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MULTIDIMENSIONAL LATIN BITRADES

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Abstract: A subset of the n-dimensional k-valued hypercube is a *unitrade* or *united bitrade* whenever the size of its intersections with the one-dimensional faces of the hypercube takes only the values 0 and 2. A unitrade is *bipartite* or *Hamiltonian* whenever the corresponding subgraph of the hypercube is bipartite or Hamiltonian. The pair of parts of a bipartite unitrade is an *n-dimensional Latin bitrade.* For the n-dimensional ternary hypercube we determine the number of distinct unitrades and obtain an exponential lower bound on the number of inequivalent Latin bitrades. We list all possible n-dimensional Latin bitrades of size less than 2^{n+1}.

A subset of the n-dimensional k-valued hypercube is a *t-fold* MDS *code* whenever the size of its intersection with each one-dimensional face of the hypercube is exactly t. The symmetric difference of two single MDS codes is a bipartite unitrade. Each component of the corresponding Latin bitrade is a switching component of one of these MDS codes. We study the sizes of the components of MDS codes and the possibility of obtaining Latin bitrades of a size given from MDS codes. Furthermore, each MDS code is shown to embed in a Hamiltonian 2-fold MDS code.

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Introduction

Put $Q_k = \{0, 1, \dots, k-1\}$ and denote by Q_k^n the set of ordered k-tuples of length n, called the *vertices.* The *Hamming distance* $d(x, y)$ between two vertices $x, y \in Q_k^n$ is the number of positions which x and y differ in. Denote by ΓQ_k^n the graph of minimal distances in the metric space $\left(Q_k^n, d\right)$. A *face* of dimension k is a subset of the hypercube Q_k^n consisting of the vertices with the same fixed values of certain $n-k$ chosen coordinates. In particular, the one-dimensional face in direction i passing through the vertex $(a_1, \dots, a_n) \in Q_k^n$ is defined as $\{(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n) \mid x \in Q_k\}$.

Refer as an MDS *code* with distance d to a set $M \subset Q_k^n$ intersecting each $(d-1)$-dimensional face in exactly one vertex. Refer to a set W as a *t-fold* MDS *code* if W intersects each face in exactly t vertices. The concept of t-fold MDS code with distance d coincides with the concept of correlation-immune function of order $n-d+1$ on Q_k^n. It is not difficult to see that $M \subset Q_k^n$ is an MDS code with distance d if and only if the size of M is k^{n-d+1} and the distance between two arbitrary distinct elements of M is at least d. A t-fold MDS code is called *splittable* whenever it is the union of t MDS codes. Unsplittable t-fold MDS codes with distance 2 are considered in [1]. In this article we consider only MDS codes (including t-fold codes) with distance 2.

A graph is called *Hamiltonian* whenever it includes a simple cycle passing through all vertices of the graph. Given an MDS code M, we can define the graph of minimal distances $\Gamma_2 M$ in which all pairs of vertices at distance 2 are joined by edges. The definition of MDS code implies that $\Gamma_2 M$ is always a Hamiltonian graph. Given a 2-fold MDS code $D \subset Q_k^n$, denote by ΓD the graph of minimal distances of D; thus, connect by edges all pairs of vertices at distance 1. In Section 3 we prove that each MDS

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code embeds in a Hamiltonian 2-fold MDS code. Observe that a 2-fold MDS code D is splittable if and only if ΓD is bipartite.

Refer to $B \subset Q_k^n$ as a *unitrade*[1)] whenever the size of the intersection of B with a one-dimensional face can take only the values 0 and 2. Call a unitrade $B \subset Q_k^n$ *bipartite* if the subgraph ΓB of ΓQ_k^n induced by the set of vertices of B is bipartite. The symmetric difference of two MDS codes is clearly a bipartite unitrade.

Since each one-dimensional face of an MDS code contains only one point, we may assume that an MDS code in Q_k^n implicitly defines a function of $n-1$ variables. The table of values of this function is an $(n-1)$-dimensional Latin cube of order k (Latin square for $n=3$), and we may regard the MDS code as the graph of this function. Refer as a *Latin bitrade* to a pair of partial Latin squares the union of whose graphs is a bipartite unitrade. We extend the term “Latin bitrade” to the multidimensional case. Furthermore, when a bipartite unitrade splits into parts uniquely (consists of one connected component), we call a *Latin bitrade* not only the pair of parts, but the unitrade itself.

