainty. Today fresh information appeared, though uncertainty still is resolved only partially. And now one of parties to an agreement realizes that according to new facts it would be beneficial to break off the contract (may be only partially), however he/she does not do it. Though it seems unreasonable, however, it can be very reasonable for strategic long-term plans, because the expected utility is applied in the latter case and a current utility in the former one. However, not all individuals behave themselves so in these circumstances and even if you are a responsible person in the sense of contractual obligations, it still does not mean that you are already the club member: it is necessary that other agents have known and believed in it... So, the contracts concluded among club members of responsible agents will be realized tomorrow for certain and everybody knows this while other contracts—depending on a situation, and nobody knows in advance what exactly will be realized. ■

9.4 Computer simulation of contractual processes in *DI*-economies

In this section some results of computer simulation for contractual processes that drives economy to the contractual allocations of different kinds are presented: private core and equilibrium, *D*-core and *D*-equilibrium, and interim concepts too.

¹³The coarsest σ -algebra, such that $p(\cdot)$ is a measurable map.

Contractual processes were suggested and studied in ([Marakulin, 2006b](#)), the obtained results were presented in Part II of this monograph. Here a standard pure exchange economy with *symmetrically* informed agents was considered as a basic model; several types of processes were suggested, the most major among others is the proper contractual one. For processes of this kind partial breaking of the contracts is allowed in the context of several basic hypotheses, determining the character of contracts' breaking process, are formulated in a general kind and for major particular cases. They are the following:

(**IB**) — instantaneous breaking of the contracts;

(**UB**) — uniform breaking of all contracts;

(**CUB**) — uniform breaking of gross within-coalitional contracts.

Combinations of these hypotheses result to proper-contractual trajectories of a different kind of a generality. Under (**IB**) and (**UB**) contractual trajectory turns out *aggregated*, under (**IB**) and (**CUB**)—*coalitional-contractual*; there are formal and mathematically reasonable definitions. Besides, concept of *trade rule* is introduced; this is a map, unequivocally determining mutually beneficial contract for the current consumption plans, having some additional good mathematical properties. By use of a trade rule, a contractual trajectory of each mentioned kinds is unequivocally determined. The special type of *benevolent* rules of trade stood out as rules which determine a new contract allowing the break of gross barter contract only if being realized *every* new mutually beneficial contract involves contracts' breaking. Just for this class of benevolent processes, the basic positive results about convergence were received. In general case, contractual processes may be cycled. However, their steady points always have specific equilibrium properties (*i.e.* it is an element of core or quasi-equilibrium, depending on the kind of process) that attracts an interest.

Computer simulation of contractual processes was considered and modeled in several student works provided under my scientific supervision; in these studies a new mutually beneficial contract and, therefore, trade rule as a whole, were determined according to one of known in bargaining theory kinds of solution. In this context Nash's and Kalai–Smorodinsky solutions were considered, see [Rubinstein, Osborne \(1990\)](#), [Thomson \(1994\)](#), that is the most interesting ones from theoretical and practical point of views. In this section we consider processes that apply so called “local Nash rule”.

The following model of pure exchange economy with asymmetrically informed agents is considered. There are 2 kinds of physically different commodities, 2 individuals and 3 states of the world. Preferences are defined by Cobb–Douglas utility functions $u_1(x)$, $x \in \mathbb{R}^6$ and $u_2(y)$, $y \in \mathbb{R}^6$ presented in the logarithmic form:

$$u_1(x) = \alpha_{a_1} \ln(x_{a_1}) + \alpha_{a_2} \ln(x_{a_2}) + \alpha_{b_1} \ln(x_{b_1}) + \alpha_{b_2} \ln(x_{b_2}) + \alpha_{c_1} \ln(x_{c_1}) + \alpha_{c_2} \ln(x_{c_2}),$$

$$u_2(y) = \beta_{a_1} \ln(y_{a_1}) + \beta_{a_2} \ln(y_{a_2}) + \beta_{b_1} \ln(y_{b_1}) + \beta_{b_2} \ln(y_{b_2}) + \beta_{c_1} \ln(y_{c_1}) + \beta_{c_2} \ln(y_{c_2}),$$

$x \gg 0$, $y \gg 0$. The coefficients of these functions are defined at the start of a modeling program:

$$\alpha = (\alpha_{a_1}, \alpha_{a_2}, \alpha_{b_1}, \alpha_{b_2}, \alpha_{c_1}, \alpha_{c_2}), \quad \beta = (\beta_{a_1}, \beta_{a_2}, \beta_{b_1}, \beta_{b_2}, \beta_{c_1}, \beta_{c_2}).$$

Initial endowments are $\mathbf{e}=(\mathbf{e}_1, \mathbf{e}_2)$, $\mathbf{e}_1 = (\mathbf{e}_1^a, \mathbf{e}_1^b, \mathbf{e}_1^c)$, $\mathbf{e}_2 = (\mathbf{e}_2^a, \mathbf{e}_2^b, \mathbf{e}_2^c)$ and information is P_1, P_2 that is presented as partitions of $\Omega = \{a, b, c\}$, it is the set of all states of the world. In the program, this information is defined as an ordered array of natural numbers; here the state a has index 1, $b=2$, $c=3$. Moreover, if for one of the agents some elements of array coincide, then it means this agent is not able to distinguish states corresponding to the elements' indexes, *i.e.* the states are placed in a common element of partition. Besides, there are parameters: $Step > 0$ for the volume of contract and $Accuracy > 0$ for the accuracy of equilibrium calculation. Program starts from the initial endowments $(x, y)^{(0)} = (\mathbf{e}_1, \mathbf{e}_2)$.

