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CLIQUE MATCHINGS IN THE k-ARY n-DIMENSIONAL CUBE

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Abstract: A *clique matching* in the k-ary n-dimensional cube (hypercube) is a collection of disjoint one-dimensional faces. A clique matching is called *perfect* if it covers all vertices of the hypercube. We show that the number of perfect clique matchings in the k-ary n-dimensional cube can be expressed as the k-dimensional permanent of the adjacency array of some hypergraph. We calculate the order of the logarithm of the number of perfect clique matchings in the k-ary n-dimensional cube for an arbitrary positive integer k as $n \to \infty$.

A perfect clique matching is called *precise* if each two-dimensional face of the hypercube includes a sole one-dimensional face of the clique matching. Precise clique matchings are particular cases of H-designs. We prove that for the existence of precise clique matchings in the k-ary n-dimensional cube it is necessary that $k = 2m$ and $n = 4m$ for some positive integer m. We propose a construction of precise clique matchings for $k = 2^t$ and $n = 2^{t+1}$ with an arbitrary positive integer t.

Keywords: perfect matching, clique matching, permanent, MDS code, H-design

Introduction

Put $Q_k = \{0, \dots, k-1\}$. Refer to Q_k^n as the *k-ary n-dimensional cube* (*hypercube*). The graph ΓQ_k^n of the minimal distances of the metric space (Q_k^n, d) is also called the hypercube, where d is the Hamming distance; i.e., $d(u,v)$ is the number of distinct entries in the tuples $u, v \in Q_k^n$. The one-dimensional face in direction i containing the vertex $(a_1, \dots, a_n) \in Q_k^n$ is defined to be the set $\{(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n) \mid x \in Q_k\}$, which is a maximal clique in ΓQ_k^n. The faces of greater dimensions of Q_k^n are defined similarly. Denote by $\widetilde{Q}_k^n$ the set of cliques (one-dimensional faces) in Q_k^n. Refer as a *clique matching* in Q_k^n to a collection of disjoint cliques. For $k = 2$ the concept of clique matching in Q_2^n coincides with the concept of matching in graph theory.

Call a clique matching B in Q_k^n *perfect* whenever B is a partition of the vertices of the hypercube into cliques: $Q_k^n = \bigcup_{b \in B} b$. It is clear that a clique matching $B \subset \widetilde{Q}_k^n$ is perfect if and only if $|B| = k^{n-1}$. Here and below $|A|$ stands for the cardinality of the set A.

The number of perfect matchings in a bipartite graph is known to be equal to the permanent of the adjacency matrix of the graph, and it is nonzero for an arbitrary regular graph (the König theorem). We show in Section 1 that the number of perfect clique matchings in Q_k^n is equal to the k-dimensional permanent of the adjacency array of the hypergraph whose k-edges are cliques in ΓQ_k^n. Moreover, we give an example showing that a regular hypergraph need not include perfect clique matchings; thus, a formal generalization of the König theorem fails.

Denote by $K_k(n)$ the set of clique matchings, and by $SK_k(n)$, the set of perfect clique matchings in Q_k^n. The definition implies a trivial upper bound $|SK_k(n)| \le n^{k^n}$ for the number of clique matchings in Q_k^n. The asymptotics $\ln |SK_2(n)| = 2^{n-1}(\ln n - 1 + o(1))$ for the logarithm of the number of perfect matchings as $n \to \infty$ is pointed out in [1] as a direct corollary of [2, 3]. In Section 2 we obtain the order of the logarithm of the number of perfect clique matchings in the n-dimensional cube for arbitrary $k > 1$. The asymptotic equality $\ln |SK_k(n)| \asymp k^n \ln n$ holds as $n \to \infty$. Moreover, we sharpen the upper bound for $|SK_k(n)|$ for even k.

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Say that a clique matching f *contains no near parallel cliques* if no pair of cliques in f lies in the same two-dimensional face. Call a perfect clique matching f *precise* if each two-dimensional face includes exactly a sole clique of f.