Say that a bipartite unitrade (Latin bitrade) B obtains from an MDS code M_1 (possibly t-fold) whenever there is an MDS code M_2 (of the same foldness) with $B = M_1 \triangle M_2$. In this case refer to the set $B \cap M_1$ as a *component* of the MDS code M_1 in accordance with the general idea of a switching component of a code as its subset which we can replace by a set of the same size while preserving the code distance. The switching components of the MDS codes M_1 and M_2 correspond to the parts of the connected components of ΓB which are the components of the Latin bitrade corresponding to the unitrade B.

This article studies the number of unitrades and Latin bitrades in the hypercube Q_k^n, the possibility of obtaining Latin bitrades from MDS codes, and the sizes of the components of MDS codes. In Section 2 for the n-dimensional ternary hypercube we determine the number (2^{2^n}) of distinct unitrades and obtain the asymptotic (as $n \to \infty$) lower bound[2)] $e^{\Omega(\sqrt{n})}$ on the number of inequivalent Latin bitrades. Nontrivial upper bounds on the number of Latin bitrades are unavailable. For $k \geq 4$ the number, and even the asymptotics of, the double logarithm of the number of unitrades and Latin bitrades also remain unknown. Some connections of the question about the number of Latin bitrades in Q_k^n to other combinatorial problems are discussed in [3].

In Section 1 we list all possible n-dimensional unitrades of size less than 2^{n+1} and prove that for each $s \in \{0, \dots, n-1\}$ there is a unique, up to equivalence, n-dimensional Latin bitrade of size $2^{n+1} - 2^{s+1}$. In Section 3 we prove that these Latin bitrades obtain from 2-fold MDS codes, and thus partially determine the spectrum of sizes of the components of 2-fold MDS codes.

The construction of [1, 4] enables us to obtain t-fold binary perfect codes from t-fold MDS codes. Some questions related to the sizes of the components of perfect binary codes and the Hamiltonian property of their graphs are considered in [5, 6]. The results of this article concerning the sizes of components of MDS codes and the Hamiltonian property of their graphs can be applied to study the properties of other classes of codes.

§ 1. Sizes of Unitrades

Proposition 1. (a) *The Cartesian product of two unitrades is a unitrade.*

(b) *The Cartesian product of two bipartite unitrades is a bipartite unitrade.*

Proof. Claim (a) is straightforward from the definitions, and claim (b), from the fact that the Cartesian product of bipartite graphs is a bipartite graph.

The following two claims are established in [3] for $k = 4$, but we can prove them in exactly the same fashion for arbitrary $k \geq 2$.

Proposition 2. *If $B \subset Q_k^n$ is a nonempty unitrade then $|B| \geq 2^n$.*

1) The term “2-code” is used in [2].

2) The equality $f(n) = \Omega(g(n))$ means that $f(n) \geq cg(n)$ for some $c > 0$ as $n \to \infty$.

Proposition 3. *Given a unitrade $B \subset Q_k^n$, the following are equivalent:*
(a) $|B| = 2^n$;
(b) *the size of the intersection of each m-dimensional face with B equals* 0 *or* 2^m;
(c) *B intersects only two facets in each direction;*
(d) *a subgraph of the graph ΓB is isomorphic to the Boolean cube ΓQ_2^n.*

Proposition 2 implies directly that Q_2^n includes the unique unitrade that coincides with the entire set Q_2^n.

Consider the question of the spectrum of sizes of unitrades.

Proposition 4. *If $B \subset Q_k^n$ is a unitrade with $2^{n+1} > |B| \geq 2^n$ then $|B| = 2^{n+1} - 2^{s+1}$, where $s \in \{0, \dots, n-1\}$.*

Proof. Induct on n. The claim for $n = 1$ is obvious. Assume it for $n-1$. If a unitrade $B \subset Q_k^n$ lies in the union of two facets in each direction then $|B| = 2^n$ by Proposition 3.

If B intersects four facets in one direction then by Proposition 3 the size of B is greater than or equal to 2^{n+1}. Suppose that B intersects three facets in one direction. If the intersection with at least one of them is of size greater than or equal to 2^n then Proposition 2 yields $|B| \geq 2^{n+1}$. Otherwise, the inductive assumption yields $|B| = 3 \cdot 2^n - 2^{s_1} - 2^{s_2} - 2^{s_3}$. Since $2^{s_1} + 2^{s_2} + 2^{s_3} > 2^n$ only when at least two of three s_i are equal to $n-1$, it follows that $|B| = 2^{n+1} - 2^s$.