Further a programmed procedure is described, it is applied for the search of private equilibrium under the endogenously going process of informational sharing.

Now I first describe the trade rule applied for our process. This rule was called the "local Nash rule" because the levels of marginal profit from contracts are defined via the Nash solution. One can show that for a pair agents coalition this rule corresponds to the principle of "moving along the bisectrix" of the angle formed by gradients of utility functions (formally for the second one needs to take an anti-gradient): it means equal contractual profit (it is calculated as an inner product of gradient and the vector of contractual flow). Let $w_1^{(n)} = \text{Pr}_W(\nabla u_1(x^{(n)}))$ and $w_2^{(n)} = \text{Pr}_W(\nabla u_2(y^{(n)}))$, where $\text{Pr}_W(\cdot)$ denotes the projection on the space $W \subseteq \mathbb{R}^6$ of *permissible contracts* (parameterized in the 1st agent consumption). Now we have:

$$v^{(n)} = (v^{(n)}, -v^{(n)}), \quad v^{(n)} := \frac{w_1^{(n)}}{|w_1^{(n)}|} - \frac{w_2^{(n)}}{|w_2^{(n)}|}.$$

Thus gradients are projected on the space of *permissible contracts* (one needs *measurability* relative to information of first and second agent) and then they are normalized. The volume of contract (length of vector) $v^{(n)} = (v_1^{(n)}, v_2^{(n)})$ is defined equal to *step*, *i.e.* $v^{(n)} := \text{step} * v^{(n)} / |v^{(n)}|$ ¹⁴. Further, the volume of the contract becomes less again (it is divided on 1.1) until the contract is mutually beneficial, or while its volume will not become less *accuracy* ($|v^{(n)}| < \text{accuracy}$). In the first case contract $v^{(n)}$ is signed and the system transits to allocation $(x, y)^{(n)} + v^{(n)}$ that can be further partially broken off. In the second case, we think that the program has found an equilibrium under presented information, and $(x, y)^{(n)}$ is written in the file. When an equilibrium relative to fixed information is already found, the next step toward realizing information sharing can be done.

Contracts breaking procedure is specified as follows. After every time when a new contract is concluded the first (or second) individual partially breaks the gross contract in the volume $\frac{|v^{(n)}|}{10}$, and if necessary this breaking is repeated while it is profitable for the agent. Let $\lambda^{(n)}$ denote the number of these breaking iterations for a current situation. Then a resulting transition is realized by the formula:

$$(x, y)^{(n+1)} = (x, y)^{(n)} + v^{(n)} + \lambda^{(n)} * \frac{|v^{(n)}|}{10} * (\mathbf{e} - (x, y)^{(n)} - v^{(n)}).$$

¹⁴It is a bit incorrect to apply a common notation for the initial and normalized contracts.

