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Infinite-Dimensional Quasigroups of Finite Orders

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Abstract—Let Σ be a finite set of cardinality $k > 0$, let $\mathbb{A}$ be a finite or infinite set of indices, and let $\mathcal{F} \subseteq \Sigma^{\mathbb{A}}$ be a subset consisting of finitely supported families. A function $f\colon \Sigma^{\mathbb{A}} \to \Sigma$ is referred to as an $\mathbb{A}$-quasigroup (if $|\mathbb{A}| = n$, then an n-ary quasigroup) of order k if $f(\overline{y}) \neq f(\overline{z})$ for any ordered families $\overline{y}$ and $\overline{z}$ that differ at exactly one position. It is proved that an $\mathbb{A}$-quasigroup f of order 4 is reducible (representable as a superposition) or semilinear on every coset of $\mathcal{F}$. It is shown that the quasigroups defined on $\Sigma^{\mathbb{N}}$, where $\mathbb{N}$ are positive integers, generate Lebesgue nonmeasurable subsets of the interval $[0, 1]$.

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1. INFINITE-DIMENSIONAL QUASIGROUPS AND NONMEASURABLE SETS

Let Σ be a nonempty finite set, and let $\mathbb{A}$ be a finite or infinite set whose elements enumerate the arguments of functions acting from $\Sigma^{\mathbb{A}}$ to Σ. Introduce a function $d\colon \Sigma^{\mathbb{A}} \times \Sigma^{\mathbb{A}} \to [0, \infty]$ in such a way that $d(\overline{y}, \overline{z})$ is the number of distinct coordinates in $\overline{y}, \overline{z} \in \Sigma^{\mathbb{A}}$. A function $f\colon \Sigma^{\mathbb{A}} \to \Sigma$ is said to be a $\mathbb{A}$-*quasigroup* (in what follows, we use simply the term *quasigroup* or, if $|\mathbb{A}| = n$, then the term n-*ary quasigroup*, see, e.g., [1]) of order $|\Sigma|$ if $f(\overline{y}) \neq f(\overline{z})$ for $d(\overline{y}, \overline{z}) = 1$.

Without loss of generality, we may assume that $\Sigma = \{0, \dots, k-1\}$. Let $\mathbb{N}$ be the set of natural numbers. Elements of the set $\Sigma^{\mathbb{N}}$ can be regarded as k-ary representations of reals $\delta \in [0, 1]$. Let us identify these reals and their k-ary representations[1].

Proposition 1. *For every quasigroup $f\colon \Sigma^{\mathbb{N}} \to \Sigma$ of finite order k and any element $a \in \Sigma$ the set $\{\delta \in [0, 1] \mid f(\delta) = a\}$ is Lebesgue nonmeasurable.*

Proof. Suppose that the set $B = \{\delta \in [0, 1] \mid f(\delta) = a\}$ is measurable. Let $\tau \in [0, 1]$ be a k-ary rational whose k-ary representation contains at most m nonzero initial symbols. Consider the half-interval $[\tau, \tau + 1/k^{m+1})$. It follows from the definition of quasigroup that

$$(B \cap [\tau, \tau + 1/k^{m+1}) + i/k^{m+1}) \cap B = \varnothing \qquad \text{for every} \quad i = 1, \dots, k-1.$$

In this case, using the invariance of the Lebesgue measure with respect to any shift of a set, we obtain the inequality

$$\mu(B \cap [\tau, \tau + 2/k^{m+1})) \leq \frac{1}{2}\, \mu([\tau, \tau + 2/k^{m+1})).$$

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[1] The fact that the correspondence is not one-to-one on a countable set is inessential for the presentation below.

Since k-ary rational points occur in an arbitrarily small neighborhood of every point $v \in (0,1)$, it follows that

$$\varliminf_{\varepsilon\to 0} \frac{\mu((v-\varepsilon, v+\varepsilon)\cap B)}{\mu((v-\varepsilon, v+\varepsilon))} \le \frac{1}{2}.$$