It is clear from the proof of Proposition 4 that a unitrade $B \subset Q_k^n$ of size less than 2^{n+1} intersects at most three facets in each direction; consequently, it embeds into the ternary hypercube.

Proposition 5. (a) *The symmetric difference of two unitrades in Q_3^n is a unitrade.*

(b) *If the symmetric difference of two Latin bitrades in Q_k^n is a unitrade, while their intersection induces a connected subgraph of Q_k^n, then the symmetric difference is a Latin bitrade.*

Proof. Claim (a) is straightforward from the definitions, and claim (b), from the fact that every connected bipartite graph splits into parts uniquely.

Proposition 6. *For every $s \in \{0, \dots, n-1\}$ there exists up to equivalence a unique unitrade $B_s \subset Q_3^n$ with $|B_s| = 2^{n+1} - 2^{s+1}$. The unitrades B_s are Latin bitrades.*

Proof. According to Proposition 5, the set $B_s = (\{0,1\}^{n-s} \triangle \{1,2\}^{n-s}) \times \{0,1\}^s$ is a Latin bitrade. It is obvious that $|B| = 2^s(2^{n-s} + 2^{n-s} - 2)$.

To prove uniqueness, consider a unitrade $B \subset Q_3^n$ with $|B| < 2^{n+1}$. As in the proof of Proposition 4, we infer that the intersections of B with the facets in some direction are of sizes 2^{n-1}, 2^{n-1}, and $2^n - 2^s$. By Proposition 3 two intersections are equivalent to Boolean cubes, and then the definition of unitrade implies that the third intersection is the symmetric difference of two Boolean cubes.

§ 2. The Number of Unitrades in Q_3^n

Regard the set of functions $g : Q_3^n \to \{0,1\}$ as the vector space $\mathbb{V}(n)$ over the field $GF(2)$. The characteristic functions of unitrades constitute the subspace $\mathscr{B}(n)$ of $\mathbb{V}(n)$ for which we can choose as a basis the characteristic functions χ^B of the hypercubes $B = \{\alpha_1, 2\} \times \dots \times \{\alpha_n, 2\}$ with $\alpha_i \in \{0,1\}$. The tuple of coefficients in the expansion of a function g in $\{\chi^B\}$ is a Boolean function of the tuple $(\alpha_1, \dots, \alpha_n)$. Below we explicitly define a transformation which associates to each Boolean function an element of $\mathscr{B}(n)$.

Define the partial order on Q_3 as follows: $0 < 2$ and $1 < 2$, while 0 and 1 are incomparable. Given $(x_1, \dots, x_n), (y_1, \dots, y_n) \in Q_3^n$, write $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ whenever $x_i < y_i$ or $x_i = y_i$ for all $i \in \{1, \dots, n\}$. Observe that $\{x \in \{0,1\}^n \mid x \leq y\}$ is a face of the n-dimensional Boolean hypercube of dimension $wt(y)$ equal to the number of symbols 2 in y. Moreover, the set of faces is in a bijective correspondence with the set of tuples $y \in Q_3^n$.

Given a Boolean function f, define $U[f] : Q_3^n \to \{0,1\}$ as $U[f](y) = \bigoplus_{x \leq y} f(x)$. Observe that $U[f]|_{\{0,2\}^n}$ is the Möbius transform of f.

Proposition 7. (a) *If* $A \subseteq \{0,1\}^n$ *then* $U[\chi^A] \in \mathscr{B}(n)$.
(b) *If* $g \in \mathscr{B}(n)$ *then* $U[g|_{\{0,1\}^n}] = g$.

Proof. (a) Take $f = \chi^A$. The definition of U implies that

$$U[f](a_1, \dots, a_{i-1}, 2, a_{i+1}, \dots, a_n) \\ = U[f](a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) \oplus U[f](a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n)$$

for all $a_j \in Q_3$. Consequently, $U[f]$ has evenly many 1's on each one-dimensional face of Q_3^n, and so $U[f] \in \mathscr{B}(n)$.

(b) We can easily establish that $U[g|_{\{0,1\}^n}](y) = g(y)$ for every $y \in Q_3^n$ by induction on the number of symbols 2 in y.

