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Characterization of polystochastic matrices of order 4 with zero permanent

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ABSTRACT

A multidimensional nonnegative matrix is called polystochastic if the sum of its entries over each line is equal to 1. The permanent of a multidimensional matrix is the sum of products of entries over all diagonals. We prove that if d is even, then the permanent of a d-dimensional polystochastic matrix of order 4 is positive, and for odd d, we give a complete characterization of d-dimensional polystochastic matrices with zero permanent.

1. Introduction

The well-known Birkhoff theorem states that every doubly stochastic matrix has a positive permanent. In [16] Taranenko conjectured that multidimensional analogues of

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doubly stochastic matrices, called polystochastic matrices, have a positive permanent if the dimension is even or the order is odd. In the present paper we prove this conjecture for polystochastic matrices of order 4.

Under a d-dimensional matrix A of order n we mean an array $A = (a_\alpha)_{\alpha \in I_n^d}$, $a_\alpha \in \mathbb{R}$, where $I_n^d = \{\alpha = (\alpha_1, \dots, \alpha_d), \alpha_i \in \{0, \dots, n-1\}\}$ is the index set of A. A *k-dimensional plane* in a matrix A is a k-dimensional submatrix obtained by fixing $d-k$ positions of indices and letting the values in other k positions vary from 0 to $n-1$. A 1-dimensional plane of the matrix A is said to be a *line*, and $(d-1)$-dimensional planes are *hyperplanes*.

A d-dimensional matrix A of order n is a *polystochastic matrix* if $a_\alpha \geq 0$ for all α and the sum of entries over each line of A is equal to 1. 2-dimensional polystochastic matrices are known as *doubly stochastic*. If all entries of a multidimensional matrix A are either 0 or 1, then A is a *$(0,1)$-matrix*, and d-dimensional polystochastic $(0,1)$-matrices are said to be *d-dimensional permutations*.

A *diagonal* in a d-dimensional matrix A of order n is a set $\{\alpha^1, \dots, \alpha^n\}$ of n indices such that each pair α^i and α^j is distinct in all components. A diagonal $\{\alpha^1, \dots, \alpha^n\}$ is *positive* for a matrix A if $a_{\alpha^i} > 0$ for all $i \in \{1, \dots, n\}$. The *permanent* of a multidimensional matrix A is given by

$$\mathrm{per} A = \sum_{p \in D(A)} \prod_{\alpha \in p} a_\alpha,$$

where $D(A)$ is the set of all diagonals of A.

It is easy to see that the number of perfect matchings in a d-partite d-uniform hypergraph with parts of size n is equal to the permanent of the d-dimensional matrix of order n representing the adjacency of parts (see, e.g., [16]). The problem of determining the positivity of the permanent of a d-dimensional $(0,1)$-matrix of order n is NP-hard because for $d = 3$ it is equivalent to the 3-dimensional matching problem (one of Karp's 21 NP-complete problems). Certain conditions on the number of 1s in lines, sufficient for the positivity of the permanent of d-dimensional $(0,1)$-matrices, were established in [1] (in the context of hypergraphs).

The permanent of multidimensional permutations is closely related to the number of transversals in latin squares and hypercubes. A d-dimensional latin hypercube Q of order n is a multidimensional matrix filled with n symbols so that each line contains all different symbols. 2-dimensional latin hypercubes are usually called latin squares. A transversal in a latin hypercube Q is a diagonal that contains all n symbols.

There is a one-to-one correspondence between d-dimensional latin hypercubes Q of order n and $(d+1)$-dimensional permutations A of order n: an entry $q_{\alpha_1,\dots,\alpha_d}$ of Q equals α_{d+1} if and only if an entry $a_{\alpha_1,\dots,\alpha_{d+1}}$ of A equals 1. Jurkat and Ryser [4] were the first to note that the number of transversals in a latin hypercube Q coincides with the permanent of the corresponding polystochastic matrix A.

The Birkhoff theorem (see, e.g., [8]) states that every doubly stochastic matrix has a positive permanent and is a convex combination of some permutation matrices. However, for $d \geq 3$ there exist d-dimensional polystochastic matrices with zero permanent, even when the matrix is a multidimensional permutation. For example, latin squares corresponding to the Cayley table of groups $\mathbb{Z}_n$ of even order n have no transversals, a fact known since Euler [3].

This observation can be extended to latin hypercubes. Let $\mathcal{Q}_n^d$ be the d-dimensional latin hypercube of order n such that $q_\alpha \equiv \alpha_1 + \cdots + \alpha_d \mod n$. In [19], Wanless showed that if n and d are even, then the latin hypercube $\mathcal{Q}_n^d$ has no transversals. This implies that the permanent of the multidimensional permutation $\mathcal{M}_n^d$ corresponding to the latin hypercube $\mathcal{Q}_n^{d-1}$ is zero when n is even and d is odd. Later, in [2] Child and Wanless proved that modifications of the matrix $\mathcal{M}_n^d$ in r consequentive hyperplanes, where $r(r-1) < n$, also produce polystochastic matrices with zero permanent for even n and odd d.

There are no known examples of latin squares of odd order with no transversals, and in 1967, Ryser [12] conjectured that every latin square of odd order has a transversal. Despite Montgomery's recent breakthrough [9], proving that every latin square of large order n has a near transversal (a partial transversal of length $n-1$), the Ryser's conjecture is still open.

In [15] Sun proved that latin hypercubes $\mathcal{Q}_n^d$ of odd dimension d have a transversal, and so all matrices $\mathcal{M}_n^d$ of even dimension d have a positive permanent. In that paper, he also conjectured that all 4-dimensional permutations have positive permanents.

On the basis of these results and computer enumeration of latin hypercubes of small order and dimension [7], Wanless conjectured the following.

Conjecture 1 *(Wanless, [19]). Every latin hypercube of odd order or odd dimension has a transversal.*

This conjecture is trivial for latin hypercubes of order 2 and easy to prove for order 3 [16]. In [17], Taranenko showed that, except for the hypercube $\mathcal{Q}_4^d$ of even dimension d, all latin hypercubes of order 4 have a transversal, and Perezhogin, Potapov, and Vladimirov in [10] proved that all latin hypercubes of order 5 have a transversal.

In [16], Taranenko extended Conjecture 1 from multidimensional permutations to all polystochastic matrices.

Conjecture 2 *(Taranenko, [16]). The permanent of every polystochastic matrix of odd order n or even dimension d is positive.*

It is straightforward to show that all polystochastic matrices of order 2 with zero permanent are matrices $\mathcal{M}_2^d$ with odd d (see [16]). It is also proved in [16] that every polystochastic matrix of order 3 has a positive permanent. Finally, in [18], Taranenko proved that the permanent of every 4-dimensional polystochastic matrix of order 4 is positive.

In the present paper, we prove that for even d, every d-dimensional polystochastic of order 4 has a positive permanent, and for odd d, we show that d-dimensional polystochastic matrices of order 4 with zero permanent are either equivalent to $\mathcal{M}_4^d$ or matrices constructed by Child and Wanless in [2]. Thus, we confirm Conjecture 2 for polystochastic matrices of order $n = 4$. Moreover, this implies that the positivity of the permanent of a d-dimensional polystochastic matrix of order 4 can be checked in polynomial time with respect to the number n^d of its entries.

2. Definitions, preliminary results, and the proof outline

2.1. Planes, support, and classes of polystochastic matrices

Let $A = (a_\alpha)_{\alpha \in I_n^d}$ be a d-dimensional matrix of order n, where $I_n^d = \{\alpha = (\alpha_1, \dots, \alpha_d),$ $\alpha_i \in \{0, \dots n-1\}\}$ is the index set of A. Given a matrix A and $1 \leq i_1 < \cdots < i_{d-k} \leq d$, we denote by $A_{i_1,\dots,i_{d-k}}^{\alpha_1,\dots,\alpha_{d-k}}$ the k-dimensional plane of *direction* $(i_1, \dots, i_{d-k})$ obtained fixing index positions $i_1, \dots, i_{d-k}$ by values $\alpha_1, \dots, \alpha_{d-k}$, respectively. If S is a subset of indices of A, then the k-dimensional plane $S_{i_1,\dots,i_{d-k}}^{\alpha_1,\dots,\alpha_{d-k}}$ is the intersection of S and the k-dimensional plane $A_{i_1,\dots,i_{d-k}}^{\alpha_1,\dots,\alpha_{d-k}}$. In particular, a hyperplane $S_i^{\alpha_i}$ is the intersection of S and the hyperplane $A_i^{\alpha_i}$.

We will say that two k-dimensional planes are *parallel* if they have the same direction, and parallel planes $A_{i_1,\dots,i_{d-k}}^{\alpha_1,\dots,\alpha_{d-k}}$ and $A_{i_1,\dots,i_{d-k}}^{\beta_1,\dots,\beta_{d-k}}$ are *diagonally located* if $\beta_j \neq \alpha_j$ for all $j = 1, \dots, d-k$. Let $(\Gamma_0, \dots, \Gamma_{n-1})$ denote the multidimensional matrix of order n with parallel hyperplanes $\Gamma_0, \dots, \Gamma_{n-1}$.

Two matrices are called *equivalent* if they are obtained from each other by permutations of positions of indices or by permutations of hyperplanes of one direction.

