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Asymptotic bounds on the numbers of vertices of polytopes of polystochastic matrices

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ABSTRACT

A multidimensional nonnegative matrix is called polystochastic if the sum of the entries in each line is equal to 1. The set of all polystochastic matrices of order n and dimension d forms a convex polytope Ω_n^d.
In the present paper, we compare known bounds on the number of vertices of the polytope Ω_n^d and prove that the number of vertices of Ω_3^d is doubly exponential in d.
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1. Introduction and definitions

Polystochastic matrices are a natural extension of doubly stochastic matrices to higher dimensions. The properties of the convex polytope of doubly stochastic matrices were extensively studied by Brualdi and Gibson in the 1970s [2–4], while there are still few results on the more complicated polytope of polystochastic matrices.

Knowledge of the vertex set of a convex polytope allows one to reveal its geometrical structure and simplify the solution of certain optimization problems. The Birkhoff theorem states that the vertices of the polytope of doubly stochastic matrices are exactly the permutation matrices, whereas the polytope of polystochastic matrices contains many other vertices for which we lack good descriptions or exact bounds on their number. The aim of the present paper is to estimate the number of vertices of the polytope of polystochastic matrices.

Let us provide the necessary definitions. A *d-dimensional matrix A of order n* is an array $(a_\alpha)_{\alpha \in I_n^d}$, $a_\alpha \in \mathbb{R}$, whose entries are indexed by α from the index set $I_n^d = \{\alpha = (\alpha_1, \dots, \alpha_d) | \alpha_i \in \{1, \dots, n\}\}$. A matrix A is called *nonnegative* if all $a_\alpha \geq 0$, and it is a *$(0,1)$-matrix* if all its entries are 0 or 1. The *support* $supp(A)$ of a matrix A is the set of all indices α for which $a_\alpha \neq 0$.

Given $k \in \{0, \dots, d\}$, a *k-dimensional plane* in A is the submatrix obtained by fixing $d-k$ positions in indices and letting the values in the remaining k positions vary from 1 to n. We will say that the set of fixed positions defines the direction of a plane. A 1-dimensional plane is said to be a *line*. Matrices A and B are called *equivalent* if one can be obtained from the other by transposes (permutations of components of indices) and permutations of parallel $(d-1)$-dimensional planes.

A multidimensional nonnegative matrix A is called *polystochastic* if the sum of its entries at each line is equal to 1. Polystochastic matrices of dimension 2 are known as *doubly stochastic*. Since doubly stochastic $(0,1)$-matrices are exactly

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the permutation matrices, for $d \geq 3$ we will say that d-dimensional polystochastic $(0, 1)$-matrices are *d-dimensional (or multidimensional) permutations*. There is a one-to-one correspondence between d-dimensional permutations of order n and $(d-1)$-dimensional latin hypercubes of order n, which are $(d-1)$-dimensional matrices of order n filled with n symbols such that each line contains exactly one symbol of each type (for more details, see, for example, [7]).

It is easy to see that the set of d-dimensional polystochastic matrices of order n is a convex polytope, which we denote as Ω_n^d and call the *Birkhoff polytope*. By the *dimension* of Ω_n^d we mean its geometric dimension as a polytope in $\mathbb{R}^{n^d}$, and its *facets* are the faces of one less dimension than the polytope itself.

A matrix $A \in \Omega_n^d$ is a *vertex* of the Birkhoff polytope Ω_n^d if there are no matrices $B_1, B_2 \in \Omega_n^d$ such that $A = \lambda B_1 + (1-\lambda) B_2$ for some $0 < \lambda < 1$. The definition also implies that for every d-dimensional polystochastic matrix A of order n there is a decomposition of the form $A = \sum\limits_i \lambda_i B_i$, where $\lambda_i > 0$, $\sum\limits_i \lambda_i = 1$, and B_i are vertices of Ω_n^d such that $supp(B_i) \subseteq supp(A)$. Let $V(n, d)$ denote the number of vertices of the polytope Ω_n^d. Note that every multidimensional permutation is a vertex in Ω_n^d.

Finally, we will say that a multidimensional matrix A is a *zero-sum* matrix if the sum of entries at each line of A is equal to 0. For example, the difference between two polystochastic matrices of the same order and dimension is a zero-sum matrix.