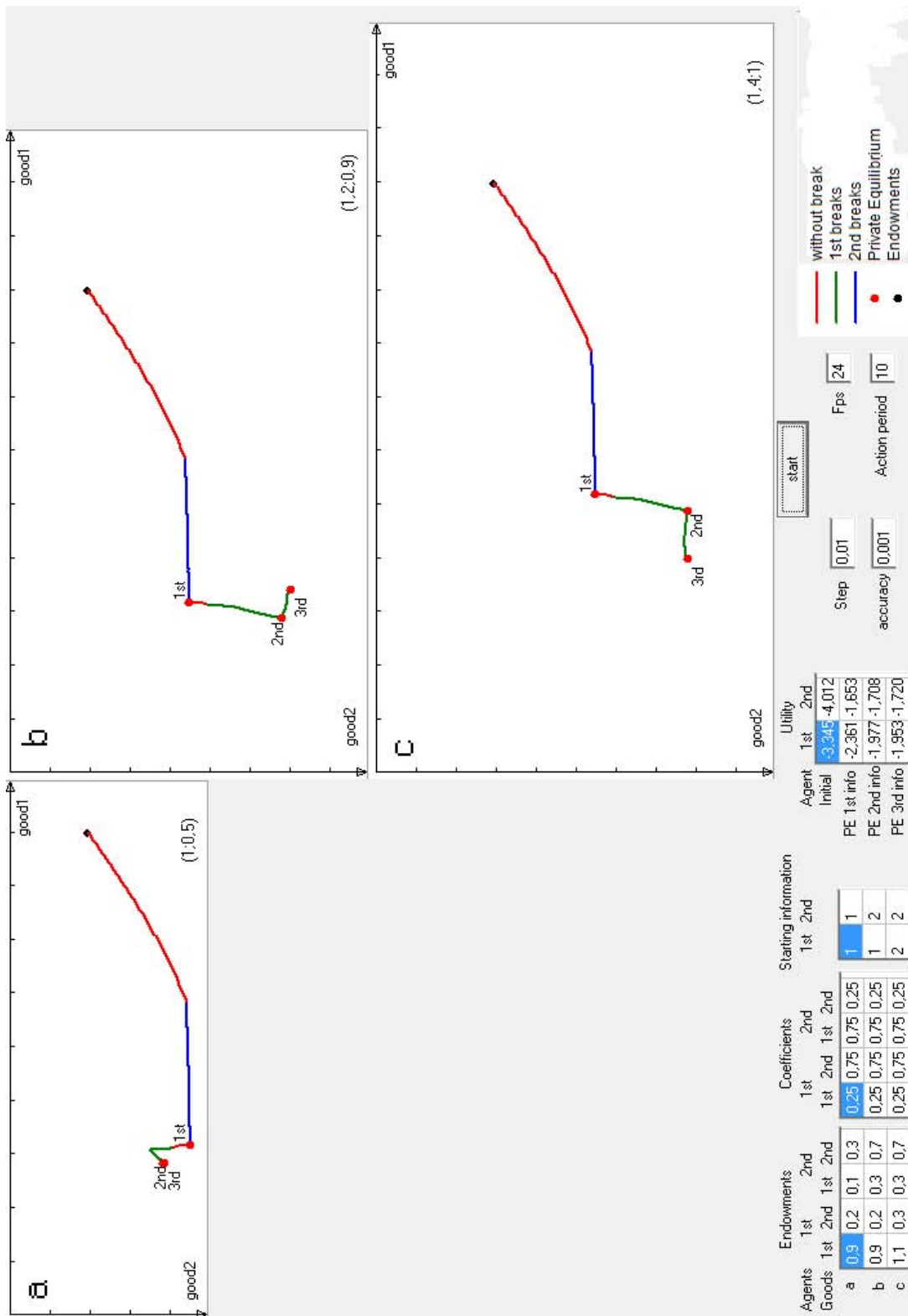


Figure 9.4.6: Dynamics of private contractual process with an informational sharing

Information sharing: at every time when equilibrium relative to presented information has found it can be profitable for agents to change information (to share). According to the analysis presented above an agent has incentives to share information (wants to inform another agent how he/she can distinguish an element of partition from another one), if the projection of his utility gradient on the space of contracts differs from the gradient (*i.e.* it is nontrivial: gradient does not belong to space,¹⁵ and therefore the map is *not measurable*). Further, supposing that trial informational sharing individuals find a new mutually beneficial contract according to the rule described above, but now already for an updated informational structure. Now if the volume of a new contract is greater of *accuracy*, then the agent really shares the information (trial sharing becomes a real one) and the stage is finished. Otherwise, the agent thinks informational sharing inefficient however he/she can also try to use another method of informational sharing (*e.g.* in sharing procedure one can apply another element of the informational partition). Finally, if both agents get nowhere, then the contractual process is finished, and the program ends the work.

The results of the program work are presented in a picture, see Figure 9.4.6. Here *three* Edgeworth boxes for the states of the world $\{a, b, c\}$ (left upper angle in a box) are presented. The contractual process starts from initial endowments (small black filled circles) and its development is presented by curves where red color denotes no breaking of contracts, green one—*1st* breaks, blue—*2nd*. Red small filled circles denote equilibria that correspond to information of *1st*, *2nd*, and *3rd* types: these structures were consequently improved during the sharing process. Here *1st* corresponds to initial information, *2nd* was obtained after sharing initiated by *2nd* agent, and then *3rd* was obtained via *1st* agent sharing. Initial model parameters can be changed, and they are presented in the bottom panel of the screen, from the left to right: 1st table presents initial endowments data (3×4 matrix); 2nd presents the factors of utility functions; 3rd is initial information; 4th presents the dynamics of utility changes over the equilibrium allocations relative to different information; further *step*, *accuracy* and number of iterations are presented.

Analyzing the obtained results, one can observe the rather exotic character of utility and allocation changes. In particular, one can see that at the state ‘*a*’ after the second stage of informational sharing equilibrium allocation does not change and the contractual process stops; another fun fact is that after the first stage of informational sharing *2nd* agent has lost because his/her utility decreased in equilibrium allocations, nevertheless he/she initiated the sharing. Clearly at the stage of sharing the agent expected that utility will increase and locally it is so, but then it decreases: this is because of specific properties of the contractual process where rational decisions are taken only in a bounded sense, not in a global one.

A similar computer simulation was conducted for contractual processes driving to *D*-quasi-equilibrium. The results are presented in Figure 9.4. It is curious that after the first act of informational sharing when *2nd* teaches *1st* to distinguish ‘*a*’ and ‘*b*’ resulting market for ‘*a*’ is separated, but nevertheless the contractual process does not develop and allocation for ‘*a*’ is the same. The similar thing for ‘*b*’ and ‘*c*’, where

¹⁵The program inspects both gradients, extending a bit the possibilities for information exchange.