Proposition 7 implies that the dimension of the subspace $\mathscr{B}(n)$ is 2^n. Consequently, we have

Proposition 8. *The hypercube* Q_3^n *includes exactly* 2^{2^n} *distinct unitrades.*

Regard the set of functions $g : Q_3^n \to R$ as the vector space $\mathbb{V}_R(n)$ over the field R of real numbers. Consider the linear subspace $\mathscr{B}_R(n)$ consisting of the functions the sum of whose values on each one-dimensional face vanishes. Take a bipartite unitrade $B \subset Q_3^n$. Define the function $h_B : Q_3^n \to \{-1, 0, 1\}$ taking the value 1 on the first part of the unitrade, -1 on the second, and 0 on the remaining vertices of the hypercube. It is clear that $h_B \in \mathscr{B}_R(n)$. Define the operator U_R on the real-valued functions by analogy with U. Given $f : Q_2^n \to R$, we have

$$U_R[f](y) = (-1)^{wt(y)} \sum_{x \le y} f(x). \tag{1}$$

The next statement is similar to Proposition 7.

Proposition 9. (a) $U_R[f] \in \mathscr{B}_R(n)$ *for every function* $f : Q_2^n \to R$.
(b) *If* $g \in \mathscr{B}_R(n)$ *then* $U_R[g|_{\{0,1\}^n}] = g$.

Consider the set $\mathscr{F}(n)$ consisting of the functions $f : Q_2^n \to \{-1, 0, 1\}$ the sum of whose values on each face equals one of three numbers: -1, 0, and 1. Refer as a *parity check function* to the Boolean function $\delta(x_1, x_2, \dots, x_n) = \bigoplus_{i=1}^n x_i$. It is not difficult to see that $(-1)^{\delta(x)} \in \mathscr{F}(n)$.

Proposition 10. (a) *If* B *is a bipartite unitrade then* $h_B|_{\{0,1\}^n} \in \mathscr{F}(n)$.
(b) *If* $f \in \mathscr{F}(n)$ *then* $U_R[f] = h_B$ *for some bipartite unitrade* $B \subset Q_3^n$.

Proof. (a) Proposition 9(b) yields $U_R[|h_B|_{\{0,1\}^n}] = h_B$. Then for every face $\{x \in \{0,1\}^n \mid x \le y\}$ we have

$$\sum_{x \le y} h_B|_{\{0,1\}^n}(x) = (-1)^{wt(y)} h_B(y) \in \{-1, 0, 1\}.$$

(b) Proposition 9(a) yields $U_R[f] \in \mathscr{B}_R(n)$. The condition $f \in \mathscr{F}(n)$ and the definition of U_R imply that $U_R[f](Q_3^n) \subseteq \{-1, 0, 1\}$. On each one-dimensional face $U_R[f]$ either takes the value 0 three times or takes the values -1, 0, and 1 once each.

Consider some constructions of functions in $\mathscr{F}(n)$.

Proposition 11. (a) *If* $f \in \mathscr{F}(n)$ *then* $f \cdot \chi^\gamma \in \mathscr{F}(n)$ *for every face* γ.
(b) *Given two faces* γ_1 *and* γ_2 *of* Q_2^n *with* $\gamma_1 \cap \gamma_2 \neq \varnothing$, *define the function* f *as*

$$f(x_1, \dots, x_n, x_{n+1}) = \begin{cases} \chi^{\gamma_1}(-1)^{\delta(x_1, \dots, x_n)} & \text{for } x_{n+1} = 0, \\ \chi^{\gamma_2}(-1)^{\delta(x_1, \dots, x_n) \oplus 1} & \text{for } x_{n+1} = 1. \end{cases}$$

Then $f \in \mathscr{F}(n+1)$.

Proposition 12. *If $F(x,y)=f(x)g(y)$ with $f\in\mathscr{F}(n)$ and $g\in\mathscr{F}(m)$ then $F\in\mathscr{F}(n+m)$.*

The proofs of Propositions 11 and 12 amount to straightforward verifications.

Observe that all unitrades in Q_3^n consist of one connected component. Therefore, all bipartite unitrades are Latin bitrades.

