# Combinatorial designs, difference sets and bent functions as perfect colorings of graphs and multigraphs

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**Abstract**

It is proved that 1) the indicator function of some onefold or multifold independent set in a regular graph is a perfect coloring if and only if the set attain the Delsarte–Hoffman bound; 2) each transversal in a uniform regular hypergraph is an independent set attaining the Delsarte–Hoffman bound in the vertex adjacency multigraph of this hypergraph; 3) combinatorial designs with parameters $t$-$(v,k,\lambda)$ and similar $q$-designs, difference sets, Hadamard matrices, and bent functions are equivalent to perfect colorings of special graphs and multigraphs, in particular, it is true in the cases of the Johnson graphs $J(n,k)$ for $(k-1)$-$(v,k,\lambda)$ designs and the Grassmann graphs $J_2(n,2)$ for bent functions.

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## Introduction

Let $G=(V,E)$ be a graph. A function mapping from the vertex set $V(G)$ to a finite set $I$ of colors is called a coloring of the graph. A coloring $f$ is called *perfect* if each collection of vertices adjacent to vertices of the same color have identical color composition. It can be formulated as follows: for each color $i \in I$ and each pair of vertices $x,y \in V(G)$ we require that $f(x)=f(y)$ follows $|f^{-1}(i)\cap S(x)| = |f^{-1}(i)\cap S(y)|$, where $S(x)=\{z\in V(G) : \{z,x\}\in E(G)\}$ is the set of vertices adjacent to $x$. Also the terms 'equitable partition' and 'partition design' are used for the set $\{f^{-1}(i) : i \in I\}$. This set is a partition of vertices of $G$ according to the colors of a perfect coloring. The matrix $P=(p_{ij})$ of size $|I|\times|I|$ is called a quotient matrix if each its entry $p_{ij}$ is equal to the number of vertices of color $j$ adjacent to each vertex of color $i$.

Perfect 2-colorings turn out to be solutions to extremal problems on graphs. The definition of 1-perfect code in an $r$-regular graph imply directly that the indicator function of the code is a perfect coloring with quotient matrix $\begin{pmatrix} 0 & r \\ 1 & r-1 \end{pmatrix}$. It is proved in [1] that all unbalanced Boolean functions reaching Fon-Der-Flaass's bound of correlation immunity are perfect 2-colorings. All Boolean functions attaining the Bierbrauer–Friedman bound for orthogonal arrays are also perfect 2-colorings (see [2]). We proved another similar result. In Theorem 1 we establish that if an onefold or multifold independent set in a regular graph attains the Delsarte–Hoffman bound then the indicator function of this set is a perfect 2-coloring.

We suggest characterizations for various well-known combinatorial configurations in terms of perfect colorings of simple graphs, hypergraphs and multigraphs. In Proposition 6 we prove that each transversal of a uniform regular hypergraph is an independent set in some multigraph attaining the Delsarte–Hoffman bound. Consequently, it is equivalent to a perfect 2-coloring of the multigraph. In Section 4 we show that combinatorial designs and $q$-designs are equivalent to perfect 2-colorings with certain parameters of the Johnson graphs and the Grassmann graphs respectively or multigraphs obtained from these graphs. In particular, the Hadamard matrices turn out to be equivalent to perfect colorings of some hypergraph. In Section 6 we prove that strongly regular graphs and partial difference sets in abelian groups are equivalent to perfect 2-colorings of the line graph of the complete graph $K_n$. At last, we prove that Boolean bent functions are equivalent to perfect 4-colorings of $J_2(n, 2)$ with certain quotient matrix (see Theorem 2).

## 1 Perfect Colorings of Multigraphs and Hypergraphs

*Multigraph* is a generalization of the notion of graph by allowing multiple edges and loops. Let us enumerate vertices of a multigraph on $n$ vertices by numbers from 1 to $n$. A square matrix $A = (a_{ij})$ of size $n \times n$ is called an adjacency matrix of a multigraph if each entry $a_{ij}$ is equal to the number of edges joining $i$th and $j$th vertices. Each adjacency matrix of multigraph is symmetric; in the case of a simple graph it consists only of the numbers 0 and 1, moreover, it has only zeros at the main diagonal. The adjacency matrix of a bipartite graph has a block structure. Eigenvalues and eigenvectors of adjacency matrices of a multigraph are called eigenvalues and eigenvectors of the multigraphs.

Let $G$ be a bipartite graph with parts $V$ and $U$. Consider the multigraph $\mathcal{M}_{12}(G)$ having $V$ as the set of vertices and vertices $v_i, v_j \in V$ are joined by $m_{ij}$ edges if there exist $m_{ij}$ distinct vertices of $U$ such that each of them is adjacent to $v_i$ and $v_j$. For convenience we assume that every vertex of $\mathcal{M}_{12}(G)$ is incident to $d$ loops where $d$ is the degree of the vertex in $G$. If all edges and loops in $\mathcal{M}_{12}(G)$ have the same multiplicity then in all statements of this section instead of $\mathcal{M}_{12}(G)$ we may consider the simple graph obtained from $\mathcal{M}_{12}(G)$ by removing loops and multiple edges.

The adjacency matrix of the bipartite graph $G$ can be represented as $M = \begin{pmatrix} 0 & Y \\ Y^* & 0 \end{pmatrix}$, where $Y$ is a matrix of size $k_1 \times k_2$ and $k_i$ is the cardinality of the $i$th part of $G$. Then the adjacency matrix of $\mathcal{M}_{12}(G)$ is $YY^*$.

Some part of the following proposition for bipartite graphs is proved in [3].

**Proposition 1** 1. *The eigenvalues of $\mathcal{M}_{12}(G)$ are nonnegative.*

2. *The restriction of every eigenfunction of $G$ with eigenvalue $\theta$ to the first part of the graph is an eigenfunction of $\mathcal{M}_{12}(G)$ with the eigenvalue $\theta^2$.*

3. *Every eigenfunction of $\mathcal{M}_{12}(G)$ with a positive eigenvalue $\theta$ can be extent to an eigenfunction of $G$ with the eigenvalue $\sqrt{\theta}$.*

4. *Every eigenfunction of $\mathcal{M}_{12}(G)$ with eigenvalue $0$ can be extended to an eigenfunction of $G$ with the same eigenvalue.*

*Proof.* 1. It follows from $YY^*$ is a nonnegative semidefinite matrix for every $Y$.

2. Let $(h, g)$ be an eigenvector of $M$ with eigenvalue $\theta$, where $h$ is a vector of length $k_1$ and $g$ is a vector of length $k_2$. Then $Yg = \theta h$ and $Y^*h = \theta g$. Therefore, $YY^*h = \theta Yg = \theta^2 h$.

