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On Extensions of Partial n-Quasigroups of Order 4

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Abstract—We prove that every collection of pairwise compatible (nowhere coinciding) n-ary quasigroups of order 4 can be extended to an $(n+1)$-ary quasigroup. In other words, every Latin $4 \times \dots \times 4 \times l$-parallelepiped, where $l = 1, 2, 3$, can be extended to a Latin hypercube.

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1. INTRODUCTION

By an *n-quasigroup of order k* we mean an algebra with the universe $\Sigma = \{0, 1, \dots, k-1\}$ and a function $f : \Sigma^n \to \Sigma$ that is invertible in each variable. The function f will be sometimes referred as an n-quasigroup, n-ary quasigroup, or quasigroup too. The value table of an n-quasigroup of order k is called a *Latin n-cube of order k*. If $n = 2$ then we call it a *Latin square*.

A (*distance* 2) MDS-*code* is a subset $M \subset \Sigma^n$ of cardinality $|\Sigma|^{n-1}$ such that different elements of M differ in at least two coordinates. For a function $f : \Sigma^n \to \Sigma$, put

$$\mathcal{M}\langle f\rangle \triangleq \Big\{\big(\overline{x}, f(\overline{x})\big) : \overline{x} \in \Sigma^n\Big\}, \quad \mathcal{M}_a\langle f\rangle \triangleq \big\{\overline{x} \in \Sigma^n : f(\overline{x}) = a\big\}, \ \ a \in \Sigma.$$

For an MDS-code $M \subset \Sigma^{n+1}$ put

$$F_i\langle M\rangle(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) = x_i$$

if and only if $(x_1, \dots, x_{n+1}) \in M$. It is immediate obvious that the mapping $\mathcal{M}\langle\cdot\rangle$ establishes a one-to-one correspondence between the set of n-quasigroups and the set of MDS-codes of length $n+1$. For every $a \in \Sigma$, the mapping $\mathcal{M}_a\langle\cdot\rangle$ takes each n-quasigroup into an MDS-code of length n. For every i with $1 \le i \le n$, the mapping $F_i\langle\cdot\rangle$ establishes a one-to-one correspondence between the set of MDS-codes of length $n+1$ and the set of n-quasigroups.

Put $[n] \triangleq \{1, \dots, n\}$. Assume that $S \subset \Sigma^n$ and, for every $\overline{v} \in S$ and $i \in [n]$, there exists a unique $\overline{u} \in S$ such that $\overline{u}$ and $\overline{v}$ differ in the i-th coordinate only. Then S is called a 2-*code*. By a 2-MDS-*code* we mean a 2-code $S \subset \Sigma^n$ of cardinality $2|\Sigma|^{n-1}$. A 2-MDS-code is said to be *prime* if it contains no proper 2-subcodes. If MDS-codes M_1 and M_2 are disjoint then $S = M_1 \cup M_2$ is a 2-MDS-code. A 2-MDS-code admitting such a representation is said to be *splittable*. The definition of a (splittable) l-fold MDS-code is similar. As is proven in [10], for $n \ge 3$ and $k \ge 4$, there exist 2-MDS-codes (and multifold codes) that are not splittable.

We say that n-quasigroups f and g are *compatible* if $f(\overline{x}) \ne g(\overline{x})$ for every $x \in \Sigma$. It is clear that n-quasigroups f and g are compatible if and only if $\mathcal{M}\langle f\rangle \cap \mathcal{M}\langle g\rangle = \varnothing$. We say that n-quasigroups f and g form an *extendable* pair if there exists an $(n+1)$-quasigroup h such that $f = h\big|_{x_{n+1}=0}$ and $g = h\big|_{x_{n+1}=1}$. For a collection of m quasigroups of order k, where $1 \le m < k$, the definition is similar. The questions on extension of a collection of n-quasigroups and splitting a multifold MDS-code are equivalent. In particular, the following proposition is immediate from the definitions.

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Proposition 1. *Compatible n-quasigroups f and g of order $|\Sigma| = 4$ form an extendable pair if and only if the* 2*-MDS-code $\Sigma^{n+1} \setminus \big(\mathcal{M}\langle f\rangle \cup \mathcal{M}\langle g\rangle\big)$ is splittable.*

A collection $f_0, \dots, f_{m-1}$ of pairwise compatible n-quasigroups with $m < |\Sigma|$ can be regarded as a partial $(n+1)$-ary quasigroup such that

$$F : \Sigma^n \times \{0, \dots, m-1\} \to \Sigma, \quad F\,\big|_{x_{n+1}=i} = f_i \text{ for } i \in \{0, \dots, m-1\}.$$

The value table of a partial $(n+1)$-ary quasigroup is called an $(n+1)$*-dimensional Latin parallelepiped*. It is clear that an $(n+1)$-dimensional Latin parallelepiped can be extended to a Latin $(n+1)$-cube if and only if the corresponding partial $(n+1)$-ary quasigroup (collection of pairwise compatible n-quasigroups) is extendable.

By the classical König–Hall Theorem [3], every Latin $(n \times m)$-rectangle with $m \le n$ can be extended to a Latin $(n \times n)$-square, i.e., every collection of pairwise compatible 1-quasigroups (permutations) of an arbitrary order k can be extended to a 2-quasigroup of order k.

In the sequel, well-known facts will be called assertions. The proof of the following assertion can be found, for example, in [5].

Assertion 2.

(a) *Every collection of $k-1$ pairwise compatible n-quasigroups of order k is extendable.*

(b) *Every n-quasigroup forms an extendable singleton collection.*

Thus, if $k \le 3$ then every collection of pairwise compatible n-quasigroups of order k is extendable.

In the present article, we prove the following

Theorem. *Every pair of compatible n-quasigroups of order* 4 *is extendable.*

By this theorem and Assertion 2, every collection of pairwise compatible n-quasigroups of order 4 is extendable. The following assertion is proven in [4, 5].

Assertion 3. *Let $k \ge 5$ and $k/2 < l < k-1$. There exists a collection of l pairwise compatible* 2*-quasigroups of order k that is not extendable.*

In [13], collections are constructed of l pairwise compatible 2-quasigroups of order k that are not extendable, where $k = 5, 6, 7, 8$ and $l = 2, 2, 3, 4$ respectively. If $n \ge 3$, $k \ge 5$, and $k/2 < l < k-1$ then there exists a collection of l pairwise compatible n-quasigroups of order k that is not extendable. This fact can be obtained from Assertion 3, see also [10]. Thus, every collection of pairwise compatible n-quasigroups of order k is extendable only if $k = 2, 3, 4$.

In [8], a method is discussed for constructing codes possessing parameters of doubly-shortened perfect codes that cannot be obtained by shortening a perfect code. This method uses pairs of compatible n-quasigroups of order 4 that are not extendable. By our main theorem, such a construction is impossible.

2. DEFINITIONS AND PRELIMINARY RESULTS

An *isotopy* in Σ^n is an n-tuple of permutations $\theta_i : \Sigma \to \Sigma$, $i \in [n]$. For an isotopy $\bar\theta = (\theta_1, \dots, \theta_n)$ and a set $S \subseteq \Sigma^n$, we put

$$\bar\theta S \triangleq \big\{(\theta_1 x_1, \dots, \theta_n x_n) : (x_1, \dots, x_n) \in S\big\}.$$

Sets $S_1 \subseteq \Sigma^n$ and $S_2 \subseteq \Sigma^n$ are *isotopic* if there exists an isotopy $\bar\theta$ with $\bar\theta S_1 = S_2$. Sets $S_1 \subseteq \Sigma^n$ and $S_2 \subseteq \Sigma^n$ are *equivalent* if there exist a permutation of coordinates $\tau : [n] \to [n]$ and an isotopy $\bar\theta$ such that

$$\chi_{S_1}(x_1, \dots, x_n) \equiv \chi_{\bar\theta S_2}\big(x_{\tau(1)}, \dots, x_{\tau(n)}\big),$$

where χ_B is the characteristic function of B. n-Quasigroups f and g are *equivalent* (*isotopic*) if the MDS-codes $\mathcal{M}\langle f\rangle$ and $\mathcal{M}\langle g\rangle$ are equivalent (isotopic). It is immediate from the definitions that the n-quasigroups $F_i\langle M\rangle$, $i \in [n]$, are pairwise equivalent for every MDS-code M. An isotopy $\bar\theta$ between n-quasigroups f and g is said to be *principal* if $f(\bar\theta\overline{x}) \equiv g(\overline{x})$.

Let $M \subset \Sigma^n$ be an MDS-code. For every $a \in \Sigma$, the set

$$M\big|_{x_i=a} = \{\overline{x} \in M : x_i = a\}$$

is called a *retract of dimension* $n-1$ of M. Fixing the values of m variables, we obtain a *retract of dimension* $n-m$, $1 \le m \le n-2$. By definition, each retract of an MDS-code is an MDS-code of a lower dimension.

Let f be an n-quasigroup and let M' be a retract of dimension m, $m \le n$, of the MDS-code $\mathcal{M}\langle f\rangle \subset \Sigma^{n+1}$. Then each $(m-1)$-quasigroup $F_i\langle M'\rangle$ is called a *retract* of f.

An n-quasigroup f is said to be *reducible* (*decomposable*) if there exist an integer m, $2 \le m < n$, an $(n-m+1)$-quasigroup h, an m-quasigroup g, and a permutation $\sigma \in S_n$ such that

$$f(x_1, \dots, x_n) \equiv h\Big(g\big(x_{\sigma(1)}, \dots, x_{\sigma(m)}\big), x_{\sigma(m+1)}, \dots, x_{\sigma(n)}\Big).$$

The following assertion is immediate from a theorem in [2].

Assertion 1. *Each reducible n-quasigroup f admits a representation*

$$f(\overline{x}) \equiv q_0\big(q_1(\widetilde{x}_1), \dots, q_m(\widetilde{x}_m)\big), \tag{2.1}$$

where each q_j is an n_j-quasigroup, $1 \le j \le m$, q_0 is an irreducible m-quasigroup, and each $\widetilde{x}_j$ is a tuple of variables x_i, $i \in I_j$, where $\{I_j\}_{j=1,\dots,m}$ is a partition of $[n]$ and $n_1, \dots, n_m$ are the cardinalities of its blocks. Moreover, if $m \ge 3$ then the partition $\{I_j\}_{j=1,\dots,m}$ is uniquely determined.

