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Construction of Hamiltonian Cycles with a Given Spectrum of Edge Directions in an n-Dimensional Boolean Cube

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Abstract—The *spectrum of a Hamiltonian cycle* (of a *Gray code*) in an n-dimensional Boolean cube is the series $a = (a_1, \dots, a_n)$, where a_i is the number of edges of the ith direction in the cycle. The necessary conditions for the existence of a Gray code with the spectrum a are available: the numbers a_i are even and, for $k = 1, \dots, n$, the sum of k arbitrary components of a is at least 2^k. We prove that there is some dimension N such that if the necessary condition on the spectrum is also sufficient for the existence of a Hamiltonian cycle with the spectrum in an N-dimensional Boolean cube then the conditions are sufficient for all dimensions n.

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INTRODUCTION

In the chapter of [8] dealing with the Gray codes (Hamiltonian cycles of a Boolean n-cube), D. Knuth pointed out the three problems unresolved at the time of publication. The first problem is to estimate the number of distinct Gray codes in a Boolean n-cube. The degree of the logarithm of this number is found in [1], and the asymptotic behavior of the logarithm of this number as $n \to \infty$ is determined in [6]. The second problem poses the question that was earlier brought up by G. Kreweras [9]: *Can each perfect matching of a Boolean n-cube be extended to a Hamiltonian cycle*? The affirmative answer to this is given by J. Fink [7]. In the case when the matching contains the edges only of a few directions, the extendability of a perfect matching to a Hamiltonian cycle is proved in [1]. In the third problem, the question is put whether the necessary conditions (formulated in the annotation) on the spectrum of a Hamiltonian cycle are also sufficient for the existence of a Gray code with such a spectrum.

In this article, some asymptotic solution is offered as a solution to the last problem. Namely, if the necessary conditions are sufficient in a Boolean n-cube for a reasonably large n then they are sufficient for all n. This result is announced in [4]. Note that several methods are available for construction of some Gray codes with various properties; in particular, in [5, 10], the Hamiltonian cycles are constructed with a totally balanced spectrum.

1. DEFINITIONS

The *Boolean n-cube* is the set Q_n of the binary words of length n. A graph GQ_n with vertices from Q_n in which two vertices are connected if and only if the corresponding words differ in exactly one position is also called a *Boolean n-cube*. Each edge $\{u, v\}$ of GQ_n has direction $i \in \{1, \dots, n\}$ if i is equal to the number of the position in which the words u and v differ. A collection P of edges of a graph G is called a *perfect matching* if each vertex of G is incident to exactly one edge from P. A *Hamiltonian cycle* of G is a cycle including each vertex exactly once. In a bipartite graph and, in particular, in GQ_n, the vertices of each Hamiltonian cycle can be decomposed into some two perfect matching. The *spectrum of a Hamiltonian cycle of the Boolean n-cube* is the series $a = (a_1, a_2, \dots, a_n)$, where a_i is the number

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of edges of the ith direction in the cycle. The spectrum of a perfect matching of GQ_n is defined in a similar fashion. It is known [2] that an arbitrary series of integers $(a_1, \dots, a_n)$ that meets the following conditions

1) a_i is a nonnegative even number for $i = 1, \dots, n$;

2) $\sum_{i=1}^n a_i = 2^{n-1}$

is a spectrum of some perfect matching of GQ_n. These conditions are not only sufficient, but necessary as well [2]. The necessary conditions for a set of integers $(a_1, \dots, a_n)$ to be a spectrum of some Hamiltonian cycle of GQ_n are also available [8]:

$(*)$ a_i is a nonnegative even number for $i = 1, \dots, n$;

$(**)$ $\sum_{i=1}^n a_i = 2^n$;

$(***)$ $\sum_{i=1}^k a_{\pi(i)} \geq 2^k$ for every permutation $\pi \in S_n$ and $k = 1, \dots, n-1$.

The conditions $(*)$ and $(**)$ are evident, and $(***)$ follows from the connectivity of the cycle.