By the Lebesgue density theorem for an arbitrary measurable set A, this limit is equal to 1 for almost all points $v \in A$. Then $\mu(B) = 0$. Moreover, it follows from the definition of quasigroup that

$$[0,1] \subset \bigcup_{i=1-k}^{1+k} \left(B + \frac{i}{k}\right).$$

Then $\mu([0,1]) = 0$. We arrived at a contradiction. $\square$

Write $\operatorname{supp}\overline{y} = \{i \in \mathbb{A} \mid y_i \neq 0\}$. Denote by $\mathcal{I}$ the family of finite subsets of the set $\mathbb{A}$. Let the set $\mathbb{A}$ be infinite. Consider

$$\mathcal{F} = \{\overline{y} \in \Sigma^{\mathbb{A}} \mid \operatorname{supp}\overline{y} \in \mathcal{I}\}.$$

It is clear that the set $\Sigma^{\mathbb{A}}$ can be represented as a disjoint union of subsets of the form $\mathcal{F}_{\overline{a}} = \overline{a} + \mathcal{F}$. Moreover, if $\mathcal{F}_{\overline{a}} \neq \mathcal{F}_{\overline{b}}$, then $d(\overline{y}, \overline{z}) = \infty$ for any $\overline{y} \in \mathcal{F}_{\overline{a}}$ and $\overline{z} \in \mathcal{F}_{\overline{b}}$. Therefore, the quasigroup f can be defined independently on every $\mathcal{F}_{\overline{a}}$. In what follows, we assume that $f\colon \mathcal{F} \to \Sigma$. As we shall see below, the quasigroups on $\mathcal{F}$ admit a constructive definition, whereas, to define a quasigroup on $\Sigma^{\mathbb{A}}$, it is necessary to choose a representative in every class $\mathcal{F}_{\overline{a}}$.

In what follows, we denote by the symbol x the family of arguments (variables) of the quasigroup and by x_L a sample of arguments with subscripts in the set L, $L \subseteq \mathbb{A}$. By a *retract* of a quasigroup f we mean a subfunction obtained from f by substituting constants into some arguments, provided that only finitely many constants can be nonzero. The number of variables of a retract is referred to as the *dimension* of the retract. By $f_L(x_L)$ (or simply by $f(x_L)$) we denote the retract of a quasigroup $f\colon \mathcal{F} \to \Sigma$ in which all arguments except for those with indices in the set L are fixed by zeros. By $f(y_{\{i\}}, x_L)$ we denote the quasigroup obtained by substituting a function, a variable, or a constant y instead of the argument x_i, $i \notin L$. The symbols I and J are used to denote finite subsets of $\mathbb{A}$ only. It is clear that every retract of the form f_J can be regarded as a $|J|$-ary quasigroup.

Denote by S_0 the group of permutations on Σ that preserve the zero element[2]. By an *isotopy* (preserving the zero) one means an element of the set $S_0 \times S_0^{\mathbb{A}}$. Two quasigroups f and g are said to be *isotopic* if $g(x_{\mathbb{A}}) = \theta_0(f(\theta_{\mathbb{A}} x_{\mathbb{A}}))$ for some $\theta_0 \in S_0$ and $\theta_{\mathbb{A}} \in S_0^{\mathbb{A}}$. If a permutation θ_0 is the identity, then the quasigroups f and g are said to be *principally isotopic*. A quasigroup f is said to be *reduced* if, when substituting every element $a \in \Sigma$ into any argument x_i, $i \in \mathbb{A}$, and simultaneously substituting the zeros into the other arguments, one obtains a, i.e., $f(a_{\{i\}}, \overline{0}_{\mathbb{A}\setminus\{i\}}) = a$.

The objective of the present paper is to classify the quasigroups of order 4. To this end, we introduce below the notions of reducibility (representability in the form of superposition) and semilinearity of a quasigroup; we claim that all quasigroups of order 4 are either semilinear or reducible. The semilinear quasigroups admit a description using Boolean functions. Reducible quasigroups can be represented by using their own retracts. However, in contrast to the similar description of quasigroups of finite dimension (see [6]), the present work does not ensure any constructive classification of infinite-dimensional quasigroups, because the tree of decomposition of a quasigroup into superpositions can turn out to be infinite.

[2]If the set $\mathbb{A}$ is finite, then one does not fix the zero element.

2. REDUCIBILITY

A quasigroup f is said to be *reducible* if it can be represented in the form of a superposition, i.e., in the form $f(x_{L_1}, x_{L_2}) = g(h(x_{L_1})_{\{j\}}, x_{L_2})$, where g and h are L_1- and $(\{j\} \cap L_2)$-quasigroups, $L_1 \cap L_2 = \varnothing$, $|L_1| \geq 2$, $|L_2| \geq 1$, $j \notin L_2$[3]. In the converse case, the quasigroup is said to be *inreducible*.