If $m \ge 3$ then (2.1) is called a *canonical decomposition* of f. Reducible quasigroups among $q_1, \dots, q_m$ can be represented as superpositions of quasigroups of lower arities, etc. It is clear that the arity of an irreducible retract of f cannot exceed the arities of all irreducible quasigroups occurring in the representation of f by a repeated superposition. Therefore, from Assertion 1 we obtain the following

Corollary 5. *Assume that a reducible n-quasigroup f possesses an irreducible retract of arity $m \ge 2$ that is contained in no irreducible retract of greater arity. Then the MDS-code $\mathcal{M}\langle f\rangle$ admits the following representation:*

$$\mathcal{M}\langle f\rangle = \Big\{\overline{x} \in \Sigma^{n+1} : q_{m+1}(x_{n+1}, \widetilde{x}_{m+1}) = q_0\big(q_1(\widetilde{x}_1), \dots, q_m(\widetilde{x}_m)\big)\Big\}. \tag{2.2}$$

Here each q_j is an n_j-quasigroup, $1 \le j \le m+1$, q_0 is an irreducible m-quasigroup, and each $\widetilde{x}_j$ is a tuple of variables x_i, $i \in I_j$, where $\{I_j\}_{j=1,\dots,m+1}$ is a partition of $[n]$ and $n_1, \dots, n_m$ are the cardinalities of its blocks.

By a *canonical decomposition* of an MDS-code $\mathcal{M}\langle f\rangle$ we mean a representation (2.2) such that the maximum of the dimensions of irreducible retracts of the MDS-code is equal to $m+1$ and the cardinality of I_{m+1} is minimal.

By definition, all 2-quasigroups are irreducible. A reducible n-quasigroup f is said to be *completely* (*commutative*) *reducible* if every retract of arity greater than 2 is reducible and every binary retract is isotopic to a commutative group.

In the sequel, we assume that $\Sigma = \{0, 1, 2, 3\}$, i.e., we consider n-quasigroups of order 4 only. As is known [1], each 2-quasigroup of order 4 is isotopic to either group $Z_2 \times Z_2$ or group Z_4. Therefore, every reducible n-quasigroup of order 4 possessing no retract of arity greater than 2 is completely reducible. In the sequel, we assume that $0 \in \Sigma$ is the neutral element. If a 2-quasigroup is isotopic to $Z_2 \times Z_2$ then we may assume that the corresponding isotopy is principal. For 2-quasigroups that are isotopic to Z_4, there exist three equivalence classes with respect to principal isotopy. The equivalence class is determined by the element of the group of order 2. In the sequel, by a group operation on Σ we mean one of the four operations mentioned above.

The following assertion is immediate from a theorem in [2].

Assertion 6. *Each completely reducible n-quasigroup $f : \Sigma^n \to \Sigma$ admits a representation*

$$f(\widetilde{x}_1, \dots, \widetilde{x}_k) \equiv q_1(\widetilde{x}_1) * \dots * q_k(\widetilde{x}_k), \tag{2.3}$$

where $*$ denotes a suitable group operation, each q_j is an n_j-quasigroup, $1 \le j \le k$, admitting no representation of the form $q_j(\widetilde{x}_j) = q'(\widetilde{x}'_j) * q''(\widetilde{x}''_j)$, and each $\widetilde{x}_j$ is a tuple of variables x_i, $i \in I_j$, where $\{I_j\}_{j=1,\dots,k}$ is a partition of $[n]$ and $n_1, \dots, n_m$ are the cardinalities of its blocks. Moreover, the operation $*$ and the partition $\{I_j\}_{j=1,\dots,k}$ are uniquely determined.

In (2.3), the n_j-quasigroups q_j and the n-quasigroup f are completely reducible. We call the operation $*$ in (2.3) the *principal* operation of f.

Let f be a completely reducible n-quasigroup admitting a representation of the form (2.3). We construct a rooted tree $T(f)$. The inner vertices of $T(f)$ are labeled with operations. The leafs of $T(f)$ are labeled with variables. If T_j is constructed for q_j, $1 \le j \le k$, then $T(f)$ consists of the root and k edges that are incident to the root. The root is labeled with $*$. The tree T_j is attached to the j-th edge. See the figure below for the tree constructed from a 9-quasigroup.

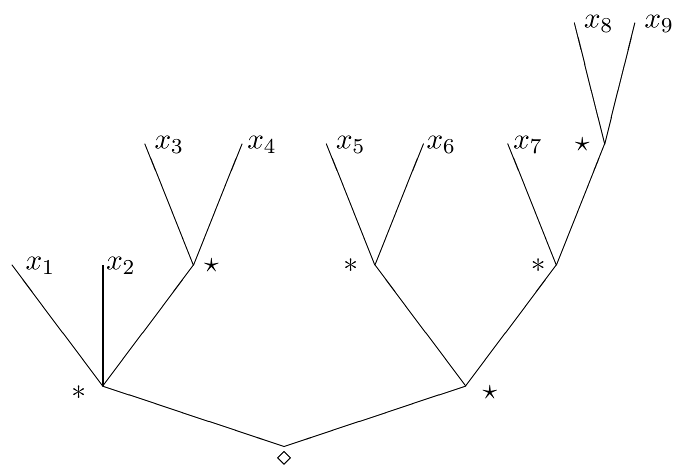

Fig. 1. The tree $T(f)$ with
$f(x_1, \dots, x_9) = \big(x_1 * x_2 * (x_3 \star x_4)\big) \diamond \big((x_5 * x_6) \star (x_7 * (x_8 \star x_9))\big)$.

A pair $\{x_i, x_j\}$ of variables is said to be *inner* with respect to an n-quasigroup f if $f(\overline{x}) \equiv g\big(\varphi(x_i, x_j), \widetilde{x}\big)$, where g is an $(n-1)$-quasigroup, φ is a 2-quasigroup, and x_i and x_j do not occur in $\widetilde{x}$. Each of the variables x_i, x_j is said to be inner too. For example, the pairs $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are inner with respect to the 9-quasigroup f, see Fig. 1. If a pair $\{x_i, x_j\}$ of variables is inner with respect to an n-quasigroup f then this pair is inner with respect to each retract of f containing x_i and x_j. Using Assertions 1 and 6 and induction on the number of variables, we easily obtain the following

Proposition 7. *Let*

$$\big\{(\overline{x}, \overline{y}) : f(\overline{x}) = g(\overline{y})\big\} = \big\{(\widetilde{x}, \widetilde{y}) : f'(\widetilde{x}) = g'(\widetilde{y})\big\},$$

where f, g, f', and g' are reducible quasigroups (in many variables), $\overline{x}, \overline{y}$ and $\widetilde{x}, \widetilde{y}$ are two partitions of the set of variables. If a pair of variables occurring in $\widetilde{x}$ is inner with respect to f then this pair is inner with respect to f' too.

The following assertion [14] shows that the notions of reducibility and complete reducibility can be naturally defined for MDS-codes.

Assertion 8. *Let $M \subset \Sigma^{n+1}$ be an* MDS*-code. Assume that $F_i\langle M\rangle$ is a reducible (completely reducible) n-quasigroup for some $i \in [n+1]$. Then, for every $j \in [n+1]$, the n-quasigroup $F_j\langle M\rangle$ is reducible (completely reducible) too.*

Let an MDS-code be defined by the equation $f(\overline{x}) = g(\overline{y})$, i.e., let

$$M = \{(\overline{x}, \overline{y}) : f(\overline{x}) = g(\overline{y})\},$$

where f is either completely reducible or depends on two variables. A pair $\{x_i, x_j\}$ of variables is said to be *inner* with respect to M if this pair is inner with respect to either f or g. Let $I(M)$ denote the set of all pairs of inner variables of M and let $\widetilde{I}(M) = \bigcup I(M)$ denote the set of all inner variables of M.

The following three propositions are immediate from the definition of a pair of inner variables of an MDS-code.

Proposition 9. *If $n \geq 4$ then each completely reducible* MDS*-code $M \subset \Sigma^n$ possesses at least two disjoint pairs of inner variables.*

Proposition 10. *A pair $\{x_i, x_j\}$ of variables is inner with respect to a reducible* MDS*-code $M \subset \Sigma^n$ if and only if $M = \{\varphi(x_i, x_j) = q(\widetilde{x})\}$, where q is an $(n-1)$-quasigroup, φ is a 2-quasigroup, and x_i and x_j do no occur in $\widetilde{x}$.*

Proposition 11. *If a pair $\{x_i, x_j\}$ of variables is inner with respect to a completely reducible* MDS*-code $M \subset \Sigma^n$ then, for every $a \in \Sigma$ and $k \in [n] \setminus \{i, j\}$, this pair is inner with respect to the retract $M\big|_{x_k=a}$.*

For a completely reducible MDS-code $M = \{(\overline{x}, \overline{y}) : f(\overline{x}) = g(\overline{y})\}$, we construct a tree $T(M)$. This tree need not be rooted. If the principal operations of f and g coincide then we identify the roots of $T(f)$ and $T(g)$. We add an edge between these roots if the principal operations of f and g are different. In $T(M)$, we preserve all labels of leafs (variables) and remove all labels of inner vertices (operations). Using induction on the number of variables and Proposition 7, it is easy to prove that this construction is independent of the equation defining the MDS-code M.

Let $W \subset V(T)$ be an arbitrary set of vertices of a tree T. We say that a vertex $w \in W$ is *extreme* in W if no path between two vertices from $W \setminus \{w\}$ passes through w. Let $M \subset \Sigma^{n+1}$ be a completely reducible MDS-code and let $S \subset [n+1]$. Let $w(s)$ be the vertex of $T(M)$ that is adjacent to the vertex labeled by x_s. We say that x_t, $t \in S$, is an *extreme* variable (in the set of variables with subscripts from S) if $w(t)$ is an extreme vertex in $\{w(s) \mid s \in S\}$. We say that x_t and x_s are *neighbors* if the vertices $w(s)$ and $w(t)$ are adjacent. For example, the variables x_7 and x_8 on Fig. 1 are neighbors.

Let $M \subset \Sigma^{n+1}$ be a completely reducible MDS-code. Considering subtrees of $T(M)$, we obtain the following propositions.

