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CODING THEORY

On the Cardinality Spectrum and the Number of Latin Bitrades of Order 3

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Abstract—By a (Latin) unitrade of order k, we call a subset of vertices of the Hamming graph $H(n,k)$ that intersects any maximal clique at either 0 or 2 vertices. A bitrade is a bipartite unitrade, i.e., a unitrade that can be split into two independent subsets. We study the cardinality spectrum of bitrades in the Hamming graph $H(n,k)$ with $k=3$ (ternary hypercube) and the growth of the number of such bitrades as n grows. In particular, we determine all possible small (up to $2.5\cdot 2^n$) and large (from $14\cdot 3^{n-3}$) cardinalities of bitrades of dimension n and prove that the cardinality of a bitrade takes only values equivalent to 0 or 2^n modulo 3 (this result can be treated in terms of a ternary Reed–Muller type code). A part of the results are valid for an arbitrary k. For $k=3$ and $n\to\infty$ we prove that the number of nonequivalent bitrades is not less than $2^{(2/3-o(1))n}$ and not greater than 2^{α^n}, $\alpha<2$ (the growth order of the double logarithm of this number remains unknown). Alternatively, the studied set B_n of bitrades of order 3 can be defined as follows: B_0 consists of three rationals $-1,0,1$; B_n consists of ordered triples (a,b,c) of elements from B_{n-1} such that $a+b+c=0$.

Key words: Latin bitrades, unitrades, Reed–Muller codes, combinatorial configurations, Boolean functions.

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1. INTRODUCTION

For combinatorial objects (configurations) of various types it is useful to define the notion of a bitrade in such a way that the definition of a bitrade is not directly based on definitions of original objects but rather involves various differences (say symmetric) of objects of this kind (see [1]). Bitrades reflect possible distinction between two combinatorial configurations of one and the same type, which is important in enumeration, description, and analysis of properties of the combinatorial configurations. There are known studies of bitrades (or trades) of combinatorial block designs [2–6], Latin squares [7], generalized designs in partially ordered sets [8], perfect codes [9,10], correlation immune Boolean functions and bent functions [11]. In the present paper we consider Latin bitrades corresponding to MDS codes with distance 2, or (equivalently) Latin hypercubes, or polyadic quasigroups. We study the spectrum of possible cardinalities of bitrades and obtain bounds on their number.

Let us pass to formal definitions. Let $Q_k=\{0,\dots,k-1\}$. Define the Hamming distance $d(u,v)$ to be the number of noncoinciding components in the tuples $u,v\in Q_k^n$. The metric space (Q_k^n,d), as well as the distance-1 graph ΓQ_k^n on the vertex set Q_k^n, is said to be a k-ary n-hypercube or a Hamming graph. The weight of a vertex $u\in Q_k^n$ is defined as $\mathrm{wt}(u)=d(u,\bar{0})$, where $\bar{0}$ is the all-zero n-tuple (below we also use the analogous notation $\bar{1}$ and $\overline{-1}$). A face in Q_k^n is a set of vertices of a hypercube obtained by fixing values of one or several coordinate(s). A set $U\subset Q_k^n$ is

said to be a *unitrade* (*of dimension* n) if the cardinalities of its intersections with one-dimensional faces (maximal cliques in ΓQ_k^n) take only two values: 0 and 2. Usually, a bitrade is defined as a pair $\{U_0, U_1\}$ consisting of two independent parts of a bipartite unitrade $U = U_0 \cup U_1$. However, it will be convenient for us to define a *bitrade* as a unitrade $U \subset Q_k^n$ such that the subgraph ΓU generated by the set of vertices U is bipartite. In the two-dimensional case ($n = 2$), any unitrade is a bitrade. Indeed, Kőnig's theorem implies that any square $(0,1)$-matrix containing the same number of ones in each column and each row contains a diagonal. Hence, the table of the characteristic function of a two-dimensional unitrade, which can be considered as a $(0,1)$-matrix, after deleting zero rows and columns contains two disjoint diagonals of ones. For $n \geq 3$ and $k \geq 3$ there are unitrades that are not bitrades. A minimal example is given below; the three two-dimensional arrays in it correspond to three parallel hyperfaces of a three-dimensional unitrade:

1	**1**	0
1	0	**1**
0	**1**	**1**

,

0	**1**	**1**
1	0	**1**
1	**1**	0

,

1	0	**1**
0	0	0
1	0	**1**

. (1)

Consider the correspondence between the above definition and the general notion of a bitrade. An *MDS code* is a subset of the hypercube Q_k^n intersecting every face of a fixed dimension r at exactly one element. One can easily see that MDS codes are codes with minimum distance $r+1$ between vertices and with the maximum cardinality k^{n-r} for this distance, i.e., codes meeting the Singleton bound. In our context, we are only interested in MDS codes with code distance 2, i.e., in the case of $r = 1$. (A function expressing the value of one coordinate of vertices of such a code through the other $n-1$ coordinates is said to be a Latin $(n-1)$-cube, a Latin square in the case of $n = 3$, and an algebraic system with this function as an operation is said to be an $(n-1)$-ary quasigroup). It is seen from the definitions that the symmetric difference of two MDS codes is a bitrade.

The isometry group of the hypercube Q_k^n is generated by the group of permutations of coordinates and the isotopy group, i.e., group of permutations of elements of Q_k in each coordinate. In the case of $k = 3$, the isometry group of Q_3^n consists only of affine transformations of Q_3^n as an n-dimensional vector space over GF(3). Subsets of the hypercube that can be interchanged by isometries of the space are said to be *equivalent*. A *retract* of a set $U \subset Q_k^n$ is a subset of the hypercube Q_k^{n-1} obtained as the intersection of U with some face of dimension $n-1$. The direction of a retract or of a hyperface is the index of the fixed coordinate in this face. The definitions immediately imply the following.

Proposition 1. *Any retract of a unitrade* (*bitrade*, *MDS code*) *is a unitrade* (*bitrade*, *MDS code*) *in a hypercube of a smaller dimension.*

Proposition 2. *The image of a unitrade* (*bitrade*, *MDS code*) *under an isometry of the hypercube is a unitrade* (*bitrade*, *MDS code*).

In this paper we almost completely confine ourselves with considering bitrades in Q_3^n. This case seems to be a keystone, since any ternary bitrade can be isometrically embedded in a hypercube Q_k^n of a larger order $k \geq 4$; thus, answering many questions in the general case seems to be no simpler than for the case of $k = 3$. On of such questions is the problem of asymptotics for the double logarithm of the number of objects when k is fixed and n grows, and for the case of $k = 3$ this problem is the most prominent: the known lower bound on the number of bitrades does not prove that this quantity is doubly exponential, whereas the upper bound is close to the trivial one 2^{2^n} (from which we slightly move away in this paper; see Theorem 8 below). For other orders, $k > 3$, a doubly exponential lower bound is obtained by a switching construction based on the possibility to arrange disjoint bitrades in the space in question (see lower bounds for the number of Latin hypercubes [12, 13]). On one hand, a limited number of "degrees of freedom" of Latin bitrades of order 3 allows for hope for constructing a consistent combinatorial-algebraic theory of such objects

(an attempt is made in the present paper); on the other hand, hardness of certain questions is completely revealed even for this small order.

For the analysis of unitrades in this paper, we use methods of linear algebra (Sections 2 and 4.2), theory of Boolean functions (Section 3), and coding theory (Sections 3 and 4.1). In particular, we show a relation between the problem of description of ternary bitrades and the problem of finding the polynomial complexity of a Boolean function [14, 15]. Also, an interrelation between ternary bitrades and almost balanced Boolean functions is known (Proposition 7).

In Section 4 we study the cardinality spectrum of ternary bitrades and of unitrades and bitrades of higher orders. We show that the cardinality of any ternary bitrade of dimension n is equivalent to either 0 or 2^n modulo 3. The minimum cardinality of a nonempty bitrade (not only ternary) of dimension n is 2^n. All possible cardinalities of bitrades no greater than $2 \cdot 2^n$ have been previously known (see [16]). In the present paper we show a relation between the cardinality spectrum of bitrades and the weight spectrum of a binary Reed–Muller code (Proposition 9). Furthermore, we find all possible cardinalities of unitrades and bitrades of dimension n from the minimum 2^n to $2.5 \cdot 2^n$ (Theorems 1 and 5 and Corollary 7). A ternary bitrade of the maximum cardinality $2 \cdot 3^{n-1}$ is a pair of disjoint MDS codes. It is known (see, e.g., [17]) that there exists a unique (up to equivalence) ternary bitrade of this cardinality. Note that even for order 4 there are exponentially many nonequivalent bitrades of the maximum cardinality $2 \cdot 4^{n-1}$ (see [13, 17]).

One of the main problems in the analysis of bitrades is finding their number as a function of the dimension n and order k. Since bitrades correspond to differences of combinatorial objects, studying the variety of bitrades opens perspectives for studying the variety of original objects and bounding their number. For Latin hypercubes (of order greater than 4) of dimension n, with which the bitrades in question are related, even the order of the growth rate of the logarithm of their number as $n \to \infty$ is still unknown (see [13]). In [16] there was obtained an almost exponential ($e^{\Omega(\sqrt{n})}$) asymptotic lower bound on the number of nonequivalent bitrades in Q_3^n. In Section 5.1 we prove the lower bound $2^{(2/3-o(1))n}$ on the number of nonequivalent bitrades in Q_3^n as $n \to \infty$ (Theorem 7) and show that a higher-than-exponential lower bound cannot be obtained by a method similar to ours (Theorem 6). In Section 5.2 we derive an upper bound of the form 2^{α^n}, $\alpha < 2$, for the number of bitrade in Q_3^n (Theorem 8), which considerably improves the trivial upper bound 2^{2^n}. However, the question of the growth rate for ternary bitrades of dimension n remains open: we do not even know whether this function is exponential or doubly exponential in n. Results of numerical experiments for small n, presented in a table in Section 4.5, show that the growth rate is faster than exponential; however, it would be hasty to conclude about the linear growth of the double logarithm.

The method of proving the upper bound uses the fact that in the hypercube Q_3^n with $n \geq 7$ we can find a set of cardinality strictly greater than $3^n - 2^n$ which contains no bitrade and even no symmetric difference of two bitrades. For a more well-known problem on the maximal subset of a hypercube containing no arithmetic progression (which in the ternary hypercube coincides with a subset containing no one-dimensional affine subspace), there was recently obtained an asymptotic upper bound of the form $o(\alpha^n)$, $\alpha < 3$ (see [18]). Finding in a ternary hypercube a subset of the maximal cardinality that contains no bitrades remains an open problem.

Concluding Section 1, let us recursively describe a class B_n of objects which is defined quite naturally and is equivalent to the class of Latin bitrades of order 3 (this formulation is merely illustrative and is never used in what follows):

$$B_0 = \{-1, 0, 1\}, \qquad B_n = \{(a, b, c) \in B_{n-1}^3 \mid a + b + c = 0\}.$$

If elements B_n are represented as n-dimensional arrays with entries $-1, 0, 1$, then the set of cells with nonzero values in such an array precisely forms a bitrade.

2. LINEAR SPACES

Let $\mathbb{F}$ be a field. Consider the set of functions $\{g\colon Q_k^n \to \mathbb{F}\}$ as a vector space over $\mathbb{F}$. Denote by $\mathbb{V}_{n,k}(\mathbb{F})$ the subspace of functions for which the sum of values over every one-dimensional face (maximal clique in the graph ΓQ_k^n) is 0. Consider a bitrade $B \subset Q_k^n$. It corresponds to a function $b[B]\colon Q_k^n \to \{0, \pm 1\}$ which takes value 1 on one part of the bitrade, value -1 on the other, and is zero at all other vertices. Consider $\{0, \pm 1\}$ as a subset of $\mathbb{F}$ (for a field of characteristic 2 we have $-1 = 1$). Clearly, $b[B] \in \mathbb{V}_{n,k}(\mathbb{F})$. The characteristic function of a unitrade in contained in $\mathbb{V}_{n,k}(\mathbb{F})$ in the case where $\mathbb{F}$ has characteristic 2.