The structure of the paper is as follows. In Section 2, using a general bound on the number of faces in polytopes, we derive an upper bound on the number of vertices of the polytope of polystochastic matrices. Then we summarize other known bounds on the number of vertices of Ω_n^d and analyze their asymptotic behavior when either the order n or the dimension d of matrices is fixed. In particular, we observe that for the number $V(3, d)$ of the polytope of d-dimensional matrices of order 3, the lower and upper bounds differ dramatically. To narrow this gap, in Section 3 we propose a construction of vertices of Ω_3^d that shows that $V(3, d)$ grows doubly exponentially.

2. Bounds on the number of vertices of Ω_n^d

We start with an upper bound on the number vertices in a general polytope. The well-known result of McMullen [13] states that cyclic polytopes have the largest possible number of faces among all convex polytopes with a given dimension and number of vertices. As a consequence, one can estimate the number of vertices of a polytope given its dimension and number of facets.

Proposition 1 *(see, e.g., [1]). The number of vertices V of a convex m-dimensional polytope with k facets, $k \geq m$, satisfies*

$$V \leq \binom{k - \lfloor \frac{m+1}{2} \rfloor}{k-m} + \binom{k - \lfloor \frac{m+2}{2} \rfloor}{k-m}.$$

The polytope of d-dimensional polystochastic matrices of order 2 has dimension 1 and only two vertices, i.e., the multidimensional permutations (see, for example, [17]). But for $n \geq 3$, the polytope Ω_n^d is nontrivial.

Proposition 2. *Let $n \geq 3$. The polytope Ω_n^d is a $(n-1)^d$-dimensional polytope with n^d facets.*

Proof. Similar to the polytope Ω_n^2 of doubly stochastic matrices (see, e.g., [2]), every face F of the polytope Ω_n^d is defined by a set of indices at which a matrix A from F takes zero values. So the facets of Ω_n^d are $\{A \in \Omega_n^d | a_\alpha = 0\}$ for some $\alpha \in I_n^d$, and there are n^d facets in Ω_n^d.

To find the dimension of Ω_n^d, it is sufficient to note that the space of d-dimensional zero-sum matrices of order n has dimension $(n-1)^d$ because every such matrix is uniquely defined by values in any d-dimensional submatrix of order $n-1$. □

From Propositions 1 and 2, we get the following upper bound on the number of vertices of Ω_n^d. For $d = 3$ it was previously stated in [10].

Theorem 1. *The number of vertices $V(n, d)$ of the polytope of polystochastic matrices Ω_n^d satisfies*

$$V(n,d) \leq \binom{n^d - \lfloor \frac{(n-1)^d+1}{2} \rfloor}{n^d - (n-1)^d} + \binom{n^d - \lfloor \frac{(n-1)^d+2}{2} \rfloor}{n^d - (n-1)^d}.$$

To our knowledge, there are no upper bounds on the number of vertices of Ω_n^d that use the specific properties of this polytope. Thus, finding any improvement to Theorem 1 is an interesting question.

A natural lower bound on the number $V(n, d)$ of vertices of Ω_n^d is the number of multidimensional permutations, since every d-dimensional permutation of order n is a vertex of Ω_n^d.

Let us study the asymptotics of the number of vertices $V(n, d)$ when d is fixed and $n \to \infty$.

When $d = 2$, the well-known Birkhoff theorem states that every vertex of the polytope of doubly stochastic matrices is a permutation matrix. So $V(n,2) = n!$ that is the number of permutation matrices of order n.

In [8], Keevash found a lower bound on the number of multidimensional permutations of fixed dimension, which, together with the upper bound by Linial and Luria [11], gives the following.

Theorem 2 *([8], [11]). The number of d-dimensional permutations of order n is* $\left(\frac{n}{e^{d-1}} + o(n)\right)^{n^{d-1}}$ *as* $d \geq 2$ *is fixed and* $n \to \infty$.

For $d \geq 3$ and $n \geq 3$, the polytope Ω_n^d has vertices other than multidimensional permutations, but we know relatively few examples and very few constructions of such vertices. Most of these constructions [5,6,12] produce vertices of Ω_n^d that have exactly two $1/2$-entries in each line. The only improvement on the lower bound on the number of vertices of the polytope of Ω_n^d of fixed dimension was obtained for $d = 3$ by Linial and Luria in [12].

Theorem 3 *([12]). If* $M(n,3)$ *is the number of* 3*-dimensional permutations of order n, then for the number* $V(n,3)$ *of vertices of* Ω_n^3 *we have*

$$V(n,3) \geq M(n,3)^{3/2-o(1)} \text{ as } n \to \infty.$$

Summarizing these results, we deduce the following asymptotic bounds for the logarithm of $V(n,d)$ when d is fixed.