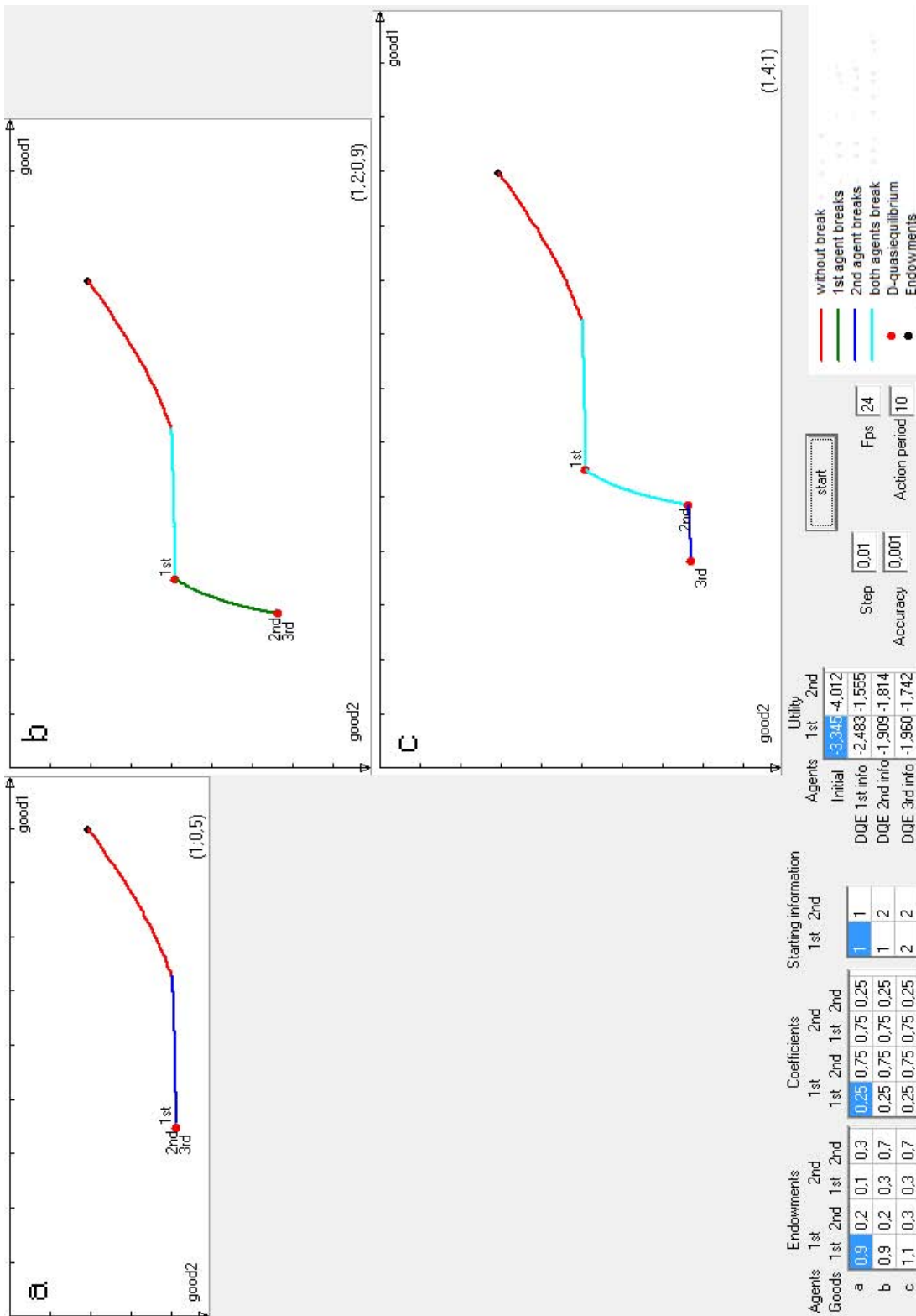


Figure 9.4.7: Dynamics of contractual process with differentiated agents and an informational sharing

after when *2nd* studied to distinguish them, the situation for ‘*b*’ does not change and a dynamic appears only in the market for ‘*c*’. In so doing the states of the world can be ordered in a natural way. It happened because for the initial information the state ‘*a*’ is a bottleneck, for new information with ‘*b*’ and ‘*c*’ it is ‘*b*’; notice that exactly these events implement *D*-quasi-equilibrium as an equilibrium relative to the presented state of the world, see Example 9.2.1. It is also curious that for differentiated agents and some initial data directly after informational sharing contractual process can develop in such a way that first the partial breaking of signed contracts is realized and new contracts are signed only later (for data presented in the picture it is not so). This fact has a simple explanation: after informational sharing in the model with differentiated agents, some new agents appeared (for an agent utility function is disintegrated into several parts), and for some of them it can be profitable to break off partially gross contract. It never can be true for a private equilibrium model.

Conclusion to Chapter 9

In this chapter contractual analysis of the various concepts of core and equilibrium was provided. It was analyzed:

- Original notions of *D*-core and *D*-equilibrium (Definitions 9.2.1 and 9.2.2).
- Original notions of interim core and interim equilibrium (Definitions 9.3.1, 9.3.2) and their relationships with the concepts of the kind *D* (Proposition 9.3.2, Lemma 9.3.1).
- A process of information sharing that drives the economy to interim equilibrium through a series of intermediate *D*-equilibria was suggested.
- Universal concepts of two-stage core and equilibrium (Definitions 9.3.3, 9.3.4), were proposed. The introduced equilibrium notion includes *WEE* and *REE* as special cases.
- The results of computer simulation are presented to demonstrate a contractual process with informational sharing.

Concepts of type *D* are based on the assumption that the individual in the future has many implementations as if he/she is “split” into parts corresponding to its information partition. These parts begin independently interact within the framework of the contractual process, and the system as a whole goes to the core and equilibria of type *D*. The results on the existence of *D*-core and of equilibria are indicated.

The notion of interim equilibrium develops the concept of Rational Expectations Equilibrium (*REE*). The concept of the interim core is also new and through interim equilibrium corresponds to *REE*. A key feature of stability, justifying the interim concepts, is the absence of agents’ intention to continue the contractual process (and therefore destroy the already reached agreements) in the position “tomorrow morning”: all contracts concluded up to moment “yesterday evening” will be implemented. The examples illustrating investigated notions are described. An example shows that

usually (generically) interim core and equilibria do not exist. However, this is the case when one can reveal information and this allows agents to sign a new contract. If interim equilibrium (element of the core) is implemented, then the contractual process is finished.