Introduce the following partial order $\preceq$ on Q_k: $k-1$ is the maximal element, and all other elements of Q_k are incomparable. Let us extend this partial order onto Q_k^n. Let $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in Q_k^n$. We denote $(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n)$ if for any $i \in \{1, \ldots, n\}$ we have $x_i \preceq y_i$. Note that the set $G_y = \{x \in Q_k^n \mid x \preceq y\}$ is a face of the hypercube Q_k^n of dimension equal to the number of symbols $k-1$ in y.

Let us show that $\dim \mathbb{V}_{n,k}(\mathbb{F}) = (k-1)^n$. Let $f \in \mathbb{V}_{n,k}(\mathbb{F})$. The sum of values of f over every face of any nonzero dimension is zero, and therefore

$$f(y) = -\sum_{x \in Q_{k-1}^n,\, x \preceq y} f(x) \quad \text{for } y \in Q_k^n \setminus Q_{k-1}^n. \tag{2}$$

Hence, to define a function $f \in \mathbb{V}_{n,k}(\mathbb{F})$, it is necessary and sufficient to define it on all minimal elements, i.e., on Q_{k-1}^n. Let us construct a family of linearly independent functions of the same cardinality. Let $x \in Q_{k-1}^n$. Consider the set $B_x = \{y \in Q_k^n \mid x \preceq y\}$. It is easily seen that the graph ΓB_x is isomorphic to the Boolean hypercube ΓQ_2^n; in particular, B_x is a bitrade. Here, to a tuple $z \in Q_2^n$ there corresponds a vertex $y \in B_x$ whose coordinates corresponding to ones in z equal $k-1$ and coordinates corresponding to zeros in z are the same as in x. Define $\widetilde{\mathrm{wt}}(y)$ to be the number of coordinates in y equal to $k-1$; i.e., $\widetilde{\mathrm{wt}}(y) = \mathrm{wt}(z)$. A function corresponding to this bitrade can be specified by the explicit formula $b_x(y) = (-1)^{\widetilde{\mathrm{wt}}(y)} \chi_{B_x}(y)$. Since $\mathrm{supp}(b_x) \cap Q_{k-1}^n = \{x\}$, the collection of functions $\{b_x \mid x \in Q_{k-1}^n\}$ is a basis in $\mathbb{V}_{n,k}(\mathbb{F})$; hence, $\dim \mathbb{V}_{n,k}(\mathbb{F}) = (k-1)^n$.

The following statement is easily proved by induction (see [16]).

Proposition 3. *Let $f \in \mathbb{V}_{n,k}(\mathbb{F})$ and $\mathrm{supp}(f) \neq \varnothing$. Then we have* (a) $|\mathrm{supp}(f)| \geq 2^n$; (b) *if $|\mathrm{supp}(f)| = 2^n$, then the graph $\Gamma(\mathrm{supp}(f))$ is isomorphic to the Boolean hypercube ΓQ_2^n.*

It is easily seen that in total there are $\binom{k}{2}^n$ variants to choose a unitrade (bitrade) with support of cardinality 2^n. As is shown above, a basis of the space $\mathbb{V}_{n,k}(\mathrm{GF}(2))$ can be composed of characteristic functions of the Boolean hypercube (more precisely, of sets inducing a subgraph isomorphic to the Boolean hypercube of dimension n). Since the number of such sets is greater than the dimension of the space for $k > 2$, a unitrade has more than one representation as a linear combination of Boolean hypercubes over GF(2). The minimum number of Boolean hypercubes in such a representation for a unitrade U will be referred to as the *rank* of the unitrade and denoted by $\mathrm{rank}(U)$.

Let us consider the ternary hypercube in more detail. Every element of the space $\mathbb{V}_{n,3}(\mathrm{GF}(2))$ is unitrade, since an even number of ones out of three possible in a one-dimensional face is either 0 or 2. The dimension of the space $\mathbb{V}_{n,3}(\mathrm{GF}(2))$ is 2^n, so in the hypercube Q_3^n there are exactly 2^{2^n} different unitrades. By induction on the dimension n, one can easily prove the following.

Proposition 4. (a) *Any pair of nonempty unitrades in Q_3^n has a nonempty intersection*;
(b) *If a nonempty unitrade in Q_3^n is a subset of another unitrade, then these unitrades coincide*;
(c) *The graph ΓU of a unitrade U is connected in Q_3^n.*

Claim (c) implies that if a unitrade $U \subset Q_3^n$ is a bitrade, then it is uniquely split in two parts.

3. BOOLEAN FUNCTIONS

Let $f\colon Q_2^n \to Q_2$ be a Boolean function. Define the function $U[f]\colon Q_3^n \to Q_2$ by $U[f](y) = \bigoplus\limits_{x\in Q_2^n,\, x\preceq y} f(x)$ (here and in what follows, the operation $\oplus$ denotes addition in the binary field GF(2)). It is seen from the definition that $U[f]|_{Q_2^n} = f$. Since the definition of $U[f]$ and equation (2) coincide over the field GF(2), $U[f]$ is the characteristic function of a unitrade in Q_3^n. Moreover, the above arguments imply that there is a one-to-one correspondence between Boolean functions and ternary unitrades.

Let $x \in Q_2^n$. Introduce the following notation: $x_i^1 = x_i$, $x_i^{-1} = x_i \oplus 1$, $x_i^0 = 1$, and if $x = (x_1, \dots, x_n)$, then $x^v = x_1^{v_1} \dots x_n^{v_n}$, where $v \in Q_3^n = \{0, \pm 1\}^n$.

A *polynomial representation* of a Boolean function f is a representation of the form $f(x) = f^A(x_1, \dots, x_n) = \bigoplus\limits_{v\in A\subset Q_3^n} x^v$. The minimum number of terms in this representation ($\min|A|$) is called the *polynomial complexity* of f (see [15]).

Denote $\{0, \pm 1\}_0 = \{0, 1\}$, $\{0, \pm 1\}_1 = \{1, -1\}$, $\{0, \pm 1\}_{-1} = \{0, -1\}$, and $\{0, \pm 1\}_v = \{0, \pm 1\}_{v_1} \times \dots \times \{0, \pm 1\}_{v_n}$. All Boolean hypercubes embedded in Q_3^n are cubes of the form $\{0, \pm 1\}_v$. It is easily seen that a restriction of the characteristic function of a hypercube $\{0, \pm 1\}_v$ onto the Boolean subcube $\{0, 1\}^n$ coincides with the monomial x^v; i.e., $\chi_{\{0,\pm 1\}_v}(x) = x^v$ for $x \in Q_2^n$. Therefore, $U[f^A] = \bigoplus\limits_{v\in A\subset Q_3^n} \chi_{\{0,\pm 1\}_v}$, and $\operatorname{rank}(U[f])$ is equal to the polynomial complexity of f. The problem of finding a minimum polynomial complexity representation for a Boolean function (minimization of exclusive-OR-sum-of-products) is considered, e.g., in [14]. It is known (see [15]) that the maximum complexity of a Boolean function in dimension 5 is 9, for dimension 6 it equals 15, and there exist Boolean functions of seven arguments with polynomial complexity 24.

The definition of the polynomial complexity implies that the polynomial complexity of a Boolean function is not greater than the sum of complexities of its two subfunctions obtained by fixing the values 0 and 1 for some variable. This implies the following.

Corollary 1. *The rank of a unitrade is not greater than the sum of ranks of its two different retracts over an arbitrary coordinate.*

A polynomial representation of a Boolean function is not unique. However, if one of the operators x^0, x^1, or x^{-1} is not used, the representation becomes unique. In Section 2 we considered a basis in the space $f \in \mathbb{V}_{n,3}(\mathrm{GF}(2))$ corresponding to the operators x^1 and x^{-1}. If we exclude the operator x^{-1} ("negation"), we come to a basis of addition and multiplication over GF(2). Namely, any Boolean function $f\colon Q_2^n \to Q_2$ can uniquely be represented in the form of a *Zhegalkin polynomial* (in *algebraic normal form*)

$$f(x_1, \dots, x_n) = \bigoplus_{y\in Q_2^n} G[f](y) x_1^{y_1} \dots x_n^{y_n},$$

where $G[f]\colon Q_2^n \to Q_2$ is a Boolean function.

The *algebraic degree* of a Boolean function f is the maximum degree of a term in its Zhegalkin polynomial; i.e., $\deg f = \max\limits_{G[f](y)=1} \mathrm{wt}(y)$.

The following fact holds true.

Proposition 5. *For any Boolean function f we have $G[f](y) = \bigoplus\limits_{x\in Q_2^n,\, x\preceq y} f(x)$.*

Thus, $G[f]$ is the Möbius transform of f over GF(2). Since $f(x) = \bigoplus\limits_{y\in Q_2^n,\, x\preceq y} G[f](y)$, we have the equality $G[G[f]] = f$ for any Boolean function f. From the definitions of the Möbius transform and of the operator $U[\,\cdot\,]$, it is seen that $U[f]|_{\{0,2\}^n}$ is the Möbius transform of the Boolean function f.

Proposition 5 immediately implies the following known fact.

Proposition 6. *A Boolean function $f\colon Q_2^n \to Q_2$ has an even number of ones in all faces of dimensions not less than m if and only if* $\deg f \le m-1$.

Vectors of values of Boolean functions $f\colon Q_2^n \to Q_2$ can be considered as elements of a Boolean cube of dimension 2^n. The set of vectors of values of Boolean functions of algebraic degree at most m is called the Reed–Muller code $\mathcal{R}(m,n)$ in $Q_2^{2^n}$. It is known that the minimum weight of a nonzero code vector in $\mathcal{R}(m,n)$, which coincides with the cardinality of the support of the corresponding Boolean function, is 2^{n-m}.

Remark 1. Note the set of elements of the space $\mathbb{V}_{n,k}(\mathrm{GF}(q))$ can similarly be considered as a linear code of length k^n and cardinality $q^{(k-1)^n}$ with code distance 2^n. In particular, unitrades in Q_3^n form a binary code of length 3^n and cardinality 2^{2^n} with code distance 2^n.

In the case of $k=q$, the space $\mathbb{V}_{n,k}(\mathrm{GF}(q))$ has a quite natural representation in terms of polynomials: it consists of all functions orthogonal to any monomial that does not essentially depend of at least one of the n variables. One can easily see that for a prime q, a basis of $\mathbb{V}_{n,q}(\mathrm{GF}(q))$ is the set of all monomials in which the degree of each variable is at most $q-2$. Thus, $\mathbb{V}_{n,q}(\mathrm{GF}(q))$ can be viewed a variants of a nonbinary generalization of Reed–Muller codes. In particular, one of the results of Section 4.2 (Corollary 2) can be treated in terms of the weight distribution of this code for $q=3$: every third component of this distribution is zero.

In [19] a relation is shown between ternary bitrades and uniformity of the distribution of ones of a Boolean function over faces. A Boolean function is said to be *almost balanced in faces* if the numbers of zeros and ones of the function differ by at most 2 in any face of any size.

Proposition 7. *Let a Boolean function f be balanced in faces, and let $p(x) = x_1 \oplus \ldots \oplus x_n$ be the evenness indicator. Then the unitrade $U[f \oplus p]$ is a bitrade.*

Proposition 1 implies that if a Boolean function f corresponds to a bitrade $U[f]$, then its subfunctions obtained by fixing some variables also correspond to bitrades in hypercubes of smaller dimensions.

4. CARDINALITY SPECTRUM OF THE SET OF BITRADES

In this section we prove properties of the cardinality spectrum of ternary bitrades, as well as bitrades and unitrades of small cardinality in arbitrary hypercubes.

4.1. Cardinalities of Unitrades and the Weight Spectra of Reed–Muller Codes

Proposition 3 implies that the minimum cardinality of a nonempty unitrade of dimension n is 2^n. In [16] the following fact was proved.

Proposition 8. *Any unitrade $U \subset Q_k^n$ with cardinality satisfying $2^{n+1} > |U| \ge 2^n$ is a bitrade and has* $\mathrm{rank}(U) = 2$ *and cardinality $|U| = 2^{n+1} - 2^{s+1}$, where $s \in \{0,\ldots,n-1\}$.*

Using the results of studying the weight spectra of Reed–Muller codes, we can substantially restrict the spectrum of hypothetically small (from 2^n to $5\cdot 2^{n-1}$) cardinalities of unitrades of dimension n.