As is well known (see, e.g., [2]), every quasigroup is isotopic to a reduced quasigroup and every reduced reducible quasigroup can be represented as a superposition of reduced quasigroups.

Proposition 2. *A quasigroup f can be represented as a superposition*

$$f(x_{L_1}, x_{L_2}) = g(h(x_{L_1})_{\{j\}}, x_{L_2})$$

if and only if, for any families of arguments y_{L_2}, y_{L_1}, y'_{L_1}, the relation $f(y_{L_1}, \overline{0}_{L_2}) = f(y'_{L_1}, \overline{0}_{L_2})$ implies the relation $f(y_{L_1}, y_{L_2}) = f(y'_{L_1}, y_{L_2})$.

Proof. Without loss of generality, we assume that the quasigroup f is reduced and $1 = j \in L_1$. The *necessity* is obvious; let us prove the *sufficiency*. Write $h = f_{L_1}$ and $g = f_{\{1\} \cup L_2}$.

Let

$$f(y_{L_1}, \overline{0}_{L_2}) = h(y_{L_1}) = a = f(a_{\{1\}}, \overline{0}_{L_1 \setminus \{1\}}, \overline{0}_{L_2})$$

for some $a \in \Sigma$. Then

$$f(y_{L_1}, y_{L_2}) = f(a_{\{1\}}, \overline{0}_{L_1 \setminus \{1\}}, y_{L_2}) = g(a_{\{1\}}, y_{L_2}) = g(h(y_{L_1})_{\{1\}}, y_{L_2}). \quad \square$$

The following assertion is an immediate corollary to a theorem in [3].

Statement 1. *Every reducible n-ary quasigroup f can be represented in the form*

$$f(x_J) = F(q_1(x_{J_1}), \dots, q_m(x_{J_m})), \tag{2.1}$$

where q_j are n_j-ary quasigroups for $j \in \{1, \dots, m\}$, F is an inreducible m-ary quasigroup, and $\{J_i\}$ is a partition of the set J into families of cardinalites $n_1, \dots, n_m$. Moreover, if $m \geq 3$, then, in this representation, the partition $\{J_i\}$ is unique.

This assertion can be restated as follows (see [4, Sec. 2]).

Corollary 1. *If a reducible n-ary quasigroup f has an inreducible retract of dimension $m > 2$ not contained in inreducible retracts of larger dimension, then*

$$\{x \mid x_0 = f_J(x_J)\} = \{x \mid q_0(x_{J_0}) = F(q_1(x_{J_1}), \dots, q_m(x_{J_m}))\}, \tag{2.2}$$

where q_j are n_j-ary quasigroups for $j \in \{0, \dots, m\}$, F is an inreducible m-ary quasigroup, and $\{J_i\}$ is the partition of the set $J \cup \{0\}$ into families of cardinalities $n_0, \dots, n_m$[4].

Let us choose some quasigroup $f\colon \mathcal{F} \to \Sigma$. We say that a set $M \subseteq \mathbb{A}$ is *inreducible* (with respect to f) if $|M| \geq 3$ and, for every finite family $J \subseteq M$, there is a finite family J', $J \subseteq J' \subseteq M$, for which the quasigroup $f_{J'}$ is inreducible.

A system of sets in which, for every pair of sets L_1, L_2, there is a set L such that $L_1 \cup L_2 \subset L$, is said to be a *direction*. We say that a direction $S \subseteq \mathcal{I}$ *converges* to a set $M \subseteq \mathbb{A}$ if, for any finite subset of M, there is an element of the direction S for which the given subset is contained in this element. By the definition of inreducibility of the set M, there is a direction S which converges to M and consists of inreducible finite families.

Proposition 3. *If a direction $S \subseteq \mathcal{I}$ converges to a set $M \subseteq \mathbb{A}$ and $S = \bigcup_{i=1}^{m} S_i$, then there is an $i \in \{1, \dots, m\}$ such that the set S_i is a direction which converges to M.*

[3] To be definite, we assume that $j \in L_1$.

[4] Here and below, we assume that x_0 is not an argument of the quasigroup f.