3. Suppose $Y^*h = \theta h$. Consider a vector $f = (\sqrt{\theta}h, Y^*h)$. Then the equation $Mf = \sqrt{\theta}f$ is equivalent to the pair of equations $YY^*h = \theta h$ and $\sqrt{\theta}Y^*h = \sqrt{\theta}Y^*h$.

4. Suppose that $YY^*h = \bar{0}$. Then $(Y^*h, Y^*h) = (YY^*h, h) = 0$, which implies that $Y^*h = \bar{0}$. Then $Mf = \bar{0}$, where $f = (h, \bar{0})$. □

The definition of a perfect coloring is naturally generalized to multigraphs and directed multigraphs. We mean that a vertex in a directed multigraph is adjacent to another vertex if there exists an arc (directed edge) from the first vertex to the second one. The definition of a quotient matrix for a perfect coloring of a directed multigraph is similar to the definition of a quotient matrix for graphs. Notice that we can consider each quotient matrix of a perfect coloring as an adjacency matrix of some directed multigraph. The number of vertices in this directed multigraph is equal to the number of colors in the perfect coloring.

Given a coloring $f : V(G) \to \{1, \dots, k\}$ of $G$. Define the matrix $F$ of size $n \times k$ as follows: every column $F_i$ is the indicator function of color $i$. More formally we have equation:

$$\chi_{f^{-1}(i)}(x) = \begin{cases} 1 & \text{if } f(x) = i, \\ 0 & \text{if } f(x) \neq i. \end{cases}$$

Notice that each row of $F$ contains only one 1. Henceforth, we represent a function mapping from the set of vertices by a column vector.

In the case of simple graphs it is well known the linear algebraic criterion for perfect colorings. Below we prove this criterion for directed multigraphs in a similar way.

**Proposition 2** *Let $M$ be the adjacency matrix of a directed multigraph $G$. The function $f$ is a perfect $k$-coloring of $G$ with the quotient matrix $S$ if and only if $MF = FS$.*

*Proof.* Let $j = 1, \dots, k$ and $v \in V(G)$. Consider entries labeled by $(vj)$ on the both sides of the equality $MF = FS$. $(MF)_{vj}$ is the number of vertices of the color $j$ adjacent to $v$. In the right-hand side the row $F_v$ contains 1 in the position corresponding to the color of $v$. This implies that $(FS)_{vj}$ is the number of vertices of the color $j$ adjacent to $v$. $\square$

The corollaries of Proposition 2 listed below can be found in [4] in the case of simple graphs. Their proofs in the case of directed multigraphs are similar.

**Corollary 1** *Let $u$ be an eigenvector of $S$. Then $Fu$ is an eigenvector of $M$ with the same eigenvalue.*

**Corollary 2** *A function $f : V(G) \to \{0, 1\}$ is a perfect coloring of an $r$-regular multigraph $G$ with quotient matrix $\begin{pmatrix} r-b & b \\ c & r-c \end{pmatrix}$ if and only if $h(x) = \begin{cases} \frac{b}{b+c} & f(x) = 0; \\ -\frac{c}{b+c} & f(x) = 1. \end{cases}$ is an eigenfunction of $G$ with the eigenvalue $r - b - c$.*

**Proposition 3** *If $f : V \cup U \to \{1, \dots, k\}$ is a perfect coloring of a bipartite graph $G$ with parts $V$ and $U$, then $f|_V$ is a perfect coloring of $\mathcal{M}_{12}(G)$.*

*Proof.* The adjacency matrix of $G$ can be represented as $M = \begin{pmatrix} 0 & Y \\ Y^* & 0 \end{pmatrix}$. Let $S = \begin{pmatrix} 0 & S_1 \\ S_2 & 0 \end{pmatrix}$ be a quotient matrix of the perfect coloring. Proposition 2 implies equality $MF = FS$ that is equivalent to the pair of equalities $YF_2 = F_1S_1$ and $Y^*F_1 = F_2S_2$, where $F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}$, and matrices $F_1$ and $F_2$ correspond to colorings of two parts of $G$. We obtain $YY^*F_1 = YF_2S_2 = F_1S_1S_2$. Observe that $YY^*$ is the adjacent matrix of $\mathcal{M}_{12}(G)$ and $S_1S_2$ is the quotient matrix of the perfect coloring $f|_V$ by Proposition 2. $\square$

The converse is false in general. A perfect coloring of $\mathcal{M}_{12}(G)$ do not need to be a restriction of any perfect coloring of $G$. In particular, we can find some examples in the case that $\mathcal{M}_{12}(G)$ is the complete graph.

A *hypergraph* is another generalization of a graph. Here we consider unordered collections of various numbers of vertices as hyperedges of the hypergraph. A hypergraph is called *k-uniform* whenever each hyperedge consists of $k$ vertices. A $(0,1)$-matrix $Y = (y_{ij})$ of size $n \times m$, where $n$ is the number of vertices and $m$ is the number of hyperedges, is called an *incidence matrix* of the hypergraph if

$$y_{ij} = \begin{cases} 1, \text{ if } v_i \in e_j \in E(G); \\ 0, \text{ otherwise.} \end{cases}$$

For any hypergraph $G$ we define the bipartite graph $\mathcal{D}(G)$. The first part of $\mathcal{D}(G)$ consists of all vertices of $G$ and the second one consists of all hyperedges of $G$. A vertex $v \in V(G)$ from the first part is adjacent to a vertex $e \in E(G)$ from the second part whenever $v \in e$. It is easy to see that the adjacency matrix of $\mathcal{D}(G)$ can be represented as $\begin{pmatrix} 0 & Y \\ Y^* & 0 \end{pmatrix}$, where $Y$ is the incidence matrix of $G$.

A coloring of $G$ is called *perfect* if it induces a perfect coloring of $\mathcal{D}(G)$. In other words, a coloring of a hypergraph is perfect whenever two vertices of the same color are incident to the equal numbers of hyperedges with any fixed color composition of vertices.

Consider the adjacency matrix $M$ of the multigraph $\mathcal{M}_{12}(\mathcal{D}(G))$. It is clear that the entry $m_{ij}$ of $M$ is equal to the number of hyperedges of $G$ containing $i$th and $j$th vertices at the same time. Then $M = YY^*$, where $Y$ is the incidence matrix of $G$.

By Proposition 3 we obtain

**Corollary 3** *Each perfect coloring of the hypergraph $G$ is a perfect coloring of $\mathcal{M}_{12}(\mathcal{D}(G))$.*

Let $G$ be a $k$-uniform hypergraph. The multigraph $\mathcal{E}(G)$ is called a *line graph* of $G$ if vertices of $\mathcal{E}(G)$ are edges of $G$ and two vertices $v, u \in \mathcal{E}(G)$ are joined by $l$ edges whenever $u \cap v$ consists of $l$ vertices of $G$.