Proposition 9. *Let U be a unitrade in Q_k^n, $k=2^\tau$. Then there exists a vector $u \in \mathcal{R}((\tau-1)n, \tau n)$ such that $|U| = \mathrm{wt}(u)$.*

Proof. Let $f = \chi_U\colon Q_k^n \to \{0,1\}$. Consider an arbitrary one-to-one map $\psi\colon Q_2^\tau \to Q_k$. Let a Boolean function F be given by

$$F = f(\psi(x_1,\ldots,x_\tau), \psi(x_{\tau+1},\ldots,x_{2\tau}), \ldots, \psi(x_{\tau(n-1)+1},\ldots,x_{\tau n})).$$

Let us check that $\deg F \le (\tau-1)n$. Consider any face Δ of the Boolean hypercube of dimension $(\tau-1)n+1$. Since Δ is obtained by fixing values of $n-1$ variables, there exists an i from 0 to $n-1$ such that the values of the variables $x_{\tau i+1},\dots,x_{\tau i+\tau}$ are not fixed in Δ. By the definition of a unitrade, under fixed values of all variables other than $x_{\tau i+1},\dots,x_{\tau i+\tau}$, the function F takes the value 1 evenly many times. Hence, it takes the value 1 evenly many times in Δ. Proposition 6 implies that $\deg F \le (\tau-1)n$. Therefore, the vector of values of F is contained in the code $\mathcal{R}((\tau-1)n,\tau n)$. Hence, to the unitrade U there corresponds some vector $u \in \mathcal{R}((\tau-1)n,\tau n)$. Now, since the functions F and f take the value 1 the same number of times, we have $|U| = \mathrm{wt}(u)$. $\triangle$

Remark 2. A $[t]$-trade in Q_k^N is defined as a pair of disjoint sets of vertices in Q_k^N such that the difference of their characteristic functions has zero sum over any $(N-t)$-dimensional face. By definition, a bitrade in Q_k^N is an $[N-1]$-trade. Similarly to Proposition 9, one can prove the following: to a bitrade in Q_k^n, $k=2^\tau$, there corresponds a $[t]$-trade in $Q_2^{\tau n}$, $t=n-1$. $[t]$-trades naturally correspond to differences of orthogonal arrays and algebraic t-designs in Hamming graphs. A study of cardinalities of small binary $[t]$-trades analogous to ours was conducted in [6]. In particular, there were constructed trades of cardinalities from the series considered by us in Section 4.6.

In [20, 21] it was shown that nonzero vertices of $\mathcal{R}(m,n)$ can only be of weights $\alpha_m 2^{n-m}$ with $\alpha_m = 2-2^{-k}$, $k=0,\dots,n-m-1$, or $\alpha_m = 2+2^{-k}$, $k=2,\dots,\left\lfloor\frac{n-m}{2}\right\rfloor$, or $\alpha_m = 2\frac{1}{2}-2^{-k}$, $k=1,\dots,n-m-1$, or $\alpha_m = 2\frac{1}{2}-2^{-k}-2^{-(k+1)}$, $k=3,\dots,n-m-2$, or $\alpha_m \ge 2\frac{1}{2}$.

Theorem 1. *The cardinality of a unitrade in Q_k^n can only have values of the form $\alpha_n 2^n$, where*

$$\begin{aligned}
&\alpha_n = 2-2^{-k}, \quad k=0,\dots,n-1, \quad \textit{or}\\
&\alpha_n = 2+2^{-k}, \quad k=2,\dots,\lfloor n/2\rfloor, \quad \textit{or}\\
&\alpha_n = 2\frac{1}{2}-2^{-k}, \quad k=1,\dots,n-1, \quad \textit{or}\\
&\alpha_n = 2\frac{1}{2}-2^{-k}-2^{-(k+1)}, \quad k=3,\dots,n-2, \quad \textit{or}\\
&\alpha_n \ge 2\frac{1}{2}.
\end{aligned}$$

Proof. Any unitrade in Q_k^n has cardinality at least 2^n (Proposition 3). The restriction of a unitrade onto a face (retract) is a unitrade by Proposition 1. Therefore, a unitrade intersecting with five parallel hyperfaces of some direction has cardinality at least $5\cdot 2^{n-1}$ (the last case in the assertion of the theorem). If a unitrade intersects with at most four parallel hyperfaces of some direction and is a subset of vertices of a subgraph isomorphic to Q_4^n, then the desired follows Proposition 9 ($\tau=2$) and the above-mentioned results of [20, 21]. $\triangle$

4.2. Ternary Bitrades as Linear Functions over GF(3)

First let us choose an appropriate basis in the space $\mathbb{V}_{n,3}(\mathrm{GF}(3))$. Here it will be convenient to assume that $Q_3 = \{0,\pm1\} = \mathrm{GF}(3)$ ($-1 \equiv 2 \bmod 3$). Let $s_0(a)=1$ and $s_1(a)=a$. Define the functions $s_\alpha\colon Q_3^n \to Q_3$, $\alpha \in Q_2^n$, by $s_\alpha(x) = s_{\alpha_1}(x_1)\dots s_{\alpha_n}(x_n)$.

Proposition 10. (a) $s_\alpha \in \mathbb{V}_{n,3}(\mathrm{GF}(3))$;
(b) $\{s_\alpha \mid \alpha \in Q_2^n\}$ *is a basis* $\mathbb{V}_{n,3}(\mathrm{GF}(3))$;
(c) $\langle s_\alpha, s_\beta\rangle_3 = \sum\limits_{x\in Q_3^n} s_\alpha(x)s_\beta(x) = 0$ *except for the case of* $\alpha=\beta=\bar{1}$.

Proof. Claim (a) follows from the fact that $\sum\limits_{a\in Q_3} s_0(a) = \sum\limits_{a\in Q_3} s_1(a) = 0$.

(b) Note that $\alpha \in \mathrm{supp}(s_\alpha)$ and that $\alpha \in \mathrm{supp}(s_\beta)$ only if $\beta \preceq \alpha$, $\alpha,\beta \in Q_2^n$. Let us arrange the functions s_α in descending (partial) order from $\bar{1}$ to $\bar{0}$. The support of each successive function contains a point that is not contained in the support of the preceding functions. Therefore, at each step

we obtain a linearly independent family of functions. As is shown above, $\dim(\mathbb{V}_{n,3}(\mathrm{GF}(3))) = 2^n$; therefore, a family of linearly independent functions of cardinality 2^n is a basis.

(c) Let the tuple α have a zero coordinate. Without loss of generality, assume that $\alpha_n = 0$. Then we have

$$\sum_{x\in Q_3^n} s_\alpha(x)s_\beta(x) = \sum s_{\alpha_1}(x_1)s_{\beta_1}(x_1)\dots s_{\alpha_{n-1}}(x_{n-1})s_{\beta_{n-1}}(x_{n-1}) \sum_{x_n\in Q_3} s_0(x_n)s_{\beta_n}(x_n)$$
$$= \sum s_{\alpha_1}(x_1)s_{\beta_1}(x_1)\dots s_{\alpha_{n-1}}(x_{n-1})s_{\beta_{n-1}}(x_{n-1}) \sum_{x_n\in Q_3} s_{\beta_n}(x_n) = 0. \quad \triangle$$

Corollary 2. *For any $f \in \mathbb{V}_{n,3}(\mathrm{GF}(3))$ we have $\langle f, f\rangle_3 \in \{0, 2^n \bmod 3\}$.*

Proof. It follows from Proposition 10 that $f = \sum\limits_{\alpha\in Q_2^n} a_\alpha s_\alpha$ and

$$\langle f, f\rangle_3 = \sum_{\alpha,\beta\in Q_2^n} a_\alpha a_\beta \sum_{x\in Q_3^n} s_\alpha(x)s_\beta(x) = a_{\bar{1}}^2 \sum_{x\in Q_3^n} s_{\bar{1}}^2 = a_{\bar{1}}^2|\mathrm{supp}(s_{\bar{1}})| = a_{\bar{1}}^2 2^n,$$

where all operations are performed in the field GF(3). $\triangle$

As is noted in Remark 1, the space $\mathbb{V}_{n,3}(\mathrm{GF}(3))$ is a ternary Reed–Muller type code, and Corollary 2 means that the weight spectrum of this code has zeros in every third component.

Theorem 2. *For any bitrade $B \subset Q_3^n$ we have $|B| \equiv 0, 2^n \bmod 3$.*

Proof. Consider the function $b\colon Q_3^n \to Q_3$ taking values 1 and -1 on two parts of the bitrade B and value 0 at other points. Then $b \in \mathbb{V}_{n,3}(\mathrm{GF}(3))$ and $|B| \bmod 3 = \langle b, b\rangle_3 \in \{0, 2^n \bmod 3\}$. $\triangle$

Corollary 3. *There is no bitrade $U \subset Q_3^n$ of cardinality 2^{n+1}.*

4.3. Some Properties of the Cardinality Spectrum of Ternary Bitrades

Let us consider several simple properties of bitrades. Let f be a Boolean function of variables $x_1, \dots, x_n$ and g a Boolean function of variables $y_1, \dots, y_m$, the two sets of variables being disjoint. Denote by $f(x)g(y)$ the Boolean function of $n + m$ variables. The following result is given in [16].

Proposition 11. *If $U[f] \subset Q_3^n$ and $U[g] \subset Q_3^m$ is a bitrade, then $U[f(x)g(y)] \subset Q_3^{n+m}$ is a bitrade and $|U[f(x)g(y)]| = |U[f]||U[g]|$.*

In particular, for g we can take a linear function $g(y) = \bigoplus y_i$ and as a result obtain the following property: if in Q_3^n there is a bitrade of cardinality a, then in Q_3^{n+m} there is a bitrade of cardinality $a2^m$. This construction can be considered as a particular case of the Cartesian product of two bitrades. The Cartesian product of two bitrades is a bitrade not only in a ternary hypercube but also in hypercubes over an arbitrary alphabet. Unitrades that can be represented as a Cartesian product will be referred to as *decomposable*.

Theorem 3 (on constricting decomposable bitrades). *Let $B \subset Q_k^n$ and $C \subset Q_k^n$ be bitrades. Then in Q_k^{n+m} there exist bitrades of cardinalities $|B| \cdot |C|$, $2^m|B|$, and $k^m|B|$.*

Proof. Bitrades of cardinalities $|B|\cdot|C|$ and $2^m|B|$ can be constructed using Cartesian products. The possibility of constructing a bitrade of cardinality $k^m|B|$ can be proved by induction starting from the case $m = 1$, which we treat separately. Let a function $b\colon Q_k^n \to \{-1, 0, 1\}$ take values 1 and -1 on two parts of some bitrade and value 0 at other points. Then the function $b'\colon Q_k^{n+1} \to \{-1, 0, 1\}$ given by $b'(x_1, \dots, x_n, x_{n+1}) = b(x_1, \dots, x_{n-1}, (x_n + x_{n+1}) \bmod k)$ defines a bitrade of dimension $n + 1$ and cardinality $k|B|$. $\triangle$

Proposition 12. *If a unitrade $U \subset Q_3^n$ has an empty retract along some direction, then it is equivalent to a unitrade $U[f]$ where f is Boolean function independent of one of the variables.*

In this case U is a bitrade if and only if any of its nonempty retracts along the same direction is a bitrade.

Proof. We may assume without loss of generality that $U \cap \{x_n = -1\} = \varnothing$. Then from the definition of a unitrade we have $U \cap \{x_n = 0\} = U \cap \{x_n = 1\} = U'$ and $U = U' \times \{0, 1\}$. Let $U' = U[f]$ $g(x_1, \ldots, x_n) = f(x_1, \ldots, x_{n-1})$. Then $U = U[g]$. Propositions 1 and 11 imply that the unitrades U and U' are bitrades. $\triangle$

Since every one-dimensional face of a hypercube intersect a unitrade in at most two vertices, a ternary unitrade U of dimension n contains at most $2 \cdot 3^{n-1}$ vertices. Furthermore, in the case of the equality $|U| = 2 \cdot 3^{n-1}$ the complement $Q_3^n \backslash U$ is an MDS code. As is known (see, e.g., [17]), in Q_3^n there is unique up to equivalence MDS code which is linear (i.e., an affine subspace over GF(3)). The complement of an affine subspace over GF(3) consists of two its cosets. Therefore, a unitrade of the maximum cardinality is unique and is a bitrade, which is reflected in the first part of the following theorem.