In this paper, we mostly consider multidimensional matrices of order 4. Let us describe partitions of such matrices into submatrices of order 2.

Firstly, note that there are three different partitions of the set $\{0, 1, 2, 3\}$ into two unordered parts, each consisting of two elements. We label them as

$$\mathcal{P}_1 = 01|23; \quad \mathcal{P}_2 = 02|13; \quad \mathcal{P}_3 = 03|12;$$

that is in partition $\mathcal{P}_i$ elements i and 0 are in the same part.

To every partition $\mathcal{P}_i$, $i = 1, 2, 3$, we assign a function $p_i : \{0, 1, 2, 3\} \to \{0, 1\}$ so that $p_i(a)$ stands for a part of $\mathcal{P}_i$ containing a. Without loss of generality, we assume that $p_i(0) = 0$ for all partitions $\mathcal{P}_i$. Then

$$\begin{aligned} p_1(0) = p_1(1) = 0; \quad p_1(2) = p_1(3) = 1; \\ p_2(0) = p_2(2) = 0; \quad p_2(1) = p_2(3) = 1; \\ p_3(0) = p_3(3) = 0; \quad p_3(1) = p_3(2) = 1. \end{aligned}$$

Then the tuple $\mathcal{E} \in \{1, 2, 3\}^d$, $\mathcal{E} = (\varepsilon_1, \dots, \varepsilon_d)$, defines a *partition* of a d-dimensional matrix of order 4 into 2^d copies of d-dimensional *subcubes* C_y, $y \in \{0, 1\}^d$ of order 2, where $\mathcal{P}_{\varepsilon_i}$ is the partition of coordinates in the ith position. In particular, each C_y is composed of indices α such that $y_i = p_{\varepsilon_i}(\alpha_i)$ for all $i \in \{1, \dots, i\}$.

The *support* $\mathrm{supp}(A)$ of a matrix A is the set of all indices α for which $a_\alpha \neq 0$. In what follows, we will denote elements of matrices from the support by $\bullet$ and elements that do not belong to the support by $\circ$. For a nonnegative multidimensional matrix (or verctor) A, the *weight* $w(A)$ is the sum of all entries of A.

Recall that a *multidimensional permutation* is a $(0, 1)$-polystochastic matrix, i.e., each line of the matrix contains exactly one nonzero entry. If a polystochastic matrix A has entries only from the set $\{0, 1/2\}$ (i.e., each line of A contains exactly two nonzero entries), then we call it a *double permutation*. We will say that a polystochastic matrix A is a *sesquialteral permutation* if every line of A contains no more than two nonzero entries. Note that multidimensional permutations and double permutations are sesquialteral permutations.

If A is a sesquialteral permutation, then we can replace all entries a_α such that $0 < a_\alpha < 1$ by $1/2$ and obtain a polystochastic matrix B with the same support and all entries equal 0, 1 or $1/2$.

We will call a multidimensional matrix A with nonnegative entries a *stochastic matrix* if the sum of its entries over lines of every direction, except for the first one, is equal to 1. Equivalently, all hyperplanes of the first direction of A are polystochastic matrices, but the matrix A is not necessarily polystochastic.

The definition of permanent and diagonally located planes imply that for every d-dimensional matrix A of order n it holds

$$\mathrm{per}(A) = \sum_{(\alpha_1,\dots,\alpha_n)} \mathrm{per}(A^{\alpha_1}_{i_1,\dots,i_k}, \dots, A^{\alpha_n}_{i_1,\dots,i_k}),$$

where sum is taken over all pairwise diagonally located k-dimensional planes $A^{\alpha_1}_{i_1,\dots,i_k}, \dots, A^{\alpha_n}_{i_1,\dots,i_k}$ of direction $(i_1, \dots, i_k)$. The proof of this formula can also be found in [16]. It yields the following statement.

Proposition 1. *Let A be a d-dimensional polystochastic matrix of order n and $1 \leq k \leq d - 1$.*

1. *The permanent of the matrix A is positive if and only if there exist pairwise diagonally located k-dimensional planes $A^{\alpha_1}_{i_1,\dots,i_k}, \dots, A^{\alpha_n}_{i_1,\dots,i_k}$ of direction $(i_1, \dots, i_k)$ such that the permanent of the stochastic matrix $(A^{\alpha_1}_{i_1,\dots,i_k}, \dots, A^{\alpha_n}_{i_1,\dots,i_k})$ is positive.*
2. *The permanent of the matrix A is zero if and only if for all pairwise diagonally located k-dimensional planes $A^{\alpha_1}_{i_1,\dots,i_k}, \dots, A^{\alpha_n}_{i_1,\dots,i_k}$ of direction $(i_1, \dots, i_k)$ we have that the permanent of stochastic matrices $(A^{\alpha_1}_{i_1,\dots,i_k}, \dots, A^{\alpha_n}_{i_1,\dots,i_k})$ is equal to zero.*

2.2. Unitrades and bitrades

Following [6], we will say that a collection of indices U, $U \subset I_n^d$, is a *unitrade*, if every line of a d-dimensional matrix of order n contains zero or two elements from U. A unitrade U is called a *bitrade* if there is a sign function $\sigma : U \to \{\pm 1\}$ such that for every α, β from the same line it holds $\sigma(\alpha) \neq \sigma(\beta)$. In other words, if we consider a unitrade U as a vertex set of a graph, where two elements are adjacent if and only if they are from the same line, then a bitrade is a bipartite unitrade. A unitrade U is called *connected* if it is not a union smaller unitrades, i.e., the graph corresponding to the unitrade is connected.

The definitions imply that for every sesquialteral permutation A the set of indices $U(A) = \{\alpha | 0 < a_\alpha < 1\}$ is a unitrade. Note that if $U(A)$ is bitrade with a sign function σ, then there is a multidimensional permutation M such that $\mathrm{supp}(M) \subseteq \mathrm{supp}(A)$, $\mathrm{supp}(M) = \{\alpha \in \mathrm{supp}(A) : a_\alpha = 1 \text{ or } \sigma(\alpha) = 1\}$. By the induction on the size of the support of A, we get that in this case the matrix A is a convex combination of multidimensional permutations.

For the future, we need the following sufficient condition for a double permutation to be a bitrade.

Lemma 1. *Let A be a d-dimensional double permutation of order n. If there exists $k \geq 2$ such that the support of all k-dimensional and $(k+1)$-dimensional planes of A are connected bitrades, then the support U of the matrix A is a connected bitrade.*

Proof. By the definition, the support U of a double permutation A is a unitrade. It is easy to see that if all k-dimensional planes of the matrix A contain a connected unitrade, then the whole unitrade U is also connected.

So it remains to prove that U is a bitrade. It is sufficient to show that the support of every $(k+2)$-dimensional plane of A is bitrade, since then we can repeat the same reasoning till $k = d - 2$.

Let a unitrade S be the support of a $(k+2)$-dimensional plane in A. By the conditions, each hyperplane of S (a $(k+1)$-dimensional plane in A) is a connected bitrade.

Fix some hyperplane S_1^0 of the first direction in the unitrade S and consider all hyperplanes S_2^i and S_3^j in S of the second and third directions. Since S_1^0, S_2^i, and S_3^j are connected bitrades, there exist sign functions γ, σ_i, and δ_j, respectively. Since all k-dimensional planes $S_{1,2}^{0,i}$ and $S_{1,3}^{0,j}$ are also connected bitrades, we can choose sign functions σ_i and δ_j so that they coincide with γ on $S_{1,2}^{0,i}$ and $S_{1,3}^{0,j}$, respectively.

Consider k-dimensional planes $S_{2,3}^{i,j}$, $i, j \in \{0, \dots, n-1\}$. Since $k \geq 2$, we see that the $(k-1)$-plane $S_{1,2,3}^{0,i,j}$ obtained as an intersection of $S_{2,3}^{i,j}$ with the hyperplane S_1^0, is a non-empty intrade. By the choice of σ_i and δ_j, we have that sign functions σ_i, δ_j, and γ coincide on $S_{1,2,3}^{0,i,j}$. Since all $S_{2,3}^{i,j}$ are connected bitrades, we obtain that we can take either σ_i or δ_j as their sign functions.

Define a function $\mu : S \to \{\pm 1\}$ so that $\mu(\alpha) = \sigma_i(\alpha) = \delta_j(\alpha)$ if $\alpha \in S^{i,j}_{2,3}$. Let us show that μ is a sign function, and so S is a bitrade. Consider two indices $\alpha, \beta \in S$ that are different in one position. Then they both belong to some hyperplane S^i_2 or S^j_3. If $\alpha, \beta \in S^i_2$, then by the construction $\mu(\alpha) = \sigma_i(\alpha) \neq \sigma_i(\beta) = \mu(\beta)$. Similarly, if $\alpha, \beta \in S^j_3$, then $\mu(\alpha) = \delta_j(\alpha) \neq \delta_j(\beta) = \mu(\beta)$. Therefore, μ is a sign function. $\square$

2.3. Main result and proof outline

Recall that $\mathcal{M}^d_n$ is the d-dimensional permutation of order n such that $m_\alpha = 1$ if $\alpha_1 + \cdots + \alpha_d \equiv 0 \mod n$. Denote by $\mathcal{L}^d_n$ the family of d-dimensional polystochastic matrices of order n obtained as a convex combination $\lambda \mathcal{M}^d_n + (1-\lambda)M$, $0 < \lambda < 1$, where M is the matrix equivalent to $\mathcal{M}^d_n$ such that $\mathrm{supp}(M) = \{\alpha : \alpha_1 + \cdots + \alpha_{d-1} + \pi(\alpha_d) \equiv 0 \mod n\}$, π is the transposition (01).