Perfect matchings without near parallel edges in the Boolean hypercube are constructed in [4–6]. More exactly, the ternary codes are exhibited in [4, 6] whose equivalence to matchings is shown in [4]. In particular, it is proved in [4] that for $n = 2^j$ with $j \geq 2$ the n-dimensional Boolean cube Q_2^n includes perfect matchings without parallel edges in the three-dimensional faces. Using the reasons of cardinality, it is not difficult to prove that for $k > 2$ the hypercube Q_k^n includes no perfect clique matchings without parallel cliques in the three-dimensional faces. In the n-dimensional Boolean cube a perfect matching is constructed in [7] whose restrictions to arbitrary faces of dimension greater than 1 and less than n are not matchings in these faces, and consequently include no near parallel edges.

Note that the precise clique matchings in Q_k^n are particular cases of the H-designs defined for the first time in [8]. A *Hanani type design* $H(n, k, w, t)$ is a collection of $(n-w)$-dimensional faces of Q_k^n such that each $(n-t)$-dimensional face of Q_k^n contains exactly a sole face of this collection (see also [9]). Therefore, a precise clique matching in Q_k^n is a Hanani type design $H(n, k, n-1, n-2)$.

In Section 3 we obtain necessary conditions for the existence of precise clique matchings in Q_k^n; namely, $n = 4m$ and $k = 2m$, where m is a positive integer. We propose a construction of precise clique matchings in Q_k^n for $k = 2^t$ and $n = 2^{t+1}$ for an arbitrary positive integer t. Moreover, we show that perfect clique matchings in Q_k^n without near parallel cliques exist for $k \leq 2^t$ and $n \geq 2^{t+1}$, where t is an arbitrary positive integer.

1. Multidimensional Permanents

As proposed in [10], the number of combinatorial objects similar to perfect clique matchings is expressed in terms of multidimensional permanents. Consider a *k-partite hypergraph* G_k with N vertices in each part, whose every k-edge consists of k vertices, with one vertex in each *part* of the hypergraph. Enumerate the vertices of each part by $1, 2, \dots, N$. Define the *adjacency array* $A(G_k) = (a_{i_1 \dots i_k})$ of G_k by $a_{i_1 \dots i_k} = 1$ if there is a k-edge of G_k consisting of the vertices with indices $i_1, i_2, \dots, i_k$ in the corresponding parts, and otherwise put $a_{i_1 \dots i_k} = 0$. Refer as a *diagonal* of the array $F = \{1, \dots, N\}^k$ to a set consisting of N elements of F distinct in all coordinates. For $k = 2$ the concept of a diagonal of the array coincides with the concept of a diagonal of a matrix. A *k-dimensional permanent* of $A(G_k)$ is

$$\operatorname{per} A(G_k) = \sum_{I \in D_N} \prod_{(i_1, \dots, i_k) \in I} a_{i_1 \dots i_k},$$

where D_N is the set of all diagonals of F. For $k = 2$ the array $A(G_2)$ is the adjacency matrix of the bipartite graph G_2 and $\operatorname{per} A(G_2)$ coincides with the permanent of $A(G_2)$. It is known (see [3] for instance) that the number of perfect matchings in a bipartite graph G_2 is equal to the permanent of $A(G_2)$.

An MDS *code* (with distance 2) is a subset of Q_k^n meeting each one-dimensional face in a sole element. Refer as a *partition* of Q_k^n into MDS codes to a collection of disjoint MDS codes $M_0, \dots, M_{k-1}$. Then $\bigcup_{i=0}^{k-1} M_i = Q_k^n$. We may regard an MDS code $M \subset Q_k^n$ as the graph of some function $\varphi_M : Q_k^{n-1} \to Q_k$ (an $(n-1)$-*ary quasigroup of order* k):

$$M = \big\{(a_1, \dots, a_{n-1}, \varphi_M(a_1, \dots, a_{n-1})) \mid (a_1, \dots, a_{n-1}) \in Q_k^{n-1}\big\}.$$

The definition of MDS code implies that $\varphi_M(\bar{a}) \neq \varphi_M(\bar{a}')$ whenever $d(\bar{a}, \bar{a}') = 1$. Note that MDS codes with distance 2, as well as partitions of Q_k^n into them, exist for all $n \geq 2$ and $k \geq 2$.

Consider the hypergraph $G_k(n)$ with the set of vertices of Q_k^n whose k-edges are the elements of the set $\widetilde{Q}_k^n$ (the maximal cliques in ΓQ_k^n). As the collection of the parts of $G_k(n)$ consider an arbitrary partition of Q_k^n into MDS codes. It is not difficult to see that a perfect clique matching in Q_k^n corresponds to a diagonal of the array $A(G_k(n))$ consisting of 1's. Therefore, the number of perfect clique matchings in Q_k^n is equal to the k-dimensional permanent of $A(G_k(n))$.