Let us now derive a lower bound on the number of inequivalent Latin bitrades. Two functions on the hypercube are called *equivalent* if they go into each other under some isometry of the hypercube. Consider the hypercube Q_2^n as a vector space over the field $GF(2)$. Refer as the *support* of $x\in Q_2^n$ to the set of positions with 1's in x. Take a tuple $z^1,\dots,z^k$ of vectors with pairwise disjoint supports and consider the subspace $V\subset Q_2^n$ spanned by $z^1,\dots,z^k$,

$$V=\Big\{\bigoplus \alpha_i z^i \mid \alpha\in Q_2^k\Big\}.$$

Given $f:Q_2^k\to\{-1,0,1\}$, define the function $G_V[f]:Q_2^n\to\{-1,0,1\}$ as

$$G_V[f](x)=\begin{cases} f(\alpha) & \text{for } x=\bigoplus\alpha_i z^i\\ 0 & \text{for } x\notin V.\end{cases}$$

Theorem 1. (a) *If $f\in\mathscr{F}(k)$ then $G_V[f]\in\mathscr{F}(n)$.*

(b) *The set $\mathscr{F}(n)$ contains at least $e^{\Omega(\sqrt{n})}$ inequivalent functions.*

(c) *The hypercube Q_3^n includes at least $e^{\Omega(\sqrt{n})}$ inequivalent Latin bitrades.*

Proof. (a) Since the supports of $z^1,\dots,z^k$ are disjoint, the sum of the values of $G_V[f]$ on the face of Q_2^n coincides with the sum of the values of f on some face of Q_2^k. Then $G_V[f]\big(Q_3^n\big)\subseteq\{-1,0,1\}$.

(b) It is known (see [8] for instance) that there exist $e^{\Omega(\sqrt{n})}$ distinct partitions of the number n into nonnegative integer terms. Each partition produces a tuple of vectors $z^1,\dots,z^k$ with pairwise disjoint supports and the corresponding subspace V. The functions $G_V[\delta]$ for distinct partitions are inequivalent since all isometries of the hypercube preserve the Hamming distances between the basis vectors.

(c) By Proposition 10(b), each function $U_R[G_V[\delta]]$ determines a Latin bitrade B. Consider the case that the supports of all $z^1,\dots,z^k$ are of size at least three and their sum equals n. Verify that the intersections of B with the facets $\gamma=\big\{x\in Q_3^n\mid x_i=2\big\}$ enjoy a certain property $(*)$ invariant under the isometries, which the intersection of B with the facets $\big\{x\in Q_3^n\mid x_i=0\big\}$ and $\big\{x\in Q_3^n\mid x_i=1\big\}$ lack. Then for every isometry φ we can construct from $\varphi(B)$ the set $\varphi(B\cap\{0,1\}^n)$. Consequently, the inequivalence of bitrades $U_R[G_V[\delta]]$ for distinct V follows from claim (b).

$(*)$ Suppose that $\gamma=\big\{x\in Q_3^n\mid x_i=a\big\}$. The inequality $a\neq 2$ holds if and only if there is a facet $\gamma'=\big\{x\in Q_3^n\mid x_j=b\big\}$ with $b\in Q_3$ and $j\neq i$ such that the intersection $\gamma\cap\gamma'\cap B$ is empty.

Without loss of generality, assume that $i=1$ and the support of z^1 contains 1 and 2. Then

$$\big\{x\in Q_3^n\mid x_1=0\big\}\cap\big\{x\in Q_3^n\mid x_2=1\big\}\cap B=\varnothing,$$

$$\big\{x\in Q_3^n\mid x_1=1\big\}\cap\big\{x\in Q_3^n\mid x_2=0\big\}\cap B=\varnothing,$$

and simultaneously (1) yields

$$y=(2,0,\dots,0)\in\big\{x\in Q_3^n\mid x_1=2\big\}\cap\big\{x\in Q_3^n\mid x_j=0\big\}\cap B,$$

$$y'=(2,0,\dots,0,2,0\dots)\in\big\{x\in Q_3^n\mid x_1=2\big\}\cap\big\{x\in Q_3^n\mid x_j=2\big\}\cap B$$

since

$$\begin{aligned}\{\overline{0}\}&=\big\{x\in\{0,1\}^n\mid x\le y\big\}\cap\{x\in\{0,1\}^n\mid G_V[\delta](x)\neq 0\}\\ &=\{x\in\{0,1\}^n\mid x\le y'\}\cap\{x\in\{0,1\}^n\mid G_V[\delta](x)\neq 0\}.\end{aligned}$$

Furthermore,

$$y''=(2,0,\dots,0,1,\dots,1,0\dots)\in\big\{x\in Q_3^n\mid x_1=2\big\}\cap\big\{x\in Q_3^n\mid x_j=1\big\}\cap B$$

since

$$\{x \in \{0,1\}^n \mid x \le y''\} \cap \{x \in \{0,1\}^n \mid G_V[\delta](x) \ne 0\} = \{(0,\dots,0,1,\dots,1,0\dots)\} = \{z^k\},$$

where the support of z^k contains j.

