On the multidimensional permanent and q-ary designs

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Let $Q_q = \{0, 1, \ldots, q-1\}$ and $Q_{q*} = Q_q \cup \{*\}$.

Q_q^n denotes the *n*-dimensional hypercube.

The set of faces of Q_q^n is in one-to-one correspondence with Q_{q*}^n and each k-dimensional face (k-face) corresponds to a codeword with k symbols $*$.

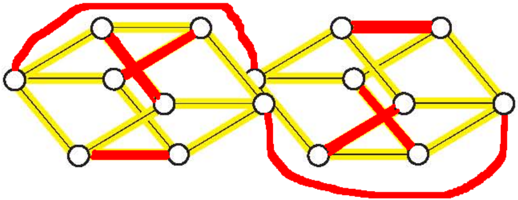
For example, the set $(0, *, 2, *) = \{(0, x, 2, y) \mid x, y \in Q_3\}$ is a 2-face of Q_3^4.

Definition

An $H(n, q, w, t)$ design (H-design) is a collection of $(n - w)$-faces of the hypercube Q_q^n that perfectly pierce all $(n - t)$-faces.

Example

The set $\{(0,0,0,*), (1,1,1,*), (1,0,*,0), (0,1,*,1), (0,*,1,0), (1,*,0,1), (*,1,0,0), (*,0,1,1)\}$
is an $H(n = 4, q = 2, w = 3, t = 2)$ design.

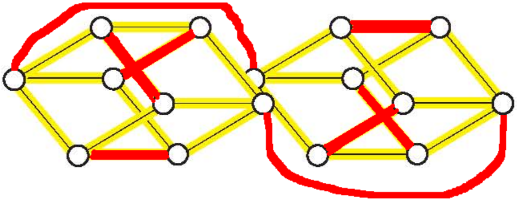


Definition

An $A(n, q, w, t)$ design (A-design) is a collection of $(n-t)$-faces of Q_q^n that perfectly cover all $(n-w)$-faces.

Example

The set $\{(0,0,0,*),(1,1,1,*),(1,0,*,0),(0,1,*,1),$
$(0,*,1,0),(1,*,0,1),(*,1,0,0),(*,0,1,1)\}$
is an $A(n=4, q=2, w=4, t=3)$ design.



If $q = 1$ then an $H(n, 1, w, t)$ design is just a Steiner system $S(t, w, n)$. Here $*$ is replaced by 0 and 0 is replaced by 1.

$$(*, 0, *, *, *, 0, 0) \Rightarrow (0, 1, 0, 0, 0, 1, 1)$$
$$H(7, 1, 3, 2) \Rightarrow S(2, 3, 7)$$

Moreover, an $A(n, 1, w, t)$ design is just a Steiner system $S(n - w, n - t, n)$. Here $*$ is replaced by 1.

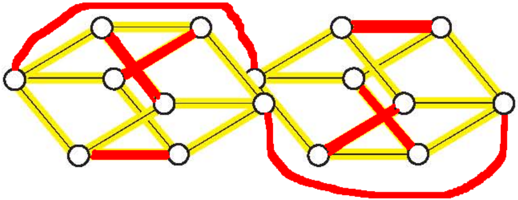
$$(*, 0, *, 0, 0, *, 0) \Rightarrow (1, 0, 1, 0, 0, 1, 0)$$
$$A(7, 1, 5, 4) \Rightarrow S(2, 3, 7)$$

A set of 1-faces is called a precise clique matching if it is both $H(n, q, n-1, n-2)$ design and $A(n, q, n, n-1)$ design. The precise clique matchings with $n = 2^{t+1}$ and $q = 2^t$ are constructed in

V. N. Potapov, Clique matchings in the k-ary n-dimensional cube, Siberian Math. J. 2011.

Example (when $t = 1$) from

P. Hamburger, R. E. Pippert and W. D. Weakley, On a leverage problem in the hypercube, Networks. 1992



H. Hanani, On some tactical configurations, Canad. J. Math. 1963.

W. H. Mills, On the existence of H design, Proceedings of the Twenty-First Southeastern Conference on Combinatorics, Graph Theory, and Computing, Congr. Numer. 1990.