Proposition 3. *If* $d \geq 4$ *is fixed, then for the number* $V(n,d)$ *of vertices of* Ω_n^d *we have*

$$n^{d-1} \ln n \cdot (1 + o(1)) \leq \ln V(n,d) \leq dn^{d-1} \ln n \cdot (1 + o(1))$$

as $n \to \infty$. *In addition, if* $d = 3$, *then*

$$\frac{3}{2} n^2 \ln n \cdot (1 + o(1)) \leq \ln V(n,3) \leq 3n^2 \ln n \cdot (1 + o(1)),$$

and if $d = 2$, *then* $\ln V(n,2) = \ln n! = n \ln n \cdot (1 + o(1))$.

Proof. For $d \geq 3$, all upper bounds follow from the standard estimate $\binom{m}{k} \leq \frac{m^k}{k!}$ for the binomial coefficients in Theorem 1 and further analysis of the expressions for large n.

For $d \geq 4$, the lower bound follows from the estimate of the number of multidimensional permutations (Theorem 2), and for $d = 3$ it is improved by Theorem 3.

Finally, the case $d = 2$ is the Birkhoff theorem for doubly stochastic matrices. □

A comparison of weaker lower and upper bounds on the number of vertices of 3-dimensional polystochastic matrices was given in [18].

Based on these bounds, we propose the following conjecture.

Conjecture 1. *For every* $d \geq 2$, *there is a constant* c_d, $1 \leq c_d \leq d$, *such that for the number* $V(n,d)$ *of vertices of* Ω_n^d *we have*

$$\ln V(n,d) = c_d n^{d-1} \ln n \cdot (1 + o(1)).$$

Let us turn to the case when the order of polystochastic matrices is fixed but the dimension grows.

As we noted before, there are only two vertices in the polytope Ω_2^d. It is also well known that for every d, the d-dimensional permutation of order 3 is unique up to equivalence, and there are $3 \cdot 2^{d-1}$ different such multidimensional permutations.

The asymptotics of the number of d-dimensional permutations of order 4 were found in [14]. It gives that $\log_2 V(4,d) \geq 2^{d-1}(1 + o(1))$.

To date, the best lower bounds on the number of d-dimensional permutations of order n, when $n \geq 5$ is fixed, were proved by Potapov and Krotov in [15]. Their results imply the following.

Theorem 4 *([15]). Let* $V(n,d)$ *be the number of vertices of the polytope of d-dimensional matrices of order n. Then* $\log_2 V(5,d) \geq 3^{(d-1)/3}(1 - o(1))$ *as* $d \to \infty$, $\log_2 V(n,d) \geq \left(\frac{n}{2}\right)^{d-1}$ *if* $n \geq 6$ *is even, and* $\log_2 V(n,d) \geq \left(\frac{n-3}{2}\right)^{\frac{d-1}{2}} \left(\frac{n-1}{2}\right)^{\frac{d-1}{2}}$ *if* $n \geq 7$ *is odd.*

Until the present work, there were no rich constructions or lower bounds on the vertices of Ω_n^d for fixed n other than those corresponding to multidimensional permutations.

Concerning the upper bound, an expansion of the binomial coefficients in Theorem 1 for fixed n gives the following.

Proposition 4. *If n is fixed, then for the logarithm of the number $V(n,d)$ of vertices of d-dimensional polystochastic matrices of order n we have*

$$\log_2 V(n,d) \leq \log_2 \frac{n}{n-1} \cdot \frac{d}{2} \cdot (n-1)^d (1+o(1)) \text{ as } d \to \infty.$$

Thus, we have a substantial gap between the lower and upper bounds for the numbers of vertices of Ω_n^d when n is fixed, and finding the asymptotics of these numbers is an interesting open problem.

3. Lower bound on the number of vertices of Ω_3^d

In this section, we prove the following theorem.

Theorem 5. *For the number of vertices $V(3,d)$ we have*

$$\log_2 V(3,d) \geq c 2^{\delta d}(1+o(1)),$$

where $c = \frac{1}{4}\log_2 9/5 \approx 0.212$ and $\delta \approx 0.047$.

The main idea of the proof is to construct a large set $\mathcal{A}$ of d-dimensional polystochastic matrices of order 3 such that for every three matrices $A_1, A_2, A_3 \in \mathcal{A}$, the faces of Ω_3^d defined by their supports do not share a common vertex. This means that every vertex from a decomposition of some matrix $A \in \mathcal{A}$ into a convex sum of vertices can appear in at most one other decomposition of some matrix $B \in \mathcal{A}$, and, therefore, we have at least $|\mathcal{A}|/2$ vertices of Ω_3^d.