It is important that in all offered and studied concepts individuals cannot lie with an advantage for themselves and, thereby, all concepts are “incentive compatible.” Certainly, it happens due to the measurability of contracts concerning private information. However, to realize *WEE*-equilibrium and allocations from the private core at the moment when tomorrow has already occurred, one needs already today to ensure the mechanism of their implementation: legally accomplished agreements and so on. It is so because if it is not an allocation from interim core then there always will exist a group of agents wishing to break off contracts “concluded at yesterday evening” and to conclude the new “at this morning”, *i.e.*, when uncertainty is resolved (cleared) at least partially. In the absence of an effective mechanism controlling the implementation of agreements, it will mean short-deliveries by the concluded contracts and, as a consequence, will imply a disbalance of the economy as a whole.

These reasons lead to the idea of “universal” (adequate) concepts of two-stage equilibrium and core described in Section 9.3.1. In other words: it was assumed that in the space of all admissible contracts measurable in the necessary measure, the special subspace is selected and contracts from it have the special exogenously given form of stability: if a contract of this type is concluded today, then by no means it will be realized tomorrow. Other contracts have not so strong stability, but agents can conclude them also: they are just preliminary agreements. Thus, these other contracts are the specific form of the optional agreement: tomorrow morning individuals can change their minds and break off the contract without any problems: this is according to the game rules. In such a way one comes to that is named by terminology from Section 1.1, page 32, complex contractual allocations. These allocations are implemented by stable webs of contracts accounting for the indicated specificity and potentially different possibilities to break contracts: only as a whole, or it is possible to do partially. In such a way specific concepts of contractual core and equilibrium can be introduced: in our case, this is two-stage notions.

Bibliography

- Aliprantis, C. D., Brown, D. J. and O. Burkinshaw** (1989). Existence and optimality of competitive equilibria. Berlin/New-York: Springer-Verlag, 284 p.
- Aliprantis, C. ed.** (2001). Economic Theory, Vol. 18. Berlin/Heidelberg: Springer-Verlag
- Allen, B.** (1981). Generic existence of completely revealing equilibria with uncertainty, when prices convey information// *Econometrica*, **49**, 1173–1199
- Allen, B.** (1991a). Market games with asymmetric information and nontransferable utility: representation results and the core, *University of Pennsylvania, CARESS Working Paper #91-09*
- Allen, B.** (1991b). Transferable utility market games with asymmetric information: representation results and the core, *University of Pennsylvania, CARESS Working Paper #91-16*
- Allen, B.** (1994). Market games with asymmetric information: the core with finitely many states of the world// *in: Munier, B. and M. J. Machina (eds.): Models and Experiments in Risk and Rationality* (Kluwer Academic Press)
- Allouch, N. and M. Florenzano** (2004). Edgeworth and Walras equilibria of an arbitrage-free exchange economy// *Economic Theory*, **23**, 353–370
- Anderson, R. M.** (1992). Non-standard analysis with applications to economics// *in: Hildenbrand, W. and H. Sonnenschein (eds.): Handbook of Mathematical Economics*, Vol. IV, 2145–2208 (Amsterdam: North-Holland)
- Anderson, R. M. and R. C. Raimondo** (2007). Incomplete markets without Hart's points// *Theoretical Economics*, **2**, 115–133
- Arrow, K. and G. Debreu** (1954). Existence of an equilibrium for a competitive economy// *Econometrica*, **22**, 3, 265–290
- Arrow, K. J. and F. H. Hahn** (1991). General competitive analysis. Amsterdam/New York/Oxford/Tokyo: North-Holland
- Aumann, R. J.** (1964). Market with a continuum of traders// *Econometrica*, **32**, 39–50

- Barten, A., P. and V. Bohm** (1982). Consumer theory// *in*: Arrow, K. J. and M. D. Intriligator (eds.): *Handbook of Mathematical Economics*, Vol. II, 381–429 (Amsterdam: North-Holland)
- Beato, P. and A. Mas-Colell** (1985). On marginal cost pricing with given tax-subsidy rules// *Journal of Economic Theory*, **37**, 356–365
- Bonnisseau, J. M. and B. Cornet** (1988). Existence of equilibria when firms follow bonded losses pricing rules// *Journal of Mathematical Economics*, **17**, 103–118
- Bonnisseau, J. M. and B. Cornet** (1990). Existence of marginal cost pricing equilibria in economies with several nonconvex firms// *Econometrica*, **58**, 661–682
- Brown, D. J.** (1991). Equilibrium analysis with non-convex technologies// *in*: Hildenbrand, W. and H. Sonnenschein (eds.): *Handbook of Mathematical Economics*, Vol. IV, 1964–1995
- Buchholz, W. and W. Peters** (2007). The Edgeworth conjecture in a public goods economy: an elementary example// *Economic Bulletin*, **8**, 6, 1–4
- de Castro, L. I. and N. C. Yannelis** (2010). Ambiguity aversion solves the conflict between efficiency and incentive compatibility// Working paper
- Clarke, F. H.** (1983). Optimization and nonsmooth analysis. New York: *Wiley-Interscience*
- Conley, J. P.** (1994). Convergence theorems on the core of a public goods economy: sufficient conditions// *Journal of Economic Theory*, **62**, 1, 161–185
- Dagan, N., Serrano, R. and O. Volij** (2000). Bargaining, coalitions and competition// *Economic Theory*, **15**(2), 279–296
- Danilov, V. I.** (1999). On Scarf theorem// *Economics and Mathematical Methods*, **35**, #3, 137–139 (in Russian)
- Debreu, G.** (1960). Topological methods in cardinal utility theory// *in*: Arrow, K. J., Karlin, S. and P. Suppes, (eds.): *Mathematical methods in the social sciences* (Stanford: Stanford University Press)
- Debreu, G. and H. E. Scarf** (1963). A limit theorem on the core of an economy// *International Economic Review*, **4**, 235–246
- Demianov, V. F., Rubinov, A. M.** (1990). Fundamentals of nonsmooth analysis and quasi-differential calculus. Moscow: *Science* (in Russian)
- Feldman, A. M.** (1973). Bilateral trading processes, pairwise optimality, and Pareto optimality// *Review of Economic Studies*, **40**, 463–473
- Filippov, A. F.** (1985). Differential equations with a discontinuous right-hand part. Moscow: Nauka, 224 p. (in Russian)