By the König theorem (see [3] for instance), an arbitrary t-regular (the degree of every vertex is equal to $t \geq 1$) bipartite graph includes a perfect matching; in other words, an arbitrary square $(0,1)$-matrix with the same (nonzero) number of 1's in every column and every row has a positive permanent. The similar claim for multidimensional permanents fails. In particular, we have

Proposition 1. *The hypergraph $G_k(n)$ for $k \geq 3$ and $n \geq 5$ includes a 2-regular subhypergraph without perfect clique matchings.*

Proof. A regular subhypergraph of degree 1 in $G_k(n)$ is a clique matching in Q_k^n. It is not difficult to see that 2-regular subhypergraphs in $G_k(n)$ correspond to the functions $h : Q_k^n \to I_n$, where I_n is the set of unordered pairs from $\{1, \dots, n\}$ satisfying

$$h(a_1, \dots, a_n) = \{i, j\} \;\Rightarrow\; \forall x \in Q_k \; i \in h(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n),$$
$$j \in h(a_1, \dots, a_{j-1}, x, a_{j+1}, \dots, a_n).$$

Refer as a *cycle* in a hypergraph $G_k(n)$ to a collection of k-edges $b_1, \dots, b_m$ such that $b_m \cap b_1 \neq \varnothing$, $b_i \cap b_{i+1} \neq \varnothing$ for arbitrary $i \in \{1, \dots, m-1\}$, and $b_i \cap b_j = \varnothing$ in the remaining cases. It is not difficult to verify that $G_3(3)$ includes a cycle of length 7 consisting of the one-dimensional faces $e_1, \dots, e_7$ with directions $i_1, \dots, i_7 = 1, 2, 3, 1, 3, 2, 3$. Define the function $h : Q_3^5 \to I_5$ as

$$h(a_1, \dots, a_5) = \begin{cases} \{i_j, i_{j'}\}, & \text{if } (a_1, a_2, a_3) \in e_j \cap e_{j'}, \\ \{j, 4\}, & \text{if } (a_1, a_2, a_3) \in e_j \setminus \bigcup\limits_{l \neq j} e_l, \\ \{4, 5\}, & \text{if } (a_1, a_2, a_3) \notin \bigcup\limits_{l=1}^{7} e_l. \end{cases}$$

By construction, h defines some 2-regular subhypergraph H in $G_3(5)$ that includes at least 9 cycles of length 7. Suppose that the 2-regular subhypergraph H includes a perfect clique matching. Then H is the union of two perfect clique matchings, one of which contains four k-edges of a length 7 cycle. Since at least two of arbitrary four k-edges of the length 7 cycle are intersect, we obtain a contradiction to the definition of clique matching. It is not difficult to see that we can embed an arbitrary 2-regular subhypergraph of $G_3(5)$ into a 2-regular subhypergraph of $G_k(n)$ with $k \geq 3$ and $n \geq 5$. Consequently, $G_k(n)$ includes a 2-regular subhypergraph without perfect clique matchings. $\square$

2. The Number of Clique Matchings

We may regard a clique matching as the function associating to each vertex of Q_k^n the direction i of the one-dimensional face of the clique matching containing the vertex, or 0 if the faces of the clique matching avoid this vertex. It is not difficult to see that $f : Q_k^n \to \{0, 1, \dots, n\}$ defines a clique matching whenever f satisfies

$$f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) = i \neq 0$$
$$\Rightarrow \;\forall x \in Q_k \; f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n) = i. \tag{1}$$

Henceforth we refer as a clique matching to its defining function as well. A clique matching f is perfect if and only if $0 \notin f(Q_k^n)$.