First, we aim to construct a rich set of polystochastic matrices. For this purpose we need several auxiliary notions and definitions.

Given indices $\alpha, \beta \in I_n^d$, let $\rho(\alpha,\beta)$ denote the Hamming distance between α and β, i.e., the number of positions in which these indices differ.

For an index $\alpha \in I_3^d$, define the set of indices $T_\alpha = \{\beta | \rho(\alpha,\beta) = d\}$. In other words, T_α is the d-dimensional submatrix of order 2 formed by indices at the maximal distance from the index α.

Let $S \subseteq I_3^d$ be a subset of indices of a d-dimensional matrix of order 3. Given ε, $0 < \varepsilon < 1$, we say that the set S is *ε-sparse* if

1. for all $\alpha, \beta \in S$, we have $\rho(\alpha,\beta) \geq \varepsilon d$;
2. for every $\alpha \in S$ there is an index $\gamma_\alpha \in I_3^d$ such that $T_{\gamma_\alpha} \cap S = \{\alpha\}$.

We are going to use ε-sparse sets as the complements of the supports of polystochastic matrices of order 3. First, we show that ε-sparse sets exist and can be quite large. For this purpose, we need a well-known statement of the coding theory that follows from the work of Shannon [16].

Proposition 5 *(see [16]). Let $0 \leq \varepsilon \leq 1/2$ and $H(\varepsilon) = -\varepsilon \log_2 \varepsilon - (1-\varepsilon)\log_2(1-\varepsilon)$ be the binary entropy. Then for every d and $N \leq 2^{d(1-H(\varepsilon))}$ there is a set $S \subseteq I_2^d$ of size N such that for every $\alpha, \beta \in S$ we have $\rho(\alpha,\beta) \geq \varepsilon d$.*

Proposition 6. *Let $0 < \varepsilon \leq 1/2$. Then for every $N \leq 2^{(1-H(\varepsilon))d}$ there exists an ε-sparse S set of size N in I_3^d.*

Proof. For shortness, we denote the index $(1,\dots,1)$ from I_3^d by $\mathbf{1}$ and let $T_{\mathbf{1}} = \{\alpha \in I_3^d : \alpha_i \in \{2,3\}\}$. By Proposition 5, for every $N \leq 2^{(1-H(\varepsilon))d}$ there is a subset S of $T_{\mathbf{1}}$ such that for all $\alpha, \beta \in S$ it holds $\rho(\alpha,\beta) \geq \varepsilon d$.

Let us show that all such sets S satisfy the second condition of the definition of ε-sparse sets. Given $\alpha \in T_{\mathbf{1}}$, consider an index γ from $T_{\mathbf{1}}$ such that $\gamma_i = 2$ if $\alpha_i = 3$, and $\gamma_i = 3$ if $\alpha_i = 2$ for every $i \in \{1,\dots,d\}$. It is easy to see that $T_\gamma \cap T_{\mathbf{1}} = \{\alpha\}$. Since $S \subseteq T_{\mathbf{1}}$, for every $\alpha \in S$ we have that $T_\gamma \cap S = \{\alpha\}$, so we can take γ as γ_α. □

Now we prove that for every ε-sparse set S (provided it is not too large and d is not too small), there exists a polystochastic matrix of order 3 whose complement of the support is exactly the set S.

Lemma 1. *Let $d \geq 14/\varepsilon$ and $S \subseteq I_3^d$ be an ε-sparse set of size $N = |S| \leq 2^{\varepsilon d/4}$. Then there exists a d-dimensional polystochastic matrix A of order 3 such that $supp(A) = I_3^d \setminus S$.*

Proof. We will look for a d-dimensional zero-sum matrix M of order 3 such that $m_\alpha = 1$ for all $\alpha \in S$ and $|m_\alpha| \leq 3/4$ for all $\alpha \notin S$. Then the matrix $A = J - \frac{1}{3}M$ is the required polystochastic matrix, where J is the d-dimensional polystochastic matrix of order 3, whose all entries are equal to $1/3$.

We construct the matrix M in two steps.