Theorem 4 (on bitrades of large cardinality). *We have the following*:

(a) *In Q_3^n there is only one up to equivalence unitrade B of the maximum cardinality $2 \cdot 3^{n-1}$, which is a bitrade and can be given by $B = \{x \in Q_3^n \mid x_1 + \ldots + x_n \not\equiv 0 \bmod 3\} = U[\ell]$, where ℓ is some symmetric Boolean function*;

(b) *In Q_3^n there are bitrades of cardinality $14 \cdot 3^{n-3}$*;

(c) *In Q_3^n there are no bitrades of intermediate cardinalities between $14 \cdot 3^{n-3}$ and $2 \cdot 3^{n-1}$.*

Proof. It remains to prove parts (b) and (c). In Q_3^3 there exists the bitrade $U[x_1x_2x_3 \oplus (x_1 \oplus 1)(x_2 \oplus 1)(x_3 \oplus 1)]$ of cardinality 14. From Theorem 3 we obtain (b).

Let us prove (c) by induction. The induction base is the case $n = 3$, which can easily be checked directly. Now let $U^{n+1} \subset Q_3^{n+1}$ be a bitrade. If three different retracts of the same direction have cardinalities less than $2 \cdot 3^{n-1}$, then $|U^{n+1}| \leq 14 \cdot 3^{n-2}$ by the induction hypothesis. Assume that in U^{n+1}, for each direction, there is one retract of cardinality $2 \cdot 3^{n-1}$. Without loss of generality we may assume that each of these retracts corresponds to the zero value of some variable ($x_i = 0$). By part (a), they are up to equivalence given by functions linear over GF(3). Then $U^{n+1} = U[f]$, where $f = \ell$ or $f = \ell \oplus x_1 \ldots x_{n+1}$. However, the second function has a non-bipartite three-dimensional retract for $n \geq 4$. Indeed, consider the three-dimensional retract obtained from $U[f]$ with $f = \ell \oplus x_1 \ldots x_{n+1}$ by fixing the coordinates $x_4 = \ldots = x_{n+1} = 1$. If $n - 2 = 0, 1 \bmod 3$, this retract is equivalent to the non-bipartite unitrade (1); if $n - 2 = 2 \bmod 3$, then the retract obtained by fixing the coordinates $x_4 = 2$ and $x_5 = \ldots = x_{n+1} = 1$ is equivalent to the non-bipartite unitrade (1). Therefore, in the case where $U[f] \subset Q_3^{n+1}$ is a bitrade, we have $f = \ell$ and $|U^{n+1}| = |U[f]| = 2 \cdot 3^n$. $\triangle$

4.4. Bitrades and Distances between Monomials

Let $B \subset Q_k^n$ be a bitrade. In Section 2 we defined a function $b[B] \colon Q_k^n \to \{0, \pm 1\}$ that takes value 1 on one part of the bitrade B, values -1 on the other part, and value 0 at other vertices. In this subsection we will consider $b[B]$ as a function acting in $\mathbb{R}$, i.e., $b[B] \in \mathbb{V}_{n,3}(\mathbb{R})$. In the case of $k = 3$ such a function can be defined in exactly two ways: $b[B]$ and $-b[B]$, since the graph ΓB is connected. In what follows, we consider only this case and usually do not specify which of the two ways was used to choose the sign of $b[B]$. For brevity, we introduce the notation $b_v = b[U[x^v]]$ and $b_V = b[U[f^V]]$, where $v \in Q_3^n$ and $V \subset Q_3^n$.

Proposition 13. *Let $B, B' \subset Q_3^n$ be bitrades and $b[B](x)b[B'](x) \neq 1$ for any $x \in Q_3^n$. Then $S = \operatorname{supp}(b[B] + b[B'])$ is a bitrade and $b[S] = b[B] + b[B']$.*

Proof. If $b[B], b[B'] \in \mathbb{V}_{n,3}(\mathbb{R})$, then $b[B] + b[B'] \in \mathbb{V}_{n,3}(\mathbb{R})$. By the condition, $b[B]b[B'] \neq 1$; hence, $(b[B] + b[B'])(Q_3^n) \subseteq \{0, \pm 1\}$. Then $S = \operatorname{supp}(b[B] + b[B'])$ is a bitrade by the definition. $\triangle$

We will say that functions b_v and b_u are *matched* if $b_u(x)b_v(x) \neq 1$ for any $x \in Q_3^n$.

Proposition 14. *Let $v, u \in Q_3^n$. If there exists a vertex $x \in Q_3^n$ such that $b_v(x)b_u(x) = -1$, then the pair of functions b_u and b_v is matched.*

Proof. The intersection of the bitrade $U[x^u]$ and $U[x^v]$ is a Boolean subcube in the face corresponding to coinciding coordinates of u and v. Since the intersection $U[x^u] \cap U[x^v]$ is connected, it follows that the unitrade $U[x^u \oplus x^v]$ is a bitrade. As was noted above, any bitrade is divided into two parts uniquely. $\triangle$

Corollary 4. *Any unitrade of rank 2 is a bitrade.*

Proposition 15. *A unitrade $U[f^V]$ is a bitrade if for each $v \in V$ it is possible to choose functions (signs of functions) b_v in such a way that the function $g = \sum\limits_{v \in V} b_v$ takes only values in the set $\{0, \pm 1\}$.*

Proof. Since $b_v \in \mathbb{V}_{n,3}(\mathbb{R})$ for any $v \in V$, we have $g \in \mathbb{V}_{n,3}(\mathbb{R})$ too. Then the condition implies that $\operatorname{supp}(g)$ is a unitrade. Clearly, $U[f^V] \subset \operatorname{supp}(g)$. Then from Proposition 4(b) we have $\operatorname{supp}(g) = U[f^V]$; i.e., $g = \pm b_V$. $\triangle$

Proposition 16. *Let $V \subset Q_3^n$ and $|V| = 3$ (i.e., $\operatorname{rank}(U[f^V]) \leq 3$). If all vertices of V differ in two coordinates only or if two vertices of V differ in one coordinate only, then $U[f^V]$ is a bitrade.*

Proof. Let $V = \{u, v, w\}$ and vertices of V differ in two coordinates only. Then with use of Proposition 12 we can pass to a two-dimensional case, where all unitrades are bitrades.

If $d(v, u) = 1$, then $x^v \oplus x^u = x^o$; i.e., $\operatorname{rank}(U[f^V]) = 2$, and the claim follows from Corollary 4. $\triangle$

We say that three vertices $\{u, v, w\} \subset Q_3^n$ *are in general position* if there exists a coordinate where they are pairwise distinct. In this case there exist points $a, a', a'' \in Q_3^n$ in which the supports of the bitrades b_u, b_v, and b_w intersect only in pairs, i.e., $a \in (\operatorname{supp}(b_v) \cap \operatorname{supp}(b_u)) \setminus \operatorname{supp}(b_w)$, $a' \in (\operatorname{supp}(b_v) \cap \operatorname{supp}(b_w)) \setminus \operatorname{supp}(b_u)$, and $a'' \in (\operatorname{supp}(b_w) \cap \operatorname{supp}(b_u)) \setminus \operatorname{supp}(b_v)$.

Let us first consider unitrades $U[f^V]$ of rank 3 where three vertices $V = \{u, v, w\}$ are not in general position.

Proposition 17. *Let $V = \{u, v, w\} \subset Q_3^n$.*

(a) *If any coordinate of w coincides with the corresponding coordinate of either u or v, then $U[f^V]$ is a bitrade.*

(b) *If every coordinate takes at most two values on u, v, and w and condition* (a) *is not satisfied, then $U[f^V]$ is not a bitrade.*

Proof. (a) Consider the case of no coordinate in which all the three vertices u, v, and w coincide. Without loss of generality we may assume that $u = \bar{0}$, $v = \bar{1}$, and $w \in \{0, 1\}^n$. By Proposition 14 the functions b_v and b_u can be chosen in such a way that the real sums $b_v + b_w$ and $b_u + b_w$ take values in the set $\{0, \pm 1\}$ only. From the condition it is seen that $U[x^{\bar{0}}] \cap U[x^{\bar{1}}] = \{\bar{1}\} \subset U[x^w]$. Therefore, $(b_v + b_w)(\bar{1}) = 0$. Hence, the function $b_v + b_u + b_w$ takes only values in the set $\{0, \pm 1\}$. Now the desired follows from Proposition 15.

The case where all the three vertices u, v, and w coincide in some coordinate can be reduced to the case considered above by Proposition 12.

(b) If every triple of coordinates in u, v, and w satisfies condition (a), then u, v, and w also satisfy this condition. Without loss of generality we may assume that condition (a) is not satisfied by the first triples of coordinates of u, v, and w, and they are, respectively, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. Since in each coordinate the vectors u, v, and w take only two values, there exists a three-dimensional face in which $x^v \oplus x^u \oplus x^w = x_1 \oplus x_2 \oplus x_3 = f'$. Direct checking shows that $U[f']$ is equivalent to a unitrade (1) and is not a bitrade. Now from Proposition 1 it follows that $U[f^V]$ is not a bitrade either. $\triangle$

Now consider unitrades $U[f^V]$ of rank 3 where three vertices $V = \{u, v, w\}$ are in general position.

Proposition 18. *Let $V = \{u, v, w\} \subset Q_3^n$.*

(a) *If the sum of pairwise distances between the vertices of V is odd, they are in general position;*

(b) *If the vertices of V are in general position, then matchedness of the pairs b_v, b_u and b_v, b_w implies matchedness of the pair b_u, b_w if and only if the sum of pairwise distances between the vertices of V is odd.*

Proof. If the vertices u, v, and w are not in general position, then each coordinate contributes either 0 or 2 to the sum $d(u, v) + d(v, w) + d(u, v)$. Therefore, in this case the sum of distances is even. Part (a) is proved.

Consider the case of $V = \{\bar{0}, \bar{1}, \overline{-1}\}$. It is easily seen that the unitrades $U[x^{\bar{0}}]$ and $U[x^{\bar{1}}]$ intersect at a single point $\bar{1}$ only. Divide the bitrade $U[x^{\bar{0}} \oplus x^{\bar{1}}]$ into two parts. Vertices of the same parity as $\bar{1}$ in the hypercubes $U[x^{\bar{0}}]$ and $U[x^{\bar{1}}]$ must belong to different parts. Hence, the vertices $\bar{0}$ and $\overline{-1}$ are in different parts. Consider the hypercube $U[x^{\overline{-1}}]$. It is clear that the vertices $\bar{0}$ and $\overline{-1}$ are in different parts if and only if n is odd. Then matchedness of the pairs $b_{\bar{0}}, b_{\bar{1}}$ and $b_{\bar{0}}, b_{\overline{-1}}$ implies matchedness of $b_{\bar{1}}, b_{\overline{-1}}$ for n odd and unmatchednees of $b_{\bar{1}}, b_{\overline{-1}}$ for n even.

The remaining part of the proof is conducted by induction. Assume that the claim is proved for $n - 1$.