Without loss of generality we can assume that the spectrum of a Hamiltonian cycle is ordered so that $a_i \leq a_j$ if $i \leq j$. Then $(***)$ can be rewritten as

$$\sum_{i=1}^k a_i \geq 2^k \quad \text{for} \quad k \leq n.$$

Define the nonnegative function μ_a of an ordered spectrum as

$$\mu_a(k) = \sum_{i=1}^k a_i - 2^k.$$

If a series of integers satisfies the above necessary conditions $(*)$–$(***)$ then we call it *admissible*. Denote the set of admissible n-dimensional series by $\mathbb{D}_n$. It is clear that each Hamiltonian cycle of GQ_n contains the edges of all directions, while a perfect matching of GQ_n can contain the edges from 1 to n directions inclusively. Let us call a matching of GQ_n that contains the edges from all n directions a *matching of full rank*. We call a Hamiltonian cycle that contains a perfect matching of full rank a *cycle of full rank*.

Proposition 1. *If each admissible integer-valued series is a spectrum of a Hamiltonian cycle of GQ_n for some n then the same holds for all m such that $2 \leq m \leq n$.*

Proof. Indeed, let $(a_1, \dots, a_m)$ be an admissible series. Then the series

$$(a_1, \dots, a_m, 2^m, \dots, 2^{n-1})$$

is also admissible. And the projection of a Hamiltonian cycle with spectrum

$$(a_1, \dots, a_m, 2^m, \dots, 2^{n-1})$$

onto the first m direction produces a Hamiltonian cycle with spectrum $(a_1, \dots, a_m)$. The proof is over. □

Each (in particular, Hamiltonian) cycle C of a graph GQ_n can be associated with a cyclical *transient word* X in the alphabet $\{1, \dots, n\}$ in which the jth letter x_j is defined as the direction of the jth edge in C. Let $S(X) = (s_1, \dots, s_n)$ denote the *series constituted* by the word X modulo 2; i.e., $s_i = 0$ if the letter $i \in \{1, \dots, n\}$ occurs evenly many times in $S(X)$ and $s_i = 1$ otherwise.

Proposition 2 [3]. *A cyclic word X in the alphabet $\{1, \dots, n\}$ defines a simple cycle of a graph GQ_n if and only if $S(X) = \overline{0}$ and $S(Y) \neq \overline{0}$. for each subword $Y \neq X$.*

From this we immediately obtain

Corollary 1. *A cyclical word X in the alphabet $\{1, \dots, n\}$ defines a Hamiltonian cycle in GQ_n if and only if the length of X equals 2^n, $S(X) = 0$, and $S(Y) \neq \overline{0}$ for each subword $Y \neq X$.*

2. CONSTRUCTION OF A HAMILTONIAN CYCLE

The purpose of this article is to prove that each admissible series is a spectrum of some Hamiltonian cycle in Boolean n-cube if this is so for a Boolean N-cube for a sufficiently large N. Let us construct a Hamiltonian cycle by using the representation of a Boolean n-cube as the Cartesian product of the cubes of dimensions k and $n-k$.

Consider a Hamiltonian cycle of GQ_k that consists of the edges from some disjoint perfect matchings P_1 and P_2. We embed P_1 into GQ_n in a natural way. Since each vertex of GQ_k in the Cartesian product $GQ_k \times GQ_{n-k}$ corresponds to a Boolean $(n-k)$-cube; therefore, each edge of P_1 can be associated with a pair of parallel $(n-k)$-cubes; i.e., one $(n-k+1)$-cube. Let us replace each edge $v \in P_1$ with a Hamiltonian cycle H_v in the $(n-k+1)$-cube containing this edge. By removing P_1 from the union of P_2 and the cycles H_v, $v \in P_1$, we obtain a new Hamiltonian cycle of

$$GQ_k \times GQ_{n-k} = GQ_n.$$

We formulate the above construction as

Lemma 1. *Let some matchings of GQ_k with spectra $(b_1, \ldots, b_k)$ and $(b'_1, \ldots, b'_k)$ make up a Hamiltonian cycle, and let there be 2^{k-1} Hamiltonian cycles in GQ_{n-k+1} with spectra*

$$\left(a^i_1, \ldots, a^i_{n-k}, c^i\right), \qquad i = 1, \ldots, 2^{k-1}.$$

Then there is a Hamiltonian cycle of GQ_n with spectrum $(d_1, \ldots, d_n)$, where