An *isotopy* is an ordered collection of n permutations $(\theta_1, \dots, \theta_n)$, $\theta_i : Q_k \to Q_k$, where $i \in \{1, \dots, n\}$. Denote by S_k^n the set of isotopies of the hypercube Q_k^n. Take $\bar\theta = (\theta_1, \dots, \theta_n) \in S_k^n$ and $f \in K_k(n)$. It is not difficult to see that the function $g(x_1, \dots, x_n) = f(\theta_1 x_1, \dots, \theta_n x_n)$ is a clique matching. Put $g = \bar\theta f$. Define the *isotopic closure* of $A \subseteq K_k(n)$ as $\overline{A} = \{\bar\theta f \mid f \in A, \bar\theta \in S_k^n\}$. We have

Proposition 2. *If $A \subseteq K_k(n)$ and $A = \overline{A}$ then the quantities $P_{A,i}(\bar x) = \frac{|\{f \in A \mid f(\bar x) = i\}|}{|A|}$ for $i \in \{0, \dots, n\}$ are independent of $\bar x \in Q_k^n$.*

If some set A of clique matchings is closed under the automorphisms of Q_k^n then the frequencies $P_{A,i}(\bar x)$ of the occurrence of the directions are equal for all $i \in \{1, \dots, n\}$ and $\bar x \in Q_k^n$.

Denote by $K_k(n,p)$ the set of clique matchings taking 0 with probability p:

$$K_k(n,p) = \left\{ f \in K_k(n,p) \mid p = \frac{|\{\bar{x} \in Q_k^n \mid f(\bar{x}) = 0\}|}{k^n} \right\}.$$

Theorem 1. *Suppose that $0 < p < 1$ and $K_k(m,p) \neq \varnothing$ for some positive integer m. Then $|K_k(n,p)| \geq n^{ck^{n-2}(1+o(1))}$ as $n \to \infty$, where $c = p^{k-1}(1-p)\ln 2$.*

Proof. Verify that

$$|K_k(n+1,p)| \geq |K_k(n,p)|^k 2^{\frac{k^{n-1}p^{k-1}(1-p)}{n}}. \tag{2}$$

Consider an arbitrary vector function $F \in (K_k(n,p))^k$, $F = (f_0, \dots, f_{k-1})$. It is not difficult to see that F defines some clique matching $\widehat{F}$ in Q_k^{n+1} by the rule $\widehat{F}(x_1, \dots, x_n, z) = f_z(x_1, \dots, x_n)$. But the clique matchings in Q_k^{n+1} constructed in this way are insufficient for the required asymptotic bound on the number $|K_k(n,p)|$. Below we describe a method for obtaining the clique matchings of another type from the clique matchings of the form $\widehat{F}$ by means of some local transformations.

Refer to a two-dimensional face $\alpha_{x_2,\dots,x_n} = \{(y, x_2, \dots, x_n, z) \mid y, z \in Q_k\}$ in Q_k^{n+1} as *shifted* by F whenever

$$(f_0(y_0, x_2, \dots, x_n), \dots, f_{k-1}(y_0, x_2, \dots, x_n)) = (0, \dots, 0, 1)$$

for some $y_0 \in Q_k$. Define the function $g_\alpha^0[F] : \alpha \to Q_k$ on an arbitrary face $\alpha = \alpha_{x_2,\dots,x_n}$ as

$$g_\alpha^0[F](y, x_2, \dots, x_n, z) = f_z(y, x_2, \dots, x_n),$$

and the function $g_\alpha^1[F] : \alpha \to Q_k$ on the shifted face $\alpha = \alpha_{x_2,\dots,x_n}$ by putting

$$g_\alpha^1[F](y, x_2, \dots, x_n, z) = \begin{cases} n+1 & \text{for } z \in Q_k \text{ and } y = y_0, \\ 0 & \text{for } \ z = k-1 \text{ and } y \neq y_0, \\ f_z(y, x_2, \dots, x_n) & \text{for } \ z \neq k-1 \text{ and } y \neq y_0. \end{cases}$$

If we can choose $y_0 \in Q_k$ in several ways then for definiteness take the minimal possible y_0. Consider an arbitrary collection Ω of the faces $\alpha_{x_2,\dots,x_n}$, $x_i \in Q_k$, shifted by a fixed vector function $F \in (K_k(n,p))^k$. It is not difficult to see that the function $h_\Omega[F] : Q_k^{n+1} \to Q_k$, defined as

$$h_\Omega[F]|_\alpha = \begin{cases} g_\alpha^0[F] & \text{for } \alpha \notin \Omega, \\ g_\alpha^1[F] & \text{for } \ \alpha \in \Omega, \end{cases}$$

is a clique matching. Moreover, $h_\Omega[F] \in K_k(n+1,p)$.