Proposition 12.

(a) *If the degree of $w(s)$ in $T(M)$ is greater than three then $I(M\big|_{x_s=a}) \subset I(M)$ for every $a \in \Sigma$.*

(b) *If $x_s \notin \widetilde{I}(M)$ and $\{x_p, x_q\} \in I(M\big|_{x_s=a}) \setminus I(M)$ then the variables x_p and x_s are neighbors as well as x_q and x_s, but the variables x_p and x_q are not.*

(c) *If $x_s \notin \widetilde{I}(M)$ and x_s is an extreme variable in S then*

$$\{\{x_p, x_q\} \mid p, q \in S\} \cap \Big(I(M\big|_{x_s=a}) \setminus I(M)\Big) = \varnothing$$

for every $a \in \Sigma$.

Proposition 13. *Let $M \subset \Sigma^{n+1}$, $n \geq 5$, be a completely reducible* MDS*-code, $x_s \in \widetilde{I}(M)$, $a \in \Sigma$, and $M' = M\big|_{x_s=a}$. We have $\{x_p, x_q\} \in I(M') \setminus I(M)$ if and only if the* MDS*-code M can be represented (up to isotopy) in the form*

$$M = \{\overline{x} : x_p *_1 (x_s *_2 x_q) = f(\widetilde{x})\}, \tag{2.4}$$

where x_s, x_p, and x_q do not occur in $\widetilde{x}$, f is an $(n-2)$-quasigroup, and the group operations $*_1$ and $*_2$ do not coincide. Moreover, if $x_p \notin \widetilde{I}(M)$ then $I(M') \setminus I(M) = \{x_p, x_q\}$.

A 2-code $S \subset \Sigma^{n+1}$ is said to be *linear* if

$$\chi_S(x_1, \dots, x_n) \equiv \chi_{S_1}(x_1) \oplus \chi_{S_2}(x_2) \oplus \dots \oplus \chi_{S_n}(x_n), \tag{2.5}$$

where $\oplus$ denotes addition modulo 2 and each S_i, $1 \le i \le n+1$, is a two-element subset of Σ. It is obvious that every linear 2-code is a 2-MDS-code.

In (2.5), we may replace each $\chi_{S_j}(x_j)$ by $\chi_{\Sigma \setminus S_j}(x_j) \oplus 1$. This allows us to choose the set containing 0 among S_j and $\Sigma \setminus S_j$. Therefore, every linear 2-code $S \subset \Sigma^{n+1}$ with $\chi_S(\overline{x}) \equiv \delta \oplus \bigoplus_{j=1}^{n+1} \chi_{\{0,\alpha_j\}}(x_j)$ is determined by the parity $\delta \in \{0,1\}$ and the tuple $(\alpha_1, \dots, \alpha_{n+1}) \in \Sigma^{n+1}$. This tuple is called the *characteristic* of the 2-code S.

We say that an MDS-code $\mathcal{M}\langle f \rangle$ is

- *semilinear* if it is contained in a linear 2-code;
- $(0,a)$-*semilinear* if it is contained in a linear 2-code of characteristic $(a, \dots, a)$, i.e., $\mathcal{M}\langle f \rangle \subset S$, where $\chi_S(\overline{x}) = 1 \oplus \bigoplus_{i=1}^{n+1} \chi_{\{0,a\}}(x_i)$;
- *anti*-$(0,a)$-*semilinear* if $\mathcal{M}\langle f \rangle \subset S$, where $\chi_S(\overline{x}) = \bigoplus_{i=1}^{n+1} \chi_{\{0,a\}}(x_i)$;
- *linear* if there exist at least two linear 2-codes containing $\mathcal{M}\langle f \rangle$.

The corresponding n-quasigroup f is said to be *semilinear*, $(0,a)$-*semilinear*, *anti*-$(0,a)$-*semilinear*, and *linear* respectively.

Let $+$ denote the operation on Σ that is equivalent to the addition of the group $Z_2 \times Z_2$. It is convenient to represent elements of Σ as two-dimensional boolean vectors $\overline{(\mu^1, \mu^2)}$, $\mu^\sigma \in \{0,1\}$, via the natural identification

$$0 = \overline{(0,0)}, \;\; 1 = \overline{(1,0)}, \;\; 2 = \overline{(1,1)}, \;\; 3 = \overline{(0,1)}.$$

Moreover, we have

$$\overline{(\mu^1, \mu^2)} + \overline{(\nu^1, \nu^2)} = \overline{(\mu^1 \oplus \nu^1, \mu^2 \oplus \nu^2)}.$$

Let $S \subset \Sigma^{n+1}$ be a linear 2-code of characteristic $(1, \dots, 1)$. Then

$$S = \left\{ \left(\overline{(\mu_1^1, \mu_1^2)}, \dots, \overline{(\mu_n^1, \mu_n^2)} \right) : \bigoplus_{i=1}^{n+1} \mu_i^2 = \delta \right\},$$

where $\delta \in \{0,1\}$. Each $(0,1)$-*semilinear* MDS-code $M \subset S$ can be represented as follows:

$$M = \left\{ \left(\overline{(\mu_1^1, \mu_1^2)}, \dots, \overline{(\mu_{n+1}^1, \mu_{n+1}^2)} \right) : \bigoplus_{i=1}^{n+1} \mu_i^2 = \delta, \; \bigoplus_{i=1}^{n+1} \mu_i^1 = \lambda_M(\mu_1^2, \dots, \mu_{n+1}^2) \right\}, \tag{2.6}$$

where λ_M is a boolean function defined on the set

$$E_\delta^{n+1} = \left\{ (\mu_1^2, \dots, \mu_{n+1}^2) : \bigoplus_{i=1}^{n+1} \mu_i^2 = \delta \right\}$$

of boolean vectors of parity δ, see [7].

Semilinear n-quasigroups f and g are said to be *opposite* if there exist $a, b \in \Sigma$ such that $\mathcal{S}_{a,b}\langle f \rangle = \mathcal{M}_a\langle f \rangle \cup \mathcal{M}_b\langle f \rangle$ is a linear 2-code and $\chi_{\mathcal{S}_{a,b}\langle f \rangle} = \chi_{\mathcal{S}_{a,b}\langle g \rangle} \oplus 1$. In particular, if f is $(0,a)$-semilinear and g is anti-$(0,a)$-semilinear then f and g are opposite. Conversely, if f is opposite to a $(0,a)$-semilinear but not linear n-quasigroup then f is anti-$(0,a)$-semilinear.

The following assertions are immediate from the definitions and can be found in [11, 14].

Assertion 14.

(a) *Each retract of a semilinear n-quasigroup is semilinear too.*

(b) *Each retract of a $(0, a)$-semilinear n-quasigroup is either $(0, a)$-semilinear or anti-$(0, a)$-semilinear.*

(c) *If $F_i\langle M\rangle$ is a semilinear (linear) n-quasigroup for some $i \in [n+1]$ then, for every $j \in [n+1]$, the n-quasigroup $F_j\langle M\rangle$ is semilinear (linear) too.*

Assertion 15.

(a) *An n-quasigroup f is semilinear if and only if there exist $a, b \in \Sigma$ such that the 2-code $\mathcal{S}_{a,b}\langle f\rangle$ is linear.*

(b) *The 2-code $\mathcal{S}_{a,b}\langle f\rangle$ is linear for every $a, b \in \Sigma$ with $a \neq b$ if there exist distinct $a, b, c \in \Sigma$ such that the 2-codes $\mathcal{S}_{a,b}\langle f\rangle$ and $\mathcal{S}_{a,c}\langle f\rangle$ are linear.*

(c) *An n-quasigroup f is linear if and only if, for every $a, b \in \Sigma$ with $a \neq b$, the 2-code $\mathcal{S}_{a,b}\langle f\rangle$ is linear.*

(d) *Each linear n-quasigroup is isotopic to the n-quasigroup $\ell(\overline{x}) = x_1 + x_2 + \cdots + x_n$.*

In the sequel, we will need the following assertions from [6, 7, 14].

Assertion 16 [6, Theorem 4.1 (c)]. *Let a 2-MDS-code $S \subset \Sigma^{n+1}$ satisfy the equality $\chi_S = \chi_{S_1} \oplus \chi_{S_2}$, where S_1 and S_2 are 2-MDS-codes of lower dimensions. Then S is splittable if and only if both S_1 and S_2 are splittable.*

Assertion 17 [14, on a unique extension of a 2-code]. *Let $S_1, S_2 \subset \Sigma^n$ be 2-MDS-codes. Assume that there exists a 2-code S_0 with $S_0 \subseteq S_1 \setminus S_2$. Then $S_1 = \Sigma^n \setminus S_2$.*

Assertion 18 [6, Theorem 4.1 (b)]. *Each 2-MDS-code can be represented as the union of a family of disjoint prime 2-codes.*

Assertion 19 [14, on a linear anti-layer]. *Let f be an n-quasigroup of order 4, let $f' = f\big|_{x_i=c}$ be a semilinear $(n-1)$-dimensional retract of f, and let $S = \mathcal{S}_{0,1}\langle f'\rangle$ be a linear set. Then there exists $d \in \Sigma$ such that the retract $f'' = f\big|_{x_i=d}$ is opposite to the retract f', i.e., $\mathcal{S}_{0,1}\langle f''\rangle = \Sigma^n \setminus \mathcal{S}_{0,1}\langle f'\rangle$.*

Assertion 20 [7, Lemma 1]. *Let an MDS-code M satisfy (2.6). Then M is reducible if and only if the function λ_M can be represented in the form $\lambda_M(\overline{\mu}', \overline{\mu}'') = \lambda'(\overline{\mu}') \oplus \lambda''(\overline{\mu}'')$, where at least two variables occur in each of the tuples $\overline{\mu}', \overline{\mu}''$.*

The following description of n-quasigroups of order 4 was obtained in [11].

Assertion 21. *Each n-quasigroup of order 4 is either reducible or semilinear.*

The proof of the main theorem is based on this description.