By the definition, the adjacency matrix of $\mathcal{E}(G)$ is $Y^*Y - kI$, where $Y$ is the incidence matrix of $G$. If $G$ is a simple graph then $\mathcal{E}(G)$ is also a simple graph.

A coloring $f'$ of $\mathcal{E}(G)$ is *induced* by a coloring $f$ of $G$ whenever we use a multiset of colors of vertices from $e \in E(G)$ as a color of $e \in \mathcal{E}(G)$, i. e., $f'(e) = \{f(v_1), f(v_2), \ldots, f(v_k)\}$, where $e = \{v_1, v_2, \ldots, v_k\}$.

**Proposition 4** *Each coloring induced by a perfect coloring of a hypergraph $G$ is perfect.*

*Proof.* Let $f$ be a perfect $t$-coloring of $G$. Consider some hyperedge $e$ of the color $\{i_1, \ldots, i_t\}$, i. e., $e$ contains $i_j$ vertices of color $j$. By the definition of the perfect coloring of hypergraphs, each vertex of the color $j$ belongs to a known number of hyperedges of each fixed color composition. The multiset of such color compositions of all vertices in $e$ is the same for hyperedges with the same color. Then the collection of vertices adjacent to vertices of $\mathcal{E}(G)$ with the same color have identical color composition. It is possible that some hyperedge $e'$ intersects $e$ by $s > 1$ vertices, then the color composition of $e'$ will be counted $s$ times. It corresponds to the fact that the vertices $e$ and $e'$ are joined by $s$ edges in $\mathcal{E}(G)$. $\square$

## 2 Transversals of Hypergraphs

A *transversal* of a hypergraph is a set of vertices that covers each hyperedge one time, i. e., the intersection between the transversal and each hyperedge consists of one vertex. An *$\ell$-fold transversal* of a hypergraph is a set of vertices which cover each edge $\ell$ times.

In particular, in a bipartite graph each part is a transversal of the graph. It is easy to see that the indicator function of each multifold transversal is a perfect 2-coloring of a $k$-uniform $r$-regular hypergraph. Moreover, the following holds:

**Proposition 5** *A function $f : V \to \{0, 1\}$ is an indicator function of an $\ell$-fold transversal in a $k$-uniform $r$-regular hypergraph $G$ if and only if $f$ is a perfect coloring of the multigraph $\mathcal{M}_{12}(\mathcal{D}(G))$ with the quotient matrix* $S = \begin{pmatrix} \ell r & (k-\ell)r \\ \ell r & (k-\ell)r \end{pmatrix}$.

*Proof.* It is clear, that each $\ell$-fold transversal generates some perfect 3-coloring of $\mathcal{D}(G)$: the first two colors are the transversal and its complement in the first part of $\mathcal{D}(G)$, all vertices of the second part of $\mathcal{D}(G)$ are colored by the third color. By Proposition 3, the restriction of this coloring to the first part of $\mathcal{D}(G)$ is a perfect 2-coloring of $\mathcal{M}_{12}(\mathcal{D}(G))$. The quotient matrix of this coloring is obtained from the following argument. Each vertex of the $\ell$-fold transversal is adjacent to itself $r$ times and also $\ell - 1$ times to other vertices of the $\ell$-fold transversal in every of $r$ incident hyperedges. Each vertex from the complement of the $\ell$-fold transversal is adjacent to $\ell$ vertices in every of $r$ incident hyperedges.

Suppose that $f : V(\mathcal{M}_{12}(\mathcal{D}(G))) \to \{0, 1\}$ is a perfect 2-coloring with quotient matrix $S$. Define $h(x) = f(x) - |f^{-1}(1)|/|V(G)|$. By Corollary 2 it follows that $h$ is an eigenvector with eigenvalue 0 of the adjacency matrix of $\mathcal{M}_{12}(\mathcal{D}(G))$. Then $YY^*h = \bar{0}$. Consequently, it holds $(Y^*h, Y^*h) = (YY^*h, h) = 0$ and $Y^*h = \bar{0}$. The last equality implies that the sum of values of $h$ in each hyperedge equals 0. Since $f$ takes only two distinct values, $-|f^{-1}(1)|/|V(G)|$ and $1 - |f^{-1}(1)|/|V(G)|$, $h$ takes every of these the same numbers of times in each hyperedge. Then $f$ satisfies the same property. $\square$

# 3 Independent Sets

A subset of vertices in a graph is called an *independent* whenever it does not contain adjacent vertices. The definition of an independent set is the same for multigraphs without loops. It is well known the Delsarte–Hoffman upper bound for the cardinality of an independent set in an $r$-regular graph (see, for instance, [5] Theorem 2.4.1). This bound is equal to $\frac{-\theta v}{r-\theta}$, where $v$ is the number of vertices in the graph and $\theta$ is the minimal eigenvalue of the graph.

**Theorem 1** *Let $\theta$ be the minimal eigenvalue of an $r$-regular multigraph $G$. If each vertices of $A \subset V(G)$ is adjacent to at most $t < r$ vertices of $A$, then $|A| \leq \frac{(t-\theta)|V(G)|}{r-\theta}$. If in addition the cardinality of $A$ equals $\frac{(t-\theta)|V(G)|}{r-\theta}$, then the indicator function $f$ of $A$ is a perfect 2-coloring of $G$ with quotient matrix $S = \begin{pmatrix} t & r-t \\ t-\theta & r-t+\theta \end{pmatrix}$.*

*Proof.* Assume that $G$ is connected, in opposite case we can consider each connected component separately. Put $n = |V(G)|$. Since the adjacency matrix $M$ of $G$ is symmetric, there exists an orthonormal basis consisting of eigenvectors $\phi_i$ of $M$. The regularity and connectedness of $G$ implies that the maximal eigenvalue $r = \theta_0$ corresponds to the unique basis vector $\phi_0 = \mathbf{1}/\sqrt{n}$. Consider the expansion of $f$ with respect to this basis as $f = \sum_i \alpha_i \phi_i$. Then

$$t(f,f) \geq (Mf,f) = \sum_i \alpha_i^2 \theta_i, \qquad (1)$$

where $\theta_i$ is the eigenvalue corresponding to $\phi_i$. By the equality $|A| = (f,f) = \sum_i \alpha_i^2$ we obtain that $\sum\limits_{i\neq 0} \alpha_i^2 = (f,f) - \alpha_0^2 = |A| - |A|^2/n$. Therefore, (1) and the minimality of $\theta$ imply that $t|A| \geq (|A| - |A|^2/n)\theta + r|A|^2/n$.
Since $r - \theta > 0$ we obtain the inequality $\frac{|A|}{n} \leq \frac{t-\theta}{r-\theta}$.