Proof. Let S_i be not a direction convergent to M. Then there is a $J_i \in \mathcal{I}$ which is not dominated by elements of S_i. Consider $J = \bigcup_{i=1}^{m} J_i$. There is an $I \in S$ such that $J \subseteq I$. By assumption, $I \in S_i$ for some $i \in \{1, \dots, m\}$. We have arrived at a contradiction. □

Proposition 4. *If a set M is inreducible, then the quasigroup f_M is inreducible.*

Proof. Suppose the contrary, i.e., let there be a representation of the quasigroup f_M in the form of a superposition

$$f_M(x_{L_1}, x_{L_2}) = g(h(x_{L_1})_{\{i_0\}}, x_{L_2}), \tag{2.3}$$

where $M = L_1 \cup L_2$, $|L_1| \geq 2$, and $|L_2| \geq 1$. Let $i_0, i_1 \in L_1$ and $i_2 \in L_2$. By the definition of inreducible set, there is a finite family J, $\{i_0, i_1, i_2\} \subset J \subseteq M$, for which the quasigroup f_J is inreducible. However, this contradicts equation (2.3). □

The inverse to this statement, which is decomposed into two subcases (Lemmas 1 and 2), is the main result of this section and the next one. It claims that, if a quasigroup $f_{\mathbb{A}}$ is inreducible, then the set $\mathbb{A}$ is inreducible.

Denote by $\mathcal{N}(f)$ the family of inreducible subsets of the set $\mathbb{A}$. Note that, if $\mathcal{N}(f) \neq \varnothing$, then the set $\mathcal{N}(f)$ contains finite elements.

Proposition 5. *Let $\mathcal{N}(f) \neq \varnothing$. For every inreducible L there is a maximal element M in $\mathcal{N}(f)$ containing L.*

Proof. Consider an arbitrary finite or infinite chain (with respect to inclusion) of inreducible sets $\{L_\beta\}$. We claim that the set $K = \cup L_\beta$ is inreducible. Let $I \subset K$ be a finite family. Then there is a β such that $I \subset L_\beta$, and hence there is a finite inreducible family J' for which $I \subseteq J' \subseteq L_\beta \subseteq K$. The desired assertion follows from Zorn's lemma. □

Assuming that the set $\mathcal{N}(f)$ is nonempty, we choose some element M_f which is maximal with respect to inclusion in $\mathcal{N}(f)$.

Proposition 6. *Let $J \in \mathcal{N}(f)$, $J' \in \mathcal{I}$, and $J \subset J'$. Then there is a representation*

$$\{x \mid x_0 = f_{J'}(x_{J'})\} = \{x \mid q_0(x_{J_0}) = F(q_1(x_{J_1}), \dots, q_m(x_{J_m}))\}, \tag{2.4}$$

where q_i are n_i-ary quasigroups for $i \in \{0, \dots, m\}$, F is an inreducible m-ary quasigroup, and $\{J_i\}$ is a partition of the set $J' \cup \{0\} \in \mathcal{I}$ into families of cardinality $n_0, \dots, n_m$. Moreover, $|(J \cup \{0\}) \cap J_i| \leq 1$ for every i, $i \in \{0, \dots, m\}$.

Proof. If J is a maximal inreducible subset of J', then we obtain the desired statement from Corollary 1. Otherwise, one should apply Corollary 1 to a maximal inreducible set J'', $J \subset J'' \subseteq J'$. □

Note that $m \geq |J|$ in (2.4); however, it is not claimed that the inequality $m \geq |J|$ holds for all representations of the set $\{x \mid x_0 = f_{J'}(x_{J'})\}$. We write $[i,j]_{f,J' \supset J}$ if in the representation (2.4) for $f_{J'}$ we have $i, j \in J_l$ for some l, $0 \leq l \leq m$.

Proposition 7. *For every $i \notin M_f$, there is a family $J(i) \in \mathcal{I}$ for which $J(i) \subseteq M_f$ and a unique $\alpha(i) \in M_f \cup \{0\}$ such that $[i, \alpha(i)]_{f, J \supset J(i)}$ for any $J \in \mathcal{I}$ and $i \in J$ with $J(i) \subset J \subseteq M_f \cup \{0, i\}$.*