3. PROOF OF THE MAIN THEOREM

We use induction on the arity of a quasigroup. The induction hypothesis (IH) is as follows: *For every natural m with $m \leq n-1$, every pair of compatible m-quasigroups of order 4 is extendable.*

By Proposition 1, assumption (IH) is equivalent to the following condition: *For every natural m with $m \leq n$, the complement of each splittable 2-MDS-code in Σ^m is a splittable 2-MDS-code too.*

In the following auxiliary propositions and lemmas, we assume that $\Sigma = \{0, 1, 2, 3\}$ and each quasigroup is of order 4.

From Assertions 14 and 21 and the definitions of semilinear quasigroups and MDS-codes we obtain the following

Proposition 22. *Assume that f is an irreducible n-quasigroup.*

(a) *Each retract $f\big|_{x_1=a}$, $a \in \Sigma$, is semilinear.*

(b) *For every $a \in \Sigma$, there exist two elements $b \in \Sigma \setminus \{a\}$ such that the retracts $f\big|_{x_1=b}$ and $f\big|_{x_1=a}$ are opposite and one element $c \in \Sigma \setminus \{a\}$ such that the retracts $f\big|_{x_1=c}$ and $f\big|_{x_1=a}$ are isotopic, i.e., $f\big|_{x_1=c} = \tau f\big|_{x_1=a}$, where τ is a permutation.*

(c) *The retracts $f\big|_{x_1=a}$ and $f\big|_{x_1=b}$ can be extended only to an irreducible n-quasigroup that is isotopic to f if and only if the retracts are opposite to each other.*

Proposition 23. *Assume that semilinear MDS-codes $\mathcal{M}\langle f\rangle$ and $\mathcal{M}\langle g\rangle$ are disjoint and the characteristics of linear 2-codes containing these codes coincide in no coordinate. Then either $\mathcal{M}\langle f\rangle$ or $\mathcal{M}\langle g\rangle$ is linear.*

Proof. The conditions of the proposition are equivalent to the following condition: There exist $a, b \in \Sigma \setminus \{0\}$, $a \neq b$, such that the sets $\mathcal{S}_{0,a}\langle f\rangle$ and $\mathcal{S}_{0,b}\langle g\rangle$ are linear and consist of prime 2-codes whose pairwise intersections are nonempty. Without loss of generality, we may assume that $\mathcal{M}\langle f\rangle$ is contained in a linear 2-code of characteristic $(1, 1, \dots, 1)$. Then $a = 1$ and the set $\mathcal{S}_{0,1}\langle f\rangle$ consists of pairwise disjoint prime codes of the form $S_\sigma = \bigotimes_{i=1}^{n}\{0,1\}^{\sigma_i}$, where $\sigma_i \in \{0,1\}$, $\{0,1\}^1 = \{0,1\}$, and $\{0,1\}^0 = \{2,3\}$.

Let $b = 2$ (the case in which $b = 3$ is similar). If $f(\mathcal{S}_{0,1}\langle f\rangle \cap \mathcal{S}_{0,2}\langle g\rangle) = \{1\}$ then f is a linear n-quasigroup. Assume that f is not linear and $f\big(\{0,1\}^n \cap \mathcal{S}_{0,2}\langle g\rangle\big) \neq \{1\}$. Since the characteristics of the linear 2-codes coincide in no coordinate, the set $P = \{0,1\}^n \cap \mathcal{S}_{0,2}\langle g\rangle$ consist of either all even or all odd boolean vectors. Hence, f is a constant function on P. We find that $f(P) = \{0\}$ and $g(P) = \{2\}$; moreover, $f\big(\{0,1\}^n \setminus P\big) = \{1\}$. Hence, $g\big(\{0,1\}^n \setminus P\big) = \{3\}$. Therefore, $\mathcal{S}_{0,3}\langle g\rangle$ is a linear 2-code. By Assertion 15 (b, c), we conclude that g is a linear n-quasigroup. □

Proposition 24. *Let a semilinear MDS-code $M \subset \Sigma^n$ satisfy (2.6). Assume that a semilinear MDS-code $M' \subset \Sigma^n$ is contained in a linear 2-code of characteristic $(\alpha_1, \dots, \alpha_n)$, where $\alpha_1 = 1$, $\alpha_{n-1} \neq 1$, and $\alpha_n \neq 1$. If $M \cap M' = \varnothing$ then the function $\lambda_M(\overline{\mu})$ does not depend essentially on the variables μ_{n-1} and μ_n.*

Proof. If $n > 3$ then we fix the values of the variables μ_i with $i \neq 1, n-1, n$. Hence, we may assume that $n = 3$. It suffices to check all possible cases. □

The following proposition is immediate from Propositions 23 and 24 and Assertion 20.

Proposition 25. *Consider two nonlinear disjoint MDS-codes. Assume that at least one of them is irreducible. Then the characteristics of linear 2-MDS-codes containing these MDS-codes either coincide of differ in one coordinate.*

The following proposition is immediate from the definition of an irreducible n-quasigroup.

Proposition 26 (on four retracts)**.** *Let $n \geq 3$ and let $2 \leq m \leq n-1$. Then every n-quasigroup possessing only four different $(n-m)$-dimensional retracts is reducible.*

Proposition 27. *Let f and g be compatible n-quasigroups. Assume that one of the following conditions holds*:

(a) $\mathcal{S}_{a,b}\langle f\rangle = \Sigma^n \setminus \mathcal{S}_{a,b}\langle g\rangle$,

(b) $\mathcal{S}_{a,b}\langle f\rangle = \mathcal{S}_{a,c}\langle g\rangle$ *for pairwise distinct $a, b, c \in \Sigma$.*

Then f and g form an extendable pair of n-quasigroups. Moreover, if condition (b) *holds then $g = \sigma f$ for a suitable permutation σ.*

Proof. Let $\{c, d\} = \Sigma \setminus \{a, b\}$.

(a) Put $\tau{=}(a,b)(c,d)$. The n-quasigroups f and g are extended to an $(n+1)$-quasigroup by the n-quasigroups τf and τg.

(b) It is easy to see that $g = \sigma f$, where $\sigma = (acdb)$. By the König–Hall Theorem [3], the identity permutation Id and the permutation σ are extended to a Latin square by suitable permutations (for example, by $\sigma' = (abdc)$ and $\sigma'' = (ad)(bc)$). Hence, the n-quasigroups f and g are extended by the n-quasigroups $\sigma' f$ and $\sigma'' f$. □

Proposition 28. *Let $M_1, M_2 \subset \Sigma^{n+1}$ be $(0,1)$-semilinear but not linear MDS-codes with*

$$M_1 \cap (\theta_1, \theta_2, \mathrm{Id}, \dots, \mathrm{Id})M_2 = \varnothing, \quad M_1 \cap (\theta_1, \mathrm{Id}, \dots, \mathrm{Id})M_2 = \varnothing,$$

where

$$\theta_1 \notin \Omega = \big\{\mathrm{Id}, (0,1), (2,3), (0,1)(2,3), (0,2)(1,3), (0,3)(1,2)\big\}, \quad \theta_2 \neq \mathrm{Id}.$$

Then M_1 and M_2 are isotopic and reducible.

Proof. Let S be a linear 2-code containing the MDS-code M_2. Then the characteristic of S is $(1,\dots,1)$. Let $\bar{\xi}$ be an isotopy. It is easy to see that the characteristic of $\bar{\xi}S$ is $(1,\dots,1)$ if and only if $\xi_i \in \Omega$ for each $i = 1,\dots,n+1$. By Proposition 25, we have $\theta_2 \in \Omega$.

Consider the n-quasigroups

$$f = F_1\langle M_1\rangle, \quad g_1 = F_1\big\langle(\theta_1,\theta_2,\mathrm{Id},\dots,\mathrm{Id})M_2\big\rangle, \quad g_2 = F_1\big\langle(\theta_1,\mathrm{Id},\dots,\mathrm{Id})M_2\big\rangle.$$

By Proposition 27 (b), we have $g_1 = \sigma_1 f$ and $g_2 = \sigma_2 f$ for suitable permutations σ_1 and σ_2 with $\sigma_1 \neq \sigma_2$. We obtain

$$\chi_{M_1}(\sigma_2^{-1}\sigma_1 x_1, x_2, x_3, \dots, x_{n+1}) \equiv \chi_{M_1}(x_1, x_2, x_3, \dots x_{n+1}). \tag{3.1}$$

Since there exists a unique linear 2-code of characteristic $(1,\dots,1)$ containing M_1, we have $\sigma_2^{-1}\sigma_1 \in \Omega$. By (3.1), we have

$$\lambda_{M_1}(x_1 \oplus 1, x_2, \dots, x_{n+1}) \equiv \lambda_{M_1}(x_1, x_2, x_3, \dots, x_{n+1}).$$

Hence, the function λ_{M_1} is defined on boolean vectors of the same parity and essentially depends on $n-1$ variables only. By Assertion 20, the MDS-code M_1 is reducible. □

Proposition 29. *Let $S \subset \Sigma^{n+1}$ be a splittable 2-MDS-code and let $\chi_S = \chi_{S_1} \oplus \chi_{S_2}$, where S_1 and S_2 are 2-MDS-codes of lower dimensions. Then the 2-MDS-code $S' = \Sigma^{n+1} \setminus S$ is splittable too.*

Proof. Let $S_1 \subset \Sigma^m$ and let $S_2 \subset \Sigma^{n-m+1}$, $1 \le m \le n$. Put $S_1' = \Sigma^m \setminus S_1$ and $S_2' = \Sigma^{n-m+1} \setminus S_2$. We have $\chi_S = \chi_{S_1} \oplus \chi_{S_2} = \chi_{S_1'} \oplus \chi_{S_2'}$. By Assertion 16, the 2-MDS-codes S_1, S_2, S_1', and S_2' are splittable. It is obvious that $\chi_{S'} = \chi_{S_1} \oplus \chi_{S_2'}$. By Assertion 16, the 2-MDS-code S' is splittable too. □

Proposition 30. *Let*

$$\Omega = \big\{\mathrm{Id}, (0,1), (2,3), (0,1)(2,3), (0,2)(1,3), (0,3)(1,2)\big\}$$

and let g and f be n-quasigroups. For every $\overline{u} \in \Sigma^{n-1}$ and $i \in [n]$, define a permutation $\xi_{i,\overline{u}}$ by the equality $g(z\overline{u}) = \xi_{i,\overline{u}} f(z\overline{u})$, where the variable z occurs in the i-th coordinate. If $\xi_{i,\overline{u}} \in \Omega$ for all $\overline{u} \in \Sigma^{n-1}$ and $i \in [n]$ then either $\mathcal{S}_{0,1}\langle f\rangle = \mathcal{S}_{0,1}\langle g\rangle$ or $\mathcal{S}_{0,1}\langle f\rangle = \mathcal{S}_{2,3}\langle g\rangle$.

