Relations between error-correcting codes and cryptographic functions

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1. Correlation Immunity and Dual Codes
2. Sets of Functions as Codes
3. Almost Perfect Nonlinear Functions and Linear Codes

1. Correlation Immunity and Dual Codes

Functions and Codes

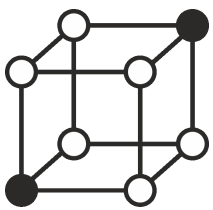
$\mathbb{F}_q$ is a finite field, where $q = p^n$, p is prime.

A function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is called a Boolean function.
A set $C \subset \mathbb{F}_2^n$ is called a binary code.
There is a one-to-one correspondence between Boolean functions and binary codes: $f^{-1}(1) = C$ or $f = \mathbf{1}_C$.

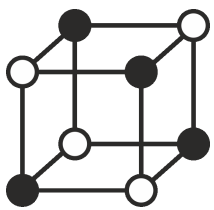
If $q > 2$ then functions $f : \mathbb{F}_q^n \to \mathbb{F}_q$ correspond to partitions $\{f^{-1}(a) : a \in \mathbb{F}_q\}$ of $\mathbb{F}_q^n$ into codes.

Correlation Immunity

An order of correlation immunity of $f : \mathbb{F}_q^n \to \mathbb{F}_q$ is equal to maximum $k = \mathrm{cor}(f)$ such that for every $a \in \mathbb{F}_q$ the cardinalities $|L \cap f^{-1}(a)|$ are the same for all $(n-k)$-dimensional face L of $\mathbb{F}_q^n$.



$\mathrm{cor}(f) = 1$

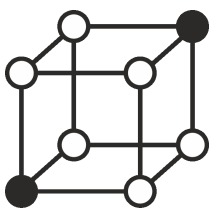


$\mathrm{cor}(f) = 2$

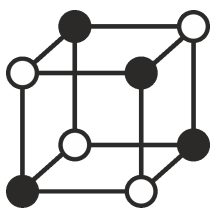
Hamming Distance

The Hamming distance $d_H(x, y)$ between $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is the number of coordinates i where $x_i \neq y_i$. The code distance $d(C)$ of code $C \subset \mathbb{F}_q^n$ is the minimum distance between codewords $d(C) = \min_{x,y \in C} d_H(x, y)$.

Code C can correct up to $\frac{d(C)-1}{2}$ errors, and it can detect up to $d(C) - 1$ errors.



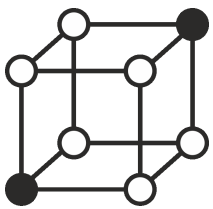
$d(C) = 3$



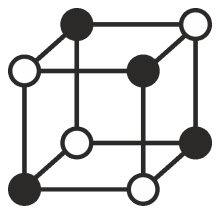
$d(C) = 2$

Dual Code

For $C \subset \mathbb{F}_q^n$ the dual code is $C^\perp = \{x \in \mathbb{F}_q^n : x_1y_1 + \cdots + x_ny_n = 0 \quad \forall y \in C\}$. If C is a linear code then $(C^\perp)^\perp = C$.



C



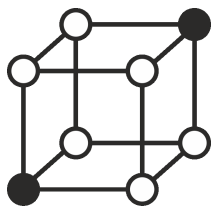
$C^\perp$

Theorem If C is a linear code then $d(C^\perp) = \mathrm{cor}(\mathbf{1}_C) + 1$ and $d(C) = \mathrm{cor}(\mathbf{1}_{C^\perp}) + 1$.

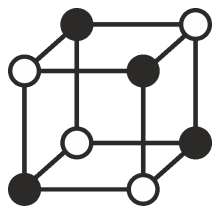
2. Sets of Functions as Codes

Set of Functions as a Code

Let $\mathcal{F} = \{f : \mathbb{F}_q^n \to \mathbb{F}_q\}$ be a set of functions. Consider f as an element of $\mathbb{F}_q^{q^n}$. Then $\mathcal{F}$ is a code in $\mathbb{F}_q^{q^n}$.



