Bounds on the size of 2-fold codes in 3-dimensional hypercube

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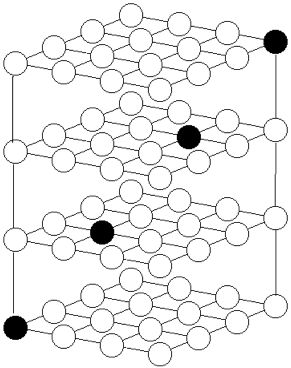
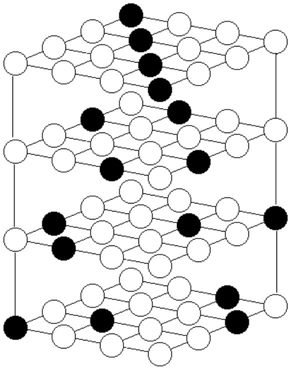
Let Q_n^3 be a 3-dimensional hypercube of order n and $d(x, y)$ be the Hamming distance between $x, y \in Q_n^3$.
Subset $C \subset Q_n^3$ is called a t-fold code if for each $x \in Q_n^3$ there exist no more than t codewords $y \in C$ such that $d(x, y) \leq 1$.
The least distance between two different codewords is named a code distance.

An MDS code with code distance 3 and cardinality n is the largest 1-fold code in Q_n^3. Such codes are named diagonals.

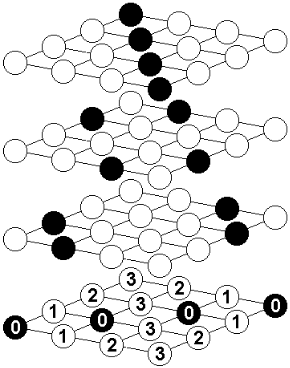


If any hyperplane contains two codewords, then there is a point on the distance 1 from both codewords.

An MDS code with code distance 2 and cardinality n^2 is the largest 3-fold code in Q_n^3. Such codes are named 3-dimensional permutations.

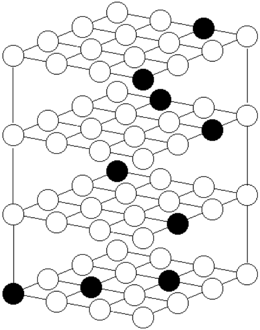


There is a natural bijection between MDS codes with code distance 2 in Q_n^3 and latin squares of order n. A graph $\{(x, f(x)) \mid x \in Q_n^3\}$ of quasigroup f is MDS code with distance 2 and Cayley table of quasigroup f is Latin square.



What is the largest code size of 2-fold code in Q_n^3?

This question can be reformulated in the following way. Consider a 3-dimensional cube as a kind of chessboard. A 2-fold code is the configuration of rooks such that none of 3 rooks attack any square of chessboard (element of lattice). What is the largest number of rooks as none of 3 rooks attack any square of 3-dimensional chessboard?



1	2	3	
		0	2
	0		1
0			3

Let $L(n)$ be the largest size of 2-fold code in Q_n^3. We recursively construct 2-fold codes $C \subset Q_n^3$ such that $|C| = 6^k$ as $n = 3^k$.

Then we obtain $L(n) \geq (3\lfloor \frac{n}{3} \rfloor)^{1+\log_3 2}$.

Iteration procedure proposed by K.Storozhuk

b	c	
	a	b
a		c

4	5		7	8				
	3	4		6	7			
3		5	6		8			
			1	2		4	5	
				0	1		3	4
			0		2	3		5
1	2					7	8	
	0	1					6	7
0		2				6		8

Consider a 3-partite graph $G(C)$, where all three part is equal to Q, undirected pairs (a, b), (a, c) and (b, c) are edges iff $(a, b, c) \in C$, i. e. all codewords are triangles in $G(C)$.

Without lost of generality, we suppose that the code distance of C is equal 2. Then all such triangles do not contain common edges.

Proposition

If graph $G(C)$ contains another triangle then C is not 2-fold.

Proof

Let (a, b), (a, c) and (b, c) be edges. But $(a, b, c) \notin C$. Then there exist codewords (a, b, f_1), (a, f_2, c) and (f_3, b, c) in C. But distances between (a, b, c) and (a, b, f_1), (a, f_2, c) or (f_3, b, c) is equal 1.

Triangle removal lemma

For any $0 < \varepsilon < 1$, there is $\delta > 0$ such that, whenever G is ε-far from being triangle-free (i.e. one has to delete at least $\varepsilon|V(G)|^2$ to destroy all triangles in it), then it contains at least $\delta|V(G)|^3$ triangles.

Corollary

$L(n) = o(n^2)$

Proof

Suppose that there is a sequence of 2-fold codes $C_n \subset Q_n^3$ with $n^2 > |C_n| \geq \varepsilon n^2$. Then graphs $G(C_n)$ are $\frac{\varepsilon}{9}$-far from triangle free. By the Triangle removal lemma graphs $G(C_n)$ contain at least $(3n)^3\delta$ triangles, where $\delta(\varepsilon) > 0$. Consequently $G(C_n)$ contains more than $|C_n|$ triangles for large n and C_n is not 2-fold.

Problem

Finding $\alpha < 2$ such that $L(n) \leq n^{\alpha}$ as $n \to \infty$.

General problem

Let G be a graph of order n and any edge of G belongs to only one triangle. How many triangles can be formed in graph G?

A function $f : F_3^n \to F_3$ is n-ary bitrade if any line in the table of f contains all values $\{0, 1, 2\}$ or only zeroes.

1	2	0
0	1	2
2	0	1

1	2	0
2	1	0
0	0	0

Let B_n be the set of n-ary bitrade.

Problem

To prove upper bound $|B_n| \leq 2^{\alpha^n}$, where $\alpha < 2$.

Consider a graph G such that $V(G) = B_n$ and $(f_0, f_1) \in E(G)$ iff there exists $f \in B_{n+1}$ such that $f(0, x) = f_0(x)$ and $f(1, x) = f_1$. Then triangles in G are equivalent to elements of B_{n+1} and any edge of G belongs to only one triangle.