Proof. Using induction, we prove that $\mathcal{S}_{0,1}\langle f\rangle \cap \mathcal{S}_{0,1}\langle g\rangle \neq \varnothing$ implies $\mathcal{S}_{0,1}\langle f\rangle = \mathcal{S}_{0,1}\langle g\rangle$. If $n = 1, 2$ then it suffices to check all possible cases. Let $n \ge 3$. Since no face of dimension $n-2$ is disjoint from the 2-MDS-code $\mathcal{S}_{0,1}\langle f\rangle$, it remains to consider the hyperfaces of Σ^n. □

Proposition 31. *Let disjoint MDS-codes $M_1, M_2 \subset \Sigma^n$ be defined by the equalities $f_1(\widetilde{x}_1) = g_1(\widetilde{x}_2)$ and $f_2(\widetilde{x}_1) = g_2(\widetilde{x}_2)$, where the tuples $\widetilde{x}_1$ and $\widetilde{x}_2$ have no common variables. Let $n_1 \ge 2$. Assume that one of the n_1-quasigroups f_1, f_2 is $(0,1)$-semilinear and the other is anti-$(0,1)$-semilinear. If* (IH) *holds then $\Sigma^{n+1} \setminus (M_1 \cup M_2)$ is a splittable 2-MDS-code.*

Proof. We have $\mathcal{M}_0\langle f_1\rangle \cup \mathcal{M}_1\langle f_1\rangle = \mathcal{M}_2\langle f_2\rangle \cup \mathcal{M}_3\langle f_2\rangle$. Assume that, for every $a \in \Sigma$, there exists $\tau(a) \in \Sigma$ with $\mathcal{M}_a\langle f_1\rangle = \mathcal{M}_{\tau(a)}\langle f_2\rangle$. Then $\tau f_1 = f_2$. By (IH) for g_1 and τg_2, we obtain the required assertion.

Assume the contrary, i.e., let either

$$\mathcal{M}_0\langle f_1\rangle \cap \mathcal{M}_2\langle f_2\rangle \neq \varnothing \text{ and } \mathcal{M}_0\langle f_1\rangle \cap \mathcal{M}_3\langle f_2\rangle \neq \varnothing$$

or (by symmetry)

$$\mathcal{M}_2\langle f_1\rangle \cap \mathcal{M}_0\langle f_2\rangle \neq \varnothing \text{ and } \mathcal{M}_2\langle f_1\rangle \cap \mathcal{M}_1\langle f_2\rangle \neq \varnothing.$$

Let the first condition hold. We have

$$\mathcal{M}_1\langle f_1\rangle \cap \mathcal{M}_2\langle f_2\rangle \neq \varnothing, \quad \mathcal{M}_1\langle f_1\rangle \cap \mathcal{M}_3\langle f_2\rangle \neq \varnothing.$$

Since $M_1 \cap M_2 = \varnothing$, we obtain

$$\mathcal{M}_0\langle f_1\rangle \cap \mathcal{M}_2\langle f_2\rangle \neq \varnothing \Rightarrow \mathcal{M}_0\langle g_1\rangle \cap \mathcal{M}_2\langle g_2\rangle = \varnothing,$$
$$\mathcal{M}_0\langle f_1\rangle \cap \mathcal{M}_3\langle f_2\rangle \neq \varnothing \Rightarrow \mathcal{M}_0\langle g_1\rangle \cap \mathcal{M}_3\langle g_2\rangle = \varnothing,$$

$$\mathcal{M}_1\langle f_1\rangle \cap \mathcal{M}_2\langle f_2\rangle \neq \varnothing \Rightarrow \mathcal{M}_1\langle g_1\rangle \cap \mathcal{M}_2\langle g_2\rangle = \varnothing,$$
$$\mathcal{M}_1\langle f_1\rangle \cap \mathcal{M}_3\langle f_2\rangle \neq \varnothing \Rightarrow \mathcal{M}_1\langle g_1\rangle \cap \mathcal{M}_3\langle g_2\rangle = \varnothing.$$

We conclude that $\mathcal{S}_{0,1}\langle g_1\rangle = \mathcal{S}_{0,1}\langle g_2\rangle$. Therefore, the MDS-code

$$\big\{(\widetilde{x}_1, \widetilde{x}_2) : f_1(\widetilde{x}_1) = \tau g_1(\widetilde{x}_2)\big\}, \text{ where } \tau = (0,1)(2,3),$$

is disjoint from the MDS-codes M_1 and M_2. □

Proposition 32. *If f and g are compatible and irreducible n-quasigroups then they form an extendable pair.*

Proof. By Assertion 21 and Proposition 25, the n-quasigroups f and g are semilinear; moreover, the characteristics of 2-MDS-codes containing the MDS-codes $\mathcal{M}\langle f\rangle$ and $\mathcal{M}\langle g\rangle$ either coincide or differ in exactly one coordinate. The required assertion follows from either Proposition 27 (a) or Proposition 27 (b). □

Proposition 33. *Assume that an n-quasigroup f is not anti-$(0,1)$-semilinear. Then at most one $(0,1)$-semilinear n-quasigroup g is compatible with f.*

Proof. Let g be a $(0,1)$-semilinear n-quasigroup. Assume that $S' \subset \mathcal{S}_{01}\langle g\rangle \setminus \mathcal{S}_{01}\langle f\rangle$, where S' is a 2-code. By Assertion 17, we have $\mathcal{S}_{01}\langle f\rangle = \Sigma^n \setminus \mathcal{S}_{01}\langle g\rangle$, i.e., the n-quasigroup f is anti-$(0,1)$-semilinear, a contradiction. Hence, no prime 2-code S' with $S' \subset \mathcal{S}_{01}\langle g\rangle$ is disjoint from $\mathcal{S}_{01}\langle f\rangle$. If an n-quasigroup g is compatible with f, then the values of g on S' are uniquely determined. By Assertion 18, there exists a unique partition of $\mathcal{S}_{01}\langle g\rangle$ into prime codes. Hence, the values of g on $\mathcal{S}_{01}\langle g\rangle$ are uniquely determined. By similar arguments, the values of g on $\mathcal{S}_{23}\langle g\rangle$ are uniquely determined too. □

Lemma 34. *Let a reducible n-quasigroup f be compatible with a semilinear n-quasigroup g, $n \geq 3$. Then either g is reducible or f is semilinear and opposite to g.*

Proof. Without loss of generality, we may assume that f is an n-quasigroup of the form $f(\overline{x}, \overline{y}) \equiv f'\big(\overline{x}, f''(\overline{y})\big)$, where f' is an n_1-quasigroup, f'' is an irreducible n_2-quasigroup, $n_1 + n_2 = n + 1$, and $1 < n_2 < n$ (see Assertions 1 and 6). Moreover, we may assume that $f''(0\overline{0}) = 0$ and $f''(1\overline{0}) = 1$. To simplify the arguments, we assume without loss of generality that g is a $(0,1)$-semilinear n-quasigroup. Consider the retracts $f'_a = f'\big|_{x_{n_1}=a}$, $a \in \Sigma$. By Assertion 19 (on a linear anti-layer), three cases are possible:

(1) there are no $(0,1)$-semilinear and no anti-$(0,1)$-semilinear retracts,

(2) there is exactly one $(0,1)$-semilinear retract and exactly one anti-$(0,1)$-semilinear retract,

(3) there are two $(0,1)$-semilinear retracts and two anti-$(0,1)$-semilinear retracts.

We consider each of these cases.

(1) Assume that the number of different retracts of the form $g\big|_{\overline{y}=\overline{v}}$, $\overline{v} \in \Sigma^{n_2}$, is greater than four. By Assertion 14 (b), each of these retracts is either $(0,1)$-semilinear or anti-$(0,1)$-semilinear. We may assume that there exist at least three different $(0,1)$-semilinear retracts of the form $g\big|_{\overline{y}=\overline{v}}$ (the case of three anti-$(0,1)$-semilinear retracts is similar). Put $\tau = (0,1)(2,3)$. Let $\overline{v} \in \Sigma^{n_2}$ and let $g\big|_{\overline{y}=\overline{v}}$ be $(0,1)$-semilinear. Since g is $(0,1)$-semilinear, there exists $\overline{u} \in \Sigma^{n_2}$ such that $g\big|_{\overline{y}=\overline{u}}$ is $(0,1)$-semilinear and $g\big|_{\overline{y}=\overline{v}} = \tau g\big|_{\overline{y}=\overline{u}}$. This equality holds if the tuples $\overline{v}$ and $\overline{u}$ differ in exactly one coordinate. Hence, the number of different $(0,1)$-semilinear retracts of the form $g\big|_{\overline{y}=\overline{v}}$ is even. By Proposition 33, each of the four $(n_1 - 1)$-quasigroups f'_a, $a \in \Sigma$, is compatible with at most one $(0,1)$-semilinear retract. Hence, there exists exactly four different $(0,1)$-semilinear retracts of the form $g\big|_{\overline{y}=\overline{v}}$; moreover, there exists a one-to-one correspondence between these retracts and the retracts f'_a, $a \in \Sigma$. For every $(0,1)$-semilinear n-quasigroup g, there exist only two different retracts of the form $g\big|_{\overline{y}=\overline{v}}$, $\overline{v} \in \{0,1\}^{n_2}$. The permutation τ takes each of them into the other one. Hence, there exist only two different retracts of the form $f\big|_{\overline{y}=\overline{v}}$, $\overline{v} \in \{0,1\}^{n_2}$, i.e., the n_2-quasigroup f'' takes only two values on the set $\overline{v} \in \{0,1\}^{n_2}$. By Assertion 17, the n_2-quasigroup f'' is $(0,1)$-semilinear. Indeed, it suffices to notice that $f''(0\overline{0}) = 0$ and

$f''(1\overline{0}) = 1$. We have $f''(\overline{v}) \in \{0,1\}$ if and only if the retract $g\big|_{\overline{y}=\overline{v}}$ is $(0,1)$-semilinear. Therefore, every $(0,1)$-semilinear retract of the form $g\big|_{\overline{y}=\overline{v}}$, $\overline{v} \in \Sigma^{n_2}$, is compatible with either f'_0 or f'_1. By Proposition 33, there exist exactly two different $(0,1)$-semilinear retracts of the form $g\big|_{\overline{y}=\overline{v}}$. We arrive at a contradiction. Hence, there exist only four different retracts of the form $g\big|_{\overline{y}=\overline{v}}$, $\overline{v} \in \Sigma^{n_2}$. By Proposition 26 (on four retracts), we obtain the required assertion.