It is not hard to prove (see [2,19]) that matrices $\mathcal{M}^d_n$ and $\mathcal{L}^d_n$ have zero permanent if d is odd and n is even. All known examples of d-dimensional polystochastic matrices of order 4 with zero permanent have odd dimension d and are equivalent to $\mathcal{M}^d_4$ or some matrix from $\mathcal{L}^d_4$.

The main result of the present paper is that $\mathcal{M}^d_4$ and matrices from the family $\mathcal{L}^d_4$ of odd dimension d are the unique (up to equivalence) polystochastic matrices of order 4 with zero permanent.

Theorem 1. *Let A be a d-dimensional polystochastic matrix of order 4 such that $\mathrm{per} A = 0$. Then d is odd and A is equivalent to $\mathcal{M}^d_4$ or some matrix from $\mathcal{L}^d_4$.*

The proof of this theorem is by consecutive narrowing the set of matrices that can have the zero permanent. Firstly, in Section 3, we show that only sesquialteral permutations can have zero permanent.

Lemma 2. *Let A be a d-dimensional polystochastic matrix of order 4. If there exists a line in A that contains at least three nonzero entries, then $\mathrm{per} A > 0$.*

Next, we prove that if a sesquialteral permutation A of order 4 has zero permanent, then the support of each 3-dimensional plane of A is equivalent to a matrix from a certain list. Thus we reduce our consideration to some class of sesquialteral permutations that we call suspicious. The similar reduction was previously used for transversals in latin hypercubes of order 5 in [10].

Using restrictions on planes of suspicious sesquialteral permutations A, in Section 4 we show that the unitrade $U(A)$ is a bitrade.

Lemma 3. *Let A be a suspicious d-dimensional sesquialteral permutation of order 4 and $U = U(A)$ be the unitrade of A. Then U is a bitrade.*

It means that if a d-dimensional polystochastic matrix A of order 4 has a zero permanent, then A is a convex combination of some multidimensional permutations with zero permanent. Such permutations were previously described in [17].

Theorem 2 *([17], Theorem 5). Let A be a d-dimensional permutation of order 4 such that* $\mathrm{per} A = 0$*. Then d is odd and A is equivalent to the matrix $\mathcal{M}_4^d$.*

Thus we obtain the following result.

Lemma 4. *Let A be a d-dimensional polystochastic of order 4 such that* $\mathrm{per} A = 0$*. Then d is odd and A is equivalent to a convex combination of matrices equivalent to $\mathcal{M}_4^d$.*

In the last section (Section 5) we show that if an odd-dimensional matrix A is equal to a convex combination of matrices equivalent to $\mathcal{M}_4^d$ and has a zero permanent, then A is equivalent to $\mathcal{M}_4^d$ or some matrix from $\mathcal{L}_4^d$.

3. Reduction to sesquialteral permutations

We start with some auxiliary results.

Proposition 2. *Let A be a d-dimensional polystochastic matrix of order 4. If there is a subcube C of order 2 such that the weight $w(C) = 0$ or $w(C) = 2^{d-1}$, then every line of A contains no more than two nonzero entries.*

Proof. Without loss of generality, we may assume that subcube C is the subcube $C_{0\cdots 0}$ for some partition $\mathcal{E}$ of A. If $w(C_{0\cdots 0}) = 0$, then all entries of $C_{0\cdots 0}$ are equal to 0. By the definition of polystochastic matrix, for every $y \in \{0,1\}^d$, $w(y) = 1$, the weight $w(C_y)$ is equal to 2^{d-1}. It implies that for every $y \in \{0,1\}^d$, $w(y) = 2$, the weight $w(C_y)$ is equal to 0. Similarly, if $w(C_{0\cdots 0}) = 2^{d-1}$, then for every $y \in \{0,1\}^d$, $w(y) = 2$, the weight $w(C_y)$ is equal to 2^{d-1}. Iterating this process, we obtain that every line contains no more than two non-zero entries located in subcubes C with $w(C) = 2^{d-1}$. □

Proposition 3. *Let A be a d-dimensional polystochastic matrix of order n. If Γ is a k-dimensional plane in A such that there exists $\alpha \in \Gamma$ for which $0 < \gamma_\alpha < 1$, then there exist a diagonally located k-dimensional plane Γ' of the same direction and $\alpha' \in \Gamma'$ such that $0 < \gamma'_{\alpha'} < 1$.*

Proof. It is sufficient to establish that for every index α such that $0 < a_\alpha < 1$ there is a diagonally located index α' with the same property. Indeed, if Γ is such that $\alpha \in \Gamma$, then we choose Γ' so that $\alpha' \in \Gamma'$.

The required statement can be easily proved by induction on d. The inductive step follows from the fact that if there is a line L in a polystochastic matrix such that $\alpha \in L$, $0 < a_\alpha < 1$, then there is another index $\beta \in L$ for which $0 < a_\beta < 1$. □

Proposition 4. *Let A be a d-dimensional polystochastic matrix of order n such that there exists a d-dimensional $(0,1)$-submatrix in A of order $n-1$. Then the matrix A is a multidimensional permutation.*

Proof. The statement can be easily proved by induction on dimension d from the definition of a polystochastic matrix. $\square$

The following claim describes 3-dimensional stochastic matrices with zero permanent in which one of the hyperplanes has a line with at least three nonzero entries.

Claim 1. *Let A and B be a pair of 2-dimensional polystochastic matrices of order 4 such that*

- *there exists a line in A with at least three nonzero entries;*
- *there exists a line in B with at least two nonzero entries;*
- *there exist permutations C_1 and C_2 such that $\operatorname{per}(A, B, C_1, C_2) = 0$;*
- *the matrix A has the minimal (by inclusion) support among all matrices with such properties.*

Then up two the equivalence there are only two types of supports of matrices A

$$(A1)\quad \begin{array}{cccc} \bullet & \circ & \circ & \circ \\ \circ & \bullet & \bullet & \bullet \\ \circ & \bullet & \circ & \bullet \\ \circ & \bullet & \bullet & \circ \end{array} \qquad (A2)\quad \begin{array}{cccc} \circ & \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ & \circ \\ \bullet & \circ & \bullet & \circ \\ \bullet & \circ & \circ & \bullet \end{array}$$

Given a matrix A with support $(A1)$, supports of the corresponding matrices B belong to List 1

List 1.

$$\begin{array}{cccc} \bullet & \circ & \circ & \circ \\ \circ & \circ & \bullet & \bullet \\ \circ & \bullet & \bullet & \circ \\ \circ & \bullet & \circ & \bullet \end{array} \quad \begin{array}{cccc} \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \bullet \\ \circ & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \bullet \end{array} \quad \begin{array}{cccc} \circ & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \bullet \\ \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \bullet \end{array} \quad \begin{array}{cccc} \bullet & \circ & \circ & \circ \\ \circ & \bullet & \bullet & \circ \\ \circ & \bullet & \bullet & \circ \\ \circ & \circ & \circ & \bullet \end{array} \quad \begin{array}{cccc} \circ & \circ & \circ & \bullet \\ \circ & \bullet & \bullet & \circ \\ \circ & \bullet & \bullet & \circ \\ \bullet & \circ & \circ & \circ \end{array}$$

and for a matrix A with support $(A2)$, supports of B belong to List 2

List 2.

$$\begin{array}{cccc} \circ & \bullet & \bullet & \circ \\ \bullet & \circ & \bullet & \circ \\ \bullet & \bullet & \circ & \circ \\ \circ & \circ & \circ & \bullet \end{array} \quad \begin{array}{cccc} \circ & \bullet & \circ & \bullet \\ \bullet & \circ & \circ & \bullet \\ \circ & \circ & \bullet & \circ \\ \bullet & \bullet & \circ & \circ \end{array} \quad \begin{array}{cccc} \circ & \circ & \bullet & \bullet \\ \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \bullet \\ \bullet & \circ & \bullet & \circ \end{array} \quad \begin{array}{cccc} \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \bullet \\ \circ & \bullet & \bullet & \circ \\ \circ & \circ & \bullet & \bullet \end{array} \quad \begin{array}{cccc} \bullet & \circ & \circ & \circ \\ \circ & \bullet & \bullet & \circ \\ \circ & \circ & \bullet & \bullet \\ \circ & \bullet & \circ & \bullet \end{array}$$

Moreover,

1. *If the support of B belongs to List 1, then the* 0*-th row and column of B contain only zeroes and ones.*
2. *If the support of A is equal to* $(A2)$ *and the support of B belongs to List 2, then for every permutations C_1, C_2 such that* $\operatorname{per}(A, B, C_1, C_2) = 0$ *we have $C_1 = C_2$.*

The claim is obtained by computer enumeration of all nonequivalent matrices A with at least three nonzero entries in some line, matrices B with at least two nonzero entries in some line, and permutations C_1 and C_2 with checking whether $\operatorname{per}(A, B, C_1, C_2) = 0$.