$$d_{k+j} = \sum_{i=1}^{2^{k-1}} a^i_j, \qquad j = 1, \ldots, n-k,$$

$$d_j = b_j + \sum_{p=s_j+1}^{s_{j+1}} (c^p - 1), \qquad j = 1, \ldots, k,$$

$$s_j = \sum_{p=1}^{j-1} b'_p.$$

Proof. Let

$$X = x_1 y_1 x_2 \ldots x_{2^{k-1}} y_{2^{k-1}}$$

be a transient word of a Hamiltonian cycle of GQ_k, where the matchings that correspond to the words $x_1 x_2 \ldots x_{2^{k-1}}$ and $y_1 y_2 \ldots y_{2^{k-1}}$ have the spectra $(b_1, \ldots, b_k)$ and $(b'_1, \ldots, b'_k)$ correspondingly. Let $y_i Z^i$ denote a transient word in a Hamiltonian cycle of GQ_{n-k+1} in the alphabet $\{k+1, \ldots, n, y_i\}$ with the spectrum

$$(a^i_1, \ldots, a^i_{n-k}, c^i), \qquad i = 1, \ldots, 2^{k-1},$$

where y_i occurs $c^i - 1$ times in Z^i.

Consider the word Z obtained from X by replacing the letters y_i with the words Z^i; i.e.,

$$Z = x_1 Z^1 x_2 Z^2 \ldots x_{2^{k-1}} Z^{2^{k-1}}.$$

Let us show that Z is a transient word of some Hamiltonian cycle of GQ_n. Consider an arbitrary subword

$$W = U_i x_{i+1} Z^{i+1} \ldots Z^{j-1} x_j V_j$$

of Z, where U_i is a suffix of the word Z^i and V_j is a prefix of Z^j (the prefix and/or suffix can be empty). If $i = j$ then $S(W) \neq \overline{0}$ since W is a subword of the transient word $y_i Z^i$. Let $i < j$ and, without loss of generality, $y_i = 1$ and $y_j \in \{1, 2\}$. Consider the case when $y_j = 1$. Let

$$0 = S(W)_t = S(x_{i+1} y_{i+1} \ldots x_j)_t = S(y_i x_{i+1} y_{i+1} \ldots x_j)_t, \qquad t = 2, \ldots, k.$$

Since either $S(x_{i+1}y_{i+1}\dots x_j)_1 = 0$ or $S(y_i x_{i+1} y_{i+1}\dots x_j)_1 = 0$; therefore, we have a contradiction to the fact that X is a transient word of a Hamiltonian cycle. The case when $y_j = 2$ can be considered in a similar way. It is easy that

$$S(Z) = S(X) = \overline{0}.$$

Therefore, the word Z satisfies the conditions of Corollary 1, and Z is a transient word of a Hamiltonian cycle of GQ_n. By construction, the Hamiltonian cycle that corresponds to Z has the required spectrum.

The proof of Lemma 1 is complete. □

Remark 1. If a matching with spectrum $(b'_1, \dots, b'_k)$ and at least one of the cycles of GQ_{n-k+1} used in the construction have full rank then we can obtain a Hamiltonian cycle of full rank as a result of the construction (Lemma 1).

3. THE MAIN RESULT

We now prove two lemmas about spectra of Hamiltonian cycles that can be produced using the above construction.

Lemma 2. *If every admissible series of integers of length $n-k+1$ is the spectrum of some Hamiltonian cycle of GQ_{n-k+1} and there is a Hamiltonian cycle of GQ_k with the spectrum $(b_1, \dots, b_k)$ then each admissible series of integers $(b_1, \dots, b_k, a_{k+1}, \dots, a_n)$ is the spectrum of a Hamiltonian cycle of GQ_n.*

Proof. Consider the lexicographical order on the set of ordered admissible integer-valued series

$$A = \{a \in \mathbb{D}_n \mid a_i = b_i,\ 1 \le i \le k\}.$$

The minimal series $(b_1, \dots, b_k, 2^k, \dots, 2^{n-1})$ in this order is the spectrum of a Hamiltonian cycle of GQ_n. Indeed, it is enough to choose $(2, 4 \dots, 2^{n-k}, 2)$ as the spectrum of $\left(a^i_1, \dots, a^i_{n-k}, c^i\right)$ for all $i = 1, \dots, 2^{k-1}$ in the above construction.

Suppose that some admissible series in A are not spectra of Hamiltonian cycles constructed by Lemma 1. Choose a lexicographically minimal $d \in A$ among these series. Consider the preceding series $d' \in A$ within the lexicographical order. It is clear that d and d' differ in the two positions $i, j \in \{k+1, \dots, n\}$, $i < j$, where

$$d'_i = d_i - 2, \qquad d'_j = d_j + 2, \qquad d'_j \ge d'_i + 4.$$

Then, for the spectrum f of one of the Hamiltonian cycles of GQ_{n-k+1} used to construct a cycle with the spectrum d', $f_j - f_i \ge 4$ holds (or $f_j - f_i = 2$ in the spectra of the two cycles). Obviously, if we change f_i with $f_i + 2$ in the integer-valued series f and f_j with $f_j - 2$ then, by hypothesis, the new series f' will also be a spectrum of a Hamiltonian cycle of GQ_{n-k+1}.