(2) Assume that f'_0 is $(0,1)$-semilinear and f'_2 is anti-$(0,1)$-semilinear (the remaining cases are similar). Consider the four retracts of the form $g'_a = g\big|_{\overline{y}=a\overline{0}}$, $a \in \Sigma$. By Assertion 14 (b), the retracts g'_0 and g'_1 are $(0,1)$-semilinear and the retracts g'_2 and g'_3 are anti-$(0,1)$-semilinear. The retract f'_1 is compatible with the $(0,1)$-semilinear (n_1-1)-quasigroups g'_1 and f'_0. By Proposition 33, we have $g'_1 = f'_0$. The proof of the equality $g'_3 = f'_2$ is similar. Since g is a $(0,1)$-semilinear n-quasigroup, we have $g'_0 = \tau g'_1 = \tau f'_0$ and $g'_2 = \tau g'_3 = \tau f'_2$. Similar arguments are valid for every tuple $\overline{u} \in \Sigma^{n_2-1}$ and the retracts $g\big|_{\overline{y}=a\overline{u}}$, $a \in \Sigma$. Hence, there exist only four different retracts of the form $g\big|_{\overline{y}=a\overline{u}}$. Each of these retracts coincides with one of the (n_1-1)-quasigroups f'_0, f'_2, $\tau f'_0$, $\tau f'_2$. By Proposition 26 (on four retracts), the n-quasigroup g is reducible.

(3) Assume that, for every $\overline{v} \in \Sigma^{n_2}$, the retract $g\big|_{\overline{y}=\overline{v}}$ is $(0,1)$-semilinear and the retract $f\big|_{\overline{y}=\overline{v}}$ is anti-$(0,1)$-semilinear. Then f is opposite to g; hence, f is semilinear. Otherwise, there exists $\overline{v} \in \Sigma^{n_2}$ such that the retracts $g\big|_{\overline{y}=\overline{v}}$ and $f\big|_{\overline{y}=\overline{v}}$ are $(0,1)$-semilinear. Without loss of generality, we may assume that $\overline{v} = 0\overline{0}$. We have

$$\tau g\big|_{\overline{y}=1\overline{0}} = g\big|_{\overline{y}=0\overline{0}} = \tau f\big|_{\overline{y}=0\overline{0}}, \quad \text{i.e.,} \quad g\big|_{\overline{y}=1\overline{0}} = f\big|_{\overline{y}=0\overline{0}} = f'_0.$$

We conclude that $\{g'_0, g'_1, g'_2, g'_3\} = \{f'_0, f'_1, f'_2, f'_3\}$. By Assertion 17, no prime subcode of the 2-MDS-code $\mathcal{S}_{0,1}\langle g\big|_{\overline{x}=\overline{0}}\rangle$ is disjoint from the set $\mathcal{S}_{0,1}\langle f\big|_{\overline{x}=\overline{0}}\rangle$ (the case in which $\mathcal{S}_{0,1}\langle f\big|_{\overline{x}=\overline{0}}\rangle = \Sigma^n \setminus \mathcal{S}_{0,1}\langle g\big|_{\overline{x}=\overline{0}}\rangle$ was considered above). Similar arguments show that every prime 2-subcode of $\mathcal{S}_{0,1}\langle g\big|_{\overline{x}=\overline{0}}\rangle$ contains $\overline{v} \in \Sigma^{n_2}$ such that the retract $g\big|_{\overline{y}=\overline{v}}$ coincides with one of the four (n_1-1)-quasigroups f'_0, f'_1, f'_2, f'_3. The same conclusion is valid for every $\overline{v} \in \Sigma^{n_2}$. Indeed, if $\overline{v} \in \Sigma^{n_2}$ and $\overline{v}$ belongs to a prime 2-subcode of $\mathcal{S}_{0,1}\langle g\big|_{\overline{x}=\overline{0}}\rangle$ then there exist only two different retracts of the form $g\big|_{\overline{y}=\overline{v}}$. The permutation τ takes them into each other. By Proposition 26 (on four retracts), the n-quasigroup g is reducible. $\square$

Proposition 35. *Let f and g be reducible and compatible n-quasigroups and let*

$$f(\overline{x}) \equiv f_0\big(q_1(\widetilde{x}_1), \dots, q_m(\widetilde{x}_m)\big),$$
$$g(\overline{x}) \equiv g_0\big(q'_1(\tilde{x}'_1), \dots, q'_{m'}(\tilde{x}'_{m'})\big),$$

where f_0 and g_0 are irreducible quasigroups, $m \ge 3$, $m \ge m'$, and $\{I_j\}_{j=1,\dots,m}$ and $\{I'_j\}_{j=1,\dots,m'}$ are distinct partitions of $[n]$. Let f_0 be a $(0,1)$-semilinear m-quasigroup and let $\tau = (01)(23)$. Then $g(\overline{x}) \ne \tau f(\overline{x})$ for every $\overline{x} \in \Sigma^n$, i.e., f and g form an extendable pair of n-quasigroups.

Proof. In $[n]$, we choose m numbers i_j such that, for every $j \in [m]$, we have $i_j \in I_j$ and some sets of the form I'_j are not singletons. Without loss of generality, we may assume that $[m]$ is the required collection of numbers. Let $\overline{y} = (x_1, \dots, x_m)$ and let $\overline{z} = (x_{m+1}, \dots, x_n)$. By construction, the retract $f\big|_{\overline{z}=\overline{u}}$ is irreducible and the retract $g\big|_{\overline{z}=\overline{u}}$ is reducible for every $\overline{u} \in \Sigma^{n-m}$. By Assertion 14 (b), $\mathcal{S}_{0,1}\langle f\big|_{\overline{z}=\overline{u}}\rangle$ is a linear 2-code. By Lemma 34, we have

$$\mathcal{S}_{0,1}\langle f\big|_{\overline{z}=\overline{u}}\rangle = \Sigma^m \setminus \mathcal{S}_{0,1}\langle g\big|_{\overline{z}=\overline{u}}\rangle \ \text{ for every } \ u \in \Sigma^{n-m}.$$

We conclude that

$$\mathcal{S}_{0,1}\langle \tau f\rangle = \mathcal{S}_{0,1}\langle f\rangle = \Sigma^n \setminus \mathcal{S}_{0,1}\langle g\rangle.$$

By Assertion 2 (a), the n-quasigroups f and g form an extendable pair. $\square$

Proposition 36. *Let disjoint* MDS*-codes* $M_1 = \mathcal{M}\langle f\rangle$ *and* $M_2 = \mathcal{M}\langle g\rangle$ *admit canonical representations*

$$q_{m+1}(x_{n+1}, \widetilde{x}_{m+1}) = q_0(q_1(\widetilde{x}_1), \dots, q_m(\widetilde{x}_m)),$$
$$q'_{m'+1}(x_{n+1}, \widetilde{x}'_{m'+1}) = q'_0(q'_1(\widetilde{x}'_1), \dots, q'_{m'}(\widetilde{x}'_{m'})),$$

where $m \ge m' \ge 3$ *and the partitions* $\{I_j\}_{j=1,\dots,m+1}$ *and* $\{I'_j\}_{j=1,\dots,m'+1}$ *of* $[n]$ *are distinct. If* (IH) *holds then* f *and* g *form an extendable pair.*

Proof. By Assertion 21, each irreducible m-quasigroup is semilinear; hence, it is equivalent to a suitable $(0,1)$-semilinear m-quasigroup. Without loss of generality, we may assume that q_0 is a $(0,1)$-semilinear m-quasigroup.

Let τ denote the permutation $(01)(23)$ and let N denote the set of solutions of the equation

$$q_{m+1}(x_{n+1}, \widetilde{x}_{m+1}) = \tau q_0(q_1(\widetilde{x}_1), \dots, q_m(\widetilde{x}_m)). \tag{3.2}$$

Then $N \subset \Sigma^{n+1}$ and N is an MDS-code.

We fix the values of the variables $\widetilde{x}_{m+1}$, i.e., put $\widetilde{x}_{m+1} = \widetilde{u}$. By the definition of a canonical representation, the lengths of the tuples $\widetilde{x}_{m+1}$ and $\widetilde{x'}_{m'+1}$ are minimal. Hence, the retracts $F_{n+1}\langle M_1\rangle\big|_{\widetilde{x}_{m+1}=\widetilde{u}}$ and $F_{n+1}\langle M_2\rangle\big|_{\widetilde{x}_{m+1}=\widetilde{u}}$ satisfy the conditions of Proposition 35. We conclude that the MDS-codes $M_1\big|_{\widetilde{x}_{m+1}=\widetilde{u}}$, $M_2\big|_{\widetilde{x}_{m+1}=\widetilde{u}}$, and $N\big|_{\widetilde{x}_{m+1}=\widetilde{u}}$ are pairwise disjoint. Since the tuple $\widetilde{u}$ of values is arbitrary, from Assertion 2(a) we obtain the required assertion. □

The proof of the following proposition is similar to that of Proposition 36.

Proposition 37. *Let* f *and* g *be* n*-quasigroups. Assume that* f *is not completely reducible,* g *is completely reducible, and* (IH) *holds. If* f *and* g *are compatible then they form an extendable pair.*

Proposition 38. *Let* f *and* g *be reducible and compatible* n*-quasigroups. Assume that* $m \ge 4$ *and*

$$\begin{aligned}\mathcal{M}\langle f\rangle &= \Big\{\overline{x} \in \Sigma^{n+1} : q_1(\widetilde{x}_1) = f_0(q_2(\widetilde{x}_2), \dots, q_m(\widetilde{x}_m))\Big\},\\ \mathcal{M}\langle g\rangle &= \Big\{\overline{x} \in \Sigma^{n+1} : q'_1(\widetilde{x}_1) = g_0(q'_2(\widetilde{x}_2), \dots, q'_m(\widetilde{x}_m))\Big\},\end{aligned} \tag{3.3}$$

i.e., the canonical representations of these quasigroups correspond to the same partition of the set of variables. If (IH) *holds then* f *and* g *form an extendable pair.*

Proof. If each $\widetilde{x}_i$, $i = 1, \dots, m$, is a singleton tuple then the required assertion is immediate from Proposition 32.