Now we are ready to prove Lemma 2.

Proof of Lemma 2. By the Birkhoff theorem, the permanent of every doubly stochastic matrix of order 4 is positive.

Let $d \geq 3$ and suppose that A is a d-dimensional polystochastic matrix of order 4 with zero permanent such that a line L of A contains at least three nonzero entries. Consider a 2-dimensional plane Γ such that $L \subset \Gamma$. Without loss of generality, we may assume that the plane Γ is the plane $A^{0,\ldots,0}_{3,\ldots,d}$. By Proposition 3, there exists a diagonally located 2-dimensional plane Γ' in which there are entries between 0 and 1. By Claim 1, we may assume that the support of plane Γ is equal to the matrix $(A1)$ or $(A2)$ (otherwise consider a matrix equivalent to A).

Suppose that a 2-dimensional plane $\Gamma = A^{0,\ldots,0}_{3,\ldots,d}$ has support equal the matrix $(A1)$. Then every 2-dimensional plane Γ' diagonally located to the plane Γ has only zeroes and ones in the 0-th lines of first and second directions (lines $A^{0,\alpha_3,\ldots,\alpha_d}_{2,3,\ldots,d}$ and $A^{0,\alpha_3,\ldots,\alpha_d}_{1,3,\ldots,d}$, where $\alpha_3, \ldots, \alpha_d \in \{1, 2, 3\}$). Indeed, if a plane Γ' has at least two nonzero entries in some line, then the support of Γ' belongs to List 1. But Claim 1(1) states that the 0-th row and the 0-th column of Γ' have only entries equal to zero and one.

Thus we have that in hyperplanes A^0_1 and A^0_2 of the matrix A there are $(0,1)$-submatrices of order 3 composed by entries $a_{0,\alpha_2,\ldots,\alpha_d}$ and $a_{\alpha_1,0,\alpha_3\ldots,\alpha_d}$, $\alpha_1, \alpha_2, \ldots, \alpha_d \in \{1, 2, 3\}$, respectively. By Proposition 4, the hyperplanes A^0_1 and A^0_2 are multidimensional permutations.

Consider now the 3-dimensional plane $A^{0,\ldots,0}_{4,\ldots,d}$. By the above, its 0-th hyperplane of the third direction has support equal to $(A1)$, and the 0-th rows and columns in other hyperplanes of this direction contain only zeros and ones. So the plane $A^{0,\ldots,0}_{4,\ldots,d}$ has support of the form

$$
\begin{array}{cccc}
\bullet & \circ & \circ & \circ \\
\circ & \bullet & \bullet & \bullet \\
\circ & \bullet & \circ & \bullet \\
\circ & \bullet & \bullet & \circ
\end{array}
\qquad
\begin{array}{cccc}
\circ & \circ & \circ & \bullet \\
\circ & \bullet & \bullet & \circ \\
\circ & \bullet & \bullet & \circ \\
\bullet & \circ & \circ & \circ
\end{array}
\qquad
\begin{array}{cccc}
\circ & \circ & \bullet & \circ \\
\circ & \bullet & \circ & \bullet \\
\bullet & \circ & \circ & \circ \\
\circ & \bullet & \circ & \bullet
\end{array}
\qquad
\begin{array}{cccc}
\circ & \bullet & \circ & \circ \\
\bullet & \circ & \circ & \circ \\
\circ & \circ & \bullet & \bullet \\
\circ & \circ & \bullet & \bullet
\end{array}
$$

Note that the 2-dimensional plane $A^{1,0,\dots,0}_{2,4,\dots,d}$ composed by the 1-th columns of this 3-dimensional plane $A^{0,\dots,0}_{4,\dots,d}$ has the support equivalent to matrix $(A1)$ with $(0,1)$-lines $A^{0,1,0,\dots,d}_{1,2,4,\dots,d}$ and $A^{1,3,0,\dots,0}_{2,3,4,\dots,d}$. Repeating the previous reasoning with respect to plane $A^{1,0,\dots,0}_{2,4,\dots,d}$ we conclude that hyperplanes A^0_1 and A^3_3 are multidimensional permutations. But the last condition contradicts the structure of the support of the plane $A^{0,\dots,0}_{4,\dots,d}$ since, e.g., the entry $a_{3,3,3,0,\dots,0}$ is between 0 and 1. So in this case $\mathrm{per} A > 0$.

Suppose now that all 2-dimensional planes that have at least three nonzero entries in some line have support equivalent to the matrix $(A2)$. As before, we also assume that the 2-dimensional plane $\Gamma = A^{0,\dots,0}_{3,\dots,d}$ has support equal the matrix $(A2)$. Then for every $1 \leq i_1 < \dots < i_{d-2} \leq d$ and $1 \leq j_1 < \dots < j_{d-1} \leq d$ all 2-dimensional planes $A^{0,\dots,0}_{i_1,\dots,i_{d-2}}$ have support of type $(A2)$ and 1-dimensional planes $A^{0,\dots,0}_{j_1,\dots,j_{d-1}}$ have three nonzero entries and one zero entry $a_{0,\dots,0}$.

Consider four pairwise diagonally located 2-dimensional planes $\Gamma_0 = A^{0,\dots,0}_{3,\dots,d}$, $\Gamma_1 = A^{1,\dots,1}_{3,\dots,d}$, $\Gamma_2 = A^{2,\dots,2}_{3,\dots,d}$, and $\Gamma_3 = A^{3,\dots,3}_{3,\dots,d}$. As before, by Proposition 3, there exists a 2-dimensional plane diagonally located with respect to Γ_0 that contains an entry between 0 and 1. By Claim 1, we get that the support of Γ_1 is a matrix from List 2, and planes Γ_2 and Γ_3 are the same permutations C. Moreover, we can simultaneously permute 1-th, 2-th, and 3-th rows and columns of planes $\Gamma_0, \dots, \Gamma_3$ so that the entry $a_{1,\dots,1}$ will be between 0 and 1.

Consider now pairs of diagonally located planes $\Gamma'_2 = A^{\alpha_3,\dots,\alpha_d}_{3,\dots,d}$ and $\Gamma'_3 = A^{\beta_3,\dots,\beta_4}_{3,\dots,d}$, $\alpha_i, \beta_i \in \{2,3\}$ for all $i \in \{3,\dots,d\}$, that are also diagonally located to Γ_0 and Γ_1. Claim 1 implies that Γ'_2 and Γ'_3 are the same permutation matrices. Therefore, the subcube, $M = \{m_\alpha : \alpha_i \in \{2,3\}, i \in \{1,\dots,d\}\}$, is a $(0,1)$-matrix. Moreover, the condition that the planes Γ'_2 and Γ'_3 are the same matrices implies that $m_\alpha = m_\beta$ for every α, β such that $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, and $\alpha_i \neq \beta_i$ for all $i \in \{3,\dots,d\}$.

As it was noted before, every 2-dimensional plane $A^{0,\dots,0,0,\dots,0}_{2,\dots,j-1,j+1,\dots,d}$ has support of type $(A2)$. Consider pairwise diagonally located planes $\Gamma''_1 = A^{1,\dots,1,1,\dots,1}_{2,\dots,j-1,j+1,\dots,d}$, $\Gamma''_2 = A^{\alpha_2,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_d}_{2,\dots,j-1,j+1,\dots,d}$, and $\Gamma''_3 = A^{\beta_2,\dots,\beta_{j-1},\beta_{j+1},\dots,\beta_d}_{2,\dots,j-1,j+1,\dots,d}$, where $\alpha_i, \beta_i \in \{2,3\}$. Note that plane Γ''_1 contains an entry between 0 and 1, namely $a_{1,\dots,1}$, and so Γ''_1 has a line with at least two nonzero entries. Therefore, by Claim 1, the support of the plane Γ''_1 belongs to List 2 and planes Γ''_2 and Γ''_3 are the same permutations. Repeating the previous reasoning with respect to planes Γ''_2 and Γ''_3, we deduce that in the subcube M it holds $m_\alpha = m_\beta$ for every α, β such that $\alpha_1 = \beta_1$, $\alpha_j = \beta_j$, and $\alpha_i \neq \beta_i$ for all $i \neq 1, j$.

Taking a sequence of 2-dimensional planes $A^{0,\dots,0}_{i_1,\dots,i_{d-2}}$ such that new planes intersect the previous ones at some line and repeating the same argument, we obtain that $m_\alpha = m_\beta$ for every α, β such that $\rho(\alpha, \beta) = d-2$ in the $(0,1)$-subcube M, where ρ is the Hamming distance between indices α and β (the number of different positions). Since for every α, β such that $\rho(\alpha, \beta) = 2$ there exists an index γ for which $\rho(\alpha, \gamma) = \rho(\gamma, \beta) = d-2$ (e.g., the distance between $(2,2,2,\dots 2)$ and $(2,2,3,\dots,3)$) is $d-2$, and the distance between $(2,2,3,\dots,3,3)$ and $(2,3,2,\dots 2,3)$) is also $d-2$), the previous condition imply that $m_\alpha = m_\beta$ for every α, β such that $\rho(\alpha, \beta)$ is even.