Proof. Since $i \notin M_f$, it follows that the quasigroup $f_{J' \cup \{i\}}$ is reducible for the finite families $J' \subset M_f$ exceeding some $J(i) \subseteq M_f$. Obviously, one can choose the family of indices $J(i)$ to be inreducible. For the sake of definiteness of our choice, we assume that the set of indices is ordered and the family $J(i)$ is lexicographically minimal among the possible families. By Proposition 6, the representation (2.4) holds. If an index $\alpha(i)$ takes distinct values $i_1, i_2 \in J(i)$ for different finite families $J', J'' \subseteq M_f$, then, substituting zero into all arguments except the arguments with indices in $\{0, i\} \cup J(i)$ in representations of the form (2.4) for the quasigroups $f_{J'}$ and $f_{J''}$, we arrive at a contradiction to the uniqueness of the representation of the quasigroup, which is ensured by Statement 1. □

Proposition 8. *For any families*

$$J', J'' \in \mathcal{I}, \quad i \in J'', \quad J' \subset M_f, \quad J(i) \subset J' \subset J'',$$

the relation $[i, \alpha(i)]_{f, J'' \supset J'}$ *holds.*

Proof. By Proposition 7, $[i, \alpha(i)]_{f, J' \cup \{i\} \supset J(i)}$. Consider the representation (2.4) for the quasigroup $f_{J''}$. In order to see that $[i, \alpha(i)]_{f, J'' \supset J'}$ holds, it suffices to substitute zero into all arguments except for the arguments whose indices belong to $J' \cup \{0, i\}$ and to apply Statement 1. □

Corollary 2. *For every finite family* $J \subset \mathbb{A} \setminus M_f$*, there is a family* $J' \in \mathcal{I}$*,* $J' \subset M_f$*, such that* $[i, \alpha(i)]_{f, J'' \supset J'}$ *for every family* $J'' \in \mathcal{I}$ *with* $J' \cup J \subset J''$ *and every* $i \in J$*.*

Proof. Consider a finite family $J' \subset M_f$ such that $f_{J'}$ is an inreducible quasigroup and $J(i) \subset J'$ for every $i \in J$. The desired statement follows from Proposition 8. □

Lemma 1. *If a set* $\mathbb{A}$ *is reducible and* $\mathcal{N}_f \neq \varnothing$*, then the quasigroup* $f \colon \mathcal{F} \to \Sigma$ *is reducible.*

Proof. Consider the set M_f; by assumption, $M_f \neq \mathbb{A}$. For every $m \in M_f$, we introduce the set of indices

$$A_m = \{i \notin M_f \mid \alpha(i) = m\} \cup \{m\}.$$

Without loss of generality, we may assume that $1 \in M_f$ and $|A_1| \geq 2$. Let $B = \mathbb{A} \setminus A_1$. We claim that, for any families of arguments y_B, y_{A_1}, y'_{A_1}, the equation $f(y_{A_1}, \overline{0}_B) = f(y'_{A_1}, \overline{0}_B)$ implies the equation $f(y_{A_1}, y_B) = f(y'_{A_1}, y_B)$. Consider arbitrary elements $y, y' \in \mathcal{F}$ and the finite family

$$J = \operatorname{supp} y_B \cup \operatorname{supp} y_{A_1} \cup \operatorname{supp} y'_{A_1}.$$

By Corollary 2, there is a family $J' \subseteq M_f$, $J \cap M_f \subset J'$, such that $[i, \alpha(i)]_{f, J'' \supset J'}$ for every finite family J'', $J' \cup J \subset J''$ and for every $i \in J \setminus M_f$. Then

$$f_{J''}(x_{A_1 \cap J''}, x_{B \cap J''}) = g(h(x_{A_1 \cap J''})_{\{1\}}, x_{B \cap J''}).$$

Let $f(y_{A_1}, \overline{0}_B) = f(y'_{A_1}, \overline{0}_B)$. Then $h(y_{A_1 \cap J''}) = h(y'_{A_1 \cap J''})$ and

$$f(y_{A_1}, y_B) = f_{J''}(y_{A_1 \cap J''}, y_{B \cap J''}) = f_{J''}(y'_{A_1 \cap J''}, y_{B \cap J''}) = f(y'_{A_1}, y_B).$$

Proposition 2 implies the desired assertion. □

Approaches similar to those used above were applied in [5] to study properties of reducible n-ary quasigroups.

3. COMPLETE REDUCIBILITY

According to the definition, all 2-ary quasigroups are inreducible. A reducible n-ary quasigroup f is said to be *completely commutatively reducible* if all its retracts of dimension greater than 2 are reducible and all retracts of dimension 2 are isotopic to commutative groups.