$(1, 0, 0, 0, 0, 0, 1)$



$(1, 0, 0, 1, 0, 1, 1, 0)$

Algebraic Normal Form

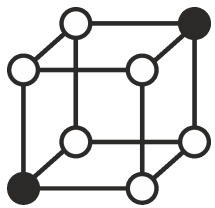
Every Boolean function can be represented in the algebraic normal form (ANF)

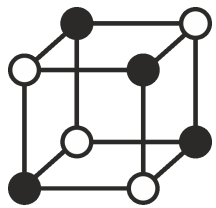
$$f(x_1, \ldots, x_n) = \bigoplus_{y \in \mathbb{F}_2^n} M_f(y) x_1^{y_1} \cdots x_n^{y_n},$$

where $x^0 = 1, x^1 = x$, $M_f : \mathbb{F}_2^n \to \mathbb{F}_2$ is the Möbius transform of f.

The weight $\mathrm{wt}(y)$ of $y \in \mathbb{F}_2^n$ is the number of nonzero coordinates of y. The algebraic degree of f is the maximum degree of the monomials in ANF, i. e., $\deg(f) = \max_{M_f(y)=1} \mathrm{wt}(y)$.

Binary Reed–Muller Code





$f_1(x_1, x_2, x_3) = 1 + x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3$
$f_2(x_1, x_2, x_3) = 1 + x_1 + x_2 + x_3$

$RM(r, n) = \{f : \mathbb{F}_2^n \to \mathbb{F}_2 : \deg(f) \leq r\}$.
Parameters (length, cardinality, code distance)$= (2^n, 2^k, 2^{n-r})$,
where $k = \sum_{i=0}^{r} \binom{n}{i}$ is the number of monomials.

Kerdock Code

n is even.
$\mathcal{F}(n)$ is a set consisting of n-variable quadratic nondegenerate forms f such that $f_1 + f_2$ is also nondegenerate for any $f_1, f_2 \in \mathcal{F}(n)$.
$K(n) = RM(1, n) + \mathcal{F}(n) \subset RM(2, n)$.
$\mathcal{A}_2(n) = RM(1, n)$ is the set of binary affine functions.

Parameters of $K(n)$ are $(2^n, 2^{2n}, 2^{n-1} - 2^{\frac{n}{2}-1})$.

Example: $f_1(x) = x_1x_2 + x_3x_4 + x_2x_3$, $f_2 = x_1x_3 + x_1x_4 + x_2x_4$, f_1, f_2 and $f_1 + f_2$ are nondegenerate quadratic forms.

Bent Functions

$f : \mathbb{F}_p^n \to \mathbb{F}_p$ is called a balanced function if $|f^{-1}(a)| = p^{n-1}$ for all $a \in \mathbb{F}_p$.
$f : \mathbb{F}_p^n \to \mathbb{F}_p$ is called a p-ary bent function if
$D_a f(x) = f(x + a) - f(x)$ is balanced for any $a \in \mathbb{F}_p^n \backslash \{\bar{0}\}$.
A function $f : \mathbb{F}_p^n \to \mathbb{F}_p$ is called maximum nonlinear if
$d_H(f, \mathcal{A}_p(n)) = \min\limits_{h \in \mathcal{A}_p(n)} d_H(f, h)$ is maximal.

Theorem(Rothaus(1975), V.Ryabov(2021)) For even n
1) every maximum nonlinear p-ary function is a bent function.
2) every binary bent function is maximum nonlinear.

Proposition Nondegenerate p-ary quadratic forms are bent functions if n is even.

Planar and Alltop Functions

The function $F : \mathbb{F}_q \to \mathbb{F}_q$ is called planar if for any $b \in \mathbb{F}_q \backslash \{0\}$ the derivative $D_b F(x) = F(x+b) - F(x)$ is bijective on $\mathbb{F}_q$.

A function $F : \mathbb{F}_q \to \mathbb{F}_q$ is called an Alltop function over $\mathbb{F}_q$ if, for any $a \in \mathbb{F}_q \backslash \{0\}$, the derivative $D_a F(x) = F(x+a) - F(x)$ is a planar function. Equivalently, for any $a, b \in \mathbb{F}_q \backslash \{0\}$, the expression

$$D_b D_a f(x) = f(x+a+b) - f(x+a) - f(x+b) + f(x)$$

is bijective on $\mathbb{F}_q$.

Examples: $f(x) = x^2$ is a planar function if $q \neq 2^t$;
$f(x) = x^3$ is an Alltop function if $q \neq 2^t$ and $q \neq 3^t$.