There are only two possibilities for d-dimensional $(0, 1)$-submatrices M of order 2 in a polystochastic matrix A of order 4 satisfying the property $m_\alpha = m_\beta$ for all α, β at even distance:

1. All entries of M are equal to 0. Then the weight of subcube M is zero, and, by Proposition 2, every line of A contains no more than two non-zero entries: a contradiction.

2. $m_\alpha = 1$ if $w(\alpha)$ is even and $m_\alpha = 0$ if $w(\alpha)$ is odd (or vice versa). Then the weight of subcube M is equal to 2^{d-1}. Using again Proposition 2 we conclude that every line of A contains no more than two non-zero entries: a contradiction.

Therefore, there are no polystochastic matrices of order 4 with at least three nonzero entries in some line that have a zero permanent. $\square$

4. Suspicious sesquialteral permutations

Firstly, we describe types of all 3-dimensional planes of order 4 that can appear in sesquialteral permutations of order 4 with zero permanent.

Claim 2. *Let A be a 4-dimensional stochastic matrix of order 4 such that each hyperplane of the first direction in A is a sesquialteral permutation. If $\mathrm{per}(A) = 0$, then the support of every hyperplane of the first direction in A is equivalent to one of the matrices from the following list*

$$(a)\quad \begin{matrix} \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \circ \\ \circ & \circ & \circ & \bullet \end{matrix} \qquad \begin{matrix} \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet \\ \circ & \circ & \bullet & \circ \end{matrix} \qquad \begin{matrix} \circ & \circ & \bullet & \circ \\ \circ & \circ & \circ & \bullet \\ \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ \end{matrix} \qquad \begin{matrix} \circ & \circ & \circ & \bullet \\ \circ & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \circ \end{matrix}$$

$$(b)\quad \begin{matrix} \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \circ \\ \circ & \circ & \circ & \bullet \end{matrix} \qquad \begin{matrix} \circ & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \circ \\ \circ & \circ & \circ & \bullet \\ \bullet & \circ & \circ & \circ \end{matrix} \qquad \begin{matrix} \circ & \circ & \bullet & \circ \\ \circ & \circ & \circ & \bullet \\ \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ \end{matrix} \qquad \begin{matrix} \circ & \circ & \circ & \bullet \\ \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \circ \end{matrix}$$

$$(c)\quad \begin{matrix} \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \circ \\ \circ & \circ & \circ & \bullet \end{matrix} \qquad \begin{matrix} \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet \\ \circ & \circ & \bullet & \circ \end{matrix} \qquad \begin{matrix} \circ & \circ & \bullet & \circ \\ \circ & \circ & \circ & \bullet \\ \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \circ & \circ \end{matrix} \qquad \begin{matrix} \circ & \circ & \circ & \bullet \\ \circ & \circ & \bullet & \circ \\ \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \circ & \circ \end{matrix}$$

$$(d)\quad \begin{matrix} \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \circ \\ \circ & \circ & \circ & \bullet \end{matrix} \qquad \begin{matrix} \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet \\ \circ & \circ & \bullet & \circ \end{matrix} \qquad \begin{matrix} \circ & \circ & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \circ & \circ \end{matrix} \qquad \begin{matrix} \circ & \circ & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \circ & \circ \end{matrix}$$

$$(e)\quad \begin{matrix} \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \end{matrix} \qquad \begin{matrix} \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \circ \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \end{matrix} \qquad \begin{matrix} \circ & \circ & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \circ & \circ \end{matrix} \qquad \begin{matrix} \circ & \circ & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \circ & \circ \end{matrix}$$

$$(f)\quad \begin{array}{cccc} \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \end{array} \qquad \begin{array}{cccc} \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \end{array} \qquad \begin{array}{cccc} \circ & \circ & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \circ & \circ \end{array} \qquad \begin{array}{cccc} \circ & \circ & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \circ & \circ \end{array}$$

$$(g)\quad \begin{array}{cccc} \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \circ \\ \circ & \circ & \circ & \bullet \end{array} \qquad \begin{array}{cccc} \circ & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \circ \\ \circ & \circ & \circ & \bullet \\ \bullet & \circ & \circ & \circ \end{array} \qquad \begin{array}{cccc} \circ & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \bullet \\ \bullet & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \circ \end{array} \qquad \begin{array}{cccc} \circ & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \bullet \\ \bullet & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \circ \end{array}$$

$$(h)\quad \begin{array}{cccc} \bullet & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \circ \\ \circ & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \bullet \end{array} \qquad \begin{array}{cccc} \circ & \bullet & \bullet & \circ \\ \circ & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \bullet \\ \bullet & \bullet & \circ & \circ \end{array} \qquad \begin{array}{cccc} \circ & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \bullet \\ \bullet & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \circ \end{array} \qquad \begin{array}{cccc} \bullet & \circ & \circ & \bullet \\ \bullet & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \circ \\ \circ & \circ & \bullet & \bullet \end{array}$$

Claim 2 is verified by computer calculation. Firstly, we enumerate all supports of 3-dimensional sesquialteral permutations of order 4 and find that they constitute 44 equivalence classes. Note that these supports are exactly the orthogonal arrays $OA(32, 3, 4, 2)$ that were previously enumerated in [13] (see case $OA(32; 2; 4^3)$). Next we choose three (possibly the same) supports of 3-dimensional sesquialteral permutations of order 4, take matrices equivalent to them as supports of three hyperplanes of the first direction in a 4-dimensional stochastic matrix A of order 4, and consider the maximal support in the last hyperplane Γ of A so that $\mathrm{per} A = 0$. To verify that there is a 3-dimensional polystochastic matrix B of order 4 such that $\mathrm{supp}(B) \subseteq \mathrm{supp}(\Gamma)$, it is sufficient to consider only sesquialteral permutations that cannot be expressed as a convex combination of 3-dimensional permutations of order 4.

Let us give more details on matrices from the list of Claim 2. Matrices (a) and (b) are supports of multidimensional permutations corresponding to the Cayley tables of groups $\mathbb{Z}_2^2$ and $\mathbb{Z}_4$ (the matrix $\mathcal{M}_4^3$), respectively. Matrices (f) and (h) are the supports of double permutations. Moreover, matrices (c)–(f) are obtained as a union of supports of two multidimensional permutations of type (a), and supports of matrices (g) and (h) are a union of supports of two permutations corresponding to a matrix type (b).

By Proposition 1, if the permanent of a d-dimensional sesquialteral permutation A of order 4 is zero, then the support of every 3-dimensional plane of A is equivalent to a matrix from the list of Claim 2. So we call sesquialteral permutations with 3-dimensional planes satisfying this property *suspicious*.

Following [5], we introduce the several operations for index subsets and multidimensional matrices.

Let $S \subseteq I_4^d$ be a set of indices. For $i \in \{1, \ldots, d\}$, let $\mathcal{T}_i(S)$ denote the union of all lines of direction i that intersect the set S. We also define the *complement of the set S in direction i* as $\backslash_i S = \mathcal{T}_i(S) \backslash S$. Note that if U is the support of a unitrade of order 4, then for every $i \in \{1, \ldots, d\}$ the set $\backslash_i U$ is also the support of a unitrade.

Given a d-dimensional unitrade U of order 4 and $y \in \{0, 1\}^d$, denote $\backslash_y U = \backslash_{j_1} \cdots \backslash_{j_k} U$, where $j_1, \ldots, j_k$ are all indices such that $y_{j_1} = \cdots = y_{j_1} = 1$. Define $E(U) =$

$\bigcup\limits_{w(y)\equiv 0 \mod 2} \setminus_y U$ to be the union of all complements of U in an even number of directions. We will say that $E(U)$ is the *even completion* of U. The definition implies that $U \subseteq E(U)$. In [5, Proposition 3.7(d)] it is proved that the even completion of every unitrade is a double permutation.

Let A be a d-dimensional double permutation of order 4. We will say that directions i and i' are *equivalent* for A if there exists a connected unitrade $S \subseteq \mathrm{supp}(A)$ such that $\setminus_i S = \setminus_{i'} S$. In [5] it is shown that this condition does define an equivalence relation $\overset{A}{\sim}$ which does not depend on the choice of S and divides the set $\{1, \dots, d\}$ into equivalence classes K_j.

At last, given a d_1-dimensional $(0,1)$-matrix $A = (a_\alpha)_{\alpha \in I_n^{d_1}}$ order n and d_2-dimensional $(0,1)$-matrix $B = (b_\beta)_{\beta \in I_n^{d_2}}$ order n, the *direct sum* $A \oplus B$ is a $(d_1 + d_2)$-dimensional matrix C of order n with entries $c_{\alpha,\beta} = a_\alpha \oplus b_\beta$, where $1 \oplus 0 = 0 \oplus 1 = 1$ and $0 \oplus 0 = 1 \oplus 1 = 0$. It is easy to see that if A and B are double permutations, then $A \oplus B$ is a double permutation.

For the future, we need the following auxiliary result.

Proposition 5 *([5], Theorem 4-1). Let A be a d-dimensional double permutation of order 4. Then there exist d_j-dimensional connected double permutations B_j, $j = 1, \dots, k$ $\sum\limits_{j=1}^{k} d_j = d$, such that $A = \bigoplus\limits_{j=1}^{k} B_j$ and the coordinate set of each B_j is the equivalence class for the relation $\overset{A}{\sim}$. Moreover, the support of the matrix A is a bitrade if and only if the support of every B_j is a bitrade.*

Since $U \subseteq E(U)$, to prove Lemma 3 it is sufficient to establish the following result.