The equality $\frac{|A|}{n} = \frac{t-\theta}{r-\theta}$ holds only in the case as $f = \phi + \alpha_0 \varphi_0$, where $\phi$ is the eigenvector corresponding eigenvalue $\theta$. By Corollary 2 we obtain that $f$ is a perfect coloring of $G$. We can find the entry $s_{21}$ of $S$ by the equation $|A|(r-t) = s_{21}(n - |A|)$, where the left and right parts of the equation are the numbers of edges incident to vertices of different colors. □

If we put $t = 0$ in Theorem 1, then we obtain the well known Delsarte–Hoffman bound on the cardinality of independent sets. Moreover, Theorem 1 implies that the indicator function of each independent set attaining the Delsarte–Hoffman bound is a perfect coloring. Conversely, it is easy to see that the first color of a coloring with quotient matrix $\begin{pmatrix} 0 & r \\ -\theta & r+\theta \end{pmatrix}$ is an independent set attaining the Delsarte–Hoffman bound.

**Proposition 6** *If $T$ is a transversal of a $k$-uniform $r$-regular hypergraph $G$, then $T$ is an independent set of the multigraph obtained from $\mathcal{M}_{12}(\mathcal{D}(G))$ by removing all loops, and it attains the Delsarte–Hoffman bound.*

*Proof.* Proposition 5 implies that each transversal of $G$ generates a perfect coloring of $\mathcal{M}_{12}(\mathcal{D}(G))$ with quotient matrix $S = \begin{pmatrix} r & (k-1)r \\ r & (k-1)r \end{pmatrix}$.

By Proposition 1, all eigenvalues of $\mathcal{M}_{12}(\mathcal{D}(G))$ are nonnegative. Every eigenvalue of a quotient matrix of a perfect coloring of the graph belongs to the spectrum of the graph (see Corollary 1). Thus the eigenvalue 0 of $S$ is the minimal eigenvalue of $\mathcal{M}_{12}(\mathcal{D}(G))$. Each vertex of $\mathcal{M}_{12}(\mathcal{D}(G))$ is incident to $r$ loops. Removing all loops, we obtain a $(k-1)r$-regular multigraph $\Gamma$ with the minimal eigenvalue $-r$. By the definition of transversal, we have that $T$ is an independent set and $|T| = |V(\Gamma)|/k = \frac{r|V(\Gamma)|}{(k-1)r+r}$. Then it attains the Delsart–Hoffman bound. □

A complete subgraph of a graph is called a *clique*. It is easy to see that each clique corresponds to an independent set in the complement of the graph and vice versa. It is known (see [5]) that the cardinality of each clique in an arc-transitive graph $r$-regular graph is at most $1 - \frac{r}{\lambda}$, where $\lambda$ is the minimal eigenvalue of the graph. A clique with cardinality $1 - \frac{r}{\lambda_{\min}}$ in an $r$-regular graph is called the *Delsarte clique*.

Consider a $(k-1)r$-regular multigraph $\Gamma$ with the minimal eigenvalue $-r$ obtained from $\mathcal{M}_{12}(\mathcal{D}(G))$ by removing all loops. It is clear that each hyperedge of $G$ corresponds to a clique of size $k$ in $\Gamma$. If $\Gamma$ is a simple graph then this clique turns out to be a Delsarte clique because of $k = 1 - \frac{(k-1)r}{-r}$. It is proved in [6], Theorem 2(ii), that the indicator function of a subset of vertices of a distance-regular graph is a perfect 2-coloring if the subset intersects each Delsarte clique in the same number of vertices. If $\Gamma$ is a distance-regular graph then this theorem coincides with Proposition 5.

# 4 Combinatorial Designs

The *Johnson graph* $J(n,k)$ is a graph whose vertices are binary $n$-tuples of weight $k$ and two vertices are joined by an edge if the Hamming distance between the corresponding $n$-tuples equals 2.

A *combinatorial design* with parameters $t$-$(n,k,\lambda)$ is a collection of $k$-elements subsets (*blocks*) of the $n$-element set such that each $t$-element subset is included into exactly $\lambda$ blocks from the collection.

The blocks of a $t$-$(n,k,\lambda)$-design can be represented as the vertices of $J(n,k)$. In the case of $\lambda = 1$, $t$-$(n,k,\lambda)$-designs are usually denoted by $S(t,k,n)$, if in addition $t = k-1$, then they are called by *Steiner systems* of order $n$, for instance, the Steiner triple system as $k = 3$, the Steiner quadruple system as $k = 4$ and so on.

A combinatorial design with parameters $t$-$(n, k, \lambda)$ can be regarded as a $\lambda$-fold transversal in the hypergraph $G_{n,k,t}$ whose vertices are all possible $k$-element blocks and hyperedges consist of blocks including a fixed $t$-element set.

In the case $t = k - 1$, the multigraph $\mathcal{M}_{12}(\mathcal{D}(G_{n,k,t}))$ without loops coincides with $J(n, k)$. Proposition 6 implies a well-known statement:

**Corollary 4** *The Steiner systems $S(k - 1, k, n)$ are maximal independent sets in $J(n, k)$.*

Denote $\mathcal{M}_{12}(\mathcal{D}(G_{n,k,t}))$ without loops by $(J(n, k))_{k-t}$. In the case $0 < t < k - 1$, we obtain $(J(n, k))_{k-t}$ from $J(n, k)$ by joining vertices of $J(n, k)$ corresponding to blocks including a joint $t$-element subset by the suitable number of edges.

Proposition 5 implies the following statement:

**Corollary 5** *A set $D$ is a $t$-$(n, k, \lambda)$-design if and only if the indicator function of $D$ is a perfect 2-coloring of $(J(n, k))_{k-t}$ with quotient matrix*
$$\begin{pmatrix} \binom{k}{t}(\lambda - 1) & \binom{k}{t}(\binom{n-k}{k-t} - \lambda + 1) \\ \binom{k}{t}\lambda & \binom{k}{t}(\binom{n-k}{k-t} - \lambda) \end{pmatrix}.$$

For the Steiner systems $S(k - 1, k, n)$ Corollary 5 was known before. By Proposition 6 we have

**Corollary 6** *Each $t$-$(n, k, 1)$-design is a maximal independent set in $(J(n, k))_{k-t}$.*

It is shown in [7, 8], that $t$-$(n, k, \lambda)$-designs exist for sufficiently large $n$ if their parameters meet the well-known arithmetic conditions. This guarantees the existence of perfect 2-colorings with corresponding parameters.

