On the Fourier Entropy–Influence Conjecture for Boolean Plateaued Functions

Vladimir N. Potapov

BFA 2026

1. Propose the following inequality for Boolean functions:

$$2 \sum_{x \in \mathbb{F}_2^n} f(x) wt(x) \geqslant wt(f)(n - \deg_{alg}(f)),$$

where $wt(x)$ denotes the Hamming weight of x and $wt(f) = |supp(f)| = |\{x \in \mathbb{F}_2^n : f(x) = 1\}|$.

2. Use this inequality to prove
the Fourier Entropy–Influence (FEI) conjecture for
Boolean plateaued functions.

FEI conjecture

Let $f : \mathbb{F}_2^n \to \mathbb{F}_2$ be a Boolean function. We use the **Walsh-Hadamard (Fourier) transform** of f, normalized by the factor 2^{-n}. It is defined by

$$W_f(z) = \frac{1}{2^n} \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus (x,z)},$$

where $(x, z) = x_1 z_1 \oplus \cdots \oplus x_n z_n$.

FEI conjecture

By Parseval's identity, $\sum_{z \in \mathbb{F}_2^n} W_f^2(z) = 1$. Therefore, the values $W_f^2(z)$ define a probability distribution on $\mathbb{F}_2^n$. The **Fourier entropy** of f is defined by $H(f) = \sum_{z \in \mathbb{F}_2^n} W_f^2(z) \log_2 \frac{1}{W_f^2(z)}$.

FEI conjecture

Let $cut(f)$ be the number of edges whose endpoints have different values of f. A proportion of such edges $I(f) = \frac{cut(f)}{n2^{n-1}}$ is called the **total influence**.

Proposition 1 For every Boolean function f

$$I(f) = \sum_{z \in \mathbb{F}_2^n} W_f^2(z) wt(z).$$

FEI conjecture

The Fourier Entropy–Influence (FEI) conjecture was introduced by Friedgut and Kalai in 1996.

Conjecture

There exists a universal constant C such that $H(f) \leqslant C \cdot I(f)$ for every Boolean function f.

FEI conjecture

Restricted versions of the FEI conjecture have been proved for symmetric functions, read-once formulas, decision trees with constant depth, and others. Gangopadhyay and Stănică proved that the FEI conjecture is true for monomial Boolean functions with $C = 4$, which is the best possible. Recently, Li and Tang constructed s-plateaued functions whose FEI constants are strictly smaller than 4, but arbitrarily close to 4.

FEI conjecture

A Boolean function is called s-**plateaued** if $W_f^2(z) = 2^{s-n}$ or $W_f^2(z) = 0$ for every $z \in \mathbb{F}_2^n$, 0-plateaued function is a **bent function**.

Proposition 2

$H(f) = n - s$ for each s-plateaued function f in n variables.

Proof. Since every nonzero Walsh coefficient satisfies $W_f^2(z) = 2^{s-n}$, and the support size equals 2^{n-s}, we obtain $H(f) = 2^{s-n} \cdot 2^{n-s} \cdot (n - s)$.

FEI conjecture

Corollary $H(f) = 2I(f)$ holds for every bent function f.

Proof. By Proposition 2 $H(f) = n$.
$I(f) = \sum_{z \in \mathbb{F}_2^n} W_f^2(z) wt(z) = 2^{-n} \sum_{z \in \mathbb{F}_2^n} wt(z) = n/2.$

FEI conjecture

Let $S_f : \mathbb{F}_2^n \to \mathbb{F}_2$ be an indicator function of $supp(W_f) = \{z \in \mathbb{F}_2^n : W_f(z) \neq 0\}$.

Proposition 3 If f is an s-plateaued function in n variables then $\deg_{alg}(S_f) \leqslant \frac{n+s}{2}$.

FEI conjecture

Theorem1 $H(f) \leqslant 4I(f)$ for every plateaued function f.

Proof. $W_f^2(z) = 2^{s-n}$ or $W_f^2(z) = 0$ for any $z \in \mathbb{F}_2^n$. By Parseval's identity $wt(S_f) = 2^{n-s}$. By $(*)$ $I(f) =$

$$= \sum_{z \in \mathbb{F}_2^n} W_f^2(z) wt(z) = 2^{s-n} \sum_{z \in \mathbb{F}_2^n} S_f(z) wt(z) \geqslant \frac{n - \deg_{alg}(S_f)}{2}.$$

By Proposition 3 $I(f) \geqslant \frac{n - \frac{n+s}{2}}{2} = \frac{n-s}{4}$. By Proposition 2 $H(f) = n - s$. We obtain $H(f) \leqslant 4I(f)$.

$$2 \sum_{z \in \mathbb{F}_2^n} S_f(z) wt(z) \geqslant wt(f)(n - \deg_{alg}(S_f)) \quad (*)$$

Proof of the inequality

Let $\geqslant$ be a natural partial order on $\mathbb{F}_2^n$.

$$T_S(a,b) = \{x \in S, x \geqslant a, x \geqslant b\}.$$

Lemma 1 For every $S \subset \mathbb{F}_2^n$ there exists a permutation $\pi : S \to S$ such that for every $y \in S$ the cardinality $|T_S(y, \pi(y))|$ is odd.

Lemma 2 Let $f : \mathbb{F}_2^n \to \mathbb{F}_2$ be a Boolean function and let $|T_{supp(f)}(y,x)|$ be odd. Then

$$wt(x) + wt(y) \geqslant n - \deg_{alg}(f).$$

Proof of the inequality

Theorem 2 Let $f : \mathbb{F}_2^n \to \mathbb{F}_2$ be a Boolean function. Then

$$2 \sum_{x \in \mathbb{F}_2^n} f(x) wt(x) \geqslant wt(f)(n - \deg_{alg}(f)).$$

Proof. $S = supp(f)$.

$$\sum_{y \in S} (wt(y) + wt(\pi(y))) \geqslant \sum_{y \in S} (n - \deg_{alg}(f))$$

$$\sum_{y \in S} 2wt(y) \geqslant wt(f)(n - \deg_{alg}(f)).$$

Proof of the inequality

Theorem 3 The following inequalities are equivalent

$$(1) \quad 2\sum_{x\in\mathbb{F}_2^n} f(x)wt(x) \geqslant wt(f)(n-\deg_{alg}(f))$$

$$(2) \quad 2\sum_{x\in\mathbb{F}_2^n} f(x)wt(x) \leqslant wt(f)(n+\deg_{alg}(f))$$

$$(3) \quad -\deg_{alg}(f)wt(f) \leqslant 2^{n-1}\sum_{i=1}^{n} W_f(u_i) \leqslant \deg_{alg}(f)wt(f)$$

for any basis $\{u_1, u_2, \ldots, u_n\}$ of $\mathbb{F}_2^n$.

Conclusion

The FEI conjecture is true for plateaued functions with a tight constant 4.

Handbook

Perfect Combinatorial Structures in Coding Theory and Cryptography

http://arxiv.org/abs/2609.09225