Varying the quasigroups q_i in (3.3), we preserve the set $\mathcal{M}\langle f\rangle$ and transform the $(m-1)$-quasigroup f_0 into an arbitrary isotopic $(m-1)$-quasigroup. Hence, we may assume that the m-quasigroups f_0 and g_0 are $(0,1)$-semilinear, i.e., we have

$$\mathcal{M}\langle f_0\rangle, \mathcal{M}\langle g_0\rangle \subset S, \text{ where } \chi_S(\overline{y}) = 1 \oplus \bigoplus_{i=1}^{m} \chi_{\{0,1\}}(y_i).$$

Without loss of generality, we may assume that the variable x_i occurs in $\widetilde{x}_i$ for each $i = 1, \dots, m$. To simplify the arguments of the proof of the proposition, the variable x_1 plays the role of the distinguished variable x_{n+1} in the canonical decomposition.

We fix the values of the variables $x_{m+1}, \dots, x_{n+1}$. Assume that there exist $i \in [m]$ and $\overline{u} \in \Sigma^{n_i-1}$ with $q_i(z\overline{u}) = \xi q'_i(z\overline{u})$, where

$$\xi \notin \Omega = \big\{\mathrm{Id}, (0,1), (2,3), (0,1)(2,3), (0,2)(1,3), (0,3)(1,2)\big\}$$

and z may occur in every coordinate. If $q_j \ne \sigma q'_j$ for suitable $j \in [m] \setminus \{i\}$ and permutation σ then the m-quasigroups f_0 and g_0 are reducible by Proposition 28. We arrive at a contradiction.

If q_j is a 1-quasigroup for every $j \in [m]$ with $j \neq i$ then the required assertion is immediate from Proposition 31. Let $q_j = \sigma q_j'$ and let the quasigroup q_j depend essentially on at least two variables. We replace coinciding functions q_j in (3.3) by a new variable. We obtain MDS-codes of lower dimensions; hence, we may use (IH). Replacing the new variable by q_j, we obtain the required assertion.

Let

$$\chi_{\widetilde{S}}(\overline{x}) = 1 \oplus \bigoplus_{j=1}^{m} \chi_{\{0,1\}}\bigl(q_j(\widetilde{x}_j)\bigr).$$

It is clear that $\widetilde{S}$ is a 2-MDS-code and $\mathcal{M}\langle f\rangle \subset \widetilde{S}$. For every $i \in [m]$ and tuple $\overline{u} \in \Sigma^{n_i-1}$, we define a permutation ξ by the equality $q_i(z\overline{u}) = \xi q_i'(z\overline{u})$. If $\xi \in \Omega$ for all $i \in [m]$ and $\overline{u} \in \Sigma^{n_i-1}$ then, by Proposition 30, we find either

$$\chi_{\{0,1\}}\bigl(q_j(\widetilde{x}_j)\bigr) = \chi_{\{0,1\}}\bigl(q_j'(\widetilde{x}_j)\bigr) \quad \text{or} \quad \chi_{\{0,1\}}\bigl(q_j(\widetilde{x}_j)\bigr) = 1 \oplus \chi_{\{0,1\}}\bigl(q_j'(\widetilde{x}_j)\bigr).$$

We conclude that either $\mathcal{M}\langle g\rangle \subset \widetilde{S}$ or $\mathcal{M}\langle g\rangle \subset \Sigma^{n+1} \setminus \widetilde{S}$. By Proposition 29 and (IH), the 2-MDS-codes $\widetilde{S}$ and $\Sigma^{n+1} \setminus \widetilde{S}$ are splittable. By Assertion 1, we obtain the required assertion. □

Proposition 39. *Let disjoint* MDS*-codes* $M_1, M_2 \subset \Sigma^n$ *be defined by the equalities*

$$\varphi_1(x_1, x_2) = f_1(x_3, \dots, x_{n+1}) \quad \textit{and} \quad \varphi_2(x_1, x_2) = f_2(x_3, \dots, x_{n+1}),$$

where $n \geq 4$. *If* (IH) *holds then* $\Sigma^{n+1} \setminus (M_1 \cup M_2)$ *is a splittable* 2-MDS*-code.*

Proof. We check possible pairs of 2-quasigroups φ_1, φ_2. We distinguish four cases.

(1) The MDS-codes $C_a^1 = \mathcal{M}_a\langle\varphi_1\rangle$ and $C_b^2 = \mathcal{M}_b\langle\varphi_2\rangle$ are disjoint for no $(a, b) \in \Sigma^2$.

(2) There exists a permutation π such that the MDS-codes C_a^1 and C_b^2 are disjoint if and only if $b = \pi(a)$.

(3) There exists a permutation π such that the MDS-codes C_a^1 and C_b^2 are disjoint if and only if $b \neq \pi(a)$.

(4) The quasigroups φ_1 and φ_2 are opposite but not equivalent.

Case (1) is impossible because the MDS-codes M_1 and M_2 are disjoint.

In case (2), we have $f_2 = \pi f_1$ because the MDS-codes M_1 and M_2 are disjoint. We transform the equality defining M_2 so that the occurrence of the permutation π is on the left-hand side. We find that M_2 is defined by the equality $\pi\varphi_2(x_1, x_2) = f_1(x_3, \dots, x_{n+1})$. The required assertion reduces to existence of extensions for compatible 2-quasigroups.

In case (3), we have $\varphi_1 = \tau\varphi_2$ for a suitable permutation τ. The required assertion reduces to existence of extensions for compatible $(n-1)$-quasigroups.

In case (4), the required assertion follows from Proposition 31. □

Proposition 40.

(a) *Let an* MDS*-code* $M \subset \Sigma^{n+1}$ *be completely reducible. Then, for every* $i \in [n+1]$, *the retracts* $M\big|_{x_i=a}$, $a \in \Sigma$, *possess a common pair of inner variables.*

(b) *Let an* MDS*-code* $M \subset \Sigma^{n+1}$ *be reducible but not completely reducible. If, for some* $i \in [n+1]$, *the retracts* $M\big|_{x_i=a}$ *and* $M\big|_{x_i=b}$ *with* $a, b \in \Sigma$ *are completely reducible then* $M\big|_{x_i=a}$ *and* $M\big|_{x_i=b}$ *possess a common pair of inner variables.*

Proof. (a) By Proposition 9, every completely reducible MDS-code of dimension at least 4 possesses at least two pairs of inner variables. By Proposition 11, if x_i does not occur in a pair of inner variable with respect to M then this pair is inner for $M\big|_{x_i=a}$, $a \in \Sigma$.

(b) By Corollary 5, an equation of the form

$$q_1(\widetilde{x}_1) = q_0\bigl(q_2(\widetilde{x}_2), \dots, q_m(\widetilde{x}_m)\bigr)$$

defines M, where q_0 is an irreducible $(m-1)$-quasigroup and $m \geq 4$. Without loss of generality, we may assume that at least two variables occur in $\widetilde{x}_1$. Since the retract is completely reducible, the variable x_i does not occur in $\widetilde{x}_1$; moreover, the quasigroup q_1 is either completely reducible or binary. Hence, no retract with respect to a variable occurring in $\widetilde{x}_1$ is completely reducible and retracts with respect to other variables preserve pairs of inner variables with respect to q_1. □

Proposition 41. *Let completely reducible* MDS-*codes* $M_1, M_2 \subset \Sigma^{n+1}$, $n \geq 5$, *be disjoint and let*

$$M_1 = \{(\widetilde{x}_1, \widetilde{x}_2) \mid f_1(\widetilde{x}_1) = g_1(\widetilde{x}_2)\},$$
$$M_2 = \{(\widetilde{x}_1, \widetilde{x}_2) \mid f_2(\widetilde{x}_1) = g_2(\widetilde{x}_2)\},$$

where at least two variables occur in each of the tuples $\widetilde{x}_1, \widetilde{x}_2$. *If* (IH) *holds then* $\Sigma^{n+1} \setminus (M_1 \cup M_2)$ *is a splittable* 2-MDS-*code.*

Proof. If either f_1 and f_2 or g_1 and g_2 possess a common pair of inner variables then the required assertion is immediate from Proposition 39. Otherwise, at least three variables occur in each of the tuples $\widetilde{x}_1$, $\widetilde{x}_2$. Without loss of generality, we may assume that $g_1(\overline{0}x_{n+1}) = g_2(\overline{0}x_{n+1}) = x_{n+1}$. Consider the MDS-codes

$$M_1' = \{(\widetilde{x}_1, y) \mid f_1(\widetilde{x}_1) = y\}, \quad M_2' = \{(\widetilde{x}_1, y) \mid f_2(\widetilde{x}_1) = y\}.$$

We have $M_1' \cap M_2' = \varnothing$; moreover, M_1' and M_2' possess no common pair of inner variables of the form $\{x_i, x_j\}$. Without loss of generality, we may assume that no common pair of the form $\{x_i, y\}$ exists (if necessary, exchange f and g). By (IH), the quasigroups f_1 and f_2 form an extendable pair. By Proposition 40, they can be extended to an irreducible quasigroup only. By Propositions 22(c) and 31, we obtain the required assertion. □

Proposition 42. *Assume that completely reducible* n-*quasigroups* f *and* g *are compatible and, for every* $a \in \Sigma$, *the pair of retracts* $f\big|_{x_1=a}$ *and* $g\big|_{x_1=a}$ *can be extended to irreducible* n-*quasigroups only. If* (IH) *holds then* f *and* g *form an extendable pair.*

Proof. By Assertion 19 (on a linear anti-layer) and Proposition 22, the retracts $f\big|_{x_1=a}$ and $g\big|_{x_1=a}$ are semilinear and opposite to each other for every $a \in \Sigma$. At least one of $f\big|_{x_1=a}$, $g\big|_{x_1=a}$ is nonlinear, for otherwise the pair can be extended to a reducible n-quasigroup.