Replacing the cycle with spectrum f in Lemma 1 with a cycle with spectrum f', we see that the series d is the spectrum of a Hamiltonian cycle; a contradiction. In the case when $f_j - f_i = 2$ in two spectra, we interchange the amounts of edges of directions i and j and apply the same construction.

The proof of Lemma 2 is complete. □

Lemma 3. *Let $4 \le s < k < n-1$; let all admissible integer-valued series of length $n-s+1$ be spectra of Hamiltonian cycles; let, for some admissible ordered series $b \in \mathbb{D}_n$,*

(i) *there be an admissible series $b' \in \mathbb{D}_s$ that is the spectrum of a Hamiltonian cycle of full rank such that $b'_i \le b_i$ for $i = 1, \dots, s$;*

and let the equations

(ii) $b_s \le 2^{n-s-1}$, (iii) $\sum_{i=s+1}^{k} b_i \ge 2^k - 2^s$

hold for k such that $2^{k-s-1} \le b_s$.

Then the series b is the spectrum of a Hamiltonian cycle.

Proof. Let $d_i = b_i - b_i'$ for $i = 1, \dots, s$ and $d = \sum_{i=1}^{s} d_i$.

From the proof of Lemma 2 it follows that the integer-valued series $(b_1', \dots, b_s', 2^s, \dots, 2^{n-1})$ is the spectrum of a Hamiltonian cycle of GQ_n constructed by the above construction with $s = k$. Let m be the least integer such that $2^{m-s-1} > b_s$. Since $d_i \le b_s < 2^{m-s-1}$, the series

$$(2, \dots, 2^{m-s-1}, 2^{m-s} - d_i, 2^{m-s+1}, \dots, 2^{n-s}, 2 + d_i)$$

is admissible for all $i = 1, \dots, s$.

Let us now interchange s Hamiltonian cycles (one for each direction) with spectrum

$$(2, 4, \dots, 2^{n-s}, 2)$$

in this construction with a Hamiltonian cycle with spectrum

$$(2, \dots, 2^{m-s} - d_i, 2^{m-s+1}, \dots, 2^{n-s}, 2 + d_i)$$

so that as a result we have a cycle with spectrum

$$(b_1, \dots, b_s, 2^s, \dots, 2^m - d, 2^{m+1}, \dots, 2^{n-1}).$$

Consider the set of ordered admissible integer-valued series

$$A = \{a \in \mathbb{D}_n \mid a_i = b_i,\ 1 \le i \le s,\ \text{the spectrum } a \text{ satisfies condition (iii)}\}.$$

It is proven above that the lexicographically least spectrum from A belongs to a Hamiltonian cycle obtained with by our construction (Lemma 1). Suppose that some of the series $a \in A$ are not spectra of so-obtained Hamiltonian cycles. Then there is the lexicographically least spectrum among them. By analogy with the proof of Lemma 2 we have a contradiction.

The proof of Lemma 3 is complete. □

Proposition 3. *Let* $2 \le s \le k \le n$, $a \in \mathbb{D}_n$, *and let k be such that*

$$\sum_{i=s+1}^{k} a_i < 2^k - 2^s, \qquad 2^{k-s-1} \le a_s. \tag{A}$$

Then $\mu_a(s) - \mu_a(k) \ge 2$ *and* $k \le 2s + \log \mu_a(s)$.

Proof. Since

$$2^{k-s-1} \le a_s \le 2^{s-1} + \mu_a(s) \le 2^{s-1}\mu_a(s),$$

we have $k \le 2s + \log \mu_a(s)$. From the first inequality in (A) it follows that

$$\sum_{i=s+1}^{k} a_i \le 2^k - 2^s - 2.$$

Therefore,

$$\mu_a(k) = \sum_{i=1}^{k} a_i - 2^k = \sum_{i=1}^{s} a_i - 2^s + \sum_{i=s+1}^{k} a_i - 2^k + 2^s \le \mu_a(s) - 2.$$

The proof is complete. □

Theorem. *There is a number N such that if all admissible integer-valued series of length N is the spectrum of some Hamiltonian cycle (of full rank in the case when* $\sum_{i=1}^{k} a_i > 2^k$ *for all* $k < N$*) then, for every integer* $n \ge 2$, *each admissible integer-valued series of length n is the spectrum of some Hamiltonian cycle of* GQ_n.