In what follows, we assume that $\Sigma = \{0, 1, 2, 3\}$, i.e., we speak only of n-ary quasigroups of order 4. As is well known (see, e.g., [1]), all 2-ary quasigroups of order 4 are isotopic to either the group $Z_2 \times Z_2$ or to the group Z_4. Therefore, reducible n-ary quasigroups of order 4 that do not contain inreducible retracts of dimension exceeding two are completely commutatively reducible. The following statement is an immediate corollary to a theorem in [3].

Statement 2. *One can represent every completely commutatively reducible n-ary quasigroup $f\colon \Sigma^n \to \Sigma$ in the form*

$$f(x_{J_1}, \dots, x_{J_k}) = q_1(x_{J_1}) * \dots * q_k(x_{J_k}), \tag{3.1}$$

where $*$ stands for a commutative group operation and q_j are n_j-quasigroups for $j \in \{1, \dots, k\}$ that cannot be represented in the form $q_j(x_{J'_j}, x_{J''_j}) = q'(x_{J'_j}) * q''(x_{J''_j})$. Moreover, in this representation, the finite families $\{J_j\}$ and the operation $*$ are determined uniquely (*for a chosen neutral element of the group*).

To a completely commutatively reducible n-quasigroup f with the representation (3.1) we assign a rooted tree $T(f)$ whose inner vertices are labeled by operations and the leaves by arguments of functions. Let us construct the tree by recurrence. Suppose that $T(g_j)$ is a tree corresponding to a function q_j, $j \in \{1, \dots, k\}$; then the tree of the n-ary quasigroup f is defined as the root labeled with the operation $*$ with k edges issuing from the root; moreover, the tree $T(q_j)$ is joined to the jth edge.

Let a quasigroup $f\colon \mathcal{F} \to \Sigma$ be such that the set $\mathcal{N}(f)$ is empty. Denote $T(f_J)$ by T_J, where $J \in \mathcal{I}$. Consider a minimal subtree in T_J which contains a pair of arguments with indices $i_1, i_2 \in J$, and denote by $C(J, i_1, i_2) \in \mathcal{I}$ the set of indices corresponding to hanging vertices of this subtree. The fact that the tree T_J and the set $C(J, i_1, i_2)$ are well defined follows from Statement 2. The set $C(J, i_1, i_2) \in \mathcal{I}$ can equivalently be defined in terms of the existence of the following representation:

$$f_J(x_J) = g((h_1(x_{C_1}) * h_2(x_{C_2}))_{\{i_1\}}, x_{J\setminus(C_1\cup C_2)}),$$

where $C_1 \cup C_2 = C(J, i_1, i_2)$ for $i_1 \in C_1$ and $i_2 \in C_2$, and the quasigroup $(h_1(x_{C_1}) * h_2(x_{C_2}))$ corresponds to the minimal subtree containing the pair of arguments with the indices $i_1, i_2 \in J$.

Proposition 9. *Let $J \subset J' \in \mathcal{I}$, $i_1, i_2 \in J$. Then $C(J, i_1, i_2) = C(J', i_1, i_2) \cap J$.*

Proof. We have

$$f_{J'}(x_{J'}) = g((h_1(x_{C_1}) * h_2(x_{C_2}))_{\{i_1\}}, x_{J'\setminus(C_1\cup C_2)}),$$

where $C_1 \cup C_2 = C(J', i_1, i_2)$ for $i_1 \in C_1$ and $i_2 \in C_2$. Substituting zero into all the arguments in $J' \setminus J$, we obtain the desired representation for the quasigroup f_J. ☐

Lemma 2. *Let a quasigroup $f\colon \mathcal{F} \to \Sigma$ be such that $\mathcal{N}(f) = \varnothing$. Then f is reducible.*

Proof. Without loss of generality, we may assume that f is reduced. If a quasigroup f is linear or f is isotopic to an iterated group of Z_4, then f is reducible. Otherwise there is a family $J = \{i_1, i_2, i_3\}$ such that $f_J(x_J) = (x_{i_1} *_1 x_{i_2}) *_2 x_{i_3}$, where $*_1$ and $*_2$ are distinct group operations. Consider the set $C = \bigcup C(J, i_1, i_2)$, where the union is taken over all finite families J containing i_1 and i_2. It follows from Proposition 9 that $i_3 \notin C$. Moreover, for every finite family $J \in \mathcal{I}$, $i_1, i_2 \in J$, we have $C(J, i_1, i_2) = C \cap J$. Indeed, if $j \in (C \cap J) \setminus C(J, i_1, i_2)$, then $j \in C(J', i_1, i_2)$ for some finite family J'. Then it follows from Proposition 9 that $j \in C(J' \cup J, i_1, i_2)$, and hence $j \in C(J, i_1, i_2)$.

