Splitting of hypercube into k-faces and DP-colorings of hypergraphs

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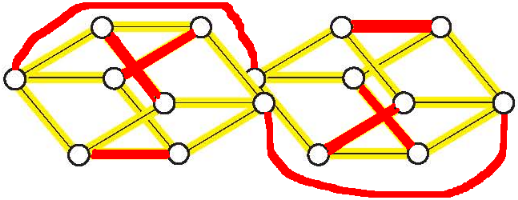
Definitions

Let Q_2^n be an n-dimensional Boolean hypercube. We consider a splitting of Q_2^n into m-dimensional axis-aligned planes or m-faces. If $m = 1$ then such splitting is equivalent to a perfect matching in Boolean hypercube. Two m-faces are called parallel if they have the same directions.

A splitting of Boolean hypercube is called an antipodal k-splitting if it consists of exactly 2^k $(n-k)$-faces and it does not contain parallel non-antipodal faces.



Definitions

The set of faces of Q_2^n is in one-to-one correspondence with n-tuples over alphabet $\{0, 1, *\}$ and each m-face corresponds to a word containing m symbols $*$.

For example, the set $(0, 0, *, 1, *) = \{(0, 0, x, 1, y) \mid x, y \in \{0, 1\}\}$ is a 2-face of Q_2^5.

An $A(n, 2, w, t)$ design (A-design) is a collection of $(n - t)$-faces of Q_2^n that perfectly covers all $(n - w)$-faces. If $w = n$ then A-design is a splitting of hypercube into $(n - t)$-faces.

Example

The set $\{(0, 0, 0, *), (1, 1, 1, *), (1, 0, *, 0), (0, 1, *, 1),$
$(0, *, 1, 0), (1, *, 0, 1), (*, 1, 0, 0), (*, 0, 1, 1)\}$
is an $A(4, 2, 4, 3)$ design.

Proposition

If there exist an antipodal k_1-splitting of $Q_2^{n_1}$ and an antipodal k_2-splitting of $Q_2^{n_2}$ then there exists an antipodal $(k_1 k_2)$-splitting of $Q_2^{n_1 n_2}$.

Proof. Let A be an antipodal k_1-splitting of $Q_2^{n_1}$ and $B = B_0 \cup B_1$ be an antipodal k_2-splitting of $Q_2^{n_2}$, where sets B_0 and B_1 do not contain parallel $(n_2 - k_2)$-faces. Consider $(a_1, \ldots, a_{n_1}) \in A$. For all a_i, if $a_i = 0$ we replace a_i by arbitrary $b \in B_0$; if $a_i = 1$ then we replace a_i by arbitrary $b \in B_1$; if $a_i = *$ then we replace a_i by $\underbrace{(*, \ldots, *)}_{n_2}$. The resulting set is an antipodal $(k_1 k_2)$-splitting of $Q_2^{n_1 n_2}$.

Example

$(0, 1, *, 1) \ \Rightarrow (000*, 01*1, ****, 111*); (000*, 01*1, ****, 01*1);$
$\ldots (10*0, 01*1, ****, 01*1); (10*0, 111*, ****, 01*1); \ldots$

Definitions

Let G be a r-uniform hypergraph on n vertices. For each $e \in E(G)$ we consider two antipodal 2-colorings $\varphi_e : e \to \{0, 1\}$ and $\overline{\varphi_e} = \varphi_e \oplus 1$. Let Φ be a collection of φ_e, $e \in E(G)$.

We say that a 2-coloring $f : V(G) \to \{0, 1\}$ avoids Φ if $f|_e \neq \varphi_e$ and $f|_e \neq \overline{\varphi}_e$ for each $e \in E(G)$.

A hypergraph G is called proper 2-colorable if there exists 2-coloring f avoiding Φ_0, where Φ_0 consist of constant maps.

A hypergraph G is called 2-DP-colorable if for every collection Φ there exists a 2-coloring f avoiding Φ.

- Dvorak Z. and Postle L. Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8, Journal of Combinatorial Theory. Series B, 2018, V. 129, P. 38–54
- Bernshteyn A. and Kostochka A. DP-colorings of hypergraphs, European Journal of Combinatorics, 2019, V. 78, P. 134–146.

A 2-coloring f of r-uniform hypergraph on n vertices is on-to-one correspond to n-tuple over alphabet $\{0, 1\}$ ($f \in Q_2^n$). Each r-hyperedge corresponds to $(n-r)$-faces of Q_2^n of some direction. For example, r-hyperedge consisting of i_1th,...,i_rth vertices corresponds to faces $(*, \ldots, *, \cdot_{i_1}, *, \ldots, *, \cdot_{i_2}, *, \ldots, \cdot_{i_r}, * \ldots, *)$. A 2-coloring f avoids $\varphi_e = (*, \ldots, 1, 0, \ldots, *)$ iff $f \notin \varphi_e$. A 2-coloring f avoids Φ if $f \notin \varphi_e \cup \overline{\varphi}_e$ for all $\varphi_e \in \Phi$.

Proposition

A non-2-DP-colorable r-uniform hypergraph with e edges and n vertices is equivalent to a covering of Q_2^n by e pairs of antipodal $(n-r)$-faces.

Corollary

Any k-uniform hypergraph with $e < 2^{k-1}$ edges is 2-DP-colorable.

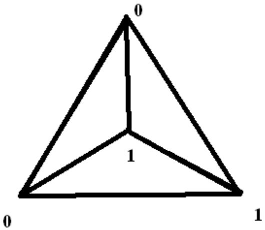
Corollary

There exists a non-2-DP-colorable k-uniform hypergraph with 2^{k-1} edges if and only if there exists an antipodal k-splitting of Q_2^n.