Lemma 5. *Let A be a suspicious d-dimensional sesquialteral permutation of order 4, $U = U(A)$ be the unitrade of A, and $E(U)$ be the double permutation equal to the even completion of U. Then $E(U)$ is a bitrade.*

Proof. By Proposition 5, there exist d_j-dimensional connected double permutations B_j, $j = 1, \dots, k$, $\sum\limits_{j=1}^{k} d_j = d$, such that the double permutation $E(U) = \bigoplus\limits_{j=1}^{k} B_j$, where the position set of each B_j is the equivalence class of $E(U)$ under the relation $\overset{E(U)}{\sim}$.

We need to show that all B_j are bitrades. For this purpose, we describe the equivalence relation $\overset{E(U)}{\sim}$ and take a closer look at planes of $E(U)$.

Firstly, note that each 3-dimensional plane of the even completion $E(U)$ of U is equivalent to a matrix (f), (h) or (i), where

$$(i) \quad \begin{array}{cccc} \bullet & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \circ \\ \circ & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \bullet \end{array} \qquad \begin{array}{cccc} \bullet & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \circ \\ \circ & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \bullet \end{array} \qquad \begin{array}{cccc} \circ & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \bullet \\ \bullet & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \circ \end{array} \qquad \begin{array}{cccc} \circ & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \bullet \\ \bullet & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \circ \end{array}.$$

To prove this fact, it is sufficient to check that the even completion of the unitrade U for every 3-dimensional sesquialteral permutation with the support from the list of Claim 2 is equivalent to a matrix (f), (h) or (i).

Then every 2-dimensional plane of $E(U)$ is equivalent to

$$(F)\quad \begin{array}{cccc} \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \end{array} \qquad \text{or} \qquad (H)\quad \begin{array}{cccc} \bullet & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \circ \\ \circ & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \bullet \end{array}$$

Moreover, if some 2-dimensional plane of $E(U)$ has type (F) (or type (H)), then all 2-dimensional planes of the same direction have type (F) (type (H)), because it is true for matrices (f), (h), and (i).

We present the types of 2-dimensional planes of the d-dimensional double permutation $E(U)$ as an edge coloring φ of the complete graph K_d, where $\varphi(i, j) = F$ if all 2-dimensional planes of direction (i, j) in $E(U)$ have type (F) and $\varphi(i, j) = H$ if all such planes have type (H).

Note that the matrix (f) corresponds to a coloring of K_3 whose all edges have color F, the matrix (h) corresponds to a coloring of edges of K_3 in color H, and the matrix (i) is a coloring of two edges of K_3 in color F, and the last edge has color H. So the condition on the 3-dimensional planes of $E(U)$ implies that the coloring φ has no subgraphs K_3 in which one edge has color F and two edges are of color H. It means that if two edges of color H are adjacent, then the third edge completing them to K_3 also has color H. Therefore, edges of color H in φ compose cliques (complete subgraphs) connected by edges of color F.

Note that if for some i and for all j an edge (i, j) has type (F) in φ, then the equivalence class containing i has size 1 and B_i is a 1-dimensional bitrade.

We show that directions $i \overset{E(U)}{\sim} j$ if the edge (i, j) in φ has type (H) and i is not equivalent to j if the edge (i, j) has type (F). Indeed, consider a connected component S of $E(U)$ and assume that all 2-dimensional planes of $E(U)$ in direction (i, j) have type (F). Then the component S has all 2-dimensional planes equivalent to the matrix

$$\begin{array}{cccc} \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \end{array}$$

and it is easy to see that $\setminus_i S \neq \setminus_{i'} S$.

Let us now show that all connected double permutations B_j are bitrades. If B_j is a 1-dimensional double permutation, then, by the definition, it is a bitrade. As we proved above, if the dimension d_j of B_j is at least 2, then every 2-dimensional plane of B_j has type (H). If $d_j \geq 3$, then the support of every 3-dimensional plane of B_j is equivalent to the matrix (h), so it is also a connected bitrade. Thus Lemma 1 implies that all B_j are bitrades.

To prove that $E(U)$ is a bitrade, it remains to apply Proposition 5. □

Remark. Claim 2 is used only to obtain that for a given suspicious sesquialteral permutation of order 4 all 2-dimensional planes of the same direction of the even completion $E(U)$ of its unitrade U have the same type. The independent proof of this fact will make unnecessary the computation of all possible 3-dimensional planes in Claim 2.

5. Convex combination of linear multidimensional permutations and proof of the main result

To characterize, when a convex combination of permutations $\mathcal{M}_4^d$ has a zero permanent, we consider a more general class of permutations, which we call block permutations. Block permutations are equivalent to latin hypercubes corresponding to semilinear quasigroups. To define and handle this class, we introduce addition notation as a modification of the notation from [17] for multidimensional permutations.

Recall that a tuple $\mathcal{E} \in \{1,2,3\}^d$, $\mathcal{E} = (\varepsilon_1, \dots, \varepsilon_d)$, is a partition of a d-dimensional matrix of order 4 into 2^d copies of d-dimensional subcubes C_y, $y \in \{0,1\}^d$ of order 2, where $\mathcal{P}_{\varepsilon_i}$ is the partition of coordinates in the ith position.

To every partition $\mathcal{P}_i$, $i = 1,2,3$, of $\{0,1,2,3\}$ we associate a parity function μ_i : $\{0,1,2,3\} \to \{0,1\}$ so that $\mu_i(0) = 0$ for all i and in each part of $\mathcal{P}_i$ the function μ takes both values. To be precise, let

$$\mu_1(0) = \mu_1(2) = 0; \quad \mu_1(1) = \mu_1(3) = 1;$$
$$\mu_2(0) = \mu_2(1) = 0; \quad \mu_2(2) = \mu_2(3) = 1;$$
$$\mu_3(0) = \mu_3(2) = 0; \quad \mu_3(1) = \mu_3(3) = 1.$$

Next, let Q_0^d be the set of Boolean vectors $y \in \{0,1\}^d$ such that the weight of y is even, and Q_1^d be the set of all Boolean vectors from $\{0,1\}^d$ with odd weights, $Q_0^d \cup Q_1^d = \{0,1\}^d$.

We will say that A is a d-dimensional *block permutation* of order 4 *with parameters* $(\mathcal{E}, \lambda, s)$, where $\mathcal{E} \in \{1,2,3\}^d$, $\mathcal{E} = (\varepsilon_1, \dots, \varepsilon_d)$, $s \in \{0,1\}$, and $\lambda : Q_s^d \to \{0,1\}$ is a Boolean function, if

$$a_\alpha = 1 \Leftrightarrow$$
$$p_{\varepsilon_1}(\alpha_1) \oplus \cdots \oplus p_{\varepsilon_d}(\alpha_d) = s \text{ and}$$
$$\mu_{\varepsilon_1}(\alpha_1) \oplus \cdots \oplus \mu_{\varepsilon_d}(\alpha_d) \oplus \lambda(p_{\varepsilon_1}(\alpha_1), \dots, p_{\varepsilon_d}(\alpha_d)) = 0.$$

Informally speaking, firstly a tuple $\mathcal{E}$ defines a partition of a d-dimensional matrix of order 4 into 2^d copies of d-dimensional subcubes C_y of order 2, $y \in \{0,1\}^d$. Next, the value s specifies which subcubes C_y contain a d-dimensional permutation of order 2: if

$s = 0$, then the subcube C_y is a permutation for every y, $w(y)$ is even, and if $s = 1$, then the same holds for y of odd weights. We will say that subcubes C_y containing a permutation are *filled*, and subcubes C_y equal to the zero matrix are *empty*. At last, the Boolean function λ is responsible for which one of two d-dimensional permutations is chosen in a subcube C_y.

It can be checked that this construction of block permutations does give multidimensional permutations. Also, if for a block permutation A a partition $\mathcal{E}$ is fixed, then s and the Boolean function λ are unique. Meanwhile, there are multidimensional permutations that have more than one presentation in a block form with different partitions $\mathcal{E}$ (their characterization is given in [11]).

Let us show that the permutation $\mathcal{M}_4^d$ can be presented as a block permutation, and, moreover, it has unique parameters.

Proposition 6. *A d-dimensional permutation $\mathcal{M}_4^d$ is a block permutation with the unique parameters $((2,\dots,2),\lambda_M,0)$, where $\lambda_M : Q_0^d \to \{0,1\}$ is such that $\lambda_M(x) = 0$ if $w(x) \equiv 0 \mod 4$ and $\lambda_M(x) = 1$ if $w(x) \equiv 2 \mod 4$.*

Proof. Note that if $\mathcal{M}_4^d$ is a block permutation with the unique parameters $((2,\dots,2),$ $\lambda_M, 0)$, then every matrix equivalent to $\mathcal{M}_4^d$ is a block permutation for some unique parameters $(\mathcal{E}', \lambda', s')$.