We can uniquely recover the vector function F from the clique matching $h_\Omega[F]$ by replacing the subfunction $g_\alpha^1[F]$ with the subfunction $g_\alpha^0[F]$ in all faces $\alpha_{x_2,\dots,x_n}$, $x_i \in Q_k$, where the clique matching $h_\Omega[F]$ takes the value $n+1$. Then the equality $h_\Omega[F] = h_{\Omega'}[F']$ implies that $F = F'$ and $\Omega = \Omega'$. Consequently,

$$|K_k(n+1,p)| \geq \sum_{F \in (K_k(n,p))^k} 2^{d_F}, \tag{3}$$

where d_F is the number of faces shifted by F.

Estimate the number of shifted faces for all functions $F \in (K_k(n,p))^k$. We have

$$|\{F \in (K_k(n,p))^k \mid F(\bar{x}) = (0, \dots, 0, 1)\}| = |K_k(n,p)|^k (P_0(\bar{x}))^{k-1} P_1(\bar{x}),$$

where

$$P_i(\bar{x}) = \frac{|\{f \in K_k(n,p) \mid f(\bar{x}) = i\}|}{|K_k(n,p)|}.$$

Since each two-dimensional face consists of k one-dimensional faces,

$$\sum_{F\in(K_k(n,p))^k} d_F \geq \frac{1}{k}|K_k(n,p)|^k \sum_{\bar{x}\in Q_k^n} (P_0(\bar{x}))^{k-1} P_1(\bar{x}). \tag{4}$$

Without loss of generality, put

$$\sum_{\bar{x}\in Q_k^n} P_1(\bar{x}) = \max_{1\leq i\leq n} \sum_{\bar{x}\in Q_k^n} P_i(\bar{x}) \geq \frac{1}{n}(1-p)k^n.$$

Since $K_k(n,p) = \overline{K_k(n,p)}$, Proposition 2 and (4) yield

$$\sum_{F\in(K_k(n,p)^k} d_F \geq \frac{1}{k}|K_k(n,p)|^k \sum_{\bar{x}\in Q_k^n} (P_0(\bar{x}))^{k-1} P_1(\bar{x}) \geq k^{n-1}|K_k(n,p)|^k \frac{p^{k-1}(1-p)}{n}.$$

By (3) and the convexity of the function $y(t) = 2^t$, we obtain (2).

Put $\beta_n = \frac{\ln|K_k(n,p)|}{k^{n-1}}$. Here we assume that $K_k(n,p) \neq \varnothing$. This holds, for instance, for $n = 2$ and $p = 1 - \frac{1}{k}$. By (2), $\beta_{n+1} \geq \beta_n + \frac{c}{kn}$; consequently, $\beta_n \geq \frac{c}{k}\ln n(1+o(1))$ as $n\to\infty$. $\square$

Corollary 1. $|SK_k(n)| \geq n^{c_n k^{n-4}}$ *as* $n\to\infty$, *where* $c_n = \frac{\ln 2(1+o(1))}{e}$.

PROOF. Take $f \in K_k(n)$. Define the function

$$\tilde{f}(x_1,\dots,x_{n+1}) = \begin{cases} f(x_1,\dots,x_n), & \text{if } f(x_1,\dots,x_n) \neq 0,\\ n+1, & \text{if } f(x_1,\dots,x_n) = 0.\end{cases}$$

It is not difficult to see that $\tilde{f} \in SK_k(n)$. Hence, $|SK_k(n)| \geq |K_k(n-1,p)|$ for arbitrary p with $0 < p < 1$. Choose $p = 1 - \frac{1}{k}$; then $p^{k-1} > 1/e$, and Theorem 1 implies the required inequality. $\square$

Since

$$\ln|SK_k(n)| \leq \ln|K_k(n)| \leq k^n \ln n,$$

we have

Corollary 2. $\ln|SK_k(n)| \asymp k^n \ln n$ *as* $n\to\infty$.

The Bregman theorem (see [3, 11]) directly implies an upper bound for the number of perfect matchings in the Boolean cube Q_2^n:

$$|SK_2(n)| \leq (n!)^{2^{n-1}/n}. \tag{5}$$

Basing on this bound, in the next statement we offer a strengthening of the trivial upper bound for the number of perfect clique matchings in the case of k even.