A matrix $H$ of size $n \times n$ is called the *Hadamard matrix* if it consists of $\pm 1$ and satisfies the equation $HH^* = nI_n$. By multiplying rows and columns of $H$ by $-1$ we can reduce $H$ to the Hadamard matrix whose first row and first column consist only of 1's. Each other row or column of such Hadamard matrix contains exactly equal numbers of 1 and $-1$. It is well known that the Hadamard matrices exist only if $n = 4m$. Consider a matrix $A_m$ of size $(4m+3) \times (4m+3)$ obtained by removing the first row and the first column of the reduced Hadamard matrix and by changing entries $-1$ by 0. We can regard all rows of $A_m$ as binary $(4m + 3)$-tuples of weight $2m + 1$. It is well known that

**Proposition 7** *The collection of rows of $A_m$ is a design with parameters $2$-$(4m + 3, 2m + 1, m)$.*

**Corollary 7** *The Hadamard matrices are equivalent to perfect 2-colorings of the corresponding multigraph with certain quotient matrix.*

Let $q$ be a prime power. Consider an $n$-dimensional vector space $X_q^n$ over the Galois field $X_q = GF(q)$. The *Grassmann graph* is a graph whose vertices are $k$-dimensional subspaces of $X_q^n$ and two vertices are joined by an edge whenever the corresponding subspaces are intersected by a $(k-1)$-dimensional subspace.

A *$q$-design* or *subspace design* with parameters $t$-$(n, k, \lambda)_q$ is a such collection of $k$-dimensional vector subspaces in $X_q^n$ that each $t$-dimensional subspace is included into exactly $\lambda$ $k$-dimensional subspaces from this collection. The subspace designs are $q$-analogs of combinatorial designs. By a similar way a subspace design with parameters $t$-$(n, k, \lambda)_q$ can be regarded as a $\lambda$-fold transversal in the hypergraph $G^q_{n,k,t}$. Vertices of $G^q_{n,k,t}$ are all possible $k$-dimensional subspaces in $X_q^n$ and hyperedges of $G^q_{n,k,t}$ are collections of subspaces which include a fixed $t$-dimensional subspace. In the case $t = k-1$, the multigraph $\mathcal{M}_{12}(\mathcal{D}(G^q_{n,k,t}))$ without loops coincides with $J_q(n, k)$.

Similar to the case of combinatorial designs above, we obtain the next corollary of Proposition 5.

**Corollary 8** *A set $D$ is a subspace design with parameters $t$-$(n, k, \lambda)_q$ if and only if the indicator function of $D$ is a perfect 2-coloring of $\mathcal{M}_{12}(\mathcal{D}(G^q_{n,k,t}))$ with quotient matrix* $\begin{pmatrix} \begin{bmatrix} k \\ t \end{bmatrix}_q(\lambda-1) & \begin{bmatrix} k \\ t \end{bmatrix}_q(\begin{bmatrix} n-k \\ k-t \end{bmatrix}_q - \lambda + 1) \\ \begin{bmatrix} k \\ t \end{bmatrix}_q\lambda & \begin{bmatrix} k \\ t \end{bmatrix}_q(\begin{bmatrix} n-k \\ k-t \end{bmatrix}_q - \lambda) \end{pmatrix}$, *where* $\begin{bmatrix} k \\ t \end{bmatrix}_q = \frac{(q^k-1)\cdots(q^{k-t+1}-1)}{(q^t-1)\cdots(q-1)}$ *are the Gaussian binomial coefficients.*

*Spreads* are subspace designs with parameters $1$-$(n, k, 1)_q$. Constructions of spreads and subspace designs with $\lambda > 1$ are studied for a long time. The first subspace design with $t > 1$ and $\lambda = 1$ was found recently. This is the subspace design with parameters $2$-$(13, 3, 1)_2$, in other words, this is a 2-analog of the Steiner triple system [9].

# 5 Difference Sets and Bent Functions

Let $K$ be a finite abelian group. A *partial difference set* with parameters $(v, k, \lambda, \mu)$ is the set $D \subseteq K$, $|K| = v$, $|D| = k$, where for each nonzero $a \in D$ there exist exactly $\lambda$ pairs $d_1, d_2 \in D$ such that $d_1 - d_2 = a$, and for each nonzero $a \in K \setminus D$ there exist exactly $\mu$ pairs $d_1, d_2 \in D$ such that $d_1 - d_2 = a$. In the case $\mu = \lambda$, every partial difference set $D \subseteq K$ is called simply a *difference set* with parameters $(v, k, \lambda)$.

A *convolution* of functions $f, g : K \to \mathbb{R}$ is defined by the equation $f * g(y) = \sum_{x \in K} f(x)g(y-x)$. Denote by $\delta : K \to \mathbb{R}$ the function taking the value $|K|$ in the identity element of $K$ and 0 in the other elements. Let $\mathbf{1}$ be the function that takes value 1 identically. It is known the following characterization of partial difference sets, see [10] for instance.

**Proposition 8** *A set $D \subseteq K$ is a partial difference set with parameters $(v, k, \lambda, \mu)$ if and only if $\chi_D * \chi_{-D} = \lambda \cdot \chi_D + \mu \cdot (\mathbf{1} - \chi_D) + (k - \mu)\delta$.*

Let $D$ be a difference set. Consider the matrix $Q_D(x, y) = \chi_D(x-y)$. By the definition, every pair of columns and every pair of rows of $Q_D$ contains the same number of pairs $(1, 1)$ as well as the other pairs $(0, 1), (1, 0), (0, 0)$. If we regard the rows of $Q_D$ as an indicator function of blocks, then the collection of the rows is a 2-$(v, k, \lambda)$-design. If, as in this case, the cardinality of 2-design is equal to its length, then the design is called *symmetric*.

As a definition of *Boolean bent function* $b : \{0, 1\}^n \to \{0, 1\}$ we can take the following property: the convolution $(-1)^b * (-1)^b$ takes the value $2^n$ in the zero vector and zeros in the remaining arguments. It is clear that $(-1)^b = \mathbf{1} - 2b$. By Proposition 8 we obtain that bent functions correspond to difference sets in $\mathbb{Z}_2^n$. It is known (see [11] for instance) that bent functions exist if and only if $n$ is even and the number of values 1 of a bent function is equal to either $2^{n-1} + 2^{n/2-1}$ or $2^{n-1} - 2^{n/2-1}$. Moreover, the following statement holds.

**Proposition 9** *Let $b$ be a Boolean bent function and let $B = \{x \in \{0, 1\}^n \mid b(x) = 1\}$, i. e., $b = \chi_B$. Then $B$ is a difference set with parameters either $(2^n, 2^{n-1} - 2^{n/2-1}, 2^{n-2} - 2^{n/2-1})$ or $(2^n, 2^{n-1} + 2^{n/2-1}, 2^{n-2} + 2^{n/2-1})$. The converse statement is also true.*

The parameters of the difference set in Proposition 9 is called the *McFarland parameters* [12]. It is easy to prove that each difference set with the McFarland parameters is possible to transform into some Hadamard matrix.