- Fisher, F. M.** (1983). *Disequilibrium foundations of equilibrium economics*. Cambridge/New York: Cambridge University Press, 236 p.
- Florenzano, M.** (1989). On the non-emptiness of the core of a coalitional production economy without ordered preferences// *Journal of Mathematical Analysis and Applications*, **141**, 484–490
- Florenzano, M.** (1990). Edgeworth equilibria, fuzzy core and equilibria of a production economy without ordered preferences// *Journal of Mathematical Analysis and Applications*, **153**, 18–36
- Florenzano, M.** (2009). Walras—Lindahl—Wicksell: What equilibrium concept for public goods provision? I — The convex case// *Document de Travail du Centre d’Economie de la Sorbonne 09009*, Université Pantheon-Sorbonne (Paris 1), Centre d’Economie de la Sorbonne, 22 p. (<http://halshs.archives-ouvertes.fr/docs/00/36/78/67/PDF/09009.pdf>)
- Florenzano, M. and E. L. Mercato** (2006). Edgeworth and Lindahl-Foley equilibria of a general equilibrium model with private provision of pure public goods// *Journal of Public Economic Theory*, **8**, 713–740
- Florenzano, M. and V. M. Marakulin** (2001). Production equilibria in vector lattices// *Economic Theory*, **17** 3, 577–598
- Foley, D., K.** (1970). Lindahl’s solution and the core of an economy with public goods// *Econometrica*, **38**, 66–72
- Gale, D.** (1955). The law of supply and demand// *Mathematica Scandinavica*, **3**, 155–169
- Gale, D.** (1978). The core of a monetary economy without trust// *Journal of Economic Theory*, **19**, 456–491
- Gale, D.** (2000). *Strategic foundations of general equilibrium—dynamical matching and bargaining games*. Cambridge: Cambridge University Press
- Geanakoplos, J.** (1990). An introduction to general equilibrium with incomplete assets markets// *Journal of Mathematical Economics*, **19**, 1–38
- Gilboa, I. and D. Schmeidler** (1989). Maxmin expected utility with non-unique prior// *Journal of Mathematical Economics*, **18**(2), 141–153
- Glycopantis, D., Muir, A. and N. C. Yannelis** (2001). An extensive form interpretation of the private core// *Economic Theory*, **18**, 293–319
- Glycopantis, D. and N. C. Yannelis** (2005). Information, efficiency, and the core of an economy: comments on Wilson’s paper// *in: Differential Information Economies*, 65–71 (Springer–Verlag)

- Glycopantis, D. and N. C. Yannelis** (2005). Differential information economies (collected papers). Berlin/Heidelberg: Springer-Verlag
- Graham, D. A. and E. R. Weintraub** (1975). On convergence to Pareto allocations// *Review of Economic Studies*, **42**, 469–472
- Graham, D. A., Jennergren, L. P., Peterson, D. P. and E. R. Weintraub** (1976). Trader-commodity parity theorems// *Journal of Economic Theory*, **12**, 443–454
- Green, J. R.** (1974). The stability of Edgeworth's recontracting process// *Econometrica*, **42**, 21–34
- Greene, W. H.** (1993). Econometric analysis. New Jersey: Prentice-Hall, Inc., 1075 p.
- Hahn, F. H.** (1982). Stability// in: K. J. Arrow and M. D. Intriligator (eds.): *Handbook of Mathematical Economics*, Vol. II, 745–794 (Amsterdam: North-Holland)
- Hahn, F. H. and T. Negishi** (1962). A Theorem on non-tâtonnement stability// *Econometrica*, **30**, 463–469
- Hahn, G. and N. C. Yannelis** (1997). Efficiency and incentive compatibility in differential information economies// *Economic Theory*, **10**, 383–411
- Hildenbrand, W.** (1974). Core and equilibria of a large economy. Princeton, NJ: Princeton University Press
- Hurwicz, L., R. Radner and S. Reiter** (1978). A stochastic decentralized resource allocation process I and II// *Econometrica*, **43**, 363–393
- Jordan, J. S.** (1983). Locally stable price mechanisms// *Journal of Mathematical Economics*, **11**, 235–259
- Kamiya, K.** (1990). A globally stable price adjustment process// *Econometrica*, **58**, 6, 1481–1485
- Kononov, A. and V. Marakulin** (2006). Equilibria without the survival assumption// *Journal of Mathematical Economics*, **42**, 198–215
- Koutsougeras, L. and N. C. Yannelis** (1993). Incentive compatibility and information superiority of the core of an economy with differential information// *Economic Theory*, **3**, 195–216
- Kozyrev, A. N.** (1982). The stable systems of contracts in an exchange economy// *Optimization*, 29(46), 66–78 (the issue of Institute of Mathematics SB AS USSR, in Russian)
- Kozyrev, A. N.** (1982). The contractual and fully contractual allocations in an abstract economy// *Institute of Mathematics SB RAS USSR*, Novosibirsk. Preprint No. 7, 44 p. (in Russian)