1. Given index $\alpha \in I_3^d$, define the matrix $F^\alpha = (f^\alpha_\beta)_{\beta \in I_n^d}$ with entries $f^\alpha_\beta = \left(-\frac{1}{2}\right)^{\rho(\alpha,\beta)}$. Note that all matrices F^α are zero-sum because each line consists of indices $\beta_1, \beta_2, \beta_3$ such that $\rho(\alpha,\beta_1) = k$ and $\rho(\alpha,\beta_2) = \rho(\alpha,\beta_3) = k+1$ for some $k \in \{0, \dots d-1\}$.

Consider the zero-sum matrix $M' = (m'_\beta)_{\beta \in I_n^d}$ such that

$$M' = \sum_{\alpha \in S} F^\alpha.$$

Let us estimate its entries. For each $\alpha \in S$, denote $\delta_\alpha = 1 - m'_\alpha$. Since S is an ε-sparse set of size N, we have that

$$|\delta_\alpha| = |1 - m'_\alpha| \le \frac{N}{2^{\varepsilon d}} \le 2^{-3\varepsilon d/4},$$

because for all $\gamma \in S$, $\gamma \neq \alpha$, each matrix F^γ has the absolute value f^γ_α in index α not greater than $1/2^{\varepsilon d}$. In particular, for $\delta = \max\limits_{\alpha \in S} |\delta_\alpha|$ we have $\delta \le 2^{-3\varepsilon d/4}$.

Suppose now that $\alpha \notin S$ and $\gamma \in S$ is an index such that $\rho(\alpha,\gamma)$ is minimal. Then from the definition of an ε-sparse set, for every other $\beta \in S$, $\beta \neq \gamma$, we have that $\rho(\alpha,\beta) \ge (\varepsilon d - 1)/2$. Using $N \le 2^{\varepsilon d/4}$, we obtain

$$|m'_\alpha| \le |f^\gamma_\alpha| + \sum_{\beta \in S, \beta \neq \gamma} |f^\beta_\alpha| \le \frac{1}{2} + \frac{N}{2^{(\varepsilon d - 1)/2}} \le \frac{1}{2} + 2^{-\varepsilon d/4 + 1/2} \le \frac{5}{8},$$

because $2^{-\varepsilon d/4 + 1/2} \le 1/8$, when $d \ge 14/\varepsilon$.

2. Now we modify the matrix M' to obtain the required matrix M.

Since the set S is ε-sparse, for every $\alpha \in S$ there is $\gamma_\alpha \in I_3^d$ such that $T_{\gamma_\alpha} \cap S = \{\alpha\}$. Let R^α be the zero-sum matrix such that for every $\beta \in T_{\gamma_\alpha}$ the entry $r^\alpha_\beta = (-1)^{\rho(\alpha,\beta)} \cdot \delta_\alpha$ and for every $\beta \notin T_{\gamma_\alpha}$ we put $r^\alpha_\beta = 0$. Consider the matrix

$$M = M' + \sum_{\alpha \in S} R^\alpha.$$

Using the definition of δ_α and the fact that for every $\alpha \in S$ it holds $T_{\gamma_\alpha} \cap S = \{\alpha\}$, we see that $m_\alpha = 1$ for all $\alpha \in S$ as required.

Suppose that $\alpha \notin S$. Using inequalities $N \le 2^{\varepsilon d/4}$, $\delta \le 2^{-3\varepsilon d/4}$, and $d \ge 14/\varepsilon$, we obtain

$$|m_\alpha| \le |m'_\alpha| + \sum_{\alpha \in S} |\delta_\alpha| \le \frac{5}{8} + N\delta \le \frac{5}{8} + 2^{-\varepsilon d/2} \le \frac{3}{4}. \quad \square$$

To construct many polystochastic matrices with the desired property, we also utilize 3-perfect hash codes that are known as trifferent codes.

Given $q \ge 2$ and $N \in \mathbb{N}$, a *q-perfect hash code C of block length N* is a collection C of words of length N over the alphabet $\{1, \dots, q\}$ such that for any distinct q words $w_1, \dots, w_q$ from C there is a position i, $i \in \{1, \dots, N\}$, for which $\{w_j(i) \mid 1 \le j \le q\} = \{1, \dots, q\}$. In what follows, we identify words from q-perfect hash codes with indices from I_q^N.

A set $C \subseteq I_3^N$ is called a *trifferent code* if it is a 3-perfect hash code. To date, the asymptotically large trifferent codes were constructed by Körner and Marton in [9].

Theorem 6 *([9], Theorem 1). For every $N \in \mathbb{N}$, there exists a trifferent code C in I_3^N of size $|C| = \left(\frac{9}{5}\right)^{N/4}$.*

Now we are ready to prove the main result of this section.