Proposition 3. *If k is even then*

$$|SK_k(n)| \leq \left(\frac{n}{e}\right)^{k^{n-1}(1+o(1))} \quad \text{as } n\to\infty.$$

PROOF. Suppose that $k = 2m$. Write the elements $a \in Q_{2m}$ as $a = (\alpha,\beta) \in Q_2 \times Q_m$. Take some MDS code M in Q_m^n. A perfect clique matching $f : (Q_2\times Q_m)^n \to \{1,\dots,n\}$ is uniquely determined by the collection of its restrictions to $Q_2^n \times \bar{c}$ with $\bar{c} \in M$. Since $|M| = m^{n-1}$, we obtain from (5) the bound $|SK_{2m}(n)| \leq (n!)^{m^{n-1}2^{n-1}/n}$. Hence, the required asymptotic inequality follows from the Stirling formula $n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}(1+o(1))$ as $n\to\infty$. $\square$

3. Precise Clique Matchings

To begin with, determine for which k and n the hypercube Q_k^n may include perfect clique matchings without near parallel cliques and precise clique matchings.

Proposition 4. (a) *If there exists a perfect clique matching in Q_k^n without near parallel cliques then $n \geq 2k$.*

(b) *A perfect clique matching in Q_k^n without near parallel cliques is precise if and only if $n = 2k$.*

PROOF. A perfect clique matching B in Q_k^n contains k^{n-1} cliques, and every clique (one-dimensional face) is included into $n-1$ two-dimensional faces. If B contains no near parallel cliques then $k^{n-1}(n-1)$ is at least the number of distinct two-dimensional faces of Q_k^n; thus,

$$k^{n-1}(n-1) \leq \frac{n(n-1)}{2} k^{n-2},$$

which implies (a). This inequality becomes an equality if and only if the perfect clique matching without near parallel cliques is precise. Hence (b) holds.

Proposition 5. *If there exists a precise clique matching in Q_k^n then $n = 4m$ and $k = 2m$ for some positive integer m.*

PROOF. Verify that a precise clique matching contains equal numbers of cliques in each direction. Denote by z_i the number of cliques in direction i, for $i \in \{1, \dots, n\}$, in a precise clique matching $B \subset \widetilde{Q}_k^n$. The number of two-dimensional faces in Q_k^n is the same for every pair of directions, and is equal to k^{n-2}. We have the system of $\frac{n(n-1)}{2}$ linear equations $z_i + z_j = k^{n-2}$, where $i \neq j$, $i, j \in \{1, \dots, n\}$. It is not difficult to see that the corresponding homogeneous system has only the zero solution for $n \geq 3$. Therefore, the original inhomogeneous system has the unique solution $z_i = \frac{k^{n-2}}{2}$ for all $i \in \{1, \dots, n\}$. Since z_i is an integer, k is divisible by 2. Then $n = 2k$ by (b) of Proposition 4. $\square$

Proceed to construct precise clique matchings in Q_k^n for $k = 2^t$ and $n = 2^{t+1}$. Enumerate the elements of the Galois field $x_i \in GF(2^{t+1})$ by the numbers $i \in \{1, \dots, n\}$ arbitrarily. We may regard the Galois field $GF(2^{t+1})$ as the $(t+1)$-dimensional vector space over the field $GF(2)$. Choose in it an arbitrary basis $\{b_1, \dots, b_{t+1}\}$. Put $Q_k = \{0, \dots, 2^t - 1\}$ into a bijective correspondence with the linear span of $\{b_1, \dots, b_t\}$ in $GF(2^{t+1})$. In order to simplify the formulas below, identify the elements of Q_k with the corresponding elements of $GF(2^{t+1})$. Define the function $f : Q_k^n \to \{1, \dots, n\}$ as

$$f(a_1, \dots, a_n) = l, \quad \text{where } x_l = \frac{\sum\limits_{i=1}^{n} x_i a_i}{b_{t+1} + \sum\limits_{i=1}^{n} a_i}, \tag{6}$$

where all arithmetic operations are in $GF(2^{t+1})$.

Theorem 2. *The function f defined in* (6) *is a precise clique matching.*

PROOF. Since b_{t+1} lies outside the linear span of $\{b_1, \dots, b_t\}$, the denominator in (6) is nonzero. Consequently, f is defined everywhere on Q_k^n.