$f : \mathbb{F}_p^n \to \mathbb{F}_p$ is called p-ary Alltop function if the derivative $D_a f(x) = f(x+a) - f(x)$ is a p-ary bent for any $a \in \mathbb{F}_p^n \backslash \{\bar{0}\}$, i.e., for any $a, b \in \mathbb{F}_p^n \backslash \{\bar{0}\}$, the second-order derivative $D_b D_a f(x)$ is balanced.

Proposition Let $q = p^n$. Every coordinate function of Alltop function over $\mathbb{F}_q$ is an p-ary Alltop function.

Generalized Kerdock Codes

A set of bent functions $K = \{f : \mathbb{F}_p^n \to \mathbb{F}_p\}$ is called the Kerdock set if for any $f_1, f_2 \in K$ the difference $f_1 - f_2$ is a bent function.

Theorem (Özbudak and P. (2025+)) If f is a p-ary Alltop function then $\mathcal{K}_f = \{D_a f : a \in \mathbb{F}_p^n, \}$ is a Kerdock set.

Theorem (Özbudak and Pelen (2020)) Let $f : \mathbb{F}_p^n \to \mathbb{F}_p$ be a bent function. Then the Hamming distance between f and any affine function is not less than
$(p-1)(p^{n-1} - p^{n/2-1})$ if n is even;
$(p-1)p^{n-1} - p^{(n-1)/2}$ if n is odd.

Consider a set $\mathcal{D}_\mathcal{K} = \mathcal{K} + \mathcal{A}_p(n)$, where $\mathcal{K}$ is a Kerdock set.

Theorem (Özbudak and P. (2025+)) $\mathcal{D}_K$ is a generalized p-ary Kerdock code with parameters $(p^n, p^{2n+1}, (p-1)(p^{n-1} - p^{\frac{n}{2}-1}))$ for even n and $(p^n, p^{2n+1}, (p-1)p^{n-1} - p^{\frac{n-1}{2}})$ for odd n.

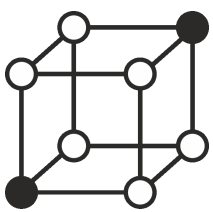
3. Almost Perfect Nonlinear Functions and Linear Codes

Parity-check Matrix

Consider a linear code $C_H = \{x \in \mathbb{F}_2^n : Hx = \bar{0}\}$ parity-check matrix H of size $k \times n$ over $\mathbb{F}_2^n$.
Rows of H belong to $C_H^{\perp}$.

Proposition If the sum of any s, $0 \leq s < S$, columns of H is not equal to zero-vector then $d(C) \geq S$.

If H consists of $2^k - 1$ different nonzero binary columns with length k then C_H is the Hamming code with parameters $(n = 2^k - 1, 2^{n-k}, 3)$.



$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Almost Perfect Nonlinear Functions

A function $F : \mathbb{F}_2^k \to \mathbb{F}_2^k$ is called an almost perfect nonlinear (APN) if for any $a \in \mathbb{F}_2^k \backslash \{\bar{0}\}$ and $b \in \mathbb{F}_2^k$ the equations $F(x) + F(x + a) = b$ have zero or two solutions $x \in \mathbb{F}_2^k$.

F is called an APN permutation if APN function F is bijective on $\mathbb{F}_2^k$.

Consider matrix M of size $2k \times (2^k - 1)$ consists of columns $\begin{pmatrix} x \\ F(x) \end{pmatrix}$, where $x \in \mathbb{F}_2^k \backslash \{\bar{0}\}$.

Theorem (Carlet, Charpin, Zinoviev (1998))
$d(C_M) = 5$ if and only if F is an APN function.

Almost Perfect Nonlinear Functions

$\mathbb{F}_2^k \sim \mathbb{F}_{2^k}$

Proposition $F(x) = x^3$ is an APN function; if k is odd then F is a permutation.

Problem Are there exist APN permutations for even $k > 6$?

Proposition APN-permutation in k variables exists if and only if there exist two Hamming codes C_1 and C_2 of length $2^k - 1$ such that $d(C_1 \cap C_2) = 5$.

Almost Perfect Nonlinear Functions

The Hamming distance between two functions is the number of arguments where values of functions are differ.

Problem Are there exist a pair of APN functions at the Hamming distance 1?

The code C of length 2^k is said a doubled Hamming code if the parity check matrix of C contains only two equal columns.

Proposition There exists a pair of APN functions at the Hamming distance 1 if and only if there exists a linear code with parameters $(n = 2^k, 2^{n-2k}, 5)$ contained a doubled Hamming code.

The End