If the retract $f\big|_{x_1=b}$ of a completely reducible n-quasigroup f is linear then, for every $a \in \Sigma$, the retract $f\big|_{x_1=a}$ is linear too. Hence, without loss of generality, we may assume that, for every $a \in \Sigma$, the retract $f\big|_{x_1=a}$ is semilinear but not linear.

Consider the characteristics of linear 2-codes containing the retracts $\mathcal{M}\langle f\big|_{x_1=a}\rangle$, $a \in \Sigma$. By Assertion 19, opposite retracts are contained in linear 2-codes of the same characteristic. If retracts are not opposite to each other then some coordinates of their characteristics coincide, for otherwise one of these retracts is linear, see Proposition 23. Without loss of generality, we may assume that the $(n+1)$-th coordinates of the characteristics coincide, i.e., the sets $\mathcal{S}_{0,1}\langle f\big|_{x_1=a}\rangle$ are linear. Since $\mathcal{M}\langle f\big|_{x_1=a}\rangle$ is a nonlinear retract, the set $\mathcal{S}_{0,b}\langle f\big|_{x_1=a}\rangle$ is nonlinear for $b = 2, 3$ and every $a \in \Sigma$, see Assertion 15. By Proposition 22(c), we have $\mathcal{S}_{0,1}\langle f\rangle = \Sigma^n \setminus \mathcal{S}_{0,1}\langle g\rangle$ because the pairs of retracts are extended to irreducible n-quasigroups only. The required assertion follows from Proposition 27(a). □

Lemma 43. *Let completely reducible* MDS-*codes* $M_1, M_2 \subset \Sigma^{n+1}$, $n \geq 5$, *be disjoint and let* $I(M_1) \cap I(M_2) = \varnothing$. *If* (IH) *holds then* $\Sigma^{n+1} \setminus (M_1 \cup M_2)$ *is a splittable* 2-MDS-*code.*

Proof. If $I(M_1\big|_{x_i=a}) \cap I(M_2\big|_{x_i=a}) = \varnothing$ for some $i \in [n+1]$ and all $a \in \Sigma$ then $M_1\big|_{x_i=a}$ and $M_2\big|_{x_i=a}$ cannot be obtained from a reducible MDS-code by a retraction with respect to one variable, see Proposition 40. In this case, the required assertion follows from (IH) and Proposition 42. Assume the contrary. Let $(*)$ denote the corresponding assumption. We consider two possible cases:

(1) $x_t \in \widetilde{I}(M_1) \cap \widetilde{I}(M_2)$,

(2) $\widetilde{I}(M_1) \cap \widetilde{I}(M_2) = \varnothing$.

In the first case, let $M_i' = M_i\big|_{x_t=a}$ and $\{x_p, x_q\} \in I(M_1') \cap I(M_2')$. If

$$\{x_p, x_q\} \in \big(I(M_1') \setminus I(M_1)\big) \cap \big(I(M_2') \setminus I(M_2)\big)$$

(i.e., the pair $\{x_p, x_q\}$ is inner for none of M_1, M_2) then, by Proposition 13, the MDS-codes M_1 and M_2 are defined by equalities of the form $g_1(x_p, x_q, x_t) = f_1(\overline{y})$ and $g_2(x_p, x_q, x_t) = f_2(\overline{y})$. The required assertion follows from Proposition 41.

If

$$\{x_p, x_q\} \notin \big(I(M_1') \setminus I(M_1)\big) \cap \big(I(M_2') \setminus I(M_2)\big)$$

then either

$$\{x_p, x_q\} \in I(M_1) \cap \big(I(M_2') \setminus I(M_2)\big)$$

or the symmetric condition holds. By Proposition 13, we have either $\{x_t, x_q\} \in I(M_2)$ or $\{x_t, x_p\} \in I(M_2)$. Without loss of generality, we may assume that $\{x_t, x_q\} \in I(M_2)$.

Let $c \in \Sigma$ be an arbitrary element. If $x_p \in \widetilde{I}(M_2)$ then, by Proposition 12(a), we have $I\big(M_2\big|_{x_p=c}\big) \subset I(M_2)$. By Proposition 13, the variable x_q occurs in every pair of $I\big(M_1\big|_{x_p=c}\big) \setminus I(M_1)$. By $(*)$, there exists $b \in \Sigma$ such that the retracts $M_2\big|_{x_p=b}$ and $M_1\big|_{x_p=b}$ possess a common pair of inner variables. By the above, this pair is $\{x_t, x_q\}$, see Fig. 2. By Proposition 13, the MDS-codes M_1 and M_2 are defined by equalities of the form $g_1(x_p, x_q, x_t) = f_1(\overline{y})$ and $g_2(x_p, x_q, x_t) = f_2(\overline{y})$. The required assertion follows from Proposition 41.

If $x_p \notin \widetilde{I}(M_2)$ then $I\big(M_2\big|_{x_q=c}\big) \setminus I(M_2) = \big\{\{x_p, x_t\}\big\}$ in view of Proposition 13. As above, the common inner pair for $M_2\big|_{x_q=c}$ and $M_1\big|_{x_q=c}$ is $\{x_p, x_t\}$. The required assertion follows from Propositions 13 and 41.

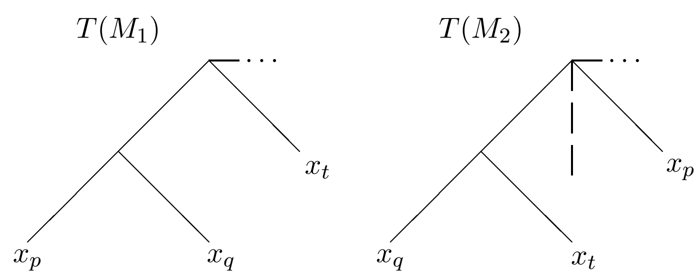

Fig. 2

We now turn to the second case. In $S = \widetilde{I}(M_1)$, we find an extreme variable x_s with respect to the tree $T(M_2)$. Let $M_i'' = M_i\big|_{x_s=b}$, $b \in \Sigma$, and let $\{x_p, x_q\} \in I(M_1'') \cap I(M_2'')$. Since x_s is an extreme variable in S, from Proposition 12(c) it follows that $\{x_p, x_q\} \notin I(M_1)$. By Proposition 13, we obtain $\{x_p, x_s\} \in I(M_1)$ $\big($or $\{x_q, x_s\} \in I(M_1)\big)$. We have $x_q \notin \widetilde{I}(M_1)$, for otherwise $x_p, x_q \in S$, which contradicts the fact that x_s is an extreme variable. Put $M_i''' = M_i\big|_{x_p=c}$. By Proposition 13, there exists a unique pair of $I(M_1''')$ containing x_s; namely, $\{x_q, x_s\}$, see Fig. 3. By Proposition 12(b), we have $\{x_q, x_s\} \notin I(M_2''') \setminus I(M_2)$ but the variable $x_s \notin \widetilde{I}(M_2)$ occurs in every pair of $I(M_2''') \setminus I(M_2)$. Hence, $I(M_2''') \cap I(M_1''') = \varnothing$. Since b is an arbitrary element of Σ, we arrive at a contradiction. $\square$

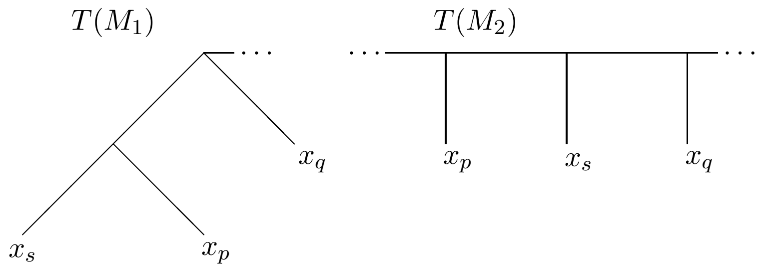

Fig. 3

We turn to the *proof of the main theorem*. We use induction on n. For $n = 1, 2, 3$, it suffices to check all possible cases, see [9]. For $n = 4$, a computer-assisted proof exists. Assume that the required assertion is valid for every natural m with $m < n$, where $n \geq 5$.

(1) Assume that f and g are compatible n-quasigroups and at least one of them admits an irreducible retract of arity at least 3.

(1a) If f and g are irreducible then is suffices to use Proposition 32.

If f admits an irreducible retract of arity m, $n > m > 2$, then, by Corollary 5, the MDS-code $\mathcal{M}(f)$ admits a canonical representation

$$q_{m+1}(x_{n+1}, \widetilde{x}_{m+1}) = q_0\big(q_1(\tilde{x}_1), \ldots, q_m(\widetilde{x}_m)\big),$$

where $\{I_j\}_{j=1,\ldots,m+1}$ is a partition of the set of variables.

Let the MDS-code $\mathcal{M}\langle g\rangle$ admit a canonical representation

$$q'_{m'+1}\big(x_{n+1}, \widetilde{x}'_{m'+1}\big) = q'_0\big(q'_1(\tilde{x}'_1), \ldots, q'_{m'}(\tilde{x}'_{m'})\big),$$

where $\{I'_j\}_{j=1,\ldots,m'+1}$ is a partition of the set of variables and $n > m' > 2$.

(1b) If the partitions $\{I_j\}_{j=1,\ldots,m+1}$ and $\{I'_j\}_{j=1,\ldots,m'+1}$ of $[n]$ are distinct then we use Proposition 36.

(1c) If the partitions $\{I_j\}_{j=1,\ldots,m+1}$ and $\{I'_j\}_{j=1,\ldots,m'+1}$ of $[n]$ coincide then we use Proposition 38.

(1d) If g is a completely reducible n-quasigroup then we use Proposition 37.

(2) Let $\mathcal{M}\langle f\rangle$ and $\mathcal{M}\langle g\rangle$ be completely reducible MDS-codes. If $I\big(\mathcal{M}\langle f\rangle\big) \cap I\big(\mathcal{M}\langle g\rangle\big) = \varnothing$ then the required assertion follows from Lemma 43; otherwise, it is immediate from Proposition 39.

This completes the proof of the theorem. $\square$

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