We prove the statement by induction on d. In the base case, the support of the matrix $\mathcal{M}_4^3$ is

$$\begin{array}{cccc} \bullet & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet \\ \circ & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \circ \end{array} \qquad \begin{array}{cccc} \circ & \circ & \circ & \bullet \\ \circ & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \circ \end{array} \qquad \begin{array}{cccc} \circ & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet \end{array} \qquad \begin{array}{cccc} \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet \\ \circ & \circ & \bullet & \circ \end{array}$$

It can be verified directly that $\mathcal{M}_4^3$ is a block permutation with parameters $((2,2,2),\lambda_M,$ $0)$ and these parameters are unique.

From the definition of $\mathcal{M}_4^d$ and the block permutations, it follows that parameters $((2,\dots,2),\lambda_M,0)$ do define the permutation $\mathcal{M}_4^d$. Since all parallel hyperplanes of a fixed direction are equivalent to $\mathcal{M}_4^{d-1}$, from the inductive assumption we get that the obtained parameters are unique. □

Let us show that a certain small modification of the permutation $\mathcal{M}_4^d$ results in a matrix with a positive permanent.

Lemma 6. *Let A be a d-dimensional block permutation equivalent to $\mathcal{M}_4^d$ and C be some filled subcube of the matrix A. Then a $(0,1)$-matrix B obtained from A by adding one new entry to the support of the subcube C has a positive permanent.*

Proof. By Proposition 6, it is sufficient to prove the statement for the matrix $\mathcal{M}_4^d$. Moreover, by Theorem 2, we may assume that d is odd.

By the definition and Proposition 6, every filled subcube C of the matrix $\mathcal{M}_4^d$ is composed by indices α such that $\alpha_1 + \cdots + \alpha_d \equiv 0$ or $2 \mod 4$. Since for every α from the support of $\mathcal{M}_4^d$ it holds $\alpha_1 + \cdots + \alpha_d \equiv 0 \mod 4$, for each new entry α in the support we have that $\alpha_1 + \cdots + \alpha_d \equiv 2 \mod 4$.

We prove that there is a positive diagonal in $\mathcal{M}_4^d$ containing any index α with the last property by induction on d. The base cases are $d = 3$ and $d = 5$.

Up to permutations of positions, there are 5 possible choices of α in a 3-dimensional case:

$$(0, 0, 2), \quad (0, 1, 1), \quad (0, 3, 3), \quad (1, 2, 3), \quad (2, 2, 2).$$

We can choose the remaining three elements from the support $\mathcal{M}_4^3$ giving a positive diagonal, for example, as follows:

$(0,0,2)$	$(0,1,1)$	$(0,3,3)$	$(1,2,3)$	$(2,2,2)$
$(1,2,1)$	$(1,0,3)$	$(1,1,2)$	$(0,3,1)$	$(0,1,3)$
$(2,3,3)$	$(2,2,0)$	$(2,2,0)$	$(2,0,2)$	$(1,3,0)$
$(3,1,0)$	$(3,3,2)$	$(3,0,1)$	$(3,1,0)$	$(3,0,1)$

For $d = 5$, there are 2 possible indices α (up to permutations of positions) that cannot be reduced to the 3-dimensional case by deleting a pair of components that sum to 0 modulo 4:

$$(1, 1, 1, 1, 2), \quad (2, 3, 3, 3, 3).$$

The remaining three elements from the support $\mathcal{M}_4^3$ completing them to a positive diagonal, for example, are the following:

$(1,1,1,1,2)$	$(2,3,3,3,3)$
$(0,0,2,2,0)$	$(0,0,1,1,2)$
$(2,2,3,0,1)$	$(1,1,0,2,0)$
$(3,3,0,3,3)$	$(3,2,2,0,1)$

It can be verified directly that for every index $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $d \geq 7$ and such that $\alpha_1 + \cdots + \alpha_d \equiv 2 \mod 4$ there are k components $i_1, \ldots, i_k$, $k \in \{2, 4\}$, for which $\alpha_{i_1} + \cdots + \alpha_{i_k} \equiv 0 \mod 4$. Without loss of generality, we assume that the first k components of α satisfy this equality.

By the inductive assumption, for the index $(\alpha_{k+1}, \ldots, \alpha_d)$ there are indices $(\alpha_{k+1}^1, \ldots, \alpha_d^1)$, $(\alpha_{k+1}^2, \ldots, \alpha_d^2)$, and $(\alpha_{k+1}^3, \ldots, \alpha_d^3)$ in $\mathcal{M}_4^{d-k}$ completing it to a positive diagonal. Also by Theorem 2, for the index $(\alpha_1, \ldots, \alpha_k)$ in $\mathcal{M}_4^k$ we can find indices

$(\alpha_1^1, \ldots, \alpha_k^1)$, $(\alpha_1^2, \ldots, \alpha_k^2)$, and $(\alpha_1^3, \ldots, \alpha_k^3)$ completing it to a positive diagonal. So the indices $(\alpha_1^1, \ldots, \alpha_d^1)$, $(\alpha_1^2, \ldots, \alpha_d^2)$, and $(\alpha_1^3, \ldots, \alpha_d^3)$ complete $\alpha = (\alpha_1, \ldots, \alpha_d)$ to a positive diagonal. $\square$

In view of this lemma, if there are two matrices equivalent to $\mathcal{M}_4^d$ such that the support of one matrix adds some new entries to a filled subcube of another matrix, then the convex combination of these matrices has a positive permanent. So we take a closer look at the intersections of filled subcubes.

Proposition 7. *Let A be a d-dimensional block permutation matrix equivalent to $\mathcal{M}_4^d$ with parameters $(\mathcal{E}, \lambda, s)$, $\mathcal{E} = (\varepsilon_1, \ldots, \varepsilon_d)$, such that exactly k elements of $\mathcal{E}$ are equal to 2.*

1. *If $0 \leq k < d$, then every filled subcube of $\mathcal{M}_4^d$ intersects 2^{d-k-1} filled subcubes of A by k-dimensional submatrices of order 2.*
2. *If $k = d$ and $s = 0$, then filled subcubes of A and $\mathcal{M}_4^d$ coincide, and if $k = d$ and $s = 1$, then the filled subcubes of A and $\mathcal{M}_4^d$ do not intersect.*

Proof. Recall that, by Proposition 6, the matrix $\mathcal{M}_4^d$ is a block permutation with parameters $((2, \ldots, 2), \lambda_M, 0)$. By the definition and Proposition 6, filled subcubes of $\mathcal{M}_4^d$ are matrices

$$C_y = \{(\alpha_1, \ldots, \alpha_d) : \alpha_i \in \{0, 2\} \text{ if } y_i = 0, \alpha_i \in \{1, 3\} \text{ if } y_i = 1\},\ w(y) \text{ is even.}$$

Also the definition of the block permutation implies that if $\mathcal{E} = (2, \ldots, 2)$ and $s = 0$, then the filled subcubes of A are the same as subcubes of $\mathcal{M}_4^d$, and if $\mathcal{E} = (2, \ldots, 2)$ and $s = 1$, then the filled subcubes of A are

$$C'_y = \{(\alpha_1, \ldots, \alpha_d) : \alpha_i \in \{0, 2\} \text{ if } y_i = 0, \alpha_i \in \{1, 3\} \text{ if } y_i = 1\},\ w(y) \text{ is odd,}$$

and so they do not intersect.

Suppose now that exactly k elements of $\mathcal{E}$ are equal to 2. Without loss of generality, we may assume that $\varepsilon_i = 2$ for $i \in \{1, \ldots, k\}$, $\varepsilon_i = 1$ for $i \in \{k+1, \ldots, \ell\}$, and $\varepsilon_i = 3$ for $i \in \{\ell + 1, \ldots, d\}$. We also suppose that $s = 0$ (for $s = 1$ the reasoning is similar).

Then the filled blocks C'_y of A, where $y \in Q_0^d$, are composed of indices $(\alpha_1, \ldots, \alpha_d)$ such that

- for $i \in \{1, \ldots, k\}$ components $\alpha_i \in \{0, 2\}$ if $y_i = 0$, and $\alpha_i \in \{1, 3\}$ if $y_i = 1$;
- for $i \in \{k + 1, \ldots, \ell\}$ components $\alpha_i \in \{0, 1\}$ if $y_i = 0$, and $\alpha_i \in \{2, 3\}$ if $y_i = 1$;
- for $i \in \{\ell + 1, \ldots, d\}$ components $\alpha_i \in \{0, 3\}$ if $y_i = 0$, and $\alpha_i \in \{1, 2\}$ if $y_i = 1$.

Without loss of generality, we study the maximal intersection of filled subcubes of A with the filled subcube $C_{0\cdots0}$ of $\mathcal{M}_4^d$. In our assumptions, it is attained on filled subcubes C'_y of A with $y_i = 0$ for all $i \in \{1, \ldots, k\}$ and consists of indices $(\alpha_1, \ldots, \alpha_d)$

such that $\alpha_i \in \{0, 2\}$ if $i \in \{1, \dots, k\}$, $\alpha_i = 0$ if $i \in \{k+1, \dots, d\}$ and $y_i = 0$ and $\alpha_i = 2$ if $i \in \{k+1, \dots, d\}$ and $y_i = 1$. It remains to note that this set is exactly the k-dimensional submatrix of order 2, and the number of such filled subcubes C'_y is 2^{d-k-1}. $\square$

Given a matrix A obtained as a convex combination of matrices B and B' equivalent to $\mathcal{M}_4^d$, let the *tesselation index* of A be the dimension of the maximal intersection of filled subcubes of B and B'. If all filled subcubes of B and B' do not intersect, we will say that the tesselation index is $-\infty$. Due to Propositions 6 and 7, the tesselation index is well defined.