A connected graph $G$ is *strongly regular* with parameters $(v, k, \lambda, \mu)$, whatever $|V(G)| = v$, the degree of each vertex equals $k$, each two adjacent vertices have $\lambda$ common neighbors, and each two non-adjacent vertices have $\mu$ common neighbors. For a group $K$ we consider a set $A \subset K$ of generator such that $A^{-1} = A$ and $\varepsilon \notin A$, where $\varepsilon$ is the identity element of $K$. The *Cayley graph* $Cay(K, A)$ is the graph whose vertices are the elements of $K$ and two vertices $x, y \in K$ are joined by an edge if $y = xa$ for some $a \in A$. The conditions $A^{-1} = A$ and $\varepsilon \notin A$ imply that $Cay(K, A)$ is a simple graph.

Consider the Cayley graph of an abelian group $K$ with a set $D$ of generators such that $D = -D$ and $0 \notin D$. The definition of a partial difference set with parameters $(v, k, \lambda, \mu)$ implies that the graph $Cay(K, D)$ is strongly regular with parameters $(v, k, \lambda, \mu)$. Indeed, vertices $x, y \in K$ are adjacent in $Cay(K, D)$ if and only if $x + a = y$ for some $a \in D$. Thus, by the definition of partial difference set, there exist exactly $\lambda$ pairs $d_1, d_2 \in D$ such that $a = d_1 - d_2$. Consequently, there exist exactly $\lambda$ vertices $b$, $b = x + d_1 = y + d_2$ adjacent to both vertices $x$ and $y$ in $Cay(K, D)$. By a similar way we can consider the case of non-adjacent vertices $x, y \in K$. It is easy to see that the converse statement is true, i. e., if $Cay(K, D)$ is a strongly regular graph then $D$ is a partial difference set in $K$.

As we showed above, each difference set $D$ with parameters $(v, k, \lambda)$ corresponds to a symmetric 2-$(v, k, \lambda)$-design. If $D = -D$ and $0 \notin D$ then $Q_D(x, y) = \chi_D(x - y)$ is the adjacency matrix of $Cay(K, D)$. The converse statement is also true: if a matrix $A$ is symmetric and has only zeros on the main diagonal, and rows of $A$ are blocks of a 2-$(v, k, \lambda)$-design, then $A$ is the adjacency matrix of some strongly regular graph with parameters $(v, k, \lambda, \lambda)$.

# 6 Difference Sets, Strongly Regular Graphs and Bent Functions as Perfect Colorings

Further we will show that each strongly regular graph and, in particular, each partial difference set corresponds to a perfect 2-coloring of some hypergraph.

Consider the complete graph $K_n$ on $n$ vertices. Define a hypergraph $\Gamma_n$ whose vertices are edges of $K_n$ and a triple of vertices constitutes a hyperedge wherever these vertices generate a triangle in $K_n$. Let $G$ be a graph on the same set of vertices. Define a coloring of $\Gamma_n$ as follows: $f(e) = 1$ if $e \in E(G)$ and $f(e) = 0$ if $e \notin E(G)$. The coloring $f$ is perfect if and only if $G$ is strongly regular. In order to see this we need to verify the property: for each two vertices $u$ and $v$ the numbers of vertices adjacent to either $u$ or $v$, adjacent to both $u$ and $v$, and adjacent to neither $u$ no $v$ in $G$ depend only on the adjacency of $u$ and $v$ in $G$. This condition coincides with the definition of a strongly regular graph. In particular, for the strongly regular graph $G$ with parameters $(n, k, \lambda, \mu)$ we obtain that if $u$ and $v$ are adjacent in $G$, then the number of vertices which are adjacent to both vertices $u$ and $v$ equals $\lambda$; the number of vertices which are adjacent to either $u$ or $v$ equals $k - \lambda$; the number of vertices which are not adjacent to $u$ or $v$ equals $v - 2 - 2k + \lambda$.

Next we will consider the abelian group $\mathbb{Z}_2^n$. Denote by $\Delta_n$ a 3-uniform hypergraph whose vertices are elements of $\mathbb{Z}_2^n$ without 0 and triples $\{a_1, a_2, a_3\}$ are hyperedges wherever $a_1 + a_2 + a_3 = 0$. Notice that the equality $a_1 + a_2 + a_3 = 0$ implies that all three elements $a_i \neq 0$, $i = 1, 2, 3$ are different. If $D$ is a partial difference set $D \subset \mathbb{Z}_2^n$ with parameters $(v, k, \lambda, \mu)$, then $\chi_D$ is a perfect 2-coloring of $\Delta_n$. Indeed, the equation $a_1 + a_2 + a_3 = 0$ is equivalent to the equation $a_1 = a_2 - a_3$ in $\mathbb{Z}_2^n$. Then each vertex $a_1 \in D$ is incident to $\lambda$ hyperedges containing three elements of $D$ and each vertex $a_1 \notin D$ is incident to $\mu$ hyperedges containing two elements of $D$. Since every pair $a, b \in \mathbb{Z}_2^n \setminus \{0\}$ belongs to exactly one hyperedge, we can calculate the numbers of hyperedges containing a vertex $a \in D$ or $a \notin D$ for all color compositions of hyperedges. These numbers depend only on the color of the vertex. Thus, $\chi_D$ is a perfect 2-coloring by the definition. The converse statement is also true. Let $D$ be the set of 1 colored vertices. It is sufficient to observe that the parameter $\lambda$ of the partial difference set $D$ is equal to the number of hyperedges of the color composition $(1, 1, 1)$ which contain a fixed vertex of color 1, and the parameter $\mu$ is equal to the number of hyperedges of the color composition $(0,1,1)$ which contain a fixed vertex of color 0.

Therefore, the next statement follows from Proposition 9.

**Proposition 10** *Boolean bent functions one-to-one correspond to perfect* 2-*colorings of* $\Delta_n$ *with certain quotient matrix.*

By Proposition 4, a perfect 2-coloring of $\Delta_n$ induces a perfect coloring of the line graph $\mathcal{E}(\Delta_n)$. Since an intersection of every two hyperedges of $\Delta_n$ consists of at most one vertex, $\mathcal{E}(\Delta_n)$ is a simple graph. Notice that a 2-coloring $\chi_D$ of $\Delta_n$ induces a 4-coloring of $\mathcal{E}(\Delta_n)$ because each hyperedge consists of three vertices and, consequently, it can contain 0, 1, 2 or 3 vertices of the first color. Consider elements of $\mathbb{Z}_2^n$ as elements of the vector space $X_2^n$. It is easy to see that hyperedges of $\Delta_n$ one-to-one correspond to 2-dimensional subspaces of $X_2^n$. Furthermore, two hyperedges of $\Delta_n$ have common vertex if and only if the corresponding subspaces meet along a 1-dimensional subspace. Thus the graph $\mathcal{E}(\Delta_n)$ is equivalent to the Grassmann graph $J_2(n,2)$ by the definition.