Let the vertices u, v, and w be pairwise distinct in each coordinate. Then the claim follows from the case considered above. Otherwise, there exists a coordinate in which not all the vertices u, v, and w are pairwise distinct. After deleting this coordinate, the shortened vectors v', u', and w' are also in general position. Without loss of generality we may assume the deleted coordinate to be the last. Obviously, $d(u', v') + d(v', w') + d(u', v') = d(u, v) + d(v, w) + d(u, v)$ if $u_n = v_n = w_n$, and $d(u', v') + d(v', w') + d(u', v') = d(u, v) + d(v, w) + d(u, v) - 2$ otherwise. Then the claim is valid for the functions $b_{v'}$, $b_{u'}$, and $b_{w'}$ by the induction hypothesis. Since the last coordinate does not take one of the values $\{0, \pm 1\}$ on the vectors u, v, and w, there exists a hyperface $x_n = \delta$ along the last direction such that $b_v(x\delta) = b_{v'}(x)$, $b_u(x\delta) = b_{u'}(x)$, and $b_w(x\delta) = b_{w'}(x)$ for all $x \in Q_3^{n-1}$. Then for the functions b_v, b_u, and b_w the desired follows from Proposition 14. $\triangle$

Corollary 5. *If the sum of pairwise distances between vertices of a set $V = \{u, v, w\} \subset Q_3^n$ is odd, then $U[f^V]$ is a bitrade.*

Corollary 6. *If vertices of a set $V = \{u, v, w\} \subset Q_3^n$ are in general position, differ in at least three coordinates, and the sum of pairwise distances between the vertices of V is even, then $U[f^V]$ is not a bitrade.*

Proof. Proposition 18 implies that the set of functions b_u, b_v, and b_w cannot be pairwise matched. If two of the three vertices are at distance 1 from each other and the three vertices are in general position, then in some coordinate all the three vertices differ, and the sum of pairwise distance between them is odd. The condition that no two of the vertices u, v, and w are at distance 1 implies that the intersections $\operatorname{supp}(b_v) \cap \operatorname{supp}(b_u)$, $\operatorname{supp}(b_v) \cap \operatorname{supp}(b_w)$, and $\operatorname{supp}(b_u) \cap \operatorname{supp}(b_w)$ are equivalent to faces of a Boolean hypercube of dimension less by at least 2 than the dimension of the hypercube, and the condition that the vertices u, v, and w differ in at least three coordinates implies that in the case of precisely less by 2 the faces corresponding to different intersections are not parallel. Therefore, the sets $\operatorname{supp}(b_w) \setminus (\operatorname{supp}(b_v) \cup \operatorname{supp}(b_u))$, $\operatorname{supp}(b_u) \setminus (\operatorname{supp}(b_v) \cup \operatorname{supp}(b_w))$, and $\operatorname{supp}(b_v) \setminus (\operatorname{supp}(b_w) \cup \operatorname{supp}(b_u))$ are connected. Then the unmatchedness implies that the unitrade is not bipartite. $\triangle$

If the vertices of $V = \{u, v, w\} \subset Q_3^n$ are in general position and the sum of pairwise distances between the vertices of V even, they cannot differ in exactly one coordinate, and if they differ in exactly two coordinates, then the rank of the unitrade $U[f^V]$ is 2.

Table

n	$N'(n)$	$N(n)$	$\ln N(n)$	$\ln\ln N(n)$	
0	2	3	1.098	0.094	
1	2	7	1.945	0.665 (+0.571)	1
2	3	31	3.433	1.233 (+0.567)	2.4 (±0.490)
3	5	403	5.998	1.791 (+0.557)	$6.448 = 2.539^2$ (±1.188)
4	13	29 875	10.304	2.332 (+0.541)	$17.960 = 2.619^3$ (±2.342)
5	92	32 184 151	17.286	2.849 (+0.517)	$50.527 = 2.666^4$ (±4.776)
6	25 493	1 488 159 817 231	28.028	3.333 (+0.483)	$142.25 = 2.695^5$ (±10.07)
7	> 2 187 260 868	6 171 914 027 409 468 739	43.266	3.767 (+0.434)	$398.17 = 2.712^6$ (±22.59)

Proposition 19. *If all pairwise distances between distinct points of a set $V \subset Q_3^n$ are odd, then $U[f^V]$ is a bitrade.*

Proof. Let us show by induction that it is possible to choose the functions b_v, $v \in V$, in such a way that all functions b_v are pairwise matched. For $|V| = 3$ this follows from Proposition 18. Assume that for sets $V \subset Q_3^n$, $|V| = k$, the claim is true. Consider a point $u \notin V$ lying at odd distance from every point of V. Choose a function b_u matched with b_v for some $v \in V$. Then it follows from Proposition 18 that b_u is matched with b_w for any $w \in V$. Thus, the functions b_v, $v \in V$, are pairwise matched for $|V| = k + 1$. If all the functions b_v, $v \in V$, are matched, then $\sum\limits_{v \in V} b_v$ takes only values in the set $\{0, \pm 1\}$. Then $U[f^V]$ is a bitrade by Proposition 15. $\triangle$

4.5. Computational Results

In this subsection we present result of computer-aided calculation of the number $N(n)$ of ternary bitrades. We could find the number of distinct bitrades up to dimension $n = 7$, and the number of nonequivalent bitrades, up to dimension $n = 6$. The computation method is not too far from direct exhaustive search, so we do not describe our algorithm in detail. To enumerate all bitrades in Q_3^n, we substituted as one of retracts one representative from each of $N'(n-1)$ equivalence classes of functions found in the preceding step. After that, for a parallel retract we performed pointwise exhaustive search for function values with an obvious check of the condition on the sum over a one-dimensional face. The obtained number of solutions was multiplied by the number of representatives in the equivalence class of the first retract. The computation of $N(7)$ took two years of CPU time (per one processor core); computations were conducted on the Computing Center of the Novosibirsk State University cluster. The computation results up to $n = 6$ were verified using the following double counting technique (see [22]): the cardinality of each equivalence class computed through the cardinality of the automorphism group of its representative coincides with the number of representatives found during the exhaustive search. The table presents the following quantities: $N(n)$ is the number of distinct (including the identical zero) $\{-1, 0, 1\}$-functions on Q_3^n whose sum over every one-dimensional face is 0 (i.e., all the three values in a face are either zero or pairwise distinct), and $N'(n)$ is the number of nonequivalent such functions. The last column shows the mean half-cardinality (the number of elements -1) of a bitrade (the mean-square deviation is given in the parentheses); the empty bitrade is excluded from this statistics.

Below we list the distribution of bitrades with respect to their cardinality. For each n we give the number of bitrades (more precisely, bipartite unitrades; i.e., the number of functions is twice as large) of cardinalities $2^n, 2^n + 2, 2^n + 4, \ldots, 2 \cdot 3^{n-1}$.

$n = 1$: 3.

$n = 2$: 9, 6.

$n = 3$: 27, 0, 54, 108, 0, 12.

$n = 4$: 81, 0, 0, 0, 324, 0, 1296, 648, 0, 3888, 2844, 0, 4536, 1296, 0, 0, 0, 0, 0, 24.

$n = 5$: 243, 0, 0, 0, 0, 0, 0, 0, 1620, 0, 0, 0, 9720, 0, 9720, 3888, 0, 0, 58320, 0, 41580, 77760, 0, 116640, 301320, 0, 259200, 660960, 0, 480816, 1368576, 0, 1156680, 2468880, 0, 1415232, 2721600, 0, 1148040, 2185056, 0, 583200, 816480, 0, 90720, 104976, 0, 10800, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 48.

$n = 6$: 729, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 7290, 0, 0, 0, 0, 0, 0, 0, 58320, 0, 0, 0, 87480, 0, 69984, 23328, 0, 0, 0, 0, 524880, 0, 0, 0, 370980, 0, 1399680, 0, 0, 699840, 2099520, 0, 4811400, 466560, 0, 2799360, 7290000, 0, 16562880, 2099520, 0, 6998400, 15244848, 0, 49968576, 19012320, 0, 46889280, 48114000, 0, 149999040, 48988800, 0, 173560320, 158793696, 0, 431451360, 203303520, 0, 593464320, 402077520, 0, 1226726208, 655983360, 0, 1759957632, 1275108480, 0, 3455693280, 1610681760, 0, 4922674560, 3332579760, 0, 8667868320, 4840793280, 0, 12263996160, 7124630400, 0, 19261521360, 10458292320, 0, 25982259840, 15546805632, 0, 37437240960, 17859890880, 0, 44159904000, 26492909760, 0, 56014493760, 28054486080, 0, 58200653952, 29447634240, 0, 63563901120, 30536701920, 0, 53914973760, 27520508160, 0, 44905723488, 18151205760, 0, 28971976320, 13573863360, 0, 17778852000, 6267067200, 0, 7903992960, 2932269264, 0, 2917632960, 1190894400, 0, 772623360, 243544320, 0, 299531520, 100077120, 0, 33592320, 41290560, 0, 5598720, 6298560, 0, 0, 1179360, 0, 3079296, 3429216, 0, 0, 0, 0, 0, 77760, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 96.

$n = 7$: 2187, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 30618, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 306180, 0, 0, 0, 0, 0, 0, 0, 612360, 0, 0, 0, 734832, 0, 489888, 139968, 0, 0, 0, 0, 0, 0, 0, 0, 3674160, 0, 0, 0, 0, 0, 0, 0, 2585520, 0, 0, 0, 14696640, 0, 0, 0, 0, 0, 14696640, 0, 22044960, 5878656, 0, 0, 48376440, 0, 9797760, 0, 0, 0, 58786560, 0, 105325920, 9797760, 0, 29393280, 252292320, 0, 48988800, 19595520, 0, 0, 264539520, 0, 258660864, 39191040, 0, 58786560, 849465792, 0, 417384576, 64665216, 0, 118599552, 1102248000, 0, 1026927720, 440899200, 0, 117573120, 3241833840, 0, 1646023680, 881798400, 0, 411505920, 5472048960, 0, 4595639328, 1293304320, 0, 1175731200, 13190234400, 0, 6642881280, 4967464320, 0, 2792361600, 23992754688, 0, 14767655616, 8897345856, 0, 5232003840, 48779617824, 0, 29422673280, 19428958080, 0, 13418032320, 86712135552, 0, 56942131680, 43070952960, 0, 29628426240, 174586285440, 0, 108140326560, 100338860160, 0, 60767667072, 333349188480, 0, 211194390960, 197052549120, 0, 133406300160, 633974103504, 0, 394756649280, 437915782080, 0, 284879669760, 1184769633600, 0, 732195313920, 918265662720, 0, 553710608640, 2237431905072, 0, 1396486980000, 1839642114240, 0, 1166266563840, 4133841505920, 0, 2463586985664, 3699825232320, 0, 2436508916352, 7740828063360, 0, 4633391621376, 7373480647680, 0, 4422630481920, 14095095607296, 0, 9008038009152, 14256009258624, 0, 8787284896320, 25936010563680, 0, 15056472533760, 27127920314880, 0, 17656455116160, 46489296385728, 0, 28595219380560, 51865158776256, 0, 30807273127680, 84076043325120, 0, 53419783072320, 96717020198400, 0, 59173324616448, 151123241747520, 0, 87927292938240, 177969306401280, 0, 112318130615040, 267008385528864, 0, 160800706809792, 321325302945024, 0, 188053645282560, 466481709832320, 0, 291050421480000, 570238841894400, 0, 343218104712000, 806560729743456, 0, 464604287555520, 996450456984192, 0, 620225396784000, 1373638417522560, 0, 816009087669552, 1708948701149760, 0, 982341376894080, 2305937678879520, 0, 1414154728014336, 2868447898270080, 0, 1691741075976000, 3783164938709184, 0, 2145564791750016, 4708186964144640, 0, 2866434717250944, 6089563693805760, 0, 3543718526510400, 7547844801104640, 0, 4235506218558720, 9574787807964288, 0, 5748041164944960, 11803431065489280, 0, 6789801853310400, 14687203311950400, 0, 8100416661309504, 17888868241614720, 0, 10588066580077056, 21840361606638720, 0, 12363491876450400, 26269417255213440, 0, 14285288100787200, 31462082237108160, 0, 18299032850230272, 37233543050766720, 0, 20709927455180544, 43720111582963200, 0, 23313370728464640, 50763763455713280, 0, 28924064989464960, 58412989843236000, 0, 31813638206316480, 66303578484210816, 0, 34565462372004480, 74583100623265920, 0, 41520901293714528, 82545422286942720, 0, 43826073580183104, 90590698165121280, 0, 45970461276122880, 97356175759100160, 0, 52673395505996160, 103853217900781440, 0, 53561231066238336, 108100043912367360, 0, 53128915099827840, 111570541229580480, 0, 58350442395913920, 112097875912081920, 0, 55765902028032000, 111406452367500864, 0, 52657506351233280, 107575556428968768, 0, 53994455165468160, 102377410528901760, 0, 48778189150406400, 94582347577421760, 0, 42669649619928576, 85745409629443200, 0, 40987358888555520, 75337864538158080, 0, 34117558361470080, 64695169565885376, 0, 27623026142341056, 53722349499008448, 0, 24230050550665344, 43420216983171840, 0, 18493690067425440, 33854862632935680, 0, 13546143606925440, 25584444523483776, 0, 10783563464378880, 18568192776336000, 0, 7350295708661760, 13016846960075520, 0, 4835426179046400,

8727631849641600, 0, 3393591789458304, 5646375105114240, 0, 2053100209063680, 3458776067268480, 0, 1173620938855680, 2041794442509696, 0, 728670130929216, 1138311067888512, 0, 375048142382592, 609672959421120, 0, 191348768439360, 307095833018880, 0, 99253170374400, 152237238241536, 0, 46874958318720, 69750838786176, 0, 19948807630080, 31988971163520, 0, 10729375110720, 13802819748480, 0, 3921475057920, 6002269439040, 0, 1657927958400, 2388145213440, 0, 1007425278720, 1073677731840, 0, 520179426144, 484107321600, 0, 144614937600, 245492674560, 0, 205312060800, 136090886400, 0, 89050752000, 73071694080, 0, 20222576640, 49468890240, 0, 45500797440, 28217548800, 0, 15872371200, 12433357440, 0, 3174474240, 2821754880, 0, 12580323840, 4938071040, 0, 3853785600, 1058158080, 0, 0, 529079040, 0, 1410877440, 0, 0, 1162667520, 0, 0, 0, 0, 0, 235146240, 0, 0, 264539520, 0, 0, 0, 0, 0, 0, 0, 0, 12700800, 0, 0, 0, 0, 0, 129330432, 120932352, 0, 71197056, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 520128, 0, 192.

