On completion of latin hypercuboids of order 4

Vladimir N. Potapov

Sobolev Institute of Mathematics, Novosibirsk State University, Novosibirsk, Russia

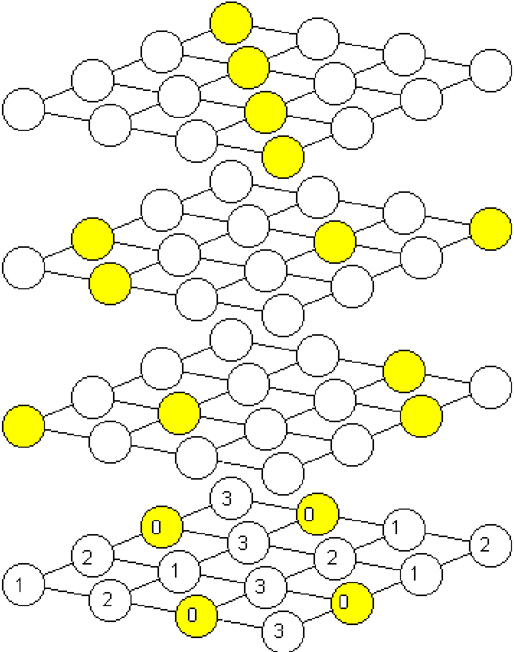
Workshop on Algebraic and Combinatorial Coding Theory (ACCT2010)

Let Σ be a nonempty set (the alphabet). A function $f : \Sigma^n \to \Sigma$ is called an *n-ary quasigroup* if $f(\overline{x}) \neq f(\overline{y})$ for all $\overline{x}, \overline{y} \in \Sigma^n$ such that $d(\overline{x}, \overline{y}) = 1$ where $d(\overline{x}, \overline{y})$ is the Hamming distance.

A function defined on $\Omega \subset \Sigma^n$ and satisfying the same condition is called a partial n-ary quasigroup. We deal only with partial n-ary quasigroups $f : \Sigma^{n-1} \times \Sigma' \to \Sigma$, where $\Sigma' \subset \Sigma$.

The table of values of an n-ary quasigroup is called a *latin hypercube* (the n-dimensional generalization of latin square) and the table of values of a partial n-ary quasigroup is called a *latin hypercuboid*.

A subset $M \subset \Sigma^{n+1}$ is called an *MDS code (with distance* 2*)* if $|M| = |\Sigma|^n$ and $d(\overline{x}, \overline{y}) \geq 2$ for all distinct $\overline{x}, \overline{y} \in M$. It is clear that the graph of an n-ary quasigroup f is an MDS code. Moreover there exists a one-to-one correspondence between the n-ary quasigroups and the MDS codes (with distance 2).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There are some constructions of perfect codes (with distance 3) that use MDS codes (with distance 2).

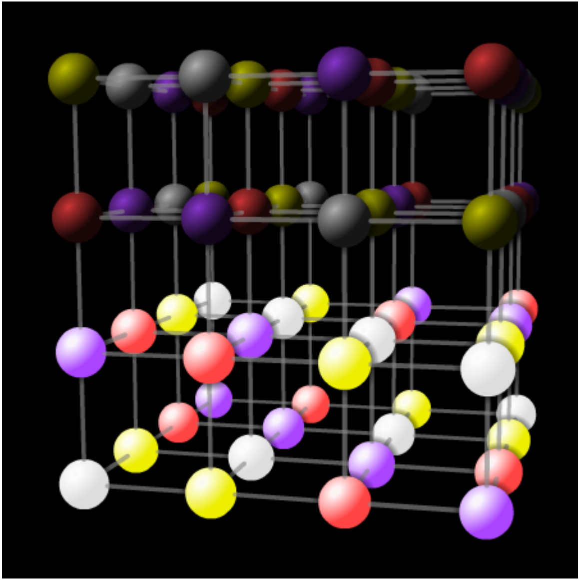
Zinov'ev V. A. Generalized cascade codes. (Russian) Problemy Peredachi Informacii **12** , no. 1, 5–15. 1976.

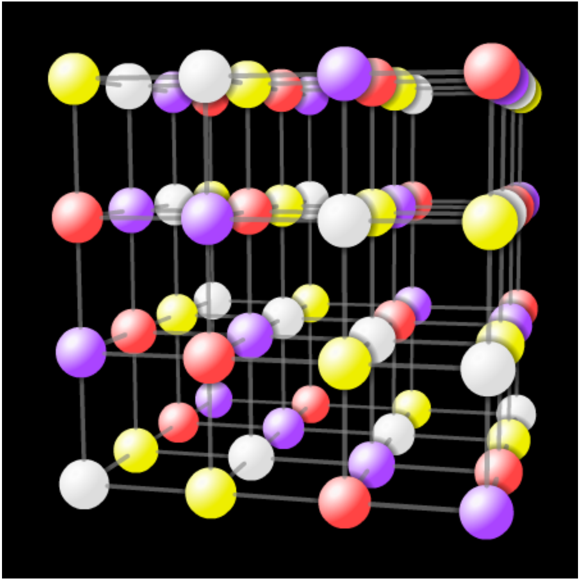
Phelps K. T. A general product construction for error correcting codes. *SIAM J. Algebraic Discrete Methods*. **5**, no. 2, 224–228. 1984.

Moreover, MDS codes are used for classifying perfect codes with rank $+2$ with respect to the linear perfect code.