T. Etzion, Optimal constant weight codes over Z_k and generalized designs, Discrete Math. 1997.

V. A. Zinov'ev and J. Rifa, On new completely regular q-ary codes, Probl. Inf. Transm. 2007.

L. Ji, An improvement on H design, J. Combin. Des. 2009.

Construction 1

Let $S \subset Q^n_{q*}$ be an $H(n, q, w, t)$ design and let $R \subset Q^w_{q'*}$ be an $H(w, q', w, w-1)$ design (MDS code). Given $(a^1, \ldots, *, \ldots, a^i, \ldots, *, \ldots, a^w) \in S$ and $(b_1, \ldots, b_w) \in R$ arrange the codeword

$$((a^1, b_1), \ldots, *, \ldots, (a^i, b_i), \ldots, *, \ldots, (a^w, b_w)) \in Q^n_{qq'*}.$$

Proposition 1

The set of all these codewords is an $H(n, qq', w, t)$ design.

As mentioned above, $H(2k, k, 2k-1, 2k-2)$ designs exist for $k = 2^t$, $t \geq 1$. Since MDS codes with distance 2 exist for all $q \geq 2$, we get

Corollary 1

For all $s, t \geq 1$ there exist $H(2^{t+1}, s2^t, 2^{t+1}-1, 2^{t+1}-2)$ designs.

Construction 2

Let $S \subset Q^n_{q*}$ be an $A(n, q, w, t)$ design. For each pair of $(a^1, \dots, *, \dots, a^i, \dots, *, \dots, a^t) \in S$ and $(b_1, \dots, b_t) \in Q^w_{q'}$ we form the codeword

$$((a^1, b_1), \dots, *, \dots, (a^i, b_i), \dots, *, \dots, (a^t, b_t)) \in Q^n_{qq'*}.$$

Proposition 2

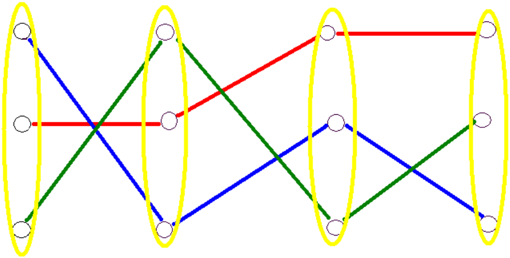
The set of all these codewords is an $A(n, qq', w, t)$ design.

As mentioned above, each Steiner system $S(n-w, n-t, n)$ is equivalent to an $A(n, 1, w, t)$ design.

Corollary 2

If there exists a Steiner system $S(n-w, n-t, n)$ then for each $q \geq 1$ there exists an $A(n, q, w, t)$ design.

Consider a *k-partite hypergraph* G_k containing N vertices in each part C_i, $i = 1, \ldots, k$. Suppose that each k-edge of G_k consists of k vertices, with one vertex in each part of the hypergraph. A set of disjoint k-edges that matches all vertices of the hypergraph is called a *perfect k-matching*.



Let each part of the hypergraph be enumerated by $1, 2, \ldots, N$. We define the adjacency array $M(G_k) = (a_{i_1 \ldots i_k})$ by the following rule: $a_{i_1 \ldots i_k} = 1$ if there exists a k-edge consisting of vertices with numbers i_1 from the first part, i_2 from the second part and so on and $a_{i_1 \ldots i_k} = 0$ otherwise. A k-element subset I of $\{1, \ldots, N\}^k$ is called a diagonal if every pair of elements of I is distinct in each position. We define the k- dimensional permanent of $M(G_k)$ as

$$\mathrm{per}_k M(G_k) = \sum_{I \in D_N} \prod_{(i_1, \ldots, i_k) \in I} a_{i_1 \ldots i_k},$$

where D_N is the set of all diagonals.

It is well known that the permanent of the adjacency matrix of a bipartite graph is equal to the number of perfect matchings of the graph. The following statement is straightforward.

Proposition 3

The number of perfect k-matchings of a hypergraph G_k is equal to $\mathrm{per}_k M(G_k)$.

Denote by $Q_q^n(t)$ the set of $(n-t)$-faces of Q_q^n. By definition each $A(n,q,w,t)$ design is a subset of $Q_q^n(t)$ such that its faces do not intersect but cover $Q_q^n(w)$. We assume that there exists a partition $A = \{A_1, \dots, A_m\}$, where $m = \binom{w}{t}$, of $Q_q^n(t)$ into $A(n,q,w,t)$ designs. Define the m-part hypergraph GA with parts $A_1, \dots, A_m$. A collection $\{\overline{a}_1, \dots, \overline{a}_m\}$, where $\overline{a}_i \in Q_q^n(t)$, is a m-edge in GA if there exists $\overline{b} \in Q_q^n(w)$, $\overline{b} = \bigcap_{i=1}^m \overline{a}_i$.

Proposition 4

The number of different $H(n,q,w,t)$ designs is equal to $\mathrm{per}_m M(GA)$.

Proposition 5

The number of different $A(n,q,w,t)$ designs is equal to $\mathrm{per}_k M(GH)$.