4.6. Refining the Cardinality Spectrum of Bitrades

Proposition 20. *Let* $\mathrm{rank}(U) \geq 4$, *and let* $U \subset Q_3^n$ *be an indecomposable unitrade of dimension* n. *Then* $|U| > 2.5 \cdot 2^n$.

Proof. Consider the ranks of retracts of a unitrade U along some coordinate. By Corollary 1, the sum of two retract is at least 4.

1. If U has an empty retract, then by Proposition 12 the unitrade U is decomposable.

2. If U has two retracts of rank at least 3 and one more retract of rank at least 1, then $|U| > 2 \cdot 2^n + 2^{n-1}$ (see Proposition 8 and Corollary 3).

3. Now let all retracts of U along any coordinate have rank 2 and $\mathrm{rank}(U) = 4$. Then $U = U[f]$ with $f = x^a \oplus x^b \oplus x^c \oplus x^e$ where all pairwise distances between the tuples a, b, c, e are at least 2. Furthermore, in any retract the distances between two disjoint pairs of tuples equal 1; therefore, in each coordinate two tuples among a, b, c, e take one value, and two others, another value. Without loss of generality we may assume that $a_i = 1$ and $(b_i, c_i, e_i) \in \{(0,0,1), (0,1,0), (1,0,0)\}$ for all $i = 1, \ldots, n$.

If some element of $E = \{(0,0,1), (0,1,0), (1,0,0)\}$ does not occur among the vectors (b_i, c_i, e_i), then the unitrade $U[f]$ is decomposable. Namely, $f = (x^u \oplus 1)(x^v \oplus 1)$ for some tuples u and v.

If all elements of E occur only once, then the U is equivalent to a unitrade (1) and has rank 3. If all elements of E occur among the vectors (b_i, c_i, e_i) and some of them occurs in coordinates i and i', $i \neq i'$, then the retract along coordinate i has rank at least 3. $\triangle$

For a set of vectors $W \subset Q_3^n$, define $r(W)$ to be the number of positions in which all vectors of W coincide if the vectors of W have at most two different values in every position. If there exists a position which contains all the three different symbols, we put $r(W) = -\infty$, i.e., $2^{r(W)} = 0$.

Proposition 21. *Let* $f(x) = \bigoplus\limits_{v \in V} x^v$, *where* $V \subset Q_3^n$. *Then*

$$|U[f]| = \sum_{t=1}^{|V|} (-2)^{t-1} \sum_{W \subset V,\, |W| = t} 2^{r(W)}.$$

Proof. A formula similar to the inclusion-exclusion formula is known:

$$|\mathrm{supp}(\chi_{A_1} \oplus \ldots \oplus \chi_{A_s})| \\ = \sum_i |A_i| - 2\sum_{i \neq j} |A_i \cap A_j| + 2^2 \sum_{i \neq j \neq k} |A_i \cap A_j \cap A_k| - \ldots + (-2)^{s-1} |A_1 \cap \ldots \cap A_s|.$$

As was noted in Section 3, we have the equalities $U[x^{v^1} \oplus x^{v^2} \oplus \ldots \oplus x^{v^s}] = U[x^{v^1}] \oplus \ldots \oplus U[x^{v^s}]$ and $U[x^v] = \chi_{\{0,\pm 1\}_v}$.

It is easily seen that $|\{0,\pm1\}_{v^1}\cap\ldots\cap\{0,\pm1\}_{v^s}| = 2^{r(\{v^1,\ldots,v^s\})}$. Then the desired expression follows by substituting this equality into the first formula. $\triangle$

Let $v^i\in\{0,\pm1\}^n$. Consider a $3\times n$ matrix $\{v^i_j\}$ whose rows are vectors $v^i\in V$, $|V|=3$. Let the matrix contain $k_1(V)$ columns of the form *acc*, $k_2(V)$ columns of the form *cac*, $k_3(V)$ columns of the form *cca*, and $k_4(V)$ columns with all symbols different. Then Proposition 21 implies the following.

Proposition 22. *Let $V\subset Q_3^n$, $\operatorname{rank}(U[f^V])=3$, and assume that all the three monomials do not coincide in any coordinate. Then $|U[x^{v^1}\oplus x^{v^2}\oplus x^{v^3}]| = 3\cdot 2^n - 2(2^{k_1(V)}+2^{k_2(V)}+2^{k_3(V)}) + 4\delta(k_4(V))$, where $\delta(k)=0$ if $k>0$ and $\delta(k)=1$ if $k=0$.*

Consider all possible bitrades of rank 3 an cardinality up to $2.5\cdot 2^n$ inclusive.

Note that if $d(u,v)=1$, then $x^u\oplus x^v = x^w$ for some w. Therefore, it suffices to consider the case where $k_i\le n-2$, where $i=1,2,3$ and $n=k_1+k_2+k_3+k_4$.

1. Let $k_1=\max\limits_{i=1,2,3}k_i = n-2$. Then the following tuples are possible.

1.1. $(k_1,k_2,k_3,k_4)=(n-2,2,0,0)$. A bitrade by Proposition 17(a). The cardinality of the bitrade is $3\cdot 2^n - 2(2^{n-2}+4+1)+4 = 2.5\cdot 2^n - 6$. For $n=3$, the rank of the bitrade is 2.

1.2. $(k_1,k_2,k_3,k_4)=(n-2,1,1,0)$. Not a bitrade by Proposition 17(b).

1.3. $(k_1,k_2,k_3,k_4)=(n-2,0,0,2)$. Not a bitrade by Corollary 6.

1.4. $(k_1,k_2,k_3,k_4)=(n-2,1,0,1)$. A bitrade by Corollary 5. The cardinality of the bitrade is $3\cdot 2^n - 2(2^{n-2}+2+1) = 2.5\cdot 2^n - 6$. For $n=3$, the rank of the bitrade is 2.

2. Let $k_1=\max\limits_{i=1,2,3}k_i = n-3$. Then the following tuples are possible.

2.1. $(k_1,k_2,k_3,k_4)=(n-3,3,0,0)$. A bitrade by Proposition 17(a). The cardinality of the bitrade is $3\cdot 2^n - 2(2^{n-3}+8+1)+4 = 2.5\cdot 2^n + 2^{n-2} - 14$. For $n=4$, the rank of the bitrade is 2; for $n=5$, it coincides with case 1.1; and for $n>5$, the cardinality is greater than $2.5\cdot 2^n$.

2.2. $(k_1,k_2,k_3,k_4)=(n-3,2,1,0)$. Not a bitrade by Proposition 17(b).

2.3. $(k_1,k_2,k_3,k_4)=(n-3,2,0,1)$. A bitrade by Corollary 5. The cardinality of the bitrade is $3\cdot 2^n - 2(2^{n-3}+4+1) = 2.5\cdot 2^n + 2^{n-2} - 10$. For $n=4$, it coincides with case 1.4; for $n=5$, the cardinality is $2.5\cdot 2^n - 2$; for $n>5$, the cardinality is greater than $2.5\cdot 2^n$.

2.4. $(k_1,k_2,k_3,k_4)=(n-3,1,1,1)$. A bitrade by Corollary 5. The cardinality of the bitrade is $3\cdot 2^n - 2(2^{n-3}+2+2) = 2.5\cdot 2^n + 2^{n-2} - 8$. For $n=4$, the cardinality is $2.5\cdot 2^n - 4$; for $n=5$, the cardinality is $2.5\cdot 2^n$; for $n>5$, the cardinality is greater than $2.5\cdot 2^n$.

2.5. $(k_1,k_2,k_3,k_4)=(n-3,0,1,2)$. Not a bitrade by Corollary 6.

2.6. $(k_1,k_2,k_3,k_4)=(n-3,0,0,3)$. A bitrade by Corollary 5. The cardinality of the bitrade is $3\cdot 2^n - 2(2^{n-3}+1+1) = 2.5\cdot 2^n + 2^{n-2} - 4$. For $n=3$, the cardinality is $2.5\cdot 2^n - 2$; for $n=4$, the cardinality is $2.5\cdot 2^n$; for $n>4$, the cardinality is greater than $2.5\cdot 2^n$.

If $\max\limits_{i=1,2,3}k_i < n-3$, the cardinality of the unitrade is greater than $2.5\cdot 2^n$.

Now consider decomposable bitrades (see Proposition 11).

3. If both factors in the Cartesian product are not Boolean hypercubes, then by Proposition 8 their cardinalities can take the values $\frac{3}{2}2^n$, $\frac{7}{4}2^n$, etc. Since $\frac{3}{2}\frac{7}{4} > \frac{5}{2}$, the cardinality 2.5 times as large as the minimum, i.e., $2.25\cdot 2^n$, $n\ge 4$, can be attained only by products of bitrades of cardinality of the form $\frac{3}{2}2^{n_1}$.

Summarizing the above cases and taking into account the possibility of Cartesian products with a Boolean hypercube (see Proposition 8), we obtain the following conclusion.

Theorem 5 (cardinalities of small ternary bitrades). *In the hypercube Q_3^{n+m} for any $m \geq 0$ there are only the following bitrades of cardinalities greater than 2^{n+m+1} but no greater than $5 \cdot 2^{n+m-1}$:*

(i) $2^m(2.5 \cdot 2^n - 6)$ *for any* $n \geq 4$ (1.1 *and* 1.4);
(ii) $2^m(2.5 \cdot 2^5 - 2)$ *for* $n = 5$ (2.3);
(iii) $2^m(2.5 \cdot 2^3 - 2)$ *for* $n = 3$ (2.4, 2.6, *and* 3);
(iv) $2^m(2.5 \cdot 2^4)$ *for* $n = 4$ (2.4 *and* 2.6).

Consider a unitrade U of cardinality at most $2.5 \cdot 2^n$ in Q_k^n, $k > 3$. If in one of directions it intersects with at least four hyperplanes, then the intersection with each of the hyperplanes is of cardinality 2^{n-1} or $3 \cdot 2^{n-2}$ (see Proposition 8). Then the cardinality of U can be $2 \cdot 2^n$, $2.25 \cdot 2^n$, or $2.5 \cdot 2^n$. As is shown above, bitrades of the cardinalities $2.25 \cdot 2^n$ and $2.5 \cdot 2^n$ exist in ternary hypercubes. Besides them, in the hypercubes Q_k^n with $k > 3$ there exist bitrades of the same cardinality consisting of two disjoint components and bitrades of the form $U = U' \times \{0,1\}^{n-2}$, where $U' \subset Q_k^2$ is a cycle of length 8 or 10. In Q_4^3 one can easily construct a bitrade of cardinality $2.25 \cdot 2^3 = 18$. Similarly to the proof of Proposition 12, one can easily deduce that in the hypercubes Q_4^n, $n \geq 3$, there are bitrades of cardinality $9 \cdot 2^{n-2}$.