Observe that we can also specify a Latin bitrade B_s (see Proposition 6) using the function $h_{B_s} = U_R[G_V[\delta]]$, where the basis for the subspace V consists of one vector z whose support is of size $n-s$.

§ 3. MDS Codes

By induction on the dimension n we can easily prove that every MDS code in Q_3^n extends to MDS codes in Q_3^{n+1} in exactly two ways, and moreover for each $n \ge 1$ all MDS codes in Q_3^n are equivalent (see [9, Exercise 13.15]). Description of all MDS codes in Q_4^n appeared in [10]. As we mentioned, the symmetric difference $M_1 \triangle M_2$ of two MDS codes M_1 and M_2 is a bipartite unitrade, and furthermore the two parts $M_1 \cap (M_1 \triangle M_2)$ and $M_2 \cap (M_1 \triangle M_2)$ are components of the corresponding MDS code. Thus, Proposition 4 yields restrictions on the possible sizes of components of MDS codes. Observe (see [3] for instance) that Q_k^n for $k \ge 4$ include MDS codes with components of size divisible by 2^{n-1} and the corresponding unitrades of size divisible by 2^n. Moreover, it is shown in [2] that every unitrade lying in a 2-fold MDS code in Q_4^n is of size divisible by 2^n. Let us discuss the possibility of obtaining from MDS codes, including these t-folds, unitrades of various sizes.

Given a function $f : Q_k^n \to Q_k$, denote its graph by $\mathscr{M}\langle f\rangle = \{(x, f(x)) : x \in Q_k^n\}$. If $\mathscr{M}\langle f\rangle$ is an MDS code then f is called an *n-ary quasigroup* of order k. Accordingly, a *partial n-ary quasigroup* of order k is a function whose graph intersects each one-dimensional face in at most one vertex.

By [11], every partial n-ary quasigroup of finite order is a restriction of an n-ary quasigroup of a greater order. This yields

Proposition 13. *Each Latin bitrade $B \subset Q_k^n$ obtains from some* MDS *code $M \subset Q_m^n$ with $m \ge k$.*

Proposition 14. *For $k \in \{3,4\}$ no Latin bitrade $B \subset Q_k^n$ with $2^{n+1} > |B| > 2^n$ obtains from an* MDS *code $M \subset Q_k^n$.*

Proof. Since for every n all MDS codes in Q_3^n are equivalent, it suffices to consider an arbitrary MDS code in Q_3^n, for instance, $M = \{x \in Q_3^n \mid x_1 + \dots + x_n = 0 \bmod 3\}$. It is not difficult to see that every unitrade obtained from M includes the whole MDS code M and is of size $2 \cdot 3^{n-1}$.

Assume that $k = 4$ and consider three-dimensional Latin bitrades. A brute-force search shows that if the bitrade B obtained from an MDS code $M \subseteq Q_4^3$ intersects some two-dimensional face in a set of size at least 6 then there is also a two-dimensional face parallel to the first which b also intersects in a set of size at least 6. Furthermore, a brute-force search shows that in this case $|B| \ge 16 = 2^4$.

Consider the Latin bitrade B' obtained from an MDS code in Q_4^n. Applying induction, we infer that if a bitrade $B' \subseteq Q_4^n$ intersects some two-dimensional face in a set of size at least 6 then $|B'| \ge 2^{n+1}$. Proposition 3 implies that if the intersection of a Latin bitrade with each two-dimensional face is of size less than 6, and so equal to 0 or 4, then the graph induced by the bitrade B' is the Boolean n-cube.

Consider the possibility of obtaining Latin bitrades of small size from 2-fold MDS codes.