Proposition 8. *Let a d-dimensional polystochastic A be a convex combination of matrices B and B' equivalent to $\mathcal{M}_4^d$. If the tessellation index of A is equal to $-\infty$, then the permanent of A is positive.*

Proof. If d is even, then the permanent of A is positive because, by Theorem 2, the permanents of matrices B and B' are positive.

Let us prove the statement for odd d. By Proposition 7, we may assume that matrices B and B' are block permutations with parameters $(\mathcal{E}, \lambda, 0)$ and $(\mathcal{E}, \lambda', 1)$, respectively. Note that the permanent of A does not depend on the partition $\mathcal{E}$ and is determined by only the Boolean functions λ and λ'.

By the definition of the tesselation index, filled subcubes of matrices B and B' do not intersect. So we may assume that the matrix A is decomposed into subcubes C_y filled by d-dimensional permutations of order 2, where the choice of a permutation is defined by the Boolean function Λ, $\Lambda(y) = \lambda(y)$ if $w(y)$ is even and $\Lambda(y) = \lambda'(y)$ if $w(y)$ is odd. It means that every line of A contains exactly two nonzero entries, and so every 2-dimensional plane of A is a doubly stochastic matrix with exactly two nonzero entries in each line. Since d is odd, for every diagonally located indices α and α' in a subcube C_y, exactly one of them belongs to the support of A.

Consider the pairwise diagonally located 2-dimensional planes $\Gamma_0 = A^{0,\dots,0}_{3,\dots,d}$, $\Gamma_1 = A^{1,\dots,1}_{3,\dots,d}$, $\Gamma_2 = A^{2,\dots,2}_{3,\dots,d}$, and $\Gamma_3 = A^{3,\dots,3}_{3,\dots,d}$ of A. There exists a pair of indices i, i', $i \neq i'$, such that $\mathrm{supp}(\Gamma_i) = I_4^2 \setminus \mathrm{supp}(\Gamma_{i'})$ because the same holds for subcubes C_y intersecting these 2-dimensional planes. Moreover, by the same reason, for j and j', $\{i, j, i', j'\} = \{0, 1, 2, 3\}$, we have $\mathrm{supp}(\Gamma_j) = I_4^2 \setminus \mathrm{supp}(\Gamma_{j'})$.

Therefore, in the 3-dimensional stochastic matrix $R = (\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3)$ every line contains exactly 2 nonzero entries, and the support of R is a support of some 3-dimensional double permutation of order 4. The exhaustive enumeration of all 3-dimensional double permutations of order 4 (see, for example, [14]) gives that all such matrices have positive permanent. By Proposition 1, the permanent of the matrix A is also positive. $\square$

Note that a matrix from class $\mathcal{L}_4^d$ has a tesselation index $d - 1$ and it is a convex combination of two matrices such that each matrix adds no new entries to the support of filled subcubes of another because otherwise, by Lemma 6, $\mathcal{L}_4^d$ should have a positive permanent. We show that matrices from $\mathcal{L}_4^d$ are unique matrices with such property.

Proposition 9. *Let A be a convex combination of matrices B and B' equivalent to $\mathcal{M}_4^d$. If the tesselation index of A is equal to $d-1$ and A is such that no new entry is added to the support of filled subcubes of B and B', then the matrix A is equivalent to a matrix from $\mathcal{L}_4^d$.*

Proof. By Proposition 7, we may assume that $B = \mathcal{M}_4^d$ and B' is a block permutation with parameters $(\mathcal{E}', \lambda', s)$, where exactly $d-1$ elements of $\mathcal{E}'$ are equal to 2. Without loss of generality, let $\mathcal{E}' = (2, \dots, 2, 3)$.

Proposition 7 also implies that every filled subcube of $\mathcal{M}_4^d$ intersects exactly one filled subcube of B' by a $(d-1)$-dimensional submatrix of order 2 and vice versa. So there is a bijection φ between filled subcubes C_y and C'_y of $\mathcal{M}_4^d$ and B'. Since there are no new entries in the supports of filled subcubes of $\mathcal{M}_4^d$ and B', the value $\lambda_M(y)$ for a filled subcube C_y uniquely defines by the value $\lambda'(\varphi(y))$ in the filled subcube $C'_{\varphi(y)}$. Thus the function λ' is unique for this partition $\mathcal{E}'$, and the matrix A coincides with some matrix $\mathcal{L}_4^d$. $\square$

From the definition of the class $\mathcal{L}_4^d$, we see that for every $A \in \mathcal{L}_4^d$, each hyperplane of direction $j \neq d$ is equivalent to some matrix from $\mathcal{L}_4^{d-1}$, and two of four hyperplanes of A of direction d are equivalent $\mathcal{M}_4^{d-1}$, while two other hyperplanes are double permutations. Moreover, if A is a matrix equivalent to some matrix from $\mathcal{L}_4^d$, then this property holds with respect to some direction j of hyperplanes. In this case, we will say that j is the *special direction* for the matrix A.

Proposition 10. *Let A be a d-dimensional polystochastic matrix of order 4 obtained as a convex combination of matrices $M_1, \dots, M_k$, $k \geq 2$, equivalent to $\mathcal{M}_4^d$ and such that no new entry is added to the support of filled subcubes of M_i. Then $k = 2$ and either the filled subcubes of M_1 and M_2 do not intersect or the matrix A is equivalent to some matrix from $\mathcal{L}_4^d$.*

Proof. Without loss of generality, we may assume that $M_1 = \mathcal{M}_4^d$.

We prove the statement by induction on d. The base of induction is $d = 3$ and it can be checked by computer enumeration.

Consider a d-dimensional polystochastic matrix A of order 4 obtained as a convex combination of matrices $M_1, \dots, M_k$, $k \geq 2$, equivalent to $\mathcal{M}_4^d$ and such that no new entry is added to the support of filled subcubes of M_i. Note that every hyperplane Γ of the matrix A is a $(d-1)$-dimensional polystochastic matrix that is either equivalent to the multidimensional permutation $\mathcal{M}_4^{d-1}$ or it satisfies the same condition as the matrix A. In the second case, by the inductive assumption, the support of hyperplane Γ is a union of two matrices equivalent to $\mathcal{M}_4^{d-1}$ such that their filled subcubes do not intersect or it is equivalent to some matrix from $\mathcal{L}_4^{d-1}$. Since $k \geq 2$, not all hyperplanes Γ of A are multidimensional permutations.

Note that if all supports of hyperplanes Γ are unions of two matrices equivalent to $\mathcal{M}_4^{d-1}$ with disjoint filled subcubes, then the same holds for the matrix A.

Assume that there exists a hyperplane Γ of direction i of the matrix A equivalent to $\mathcal{L}_4^{d-1}$ having a special direction j. Since every hyperplane Γ' of direction $\ell \neq i, j$ intersects Γ by $(d-2)$-dimensional planes equivalent to a matrix from $\mathcal{L}_4^{d-2}$, we deduce that all such hyerplanes Γ' are also equivalent to matrices from $\mathcal{L}_4^{d-1}$ and, moreover, their special direction is j. Then A is the convex combination of two matrices M_1 and M_2 equivalent to $\mathcal{M}_4^d$ such that two of four hyperplanes of direction j are equivalent to $\mathcal{M}_4^{d-1}$ and the other two hyperplanes of this direction are double permutations.

Recall that, by Proposition 6, the first two hyperplanes of direction j in A have the unique parameters as block permutations. Then the matrix M_2 has parameters $(\mathcal{E}, \lambda, s)$, where $\mathcal{E}$ has an element nonequal to 2 only at position j because otherwise M_1 and M_2 (and their filled subcubes) cannot coincide in some hyperplanes of direction j (see Proposition 7). Thus the tesselation index of the matrix A is equal to $d-1$ and, by Proposition 9, A is equivalent to some matrix $\mathcal{L}_4^d$. $\square$

At last, we are ready to prove the main result of the paper.

Proof of Theorem 1. Let A be a d-dimensional polystochastic of order 4 such that $\mathrm{per} A = 0$. By Lemma 4, d is odd and there are different permutation matrices $M_1, \dots, M_k$ equivalent to $\mathcal{M}_4^d$ such that A is a convex combination of $M_1, \dots, M_k$. If $k = 1$, then A is equivalent to $\mathcal{M}_4^d$.

Suppose that $k \geq 2$. If there is a matrix M_i such that some of another matrix M_j adds a new entry to a filled subcube of M_i, then, by Lemma 6, the permanent of A is nonzero. Thus, no new entry is added to the support of filled subcubes of M_i, $i = 1, \dots, k$. By Proposition 10, $k = 2$ and either the filled subcubes of M_1 and M_2 do not intersect or the matrix A is equivalent to some matrix from $\mathcal{L}_4^d$. In the first case, the tessellation index of the matrix A is $-\infty$, and so by Proposition 8, we have that $\mathrm{per} A \neq 0$. So the unique remained possibility for A to have the zero permanent is to be equivalent to some matrix from $\mathcal{L}_4^d$. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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