From the above-said, we have the following.

Corollary 7 (cardinalities of small bitrades). *Possible small cardinalities (at most $2.5 \cdot 2^n$) of bitrades in hypercubes Q_k^n with $k > 3$ are covered by the same list as in Theorem 5 and the additional cardinality 2^{n+1}.*

5. NUMBER OF BITRADES

5.1. Lower Bound on the Number of Bitrades

First let us find out what is the maximum cardinality of a subset of a hypercube Q_k^n if all pairwise distances between its elements are odd. We start with arguments concerning a set of vertices in a Euclidean space.

Let $\{v_1, \ldots, v_n\} \subset \mathbb{R}^m$, and let squared pairwise Euclidean distances between vectors v_i and v_j are $d_{ij} = \|v_i - v_j\|_2^2$. The Cayley–Menger determinant is

$$\det \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & d_{11} & d_{12} & \ldots & d_{1n} \\ 1 & d_{21} & d_{22} & \ldots & d_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & d_{n1} & d_{n2} & \ldots & d_{nn} \end{pmatrix} = (-1)^{n+1} 2^n (n!)^2 (\mathrm{Vol}_{n-1})^2,$$

where Vol_{n-1} is the $(n-1)$-dimensional volume of the convex hull of $\{v_1, \ldots, v_n\}$. A proof of this volume formula through the determinant can be found, e.g., in the monographs [23, Section 40; 24, Section 4.7]. For us it is important that if $n - 1 > m$, the determinant is zero, since $\mathrm{Vol}_{n-1} = 0$. Properties of determinants imply the following known lemma.

Lemma. *Let $A \subset \mathbb{R}^m$, all pairwise squared Euclidean distance between points of A being odd integers, and $|A| = m + 2$. Then $(m + 2) \equiv 0 \bmod 4$.*

Proof. Let $n = |A| = m + 2$. Let us make several additions of rows and columns, which do not change the determinant. Subtract the first row of the matrix from the others. We obtain

$$\det \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & -1 & c_{12} & \ldots & c_{1n} \\ 1 & c_{21} & -1 & \ldots & c_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & c_{n1} & c_{n2} & \ldots & -1 \end{pmatrix} = 0,$$

where the numbers $c_{ij} = d_{ij} - 1$ are even. Now we add the sum of columns from the second to the $(n+1)$st to the first column, and then the sum of rows from the second to the $(n+1)$st to the first row. We obtain

$$\det \begin{pmatrix} b & a_1 & a_2 & \dots & a_n \\ a_1 & -1 & c_{12} & \dots & c_{1n} \\ a_2 & c_{21} & -1 & \dots & c_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_n & c_{n1} & c_{n2} & \dots & -1 \end{pmatrix} = 0,$$

where $a_i = \sum\limits_j c_{ij} = \sum\limits_j c_{ji}$ and $b = n + \sum\limits_i a_i = n + 2\sum\limits_{i<j} c_{ij}$. Any diagonal of the matrix except for the main one contains at least two even numbers, so the product of elements of a diagonal is a multiple of 4. Hence, the product of elements of the main diagonal must also be a multiple of 4. Then $n \equiv b \equiv 0 \bmod 4$. $\triangle$

Corollary 8. *Let $A \subset \mathbb{R}^m$, and let all squared pairwise Euclidean distances between points of A be odd integers. Then $|A| \le m + 2$.*

Proof. We prove this by contradiction. Let us have such a set of $m+3$ points in $\mathbb{R}^m$. Then it is also contained in $\mathbb{R}^{m+1}$ and satisfies the conditions of the lemma, i.e., $m+3 \equiv 0 \bmod 4$. Besides, its subset of $m+2$ points in $\mathbb{R}^m$ also satisfies the conditions of the lemma, and $m+2 \equiv 0 \bmod 4$. $\triangle$

Corollary 9. (a) *Let $A \subset Q_k^m$, and let all pairwise Hamming distances between points of A be odd. Then $|A| \le (q-1)m + 2$.*

(b) *Let $A \subset Q_k^m$, and assume that for any three points of A the sum of their pairwise distances is odd. Then $|A| \le (q-1)m + 3$.*

Proof. (a) Let us encode elements of Q_k by real vectors of length $k-1$ with pairwise Euclidean distances 1 (vertices of a simplex). Then words of Q_k^m correspond to vectors in a $(q-1)m$-dimensional Euclidean space, and the Hamming distance between words equals the squared Euclidean distance between the corresponding vectors.

(b) Consider an arbitrary point $a \in A$. Denote by A' the set of points of A lying at odd distances from a, and by A'', the set of points of $A \setminus \{a\}$ lying at even distances from a. It is easily seen that the distance between b and c is odd if either $b, c \in A'$ or $b, c \in A''$, and is even if ether $b \in A'$ and $c \in A''$ or $c \in A'$ and $b \in A''$. We append 1 to all vectors in $A' \cup \{a\}$ and append 0 to all vectors in A''. We obtain a set $B \subset Q_k^m \times Q_2$ with pairwise odd distances. Now, by applying the techniques analogous to (a), we obtain a set of points in a Euclidean space with odd squared pairwise distances. Since encoding the values in the last coordinate requires only one Euclidean coordinate, the bound is only greater by 1 than in case (a). $\triangle$

For the case where q is a prime, the following fact is commonly known.

Proposition 23. *In a hypercube Q_q^m with $m = \dfrac{q^t - 1}{q-1}$ there exists an equidistant code H_t of cardinality $(q-1)m + 1 = q^t$ with code distance q^{t-1} dual to the Hamming code.*

Theorem 6. (a) *For $n = \dfrac{3^t - 1}{2}$ there exists a set $V \subset Q_3^n$ such that for any $W \subseteq V$ the unitrade $U[f^W]$ is a bitrade and $|V| = 2n + 1$.*

(b) *Let $V \subset Q_3^n$, $n \ge 3$, and let for any $W \subseteq V$ the unitrade $U[f^W]$ is a bitrade. Then $\operatorname{rank}(U[f^V]) \le 3n$.*

Proof. Part (a) follows from Propositions 19 and 23. Let us prove part (b).

Without loss of generality we assume that $|V| = \operatorname{rank}(U[f^V])$. Put $\pi(1) = 3$ and $\pi(2) = 6$. Denote by $\pi(n)$, $n \ge 3$, the maximum cardinality of a set $V \subset Q_3^n$ in which every triple of vertices generates a bitrade. We prove the inequality $\pi(n) \le 3n$ by induction. For $n = 3$ the inequality can

be checked directly. If any triple of vertices in V is in general position, the desired claim follows from Proposition 18(b) and Corollary 9(b). Now let us prove several auxiliary facts about the structure of a set V in the case where not all triples in it are in general position.

By Proposition 17 any triple of vertices in V which is not in general position must satisfy the condition of Proposition 17(a), i.e., one of the vertices must lie between two others. We will call this property the orderliness of the triple.

Consider a maximal set $v^0, \dots, v^m \in V$ such that every triple of vertices in it is ordered. Without loss of generality (applying, if necessary, isometries of the hypercube) we may assume[1] that $v^0 = \bar{0}$, $v^i = (2 \dots 20 \dots 0)$, and $v^0 \prec v^1 \prec \dots \prec v^m$, where $m > 1$. We denote the set of coordinates j with $v^m_j = 2$ by M. The set of coordinates in which the vertex v^i differs from v^{i+1} will be denoted by M_i.

In the set $V \setminus V_0$, where $V_0 = \{v^0, \dots, v^m\}$, there are no vertices having only 0 and 2 in the positions from M, since this contradicts either the maximality of V_0 or Proposition 17.

Let there exist a vertex $u \in V$ such that all the three triples of vertices $\{v^{i_1}, v^{i_2}, u\}$, $\{v^{i_1}, v^{i_3}, u\}$, and $\{v^{i_2}, v^{i_3}, u\}$ are in general position. Since $d(v^{i_1}, v^{i_2}) + d(v^{i_2}, v^{i_3}) = d(v^{i_1}, v^{i_3})$, one can easily check that the sum of side lengths in one of the three triangles is even. By Corollary 6, we have a contradiction. We will refer to this observation as property $(*)$.

Let us show that a vertex $u \in V \setminus V_0$ contains 1 in only one of the blocks M_i, $i = 0, \dots, m-1$. If u contains 1 in two coordinates, from M_i and M_j, $i < j$, then u is in general position with any two of the three vertices v^0, v^i, and v^m. We have obtained a contradiction by property $(*)$. Denote by U_i the set of vertices in V having at least one 1 in the block M_i.

Let us show that every vertex u in U_j has equal symbols, either zeros or twos, in coordinates of each block M_i, $i < j$. Indeed, otherwise the triple of vertices v^0, v^{j-1}, u is not in general position but is not ordered and therefore by Proposition 17(b) does not generate a bitrade. Similarly, any vertex u in U_j has equal symbols in coordinates of each of the blocks M_i, $i > j$ (it suffices to consider the triple v^m, v^j, u).

Let a vertex $u^0 \in U_0$ contain 2 in a coordinate of some block M_i, $i > 0$. Then the triple of vertices v^1, v^m, u^0 is not in general position and is not ordered and therefore by Proposition 17(b) does not generate a bitrade. Hence, $u^0 \in U_0$ contains only zeros in the coordinates of any blocks M_i, $i > 0$. Let us show that no vertex $u^j \in U_j$, $j > 0$, contains 0 in the block M_0. Indeed, in this case u^j is in general position with any two of the three vertices u^0, v^1, and v^m. We have arrived at a contradiction by property $(*)$. Similarly, by considering the case where the vertex $u^{m-1} \in U_{m-1}$ contains 0 in a coordinate from some block M_i, $i < m-1$, and the cases where the vertex $u^j \in U_j$, $m-1 > j > 0$, contains symbol 0 in the blocks M_i, $i < j$, or symbol 2 in the blocks M_i, $i > j$, we conclude that the only consistent possibility is as follows: the vertex $u^j \in U_j$, $j = 0, \dots, m-1$, contains only twos in the blocks M_i with $i < j$ and only zeros in the blocks M_i with $i > j$.

Let us show by contradiction that two vertices $u^i \in U_i$ and $u^j \in U_j$, $i < j$, cannot have two different nonzero kth coordinates for $k > |M|$. Let $u^i(k) = 1$ and $u^j(k) = 2$. Consider the triple of vertices v^0, v^i, u^j. It is ordered (the condition of Proposition 17(a) is fulfilled), but any pair of vertices of this triple is in general position with u^i. This is a contradiction by property $(*)$.

We have shown that $u^i(k) = u^j(k)$, or $u^i(k) = 0$, or $u^j(k) = 0$ for $k > |M|$. Now let us show that the case $u^i(k) = u^j(k) \neq 0$ for $k > |M|$ is impossible. Indeed, in this case the triple of vertices v^i, u^i, u^j, $i < j$, is not in general position and is not ordered and therefore by Proposition 17(b) does not generate a bitrade. Thus, the coordinates in the complement to M can be divided into disjoint groups $N_0, \dots, N_{m-1}$ in such a way that for any vertex $u^i \in U_i$ nonzero coordinates are only those from the set N_i.

[1] It is more convenient to use here the alphabet $\{0, 1, 2\}$ with the partial order $0 \prec 2$ and $1 \prec 2$ defined at the beginning of the paper.

Consider the restriction of the set of vertices $W_i = \{v^0, v^m\} \cup U_i$ onto the coordinates $M_i \cup N_i$. It is easily seen that a triple of vertices from W_i is in general position if and only if its restriction onto the coordinates $M_i \cup N_i$ is in general position. In this case the parity of the sum of distances between the vertices of the triple over all coordinates and that of the sum over the set $M_i \cup N_i$ only coincide. If the restriction of the triple of vertices onto $M_i \cup N_i$ is not in general position and is not ordered, then the triple of vertices is not ordered on the set of all coordinates too. Hence, the triple of vertices from W_i generates a bitrade if only if it generates a bitrade on the restriction.