Verify that f determines a clique matching. It suffices to prove that $f(a_1, \dots, a_l, \dots, a_n) = l$ implies that $f(a_1, \dots, a_{l-1}, c, a_{l+1}, \dots, a_n) = l$ for arbitrary $c \in Q_k$. By (6),

$$x_l \left(b_{t+1} + \sum_{i=1}^{n} a_i \right) = \sum_{i=1}^{n} x_i a_i.$$

Adding $x_l(c - a_l)$ to both sides, we obtain $f(a_1, \dots, a_{l-1}, c, a_{l+1}, \dots, a_n) = l$.

Verify that the clique matching f contains no near parallel cliques; i.e., for arbitrary $c \in Q_k$, $c \neq a_m$, and $m \neq l$ we have $f(a_1, \dots, a_{m-1}, c, a_{m+1}, \dots, a_n) \neq l$ whenever $f(a_1, \dots, a_l, \dots, a_n) = l$. Suppose that

$$f(a_1, \dots, a_{m-1}, c, a_{m+1}, \dots, a_n) = f(a_1, \dots, a_l, \dots, a_n) = l.$$

Then (6) yields

$$x_l\left(b_{t+1} + \sum_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} x_i a_i,$$

$$x_l(c - a_m) + x_l\left(b_{t+1} + \sum_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} x_i a_i + x_m(c - a_m).$$

Taking the difference, we obtain $x_l(c - a_m) = x_m(c - a_m)$. If $a_m \neq c$ then $x_m = x_l$. We arrive at a contradiction.

Claim (b) of Proposition 4 implies that the clique matching f is precise. $\square$

For $t = 1$ the matching determined by (6) coincides with that of [5].

Consider the question of constructing perfect clique matchings without near parallel cliques.

Proposition 6. *For $k = 2^t$ and $n = 2^{t+1}$, where t is a positive integer, in the hypercube Q_k^n there exists a partition of the set of cliques $\widetilde{Q}_k^n$ into n precise clique matchings.*

Proof. Define the clique matchings f_j for $j \in \{1, \dots, n\}$ as

$$f_j(a_1, \dots, a_n) = l, \quad \text{where } x_l + x_j = \frac{\sum\limits_{i=1}^{n}(x_i + x_j)a_i}{b_{t+1} + \sum\limits_{i=1}^{n} a_i}.$$

By Theorem 6, the clique matchings f_j are precise for $j \in \{1, \dots, n\}$. It is clear that the equalities $f_j(a_1, \dots, a_n) = f_{j'}(a_1, \dots, a_n)$ yield $x_j = x_{j'}$. Therefore, the collection $\{f_j\}_{j=1,\dots,n}$ amounts to a partition of $\widetilde{Q}_k^n$ into disjoint precise clique matchings. $\square$

Proposition 7. *For $2 \leq k \leq 2^t$ and $n \geq 2^{t+1}$, where t is a positive integer, in the hypercube Q_k^n there exists a perfect clique matching without near parallel cliques.*

Proof. We proved in Proposition 6 that for $m = 2^t$ there exists a partition of $\widetilde{Q}_m^{2m}$ into $2m$ perfect clique matchings without near parallel cliques. Express an arbitrary m of these $2m$ clique matchings as functions $f_0, \dots, f_{m-1} : Q_m^{2m} \to \{1, \dots, 2m\}$. Take some $(n - 2m)$-ary quasigroup $\varphi : Q_m^{n-2m} \to Q_m$. Then the function $F : Q_m^{2m} \times Q_m^{n-2m} \to \{1, \dots, 2m\}$ defined as

$$F(c_1, \dots, c_{2m}, a_1, \dots, a_{n-2m}) = f_{\varphi(a_1,\dots,a_{n-2m})}(c_1, \dots, c_{2m})$$

corresponds to a perfect clique matching without near parallel cliques in Q_m^n.

Suppose that $k < 2m$. Then the restriction $g = F|_{Q_k^n}$ is also a perfect clique matching without near parallel cliques. $\square$

Therefore, the questions of the existence of precise clique matchings in Q_k^n for $k = 2m$ and $n = 4m$, where m is not a power of 2, and perfect clique matchings without near parallel cliques for k and n satisfying $2^{t-1} < k < 2^t$ and $2k \leq n < 2^{t+1}$, remain open.

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