Corollary

There exist non-2-DP-colorable k-uniform hypergraphs with 2^{k-1} vertices where $k = 3^t$.

Consider a tetrahedron as a 3-uniform hypergraph. There exists a proper 2-coloring of tetrahedron.



But tetrahedron is not 2-DP-colorable because there exists a 3-antipodal splitting of Q_2^4.
$\Phi = \{(0, 0, 0, *), (1, 1, 1, *), (1, 0, *, 0), (0, 1, *, 1),$
$(0, *, 1, 0), (1, *, 0, 1), (*, 1, 0, 0), (*, 0, 1, 1)\}$

Theorem

There are no antipodal 5-splittings of Q_2^n.

Corollary

Every 5-uniform hypergraph with 16 (or less) edges is 2-DP-colorable.

Definitions

A pair $\{T_0, T_1\}$ of disjoint collections of k-subsets (blocks) of a set V, $|V| = n$, is called a bitrade (more specifically, a $(k-1)-(n,k)$ bitrade) if every $(k-1)$-subset of V is contained in the same number of blocks of T_0 and T_1. Bitrades correspond to possible difference between two Steiner designs. A collection U of k-subsets (blocks) of a set V of cardinality n is called a k-unitrade if every $(k-1)$-subset of V is contained in each block of U even times. It is easy to see that if $\{T_0, T_1\}$ is a bitrade then $T_0 \cup T_1$ is a unitrade.

Examples of unitrades

$W_5 = \{\{1,2,3,4,5\},\{2,3,4,5,6\},\{1,3,4,5,6\},$
$\{1,2,4,5,6\},\{1,2,3,5,6\},\{1,2,3,4,6\}\}$
$R_5 =$
$\{\{2,3,4,5,6\},\{1,3,4,5,6\},\{1,2,4,5,6\},\{1,2,3,5,6\},\{1,2,3,4,6\},$
$\{2,3,4,5,7\},\{1,3,4,5,7\},\{1,2,4,5,7\},\{1,2,3,5,7\},\{1,2,3,4,7\}\}$

Proposition

An antipodal k-splitting of Q_2^n corresponds to a k-unitrade with cardinality 2^{k-1} on n-element set.

Example of unitrade

$\{(0,0,0,*),(1,1,1,*),(1,0,*,0),(0,1,*,1),$
$(0,*,1,0),(1,*,0,1),(*,1,0,0),(*,0,1,1)\}$

$$\Rightarrow$$

$\{\cdot,\cdot,\cdot,*\},\{\cdot,\cdot,*,\cdot\},\{\cdot,*,\cdot,\cdot\},\{*,\cdot,\cdot,\cdot\}$

$$\Rightarrow$$

$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$

characterization of 5-unitrades of cardinality 16

Theorem

Up to equivalence, all 5-unitrades of cardinality 16 are exhausted by the following list:
1) disjoint unions of W_5 and R_5;
2) $E = \{\{1,2,3,4,5\},\{2,3,4,5,6\},\{1,3,4,5,7\},\{1,2,4,5,7\},$
$\{1,2,3,5,8\},\{1,2,3,4,8\},\{2,3,5,7,8\},\{1,3,5,7,8\},$
$\{1,2,5,7,8\},\{2,3,4,7,8\},\{1,3,4,7,8\},\{1,2,4,7,8\},$
$\{3,4,5,6,7\},\{2,4,5,6,7\},\{2,3,5,6,7\},\{2,3,4,6,7\}\}$;
3) $F = \{\{1,2,3,5,7\},\{1,2,4,5,7\},\{1,3,4,5,7\},\{2,3,4,5,7\},$
$\{1,2,3,6,7\},\{1,2,4,6,7\},\{1,3,4,6,7\},\{2,3,4,6,7\},$
$\{1,2,3,6,8\},\{1,2,4,6,8\},\{1,3,4,6,8\},\{2,3,4,6,8\},$
$\{1,2,3,5,8\},\{1,2,4,5,8\},\{1,3,4,5,8\},\{2,3,4,5,8\}\}$.

But no 5-unitrade corresponds to an antipodal 5-splitting.

$F =$

$$\begin{array}{cccc}
(\cdot,\cdot,\cdot,*,\cdot,*,\cdot,*) & (\cdot,\cdot,\cdot,*,*,\cdot,\cdot,*) & (\cdot,\cdot,\cdot,*,*,\cdot,*,\cdot) & (\cdot,\cdot,\cdot,*,\cdot,*,*,\cdot) \\
(\cdot,\cdot,*,\cdot,\cdot,*,\cdot,*) & (\cdot,\cdot,*,\cdot,*,\cdot,\cdot,*) & (\cdot,\cdot,*,\cdot,*,\cdot,*,\cdot) & (\cdot,\cdot,*,\cdot,\cdot,*,*,\cdot) \\
(\cdot,*,\cdot,\cdot,\cdot,*,\cdot,*) & (\cdot,*,\cdot,\cdot,*,\cdot,\cdot,*) & (\cdot,*,\cdot,\cdot,*,\cdot,*,\cdot) & (\cdot,*,\cdot,\cdot,\cdot,*,*,\cdot) \\
(*,\cdot,\cdot,\cdot,\cdot,*,\cdot,*) & (*,\cdot,\cdot,\cdot,*,\cdot,\cdot,*) & (*,\cdot,\cdot,\cdot,*,\cdot,*,\cdot) & (*,\cdot,\cdot,\cdot,\cdot,*,*,\cdot)
\end{array}$$