- Krasa, S. and N. C. Yannelis** (1994). The value allocation of an economy with differential information// *Econometrica*, **62**, 881–900
- Kreps, M. D.** (1977). A note on ‘fulfilled expectations’ equilibria// *Journal of Economic Theory*, **14**, 32–43
- Kunimoto, T. and R. Serrano** (2004). Bargaining and competition revisited// *Journal of Economic Theory*, **115** (1), 78–88
- Liu, Z. and N. C. Yannelis** (2013). Implementation under ambiguity: the maximin core// *mimeo*, 29 p.
- Madden, P. J.** (1975). Efficient sequences of non-monetary exchange// *Review of Economic Studies*, **42**, 581–595
- Magill, M. and M. Quinzii** (2002). Theory of incomplete markets. Cambridge/Massachusetts/London: The MIT Press
- Magill, M. and W. Shafer** (1991). Incomplete markets// *in*: W. Hildenbrand, H. Sonnenschein (eds.): *Handbook of Mathematical Economics*, Vol. IV, 1523–1614 (Amsterdam: North-Holland)
- Makarov, V. L.** (1980). On the notion of contract in an abstract economy// *Optimization* 24(41), 5–17 (the issue of Institute of Mathematics SB AS USSR, in Russian)
- Makarov, V. L.** (1982). Economic equilibrium: the existence and extremal properties// *Sovremennye problemy matematiki. Itogi nauki i tehniki*, Vol. **19**, 23–58 (in Russian)
- Mantel, R.** (1979). Equilibrio con rendimiento crecientes a escala// *Anales de la Asociation Argentine de Economia Politica*, **1**, 271–283
- Marakulin, V. M.** (1999). Incomplete markets: the concept of generalized equilibrium and its existence theorem// *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales*, **93** 4, 489–498
- Marakulin, V. M.** (2003). Contracts and domination in incomplete markets// *Economic Education and Research Consortium*. Working Paper Series, no 02/04, 104 p. URL <http://eerc.ru:8088/details/EERCWorkingPaper.aspx?id=253> or www.math.nsc.ru/~mathecon/marakulinENG.html
- Marakulin, V. M.** (2006a). Equilibrium analysis in Kantorovich spaces// *Journal of Mathematical Sciences*, **133** 4, 1477–1490.
- Marakulin, V. M.** (2006b). On the convergence of contractual trajectories in pure exchange economies// *Economic Education and Research Consortium*. Working Paper Series, No 06/07, 78 p.

- Marakulin, V. M.** (2009). Economies with asymmetrically informed agents: the concept of limit information// *Journal of the New Economic Association*, **1–2**, 62–85 (in Russian and English transl.)
- Marakulin, V. M.** (2010). Differential information economies: contract-based approach// *Economic Education and Research Consortium*. Working Paper Series, No 10/02E, 68 p. (in English); *Social Science Research Network*, Working paper December 2010, <http://hq.ssrn.com/author=1584377>
- Marakulin, V. M.** (2011). Contracts and domination in competitive economies// *Journal of the New Economic Association*, **9**, 10–32 (in Russian)
- Marakulin, V. M.** (2012). Abstract equilibrium analysis in mathematical economics, Novosibirsk: *SB Russian Academy of Science Publisher*, 348 p. (in Russian), ISBN 978-5-7692-1242-0
<http://www.math.nsc.ru/~mathecon/marakulinRUS.html>
- Marakulin, V. M.** (2013). On the Edgeworth conjecture for production economies with public goods: A contract-based approach// *Journal of Mathematical Economics*, **49**, 3 (May 2013), 189–200; ISSN 0304–4068
- Marakulin, V. M.** (2014). On contractual approach for Arrow–Debreu–McKenzie economies// *Economics and Mathematical Methods*, **50**, #1, 61–79 (in Russian)
- Marakulin, V. M.** (2016). Contracts and domination in incomplete markets: what is a true core?// *Economic Theory Bulletin*, **2016**, 28 pages, DOI 10.1007/s40505-016-0105-0
- Marakulin, V. M.** (2021). On contractual approach for non-convex production economies. — *In book: Mathematical Optimization Theory and Operations Research. MOTOR 2022. Lecture Notes in Computer Science*, vol 12755, p. 410–429. Springer, Cham. https://doi.org/10.1007/978-3-030-77876-7_28
- Marakulin, V. M.** (2022). On the existence of a fuzzy core in an exchange economy. — *In: Pardalos P., Khachay M., Mazalov V. (eds) Mathematical Optimization Theory and Operations Research. MOTOR 2022. Lecture Notes in Computer Science*, vol 13367. Springer, Cham. 210–217. https://doi.org/10.1007/978-3-031-09607-5_15
- Mas-Colell, A., Whinston, M. D., and J. R. Green** (1995). *Microeconomic Theory*. New York/Oxford: Oxford University Press, 981 p.
- McKenzie, L. W.** (1954). On equilibrium in Graham’s model of world trade and other competitive systems// *Econometrica*, **22**, 2, 147–161
- Michael, E.** (1956). Continuous Selections I// *Annals of Mathematics*, **63**(2), 361–82
- Milleron, J.-C.** (1972). Theory of value with public goods: a survey article// *Journal of Economic Theory*, **5**, 419–477