Proof. Take $a \in \mathbb{D}_n$. It is easy to prove by simple exhaustive search through the admissible series that $a_2 \geq 4$ and $a_4 \geq 6$, except for the case when $a_1 = a_2 = 2$ and $a_1 = a_2 = a_3 = a_4 = 4$. The Hamiltonian cycle of GQ_4 with the transient series 1213414243212343 has the spectrum $(4, 4, 4, 4)$. The existence of a Hamiltonian cycle with the spectrum $a \in \mathbb{D}_n$ for $n \geq 5$ and $a_1 = a_2 = a_3 = a_4 = 4$ $(a_1 = a_2 = 2)$ follows from Lemma 2.

Assume that $a_2 \geq 4$ and $a_4 \geq 6$. It is easy that $a_1 \geq 2$ and $a_3 \geq 4$ in every ordered admissible sequence. There is a Hamiltonian cycle of full rank in GQ_4 with the transient series 4212312141312313 and the spectrum $(2, 4, 4, 6)$. Thus, if $s = 4$ then the condition (i) of Lemma 3 and the hypothesis of Lemma 2 hold. Consider the conditions (ii) and (iii) of Lemma 3 for $s = 4$. Let $n \geq 35$, then

$$a_4 \leq \frac{2^n}{n-3} \leq 2^{n-5}.$$

Suppose that the condition (iii) fails; i.e., for some $k \geq 5$, we have

$$2^{k-5} \leq a_4. \qquad \sum_{i=5}^{k} a_i < 2^k - 2^4.$$

Then

$$(k-4)2^{k-5} \leq (k-4)a_4 < 2^k - 2^4,$$

whence it follows that $k \leq 35$ and $a_4 < \frac{2^{35}-2^4}{31}$. Therefore,

$$\mu_a(4) < \mu^* = 4 \cdot \frac{2^{35} - 2^4}{31}.$$

Take

$$n \geq N = 2^{2^{\mu^*/2}(4+\log \mu^*)}$$

and induct. Suppose that, for $m < n$, all admissible series is a spectrum of a Hamiltonian cycle (I) and if $m > 4$ and $\sum_{i=1}^{k} a_i > 2^k$ for any $k < m$ then the series $a \in \mathbb{D}_m$ is the spectrum of a Hamiltonian cycle of full rank (II). To perform the inductive step, we check that the hypotheses of Lemma 2 and 3 are met. The condition (i) of Lemma 3 is true by inductive hypothesis. It is easy that

$$a_s \leq \frac{2^n}{n-s+1}.$$

Therefore, the inequality (ii) $a_s \leq 2^{n-s-1}$ holds for

$$s \leq 2^{\mu^*/2}(4 + \log \mu^*) - 1.$$

If $\mu_a(4) \geq \mu^*$ then, as shown above, the condition (iii) holds. Take $\mu_a(4) < \mu^*$. We define the sequence of numbers s_i recurrently. Let $s_0 = 4$. If s_i is already selected then, for s_{i+1}, we select minimal k such that the condition (iii) of Lemma 3 fails; i.e.,

$$\sum_{i=s_i+1}^{s_{i+1}} a_i < 2^{s_{i+1}} - 2^{s_i}.$$

By Proposition 3, there are at most $M = \mu^*/2$ elements in the sequence $s_1 < s_2 < \cdots < s_M$ such that $\mu_a(s_i) > 0$. Moreover, by Proposition 3,

$$s_M \leq 2^M(4 + \log \mu^*) - 1.$$

If the next s_j cannot be selected then we use Lemma 3 to construct the desired Hamiltonian cycle; and if $\mu_a(s_j) = 0$ then we use Lemma 2. Thus, the inductive hypothesis (I) is proven. The hypothesis (II) follows from Remark 1.

Proposition 1 completes the proof of the theorem. □

To obtain the complete solution for the problem of spectra of the Gray codes we need to provide the base of induction to apply the theorem. The Hamiltonian cycles of GQ_n for $n \leq N$ with arbitrary admissible spectra can be built using the above construction as well as other known constructions such as Bakos's construction [10].

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