We claim that $f(x_{\mathbb{A}}) = g(h(x_C)_{\{i_1\}}, x_{\mathbb{A}\setminus C})$. By Proposition 2, it suffices to prove that the equation $f(y_C, \overline{0}_{\mathbb{A}\setminus C}) = f(y'_C, \overline{0}_{\mathbb{A}\setminus C})$ implies the equation $f(y_C, y_{\mathbb{A}\setminus C}) = f(y'_C, y_{\mathbb{A}\setminus C})$ for every arguments $y_{\mathbb{A}\setminus C}$, y_C, and y'_C. Consider arbitrary $y, y' \in \mathcal{F}$; write $J = \operatorname{supp} y \cup \operatorname{supp} y' \cup \{i_1, i_2, i_3\}$. Then

$$f(y) = f_J(y_J) = g(h(y_{J\cap C})_{\{i_1\}}, y_{J\setminus C}).$$

It follows from the equation $f(y_C, \overline{0}_{\mathbb{A}\setminus C}) = f(y'_C, \overline{0}_{\mathbb{A}\setminus C})$ that $h(y_{J\cap C}) = h(y'_{J\cap C}) = a$ for some $a \in \Sigma$. Then

$$\begin{aligned} f(y_C, y_{\mathbb{A}\setminus C}) &= f_J(y_C, y_{\mathbb{A}\setminus C}) = g(a_{\{i_1\}}, y_{\mathbb{A}\setminus C}) \\ &= g(h(y'_{J\cap C})_{\{i_1\}}, y_{\mathbb{A}\setminus C}) = f_J(y'_C, y_{\mathbb{A}\setminus C}) = f(y'_C, y_{\mathbb{A}\setminus C}). \end{aligned}$$

☐

4. SEMILINEARITY

A quasigroup $f\colon \mathcal{F} \to \Sigma$ of order 4 is said to be *semilinear* if it satisfies the equation

$$f(\{0,1\}^{\mathbb{A}} \cap \mathcal{F}) = \{0,1\}$$

or if f is isotopic to a quasigroup satisfying this equation. In particular, a *reduced linear* quasigroup $f(x_{\mathbb{A}}) = \sum_{\beta\in\mathbb{A}} x_\beta$ is semilinear. Here the addition which is assumed is isomorphic to the group operation in $Z_2 \times Z_2$. The quasigroups isotopic to the reduced linear quasigroup are also referred to as *linear* ones.

It can readily be seen (see also [2]) that the semilinear quasigroups can be partitioned into three classes Q_1, Q_2, Q_3, in such a way that Q_a contains the quasigroups principally isotopic to quasigroups satisfying the equation $f(\{0,1\}^{\mathbb{A}} \cap \mathcal{F}) = \{0,a\}$.

The following statement was proved in [2].

Statement 3. *For $a \neq b$, the set $Q_a \cap Q_b$ coincides with the set of linear n-ary quasigroups.*

A description of the n-quasigroups of order 4 in the above terms was obtained in [6]. Namely, the following statement was proved.

Statement 4. *For every finite n, every n-ary quasigroup of order 4 is either reducible or semilinear.*

Using Statement 4, let us prove the following lemma for infinite-dimensional quasigroups.

Lemma 3. *Let f be a quasigroup of order 4 and $M \in \mathcal{N}(f)$. Then the quasigroup f_M is semilinear.*

Proof. Without loss of generality, we may assume that f is reduced. If M is finite, then the desired assertion follows from Statement 4. Below, we assume that the set M is infinite. Since $M \in \mathcal{N}(f)$, there is a direction $S \subseteq \mathcal{I}$ which converges to M. Let

$$B_a = \{J \in \mathcal{I} \mid J \in S,\ f_J \in Q_a\}.$$

It follows from Proposition 3 that at least one of the sets B_a, $a \in \{1,2,3\}$, forms a direction convergent to M. As is well known (see, e.g., [2]), if $J \in B_a$ $J' \subset J$, then $J' \in B_a$. Hence, B_a consists of all finite subsets of M.