Proof of Theorem 5. Recall that we aim to construct a large set $\mathcal{A}$ of d-dimensional polystochastic matrices of order 3 such that for every three matrices $A_1, A_2, A_3 \in \mathcal{A}$ there are no vertices B of Ω_3^d for which $supp(B) \subseteq supp(A_1) \cap supp(A_2) \cap supp(A_3)$.

Let $\varepsilon \le 1/2$ be a solution of $1 - H(\varepsilon) = \varepsilon/8$, $\varepsilon \approx 0.3735$, and put $\mu = \varepsilon/8$. Then by Proposition 6, there exists an ε-sparse set S in I_3^d with cardinality $N = \lfloor 2^{\mu d} \rfloor$, and by Theorem 6, there is a trifferent code C in I_3^N such that $|C| = \left(\frac{9}{5}\right)^{N/4}$.

Using the ε-sparse set S in I_3^d and the trifferent code C, let us now construct many sparse sets in I_3^{d+1}. Since a word $x \in C$ has length N, we may assume that its positions are indexed by $\alpha \in S$. For every $x \in C$, consider the set S_x in I_3^{d+1} such that $S_x = \{(\alpha, x_\alpha) : \alpha \in S\}$, $|S_x| = |S|$. Also note that the number of sets S_x is equal to $|C|$.

Let us prove that all sets S_x are $\varepsilon/2$-sparse. Let $\beta = (\alpha, x_\alpha)$ and $\beta' = (\alpha', x_{\alpha'})$, where $\beta, \beta' \in S_x$. Then $\rho(\beta, \beta') \ge \rho(\alpha, \alpha')$. Since S is an ε-sparse set, we have that $\rho(\alpha, \alpha') \ge \varepsilon d \ge \frac{\varepsilon}{2}(d+1)$, and so $\rho(\beta, \beta') \ge \frac{\varepsilon}{2}(d+1)$.

Next, by the condition on the ε-sparse set S, for every $\alpha \in S$ there is $\gamma_\alpha \in I_3^d$ such that $T_{\gamma_\alpha} \cap S = \{\alpha\}$. Given $\beta \in S_x$, $\beta = (\alpha, x_\alpha)$, consider index $\gamma_\beta = (\gamma_\alpha, j)$ from I_3^{d+1}, where $j \in \{1, 2, 3\}$ is different from x_α. Then the construction of sets S_x implies that $T_{\gamma_\beta} \cap S_x = \{\beta\}$. Therefore, sets S_x are $\varepsilon/2$-sparse for all $x \in C$.

Using Lemma 1 and the fact that $N \le 2^{\varepsilon d/8}$, for all $d \ge 28/\varepsilon$ and each $x \in C$ there is a $(d+1)$-dimensional polystochastic matrix A_x of order 3 such that $supp(A_x) = I_3^{d+1} \setminus S_x$. Define a collection of polystochastic matrices $\mathcal{A} = \{A_x : x \in C\}$.

Since C is a trifferent code, for all words $x^1, x^2, x^3 \in C$ there is a position in which all these three words are different. In our construction, this position corresponds to a line ℓ of direction $d+1$ in I_3^{d+1} such that $supp(A_{x^1}) \cap supp(A_{x^2}) \cap supp(A_{x^3}) \cap \ell = \emptyset$. Therefore, there are no vertices B of the Birkhoff polytope Ω_3^{d+1} for which $supp(B) \subseteq supp(A_{x^1}) \cap supp(A_{x^2}) \cap supp(A_{x^3})$.

Denote by $\mathcal{B}$ the set of all vertices B of the polytope Ω_3^{d+1} such that $supp(B) \subseteq A_x$ for some $A_x \in \mathcal{A}$. The obtained property of the set $\mathcal{A}$ means that for every $B \in \mathcal{B}$ there are at most two matrices A_x such that $supp(B) \subseteq A_x$. Therefore,

$$|\mathcal{B}| \ge \frac{|C|}{2} = \frac{1}{2}\left(\frac{9}{5}\right)^{N/4} \ge \frac{1}{2}\left(\frac{9}{5}\right)^{\frac{\lfloor 2^{\mu d} \rfloor}{4}}$$

that implies the statement of the theorem. □

Remark. The same reasoning can be applied to construct a quite rich families of vertices of polytopes Ω_n^d for any fixed $n \ge 3$ and large d. However, for $n \ne 3$ the numbers of such vertices will be much less than the lower bounds on the numbers of multidimensional permutations from Theorem 4.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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