- Moulin, H.** (1988). Axioms of cooperative decision making. Cambridge: Cambridge University Press
- Muench, T.** (1972). The core and Lindahl equilibrium of an economy with a public good// *Journal of Economic Theory*, **4**, 241–255
- Mukherji, A.** (1974). The Edgeworth–Uzawa barter stabilizes prices// *International Economic Review*, **15**, 236–241
- Mukherji, A.** (1995). A locally stable adjustment process// *Econometrica*, **63** (2), 441–448
- Mukherji, A.** (2003). Competitive equilibria: convergence, cycles or chaos// *Discussion Paper* No 591. (The Institute of Social and Economic Research, Osaka University, Japan, 286 p.) Available at SSRN: <https://ssrn.com/abstract=428920> or <http://dx.doi.org/10.2139/ssrn.428920>
- Muthoo, A.** (1999). Bargaining theory with applications. Cambridge: Cambridge University Press
- Negishi, T.** (1962). The stability of a competitive economy: A survey article// *Econometrica*, **30**, 635–669
- Penrose, R. A.** (1955). A general inverse for matrices// *Proceedings of the Cambridge Philosophical Society*, **51**, 406–413
- Polovinkin, E. S.** (1985). On the question of differentiation of multivalued mappings// *Some problems of modern mathematics and their applications to problems of math. physics.* — Moscow: Moscow Institute of Physics and Technology, 90–97
- Polovinkin, E. S. and M. V. Balashov** (2004). Elements of convex and strongly convex analysis. Moscow: *Fizmatlit*, 416 p.
- Polterovich, V. M.** (1970). Mathematical models of resources allocations// *Central Economic and Mathematical Institute*, Moscow, 107 p. (in Russian)
- Polterovich, V. M. and V. A. Spivak** (1982). Mappings with gross substitution in economic equilibrium theory// *Sovremennye problemy matematiki. Itogi nauki i tehniki*, Vol. **19**, 111–154 (in Russian)
- Radner, R.** (1968). Competitive equilibrium under uncertainty// *Econometrica*, **36**, 31–58
- Radner, R.** (1978). Rational expectation equilibrium: generic existence and information revealed by prices, *Econometrica*, **47**, 655–678
- Radner, R.** (1982). Equilibrium under uncertainty// *in*: Arrow, K. J. and M. D. Intriligator (eds.): *Handbook of Mathematical Economics*, Vol. II (Amsterdam: North-Holland), 923–1006

- Repullo, R.** (1988). The core of an economy with transactions costs// *Review of Economic Studies*, **55**, 447–458
- Roberts, D.** (1974). A note on returns to group size and the core with public goods// *Journal of Economic Theory*, **9**, 350–356
- Rockafellar, R. T.** (1979). *Convex Analysis*, Princeton University Press, xviii+451 p.
- Rubinstein A. and M. J. Osborne** (1990). *Bargaining and Markets*. San Diego: Academic Press, California, USA
- Ruys, P. H. M.** (1974). *Public goods and decentralization*. Tilburg: Tilburg University Press, The Netherland, 235 p.
- Saari, D. G. and C. T. Simon** (1978). Effective price mechanisms// *Econometrica*, **46** (5), 1097–1125
- Samuelson, P. A.** (1941). The stability of equilibrium// *Econometrica*, **9**, 97–120
- Scarf, H.** (1967). The core of an N person game// *Econometrica*, **35**, 50–69
- Schwalbe, U.** (1999). *The core of economies with asymmetric information*. Berlin/Heidelberg/New York: Springer-Verlag, 141 p.
- Searle, S. R. and W. H. Hausman** (1970). *Matrix algebra for business and economics*. New York: Wiley, 1075 p.
- Shaher, W. and H. Sonnenschein** (1982). Market demand and excess demand functions// *in: Arrow, K. J. and M. D. Intriligator (eds.): Handbook of Mathematical Economics*, Vol. II, 671–693 (Amsterdam: North-Holland)
- Smale, S.** (1976). A convergent process of price adjustments and global Newton Methods// *Journal of Mathematical Economics*, **3**, 107–120
- Smale, S.** (1981). Global analysis and economics// *in: Arrow, K. J. and M. D. Intriligator (eds.): Handbook of Mathematical Economics*, Vol. I, Chapter 8 (Amsterdam: North-Holland)
- Thomson, W.** (1994). Cooperative models of bargainings// *in: Aumann, R. J. and S. Hart (eds.), Handbook of Game Theory*, Vol. II, (Amsterdam/Lausanne/New York/Oxford/Shannon/Tokyo: Elsevier Science) 1237–1284
- Uzawa, T.** (1962). On the stability of Edgeworth's barter process// *International Economic Review*, **3**, 218–232
- Vasil'ev, V. A.** (1996). On Edgeworth equilibria for some types of nonclassical markets// *Siberian Advances in Mathematics*, **6**, 3, 96–150 [translation from *Trudy Sobolev Inst. Mat.* **28** (1994), 3–52]

- Vasil'ev, V., Weber, S. and H. Wiesmeth** (1995). Core equivalence with congested public goods// *Economic Theory*, **6**, 373–387
- Vind, K.** (1995). Perfect competition or the core// *European Economic Review*, **39**, 1733–1745
- Weber, S. and H. Wiesmeth** (1991). The equivalence of the core and cost share equilibria in an economy with public goods// *Journal of Economic Theory*, **54**, 180–197
- Wilson, R.** (1978). Information, efficiency, and the core of an economy// *Econometrica*, **46**, 807–816
- Yannelis, N. C.** (1991). The core of an economy with differential Information// *Economic Theory*, **1**, 183–195
- Yildiz, M.** (2003). Walrasian bargaining// *Games and Economic Behavior*, **45**(2), 465–487