If there are $a \neq b$ such that the directions B_a and B_b converge to M, then all quasigroups f_J, $J \in S$, are linear by Statement 3. Then the quasigroup f_M is linear. If only one of the directions B_a, $a \in \{1,2,3\}$, converges to M, then we set $c_0 = a$.

We similarly define c_i for every $i \in M$. Namely, let us define the quasigroup $f_M^{(i)}$ as the inversion of the quasigroup f with respect to the ith argument, i.e., in such a way that

$$\{(x,y) \mid x_i = f_M^{(i)}(x_{M\setminus\{i\}}, y_{\{i\}})\} = \{(x,y) \mid y = f_M(x_M)\}.$$

After this, we find an element $c_0 \in \{1,2,3\}$ for the quasigroup $f_M^{(i)}$ and denote this element by c_i.

Let $\theta_i = (1, c_i)$. Consider the quasigroup $g_M(x_M) = \theta_0 f_M(\theta_M x_M)$. It can readily be seen that $g_J(\{0,1\}^{|J|}) = \{0,1\}$ for every $J \subseteq M$, $J \in \mathcal{I}$. Then the quasigroups g_M and f_M are semilinear. $\square$

As is well known (see, e.g., [2]), every n-ary quasigroup of order 2 is of the form

$$f(x_1,\dots,x_n) = x_1 + \dots + x_n + \sigma \mod 2,$$

where $\sigma \in \{0,1\}$, and every n-ary quasigroup of order 3 is isotopic to the n-quasigroup

$$f(x_1,\dots,x_n) = x_1 + \dots + x_n + \sigma \mod 3.$$

Using an approach similar to that used above, one can readily prove the following proposition.

Proposition 10. (a) *Let* $f\colon \mathcal{F} \to \{0,1\}$ *be a quasigroup of order* 2. *Then* $f(x_{\mathbb{A}}) = p(x_{\mathbb{A}}) + \sigma$, *where* $\sigma \in \{0,1\}$ *and*

$$p(x_{\mathbb{A}}) = \sum_{i\in\mathbb{A}} x_i \bmod 2.$$

(b) *Let* $f\colon \mathcal{F} \to \{0,1,2\}$ *be a quasigroup of order* 3. *Then* f *is isotopic to a quasigroup* g, *for which*

$$g(x_{\mathbb{A}}) = \sum_{i\in\mathbb{A}} x_i \bmod 3.$$

The function p is referred to as the *parity check function*. Let $J \in \mathcal{I}$. Introduce a function $\chi\colon \mathcal{I} \to \{0,1\}^{\mathbb{A}}$ by the equation $\chi(J) = \overline{\delta}$, where $\delta_i = 1$ if and only if $i \in J$.

Proposition 11. *Let* g *be a semilinear quasigroup of order* 4. *Then* g *is isotopic to some quasigroup* f *whose restrictions* $f^J = f|_{\{2,3\}^J \times \{0,1\}^{\mathbb{A}\setminus J}}$ *are of the form*

$$f^J(x) = 2p(\chi(J)) + (p(x \bmod 2) + \sigma_J) \bmod 2.$$

Proof. By the definition of semilinearity, the quasigroup g is isotopic to a quasigroup f satisfying the equation $f(\{0,1\}^{\mathbb{A}} \cap \mathcal{F}) = \{0,1\}$. Then, by the definition of quasigroup,

$$f(\{2,3\}^J \times \{0,1\}^{\mathbb{A}\setminus J} \cap \mathcal{F}) = \begin{cases} \{0,1\} & \text{for } p(\chi(J)) = 0, \\ \{2,3\} & \text{for } p(\chi(J)) = 1. \end{cases}$$

The desired assertion follows from Proposition 10(a). ☐

5. MAIN RESULT

Theorem 1. *Let* $f\colon \mathcal{F} \to \Sigma$ *be a quasigroup of order* 4.

(a) *The set* $\mathbb{A}$ *is irreducible if and only if the quasigroup* $f_{\mathbb{A}}$ *is.*

(b) *The quasigroup* f *is either reducible or semilinear.*

Proof. (a) If $\mathbb{A}$ is finite, then the assertion is obvious. If $\mathbb{A}$ is infinite, then the assertion follows from Proposition 4 and from Lemmas 1 and 2.

(b) If $\mathbb{A}$ is finite, then we apply Statement 4. If $\mathbb{A}$ is infinite, then the assertion of part (b) follows from part (a) and from Lemma 3. ☐

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