By Proposition 9, we obtain that a Boolean bent function generates a difference set in $\mathbb{Z}_2^n$. As proved above, this set induces a perfect 4-coloring of $J_2(n,2)$. In the following theorem we prove that the converse holds: each perfect coloring of $J_2(n,2)$ with certain quotient matrix determines a Boolean bent function. We consider only the case as bent functions with $2^{n-1}+2^{n/2-1}$ values 1, the case as bent functions with $2^{n-2}-2^{n/2-1}$ values 1 is similar.

**Theorem 2** *Bent functions* $b:\{0,1\}^n\to\{0,1\}$ *with* $|\mathrm{supp}(b)|=2^{n-1}+2^{n/2-1}$ *and* $b(\bar{0})=1$ *one-to-one correspond to perfect colorings of* $J_2(n,2)$ *with quotient matrix*

$$\begin{pmatrix} 3(2^{n-3}-2^{\frac{n}{2}-2}-1) & 3\cdot 2^{n-2}-3 & 3(2^{n-3}+2^{\frac{n}{2}-2}) & 0 \\ 2^{n-2}-2^{\frac{n}{2}-1} & 5\cdot 2^{n-3}-2^{\frac{n}{2}-2}-5 & 2^{n-1}+2^{\frac{n}{2}-1} & 2^{n-3}+2^{\frac{n}{2}-2}-1 \\ 2^{n-3}-2^{\frac{n}{2}-2} & 2^{n-1}-2^{\frac{n}{2}-1}-1 & 5\cdot 2^{n-3}+2^{\frac{n}{2}-2}-3 & 2^{n-2}+2^{\frac{n}{2}-1}-2 \\ 0 & 3(2^{n-3}-2^{\frac{n}{2}-2}) & 3\cdot 2^{n-2} & 3(2^{n-3}+2^{\frac{n}{2}-2}-2) \end{pmatrix}.$$

*Proof.* At first we obtain a coloring of $J_2(n,2)$ from a given bent function. We determine a vertex color of $J_2(n,2)$ as the number of values 1 of the bent function on the corresponding 2-dimensional subspace. Since $b(\bar{0})=1$, there exit the 1st, 2nd, 3rd and 4th colors. We need to prove that this coloring is perfect.

By the definition of bent function it follows that

$$|\{x\in\{0,1\}^n : b(x)\oplus b(x+y)=0\}| = |\{x\in\{0,1\}^n : b(x)\oplus b(x+y)=1\}| \quad (2)$$

for each $y\neq\bar{0}$.

Consider two 2-dimensional subspaces meeting along $\{\bar{0},y\}$. Denote by $A_{\alpha\beta}$ the number of pairs of values $(\alpha,\beta)$ among all pairs $(b(x),b(x+y))$. Then $A_{01}+A_{10}=A_{00}+A_{11}$ by (2). Moreover, $A_{01}+A_{10}+A_{00}+A_{11}=2^n$,

$A_{01} = A_{10}$ from the symmetry and $A_{10} + A_{11} = |\mathrm{supp}(b)| = 2^{n-1} + 2^{n/2-1}$. Hence, it is easy to calculate that $A_{01} = A_{10} = 2^{n-2}$, $A_{11} = 2^{n-2} + 2^{\frac{n}{2}}$ and $A_{00} = 2^{n-2} - 2^{\frac{n}{2}-1}$.

If a color of some subspace is known then we can calculate the number of subspaces of each color adjacent to it in $J_2(n,2)$. For instance, the 4th color subspace is not adjacent to subspaces of the first color; it is adjacent to $3A_{00}/2$ subspaces of 2nd color because each pair of vectors $(x, x+y)$, $(b(x), b(x+y)) = (0,0)$, is counted twice $(x, x+y)$ and $(x+y, x)$; it is adjacent to $(3A_{11} - 12)/2$ subspaces of the 4th color because the initial subspace contains four vectors of the value 1 and by fixing every of three nonzero vectors we have $(A_{11} - 4)/2$ adjacent subspaces of the 4th color. Then the coloring of $J_2(n,2)$ is perfect by the definition. Furthermore, we can calculate the quotient matrix of the coloring

$$\frac{1}{2}\begin{pmatrix} 3A_{00}-6 & 6A_{01}-6 & 3A_{11} & 0 \\ 2A_{00} & 4A_{01}+A_{00}-10 & 2A_{01}+2A_{11} & A_{11}-2 \\ A_{00} & 2A_{01}+2A_{00}-2 & 4A_{10}+A_{11}-6 & 2A_{11}-4 \\ 0 & 3A_{00} & 6A_{01} & 3A_{11}-12 \end{pmatrix}.$$

Suppose that a perfect 4-coloring of $J_2(n,2)$ with this quotient matrix is given. Denote $b$ as follows: let $b(\bar{0}) = 1$ and $b(x) = 1$ at all points $x \in X_2^n$ belonging to subspaces of the 4th color. At the remaining points $x \in X_2^n$ we put $b(x) = 0$. At the last part of the proof we verify that the resulting $b$ is a bent function.

Denote by $M(x)$ the number of the 4th color subspaces containing a vector $x \in B = \{x \neq \bar{0} : b(x) = 1\}$. By the entry (1st row, 4th column) of the quotient matrix we conclude that subspaces of the 1st color do not contain $x$. Moreover, all vectors of a subspace belong to $B$ if and only if it is the 4th color subspace because subspaces of other colors are adjacent to subspaces of the 1st color. By the entry (4th row, 4th column) of the quotient matrix we conclude that the average of $M(x)$ over an arbitrary subspace of the 4th color equals $\frac{A_{11}}{2} - 1$, where $A_{11} = 2^{n-2} + 2^{\frac{n}{2}}$. Suppose that there is a such $x \in B$ that $M(x) \neq \frac{A_{11}}{2} - 1$. Let us take $x \in B$ such that the difference $|M(x) - (\frac{A_{11}}{2} - 1)|$ attains the maximum.