Proposition 15. *Each n-dimensional Latin bitrade of size less than 2^{n+1} obtains from a 2-fold* MDS *code in Q_4^n.*

Proof. To begin with, verify that the bitrade $B_0^k = \{0,1\}^k \triangle \{1,2\}^k$ obtains from a 2-fold MDS code in Q_4^k. Define a function $g : Q_k^n \to \{0,1\}$ as $g(a_1,\dots,a_k) = \sum_{i=1}^k a_i \bmod 2$. It is not difficult to see that g, $g_1 = g \oplus \chi^{\{1,2\}^k}$, and $g_2 = g \oplus \chi^{\{0,1\}^k}$ are the characteristic functions of certain 2-fold MDS codes M_0, M_1, and M_2. Since $g_1 \oplus g_2 = \chi^{\{1,2\}^k} \oplus \chi^{\{0,1\}^k}$, we have $B_0^k = M_1 \triangle M_2$.

Consider the Boolean-valued functions

$$f_1(a_1,\dots,a_k,y)=\begin{cases} g_1(a_1,\dots,a_k) & \text{for } y=0,\\ g_2(a_1,\dots,a_k) & \text{for } y=1,\\ g_1(a_1,\dots,a_k)\oplus 1 & \text{for } y=2,\\ g_2(a_1,\dots,a_k)\oplus 1 & \text{for } y=3;\end{cases} \tag{2}$$

$$f_2(a_1,\dots,a_k,y)=\begin{cases} g_2(a_1,\dots,a_k) & \text{for } y=0,\\ g_1(a_1,\dots,a_k) & \text{for } y=1,\\ g_1(a_1,\dots,a_k)\oplus 1 & \text{for } y=2,\\ g_2(a_1,\dots,a_k)\oplus 1 & \text{for } y=3.\end{cases} \tag{3}$$

It is easy to see that f_1 and f_2 are the characteristic functions of 2-fold MDS codes; furthermore, $f_1\oplus f_2=\chi^{B_0^k\times\{0,1\}}$.

Inserting f_1 and f_2 instead of g_1 and g_2 into (2) and (3), we find that the Latin bitrade $B_0^k\times\{0,1\}^2$ also obtains from a 2-fold MDS code in Q_4^{k+2}, and so on. In order to complete the proof, it remains to apply Propositions 4 and 6.

Let us present some tables of the functions g, g_1, and g_2 for $n=2$:

$$\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix},\quad \begin{pmatrix}0&1&0&1\\1&1&0&0\\0&0&1&1\\1&0&1&0\end{pmatrix},\quad \begin{pmatrix}1&0&0&1\\0&1&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}.$$

We can show that the 2-fold MDS codes constructed in Proposition 15 are unsplittable for $n\ge 3$.

Let us now study when the graphs of minimal distances of 2-fold MDS codes are Hamiltonian.

Proposition 16. *For every* MDS *code* $M\subset Q_k^n$ *there is a* 2*-fold* MDS *code* D *with* $M\subset D$ *such that the graph of minimal distances* ΓD *is Hamiltonian.*

Proof. Define an MDS code $M'\subset Q_k^n$ as

$$M'=\{(x_1+1 \bmod k, x_2,\dots,x_n)\mid (x_1,\dots,x_n)\in M\}.$$

Consider the 2-fold MDS code $D=M\cup M'$. Verify by induction that the graph ΓD is Hamiltonian, and moreover for each edge there is a Hamiltonian cycle passing through it. For $n=2$ the graph ΓD is a Hamiltonian cycle consisting of alternating edges in two directions.

Suppose that the inductive assumption holds for $n-1$. Consider an arbitrary edge in ΓD, for instance, the edge v connecting $(0,\dots,0,c)$ and $(0,\dots,0,c')$. The graph $\Gamma D'$, where

$$D'=\{(x_1,x_n)\mid (x_1,0,\dots,0,x_n)\in D\},$$

is a simple cycle H' and includes the edge v. By the inductive assumption, the graph ΓD_a for

$$D_a=\{(x_1,\dots,x_{n-1})\mid (x_1,\dots,x_{n-1},a)\in D\}$$

includes a Hamiltonian cycle H_a for every $a\in Q_k$, and moreover we can choose a Hamiltonian cycle H_a passing through the edge belonging to H'.

If two simple cycles intersect along precisely one edge then, removing this edge from the union of two cycles, we obtain a simple cycle. Thus, the graph $H=H'\triangle\bigcup_a H_a$ is a simple cycle. Furthermore, H passes through all vertices of ΓD and includes the edge v.

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V. N. Potapov
Sobolev Institute of Mathematics
Novosibirsk State University, Novosibirsk, Russia
E-mail address: vpotapov@math.nsc.ru