Avgustinovich S. V., Heden O., Solov'eva F. I. The classification of some perfect codes. *Des. Codes Cryptogr.* **31**, no. 3, 313–318. 2004.

A partial n-ary quasigroup f is *extendable* if $f = q|_{\Sigma^{n-1}\times\Sigma'}$, where q is an n-ary quasigroup.
By the definitions a latin hypercuboid can be completed to a latin hypercube if and only if the partial n-ary quasigroup is extendable.





The fact that every latin rectangle can be completed to a latin square is a simple consequence of Konig's theorem. It is well known that every partial n-ary quasigroup $f : \Sigma^{n-1} \times \Sigma' \to \Sigma$ is extendable if $|\Sigma'| = 1$ or $|\Sigma'| = |\Sigma| - 1$. Therefore, all partial n-ary quasigroups of order $M = |\Sigma| \le 3$ are extendable.

Kochol proved that for all k and M satisfying $M/2 < k < M - 2$ there is an $M \times M \times k$ latin cuboid that cannot be completed to a latin cube.

Kochol M. Relatively narrow latin parallelepipeds that cannot be extended to a latin cube. *Ars Comb.*, **40**, 247–260, 1995.

Kochol M. Latin $(n \times n \times (n-2))$-parallelepipeds not completing to a latin cube. *Math. Slovaka,* **39**, 3–9, 1989.

Examples of non completable $M \times M \times k$ latin cuboids for $M = 5, 6, 7, 8$ and $k = 2, 2, 3, 4$ respectively are constructed by McKay and Wanless.

McKay B. D., Wanless I. M. A census of small Latin hypercubes. *SIAM J. Discrete Math.*, **22**, (2), 719–736, 2008.

Using these facts it is easy to show that there are n-dimensional latin hypercuboids of order M that cannot be completed to a latin hypercube for $n \geq 3$ and $M \geq 5$. However the question of existence of a non completable latin hypercuboid for $M = 4$ was open.

Main result

Theorem. *Every latin hypercuboid of order 4 is completable to a latin hypercube.*

As mentioned above we need to complete only $4 \times \cdots \times 4 \times 2$ latin hypercuboids because the other cases, $4 \times \cdots \times 4 \times 1$ and $4 \times \cdots \times 4 \times 3$, are trivial. In other words, we have a pair

of $(n-1)$-ary quasigroups f_0 and f_1 such that $f_0(\overline{x}) \neq f_1(\overline{x})$ for all $\overline{x} \in \Sigma^{n-1}$ and we want to extend the pair to an n-ary quasigroup.

The proof of the theorem is based on the classification theorem of n-ary quasigroups of order 4.

Theorem (classification of n-ary quasigroups of order 4). *Every n-ary quasigroup of order 4 is permutably reducible or semilinear.*

Krotov D. S. and Potapov V. N. n-Ary quasigroups of order 4. *SIAM J. Discrete Math.*, **23** (2), 561–570, 2009.

An n-quasigroup f is termed *permutably reducible* if there exist $m \in \{2, \dots, n-1\}$, an $(n-m+1)$-quasigroup h, an m-quasigroup g, and a permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ such that

$$f(x_1, \dots, x_n) \equiv h(g(x_{\sigma(1)}, \dots, x_{\sigma(m)}), x_{\sigma(m+1)}, \dots, x_{\sigma(n)}).$$

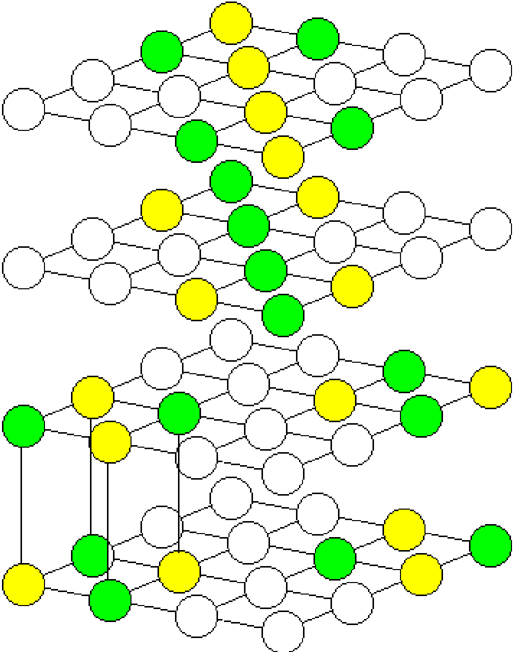
Let $\Sigma = \{0, 1, 2, 3\}$. Consider the 2-fold MDS code (the union of two disjoint MDS codes) $L \subset \Sigma^n$ defined by the indicator function

$$\chi_L(x_1, \dots, x_n) \equiv \chi_{0,1}(x_1) \oplus \dots \oplus \chi_{0,1}(x_n).$$

We say that a 2-MDS code is *linear* if it is isotopic to the 2-MDS code L. Let f be an n-ary quasigroup and let $a, b \in \Sigma, a \neq b$. Then the set

$$S_{a,b}(f) = \{(x_1, \dots, x_n) \in \Sigma^n \mid f(x_1, \dots, x_n) \in \{a, b\}\}$$

is a 2-fold MDS code. An n-ary quasigroup f is called *semilinear* if for some $a, b \in \Sigma$ the set $S_{a,b}(f)$ is linear.



By analyzing the various cases of types of pairs of $(n-1)$-ary quasigroups f_0 and f_1, we can prove the theorem.