Hence it follows that the cardinality of the set $W_i = \{v^0, v^m\} \cup U_i$ is not greater than $\pi(|M_i|+|N_i|)$ if $|M_i|+|N_i| \geq 3$. For $|M_i|+|N_i| = 1$, the inequality $|W_i| \leq 3 = \pi(1)$ is obvious. If $|M_i|+|N_i| = 2$, the inequality $|W_i| \leq 6 = \pi(2)$ can easily be proved by considering restrictions of vertices onto a set of three coordinates containing the sets N_i and M_i. Then by the induction hypothesis we have

$$|V| = |V_0| + \sum_{i=0}^{m-1} |U_i| \leq m + 1 + \sum_{i=0}^{m-1} (\pi(|M_i| + |N_i|) - 2) \leq 3n. \quad \triangle$$

Now we will need the code H_t with $q = 3$. We will bound from below the number of nonequivalent bitrades by choosing sets of vertices of the code with pairwise odd distances to generate a bitrade. Theorem 6 shows that our bound almost exhaust the possibilities of this way of constructing bitrades of large cardinality.

Proposition 24. *Let D be the code distance of a set $V_i \subset Q_3^n$ and let $|V_i| \leq 2^{D-3}$ for $i = 1, 2$. Then equivalence of unitrades $U[f^{V_1}]$ and $U[f^{V_2}]$ implies equivalence of the sets V_1 and V_2.*

Proof. Let a unitrade $U[f^V]$ be equivalent to a unitrade U'. Then $U' = U[f^{V'}]$, where the set V' is equivalent to V. However, this correspondence is not unique, i.e., in the general case there are other sets $W \subset Q_3^n$ for which $U' = U[f^W]$.

It suffices to show that if $2^{n-2} > |V|2^{n-D+1}$, then the set V with code distance D is uniquely recovered from the unitrade $U[f^V]$. Consider an arbitrary subcube $U[x^v]$. We have $v \in V$ if and only if $|U[f^V] \cap U[x^v]| \geq 2^n - 2^{n-2}$. Indeed, since $|U[x^w] \cap U[x^v]| = 2^{n-d(v,w)}$ for any $w \in Q_3^n$, we have the inequalities

(1) $|U[f^V] \cap U[x^v]| \geq 2^n - |V|2^{n-D}$ for $v \in V$;

(2) $|U[f^V] \cap U[x^w]| < 2^{n-1} + |V|2^{n-D+1}$ for $w \notin V$.

We have $2^n - |V|2^{n-D} \geq 2^{n-1} + |V|2^{n-D+1}$ for $|V| \leq 2^{D-3}$. $\triangle$

Denote by $\mathrm{sp}(v)$ the composition of a vector v, for instance, $\mathrm{sp}(0, 1, 1, 0, -1) = (2, 2, 1)$. We say that the composition of a vector is *unique* for some linear space if it has no vectors with the same composition.

Proposition 25. *Let $W \subset Q_k^n$ be a linear subspace over $\mathrm{GF}(k)$, and assume that in W there is a basis B consisting of vectors with unique composition. Then in W there are at least $2^{|W|-\dim W-1}/|W|$ nonequivalent subsets of vectors.*

Proof. Consider subsets $C \subset W$ that contain the zero vector and the basis, i.e., $B \subset C$ and $\bar{0} \in C$. Let $\varphi_{\pi,a}$ be an isometry taking one such set C to another set C', i.e., $\varphi_{\pi,a}(C) = \pi(C) + a = C'$. Since $\pi(\bar{0}) = \bar{0}$, we have $a \in C' \subset W$. Consider a basis vector $v \in C \cap C'$ with a unique composition. From the equality $\mathrm{sp}(\pi(v)) = \mathrm{sp}(v)$ and uniqueness of the composition, we have $\pi(v) = v$. The linearity of the autotopy π, i.e., the equality $\pi(\alpha u + \beta w) = \alpha\pi(u) + \beta\pi(w)$, implies that π acts identically on W. Then $\varphi_{\pi,a}(u) = u + a$, where $a \in W$. Clearly, the number of subspaces in W containing some basis and the zero vector is $2^{|W|-\dim W-1}$, an every equivalence class of subsets contains at most $|W|$ elements. $\triangle$

Consider a generator matrix A of the code H_t of dimension $t > 1$ (see Proposition 23). The matrix A contains an identity submatrix. Let us make the following transformation of the generator

matrix A: add to it columns of the identity matrix, namely 2^{k-1} copies of the kth column for $k = 2, \dots, t$. Denote by H'_t the linear code generated by the matrix thus transformed. All vectors of H_t (except for the zero vector) are of the same odd weight. Therefore, the difference between the number of coordinates equal to 1 and -1 is odd, and hence adding evenly many columns to the generator matrix cannot result in equal composition of pairs of collinear vectors from H'_t. Noncollinear vectors from H'_t have different weight by the construction. Therefore, H'_t has a basis (rows of the generator matrix A) of vectors with unique composition. The distance between any pair of vectors from H_t is odd. Since we have added evenly many copies of unit columns to the generator matrix A, the distance between any pair of vectors from H'_t is also odd. The length of the code H'_t is $2^t - 2 + \dfrac{3^t - 1}{2}$.

Theorem 7 (lower bound). *The number of nonequivalent bitrades of dimension n is at least $2^{(2/3-o(1))n}$ as $n \to \infty$.*

Proof. For $2^t - 2 + \dfrac{3^t - 1}{2} \le n < 2^{t+1} - 2 + \dfrac{3^{t+1} - 1}{2}$, consider the set H'_t. By Proposition 23 the code distance of H'_t is $D = 3^t$. For t large enough, we have $|H'_t| 2^{n-D+1} = 3^t 2^{n-D+1} \le 3^t 2^{2^t - 1 - \frac{3^t+1}{2}} < 2^{n-2}$. Proposition 24 implies that equivalence of two bitrades $U(f^V)$ and $U(f^W)$ is equivalent to that of the sets $V, W \subset H'_t$. The desired bound on the number of such subsets follows from Proposition 25. $\triangle$

5.2. Upper Bound on the Number of Bitrades

We say that a family of functions $\mathcal{A} = \bigcup\limits_{n=1}^{\infty} \mathcal{A}_n$, $\mathcal{A}_n \subseteq \{f \colon Q_k^n \to S\}$, $n \in \mathbb{N}$, is *hereditary* if any set of functions $\mathcal{A}_n$ is closed with respect to the action of isometries of Q_k^n on arguments of the functions and any retract of any function in $\mathcal{A}_n$ belongs to $\mathcal{A}_{n-1}$. A set $T \subset Q_k^m$ is said to be *testing* for a set of functions $\mathcal{A}_m$ if for any $f, g \in \mathcal{A}_m$, $f|_T = g|_T$ implies $f = g$. A set T is a testing set for a set of functions $\mathcal{A}_m$ if and only if $\operatorname{supp}(f - g) \cap T \ne \varnothing$ for any f and g in $\mathcal{A}_m$, i.e., its complement $Q_k^m \setminus T$ does not contain the support of the difference of any two functions from $\mathcal{A}_m$. Since the difference of two characteristic functions of some combinatorial configurations is a bitrade (in a wide sense), searching for testing functions is equivalent to searching for sets that do not contain bitrades.

Proposition 26. *Let a family $\mathcal{A} = \bigcup\limits_{n=1}^{\infty} \mathcal{A}_n$, $n \in \mathbb{N}$, be hereditary. Let $T \subset Q_k^m$ be a testing set for $\mathcal{A}_m$. Then the Cartesian product of testing sets $T^\ell \subset Q_k^{\ell m}$ is a testing set for $\mathcal{A}_{\ell m}$.*

Proof. We prove the proposition by induction. Let $f|_{T^\ell} = g|_{T^\ell}$. Then by the induction hypothesis, for any $v \in T$, $f|_{T^{\ell-1} \times \{v\}} = g|_{T^{\ell-1} \times \{v\}}$ implies $f|_{Q_k^{(\ell-1)m} \times \{v\}} = g|_{Q_k^{(\ell-1)m} \times \{v\}}$. Hence, for any $w \in Q_k^{(\ell-1)m}$ we have $f|_{\{w\} \times T} = g|_{\{w\} \times T}$. The set $\{w\} \times T$ is testing for retracts on $\{w\} \times Q_k^m$, since the family $\mathcal{A}_n$ is hereditary. Then $f|_{\{w\} \times Q_k^m} = g|_{\{w\} \times Q_k^m}$ for any $w \in Q_k^{(\ell-1)m}$. $\triangle$

The definition of a testing set and Proposition 26 imply the following.

Proposition 27. *Let $\mathcal{A} = \bigcup\limits_{n=1}^{\infty} \mathcal{A}_n$ be a hereditary family of functions and $T \subset Q_k^m$ be a testing set for $\mathcal{A}_m$. Then $|\mathcal{A}_{\ell m}| \le |S|^{|T|^\ell}$.*

Below we will identify unitrades and their characteristic functions. Families of bitrades and unitrades are hereditary (see Propositions 1 and 2). As follows from equation (2), any subset in Q_3^n inducing a subgraph isomorphic to a Boolean hypercube is a testing set for the family of ternary unitrades (and bitrades). Let T be a testing set for the family of unitrades in Q_3^n. Since the number of unitrades in Q_3^n is 2^{2^n}, it follows from Proposition 27 that $|T| \ge 2^n$. Note that for any testing

set T its complement $Q_3^n \setminus T$ contains no (nonempty) unitrades, and vice versa, if $Q_3^n \setminus T$ contains a unitrade, then T is not a testing set for unitrades. Therefore, the maximum cardinality of a subset in Q_3^n containing no unitrades is $3^n - 2^n$. For the family of bitrades, the similar question remains open. Below we in fact prove that there exists a subset in Q_3^n of cardinality greater than $3^n - 2^n$ that does not contain symmetric differences of bitrades.

Proposition 28. *If there exists a unitrade $U \subset Q_3^m$ whose characteristic function is not a (modulo 2) sum of two bitrades, then for bitrades in Q_3^m there exists a testing set of cardinality $2^m - 1$.*

Proof. To each vertex $v \in Q_3^m$ we assign a variable x_v. Consider the following system of Boolean equations uniquely determining a unitrade U:

(i) $x_a \oplus x_b \oplus x_c = 0$, for each one-dimensional face $\{a, b, c\}$ in Q_3^m;

(ii) $x_v = 0$, for each v in $Q_3^m \setminus U$.

From equations of type (i) we select an independent subsystem (I), and then from equations of type (ii), a maximal independent subsystem (II) which is also independent with equations of type (i). The set of solutions of subsystem (I) of equations of type (i) has dimension 2^m, and the joint system has dimension 1, since (see Proposition 4) no unitrade is a subset of another, i.e., zeros of one function cannot be a subset of zeros of another characteristic function of a unitrade. Therefore, there are $2^m - 1$ equations in subsystem (II), which are given by points v of some set $T \subset Q_3^m$, $|T| = 2^m - 1$.

Let us show that T is a testing set for bitrades in Q_3^m. Let two characteristic functions χ_A and χ_B of different bitrades A and B coincide on T; then $\chi_U = \chi_A \oplus \chi_B$, since the function $\chi_A \oplus \chi_B$ is a solution of the system of equations defining the unitrade U. This contradicts the condition. $\triangle$

The computational experiment (see the table in Section 4.5) shows that the number of bitrades in Q_3^7 is at most $2^{2^6} = \sqrt{2^{2^7}}$, i.e., the square root of the number of unitrades in Q_3^7. Then the number of pairs of bitrades in Q_3^7 is less than that of unitrades. Thus, for $n = 7$, conditions of Proposition 28 are fulfilled. Hence we obtain the following.

Corollary 10. *The number of bitrades in $Q_3^{7\ell}$ is at most $2^{\alpha^{7\ell}}$, where $\alpha = (2^7 - 1)^{1/7} < 2$.*

Let $\beta(n)$ be the number of bitrades in Q_3^n. Then $\beta(n + m) \le (\beta(n))^{2^m}$, $m = 1, \ldots, 6$. Hence, there is a similar bound on the number of bitrades for arbitrary $n > 7$.

Theorem 8 (upper bound). *The number of bitrades in Q_3^n is at most $2^{\alpha_1^n}$, where $\alpha_1 < 2$.*

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