Suppose $M(x) > \frac{A_{11}}{2} - 1$ and this difference is greater than the difference $|M(u) - (\frac{A_{11}}{2} - 1)|$ for each $u \in B$ such that $M(u) < \frac{A_{11}}{2} - 1$. We will show that there exist one more vector $v \in B$ such that $M(v) \geq A_{11}/2$. Consider some subspace containing $x$. For both vectors $z \neq x$ and $x + z$ we have that $M(z) < \frac{A_{11}}{2} - 1$ and $M(x+z) < \frac{A_{11}}{2} - 1$ because the average value of $M(y)$ over every subspace of the 4th color is $\frac{A_{11}}{2} - 1$ but the difference between $M(x)$ and $\frac{A_{11}}{2} - 1$ is strongly maximal by the assumption. If $M(z) > 1$ or $M(x+z) > 1$ than there exist at least two distinct subspaces, and each of them contains a vector $v \in B$ such that $M(v) \geq A_{11}/2$. If $M(z) = 1$ and $M(x+z) = 1$ then $M(x) = \frac{3A_{11}}{2} - 6$ and each subspace containing $x$ is adjacent to $\frac{3A_{11}}{2} - 6$ subspaces of the 4th color. It contradicts the definition of the quotient matrix.

We have shown that there exist at least two vectors $x, y \in B$ such that $M(x) \geq M(y) \geq A_{11}/2$. Consider the subspace $\{\bar{0}, x, y, x+y\}$. We obtain that $x + y \notin B$ because $x + y \in B$ contradicts the choice of $x$. But there are no subspaces of colors 1, 2 or 3 adjacent to $M(x) + M(y) - 1 \geq A_{11} - 1$ subspaces of the 4th color.

Next we suppose that the difference $|M(x) - (\frac{A_{11}}{2} - 1)|$ is maximal (not necessarily strictly maximal) for some $x \in B$ such that $M(x) < \frac{A_{11}}{2} - 1$. Consider some 4th color subspace containing $x$. In this subspace there exists a vector $z \in B$ such that $M(z) > \frac{A_{11}}{2} - 1$ because the average number of $M(v)$ over every subspace of the 4th color is the constant. By a similar way we conclude that every subspaces (among $M(z)$ 4th color subspaces) containing $z$ contains vectors $y \in B$ such that $M(y) < \frac{A_{11}}{2} - 1$. We will use three distinct $x, y, y' \in B$ satisfied the last inequality.

Consider the subspace $\Omega = \{\bar{0}, x, y, x+y\}$. This subspace is adjacent to $M(x)+M(y)$ subspaces of color 4 because the opposite assumption $x+y \in B$ contradicts maximality of the difference $|M(x) - (\frac{A_{11}}{2} - 1)|$. The color of $\Omega$ does not equal 4 because of $x + y \notin B$ and it does not equal 1 because the 1st color subspaces are not adjacent to subspaces of color 4. The color of $\Omega$ does not equal 3 because $M(x) + M(y) < A_{11} - 2$, i. e., the 3rd color of $\Omega$ contradicts the quotient matrix. It leaves the last possibility that the color of $\Omega$ is equal to 2. By the quotient matrix, $\Omega$ is adjacent to $A_{00} = 2^{n-2} - 2^{\frac{n}{2}-1}$ subspaces of the 1st color. These subspaces do not contain $x$ or $y$ because subspaces of colors 1 and 4 are not adjacent.

Consider subspace $\Omega' = \{\bar{0}, x, y', x+y'\}$. By the same way we obtain that $x + y'$ belongs to $A_{00}$ subspaces of color 1. Then the subspace $\{\bar{0}, x + y, x + y', y + y'\}$ is adjacent to $2A_{00} - 1$ subspaces of color 1 at least. By the quotient matrix such subspace does not exist. By this contradiction we proved that $M(x) = \frac{A_{11}}{2} - 1$ for each $x \in B$.

By the 4th column of the quotient matrix we obtain that every subspace of color $i$ contains $i - 1$ vectors of $B$. Consequently, the coloring of $J_2(n, 2)$ induces a perfect 2-coloring of $\Delta_n$. It is easy to see that parameters of the coloring of $\Delta_n$ correspond to parameters of a coloring induced by a bent function. Therefore, the coloring of $J_2(n, 2)$ with the quotient matrix determined above induces a bent function by Proposition 10. $\square$

It is not difficult to verify that, combining pairwise even and odd colors in the above 4-coloring of $J_2(n, 2)$, we obtain a perfect coloring with quotient matrix

$$\begin{pmatrix} 3 \cdot 2^{n-2} - 3 & 3 \cdot 2^{n-2} - 3 \\ 3 \cdot 2^{n-2} & 3 \cdot 2^{n-2} - 6 \end{pmatrix}.$$

But these perfect 2-colorings can not one-to-one correspond to bent functions. When we add an affine function to a bent function, we will get a new bent function but the 2-coloring will remain the same.

## References


[1] D.G. Fon-Der-Flaass, A bound on correlation immunity. *Sib. Electr. Math. Reports*, 2007. vol. 4, P. 133–135.

[2] V.N. Potapov, On perfect 2-colorings of the $q$-ary $n$-cube. *Discrete Math.*, 2012. vol. 312(6), P. 1269–1272.

[3] S.V. Avgustinovich, I.Yu. Mogilnykh, Perfect 2-colorings of Johnson graphs J(8,3) and J(8,4). *J. Appl. Industr. Math.*, 5:1 (2011), 19-30.

[4] D.G. Fon-Der-Flaass, Perfect 2-coloring of a hypercube. *Siberian Math. J.*, 48:4 (2007), 740-745.

[5] C. Godsil and K. Meagher, *Erdos-Ko-Rado Theorems: Algebraic Approaches.* Cambridge University Press, 2016.

[6] D.S. Krotov, I.Yu. Mogilnykh, V.N. Potapov. To the theory of $q$-ary Steiner and other-type trades. *Discrete Math.*, 2016. vol. 339(3), P. 1150-1157.

[7] P. Keevash, The existence of designs, arXiv preprint, https://arxiv.org/abs/arXiv:1401.3665v3[math.CO], 2019 - arxiv.org

[8] S. Glock, D. Kühn, A. Lo, and D. Osthus, The existence of designs via iterative absorption: hypergraph $F$-designs for arbitrary $F$. *Mem. Amer. Math. Soc.*, 284 (2023), no. 1406, v+131 pp.

[9] M. Braun, T. Etzion, P.R.J. Östergard, A. Vardy, A. Wassermann, Existence of $q$-analogs of steiner systems. *Forum of Mathematics Pi*, 2016. vol. 4, e7.

[10] S.L. Ma, A survey of partial difference sets. *Designs, Codes and Cryptography.* 1994. vol. 4, P. 221–261.

[11] S. Mesnager, *Bent Functions: Fundamentals and Results.* Springer International Publishing Switzerland, 2016.

[12] R.L. McFarland, A family of difference sets in non-cyclic groups. *J. Combin. Theory. Ser. A*, 1973. vol. 15(1), P. 1–10.