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The John equation for tensor tomography in three-dimensions*

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Abstract

John proved that a function φ on the manifold of lines in $\mathbb{R}^3$ belongs to the range of the x-ray transform if and only if φ satisfies some second order differential equation and obeys some smoothness and decay conditions. We generalize the John equation to the case of the x-ray transform on arbitrary rank symmetric tensor fields: a function φ on the manifold of lines in $\mathbb{R}^3$ belongs to the range of the x-ray transform on rank m symmetric tensor fields if and only if φ satisfies some differential equation of order $2(m+1)$ and obeys some smoothness and decay conditions.

Keywords: tensor fields tomograhy, John equation, Radon transform

1. Introduction

The famous 1938 paper [4] by John gives a characterization of the range of the x-ray transform in $\mathbb{R}^3$ in terms of an ultrahyperbolic equation in four variables parameterizing (locally) the tangent bundle of the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Afterwards, this result was generalized to arbitrary dimensions [3] and to the case of arbitrary rank symmetric tensor fields [6]. Note, however, that these generalizations are formulated in terms of the tangent bundle of the whole Euclidean space $\mathbb{R}^n$ rather than that of the unit sphere. This naturally leads to more variables and more equations. For example, in three-dimensions even for the scalar functions

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case, one gets three equations in six variables instead of the single John equation in four variables. This difference becomes more essential in the case of symmetric tensor fields. Therefore the following question arises in the most important case of three-dimensional tensor tomography: Can one formulate the corresponding conditions on using four (local) coordinates on the tangent bundle of the unit sphere? The present paper gives the positive answer to the question.

First of all we recall the definition of the x-ray transform. Given a continuous rank m symmetric tensor field f on $\mathbb{R}^n$, the *x-ray transform* of f is defined by

$$(If)(x,\xi)=\sum_{i_1,\dots,i_m=1}^{n}\int_{-\infty}^{\infty} f_{i_1\dots i_m}(x+t\xi)\xi_{i_1}\dots\xi_{i_m}\,\mathrm{d}t \quad (x\in\mathbb{R}^n,\ 0\neq\xi\in\mathbb{R}^n) \tag{1.1}$$

under the assumption that f decays at infinity so that the integral converges. Let $\mathcal{S}(S^m\mathbb{R}^n)$ be the space of all rank m symmetric tensor fields on $\mathbb{R}^n$ whose all components belong to the Schwartz space. We restrict ourselves by considering tensor fields $f\in\mathcal{S}(S^m\mathbb{R}^n)$. For such a tensor field, $\psi(x,\xi)=(If)(x,\xi)$ is a C^∞-smooth function on $\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$ and satisfies the following conditions

$$\psi(x,t\xi)=\frac{t^m}{|t|}\psi(x,\xi)\quad(0\neq t\in\mathbb{R}),\qquad \psi(x+t\xi,\xi)=\psi(x,\xi) \tag{1.2}$$

which mean that $(If)(x,\xi)$ depends actually on the line through the point x in direction ξ.

Let $\langle\cdot,\cdot\rangle$ be the standard dot-product on $\mathbb{R}^n$. We parameterize the manifold of oriented lines in $\mathbb{R}^n$ by points of the tangent bundle

$$T\mathbb{S}^{n-1}=\{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n \mid \|\xi\|=1,\ \langle x,\xi\rangle=0\}\subset\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$$

of the unit sphere $\mathbb{S}^{n-1}\subset\mathbb{R}^n$. Let $\mathcal{S}(T\mathbb{S}^{n-1})$ be the space of C^∞-smooth functions $\chi(x,\xi)$ on $T\mathbb{S}^{n-1}$ such that all their derivatives decay rapidly in the first argument, where derivatives are taken with respect to Cartesian coordinates on $\mathbb{R}^n\times\mathbb{R}^n$ while the function χ is extended to a neighborhood of $T\mathbb{S}^{n-1}$ in $\mathbb{R}^n\times\mathbb{R}^n$ by the homogeneity: $\chi(x,\xi)=\chi(x,\xi/\|\xi\|)$. For a tensor field $f\in\mathcal{S}(S^m\mathbb{R}^n)$, the restriction $\chi=\psi|_{T\mathbb{S}^{n-1}}$ of the function $\psi=If$ to the manifold $T\mathbb{S}^{n-1}$ belongs to $\mathcal{S}(T\mathbb{S}^{n-1})$. Moreover, the function ψ is uniquely recovered from χ by the formula

$$\psi(x,\xi)=\|\xi\|^{m-1}\chi\left(x-\frac{\langle x,\xi\rangle}{\|\xi\|^2}\xi,\ \frac{\xi}{\|\xi\|}\right) \tag{1.3}$$

that follows from (1.2). Thus, the x-ray transform can be considered as the linear continuous operator

$$I:\mathcal{S}(S^m\mathbb{R}^n)\to\mathcal{S}(T\mathbb{S}^{n-1}). \tag{1.4}$$

Let us cite theorem 2.10.1 of [6].

Theorem 1.1. *A function $\chi\in\mathcal{S}(T\mathbb{S}^{n-1})$ $(n\geqslant 3)$ belongs to the range of operator (1.4) if and only if the following two conditions hold:*

(1) $\chi(x,-\xi)=(-1)^m\chi(x,\xi)$;
(2) being defined by (1.3), the function $\psi\in C^\infty(\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\}))$ satisfies the equations

$$\left(\frac{\partial^2}{\partial x_{i_1}\partial\xi_{j_1}}-\frac{\partial^2}{\partial x_{j_1}\partial\xi_{i_1}}\right)\dots\left(\frac{\partial^2}{\partial x_{i_{m+1}}\partial\xi_{j_{m+1}}}-\frac{\partial^2}{\partial x_{j_{m+1}}\partial\xi_{i_{m+1}}}\right)\psi=0 \tag{1.5}$$

that are written for all indices $1\leqslant i_1,j_1,\dots,i_{m+1},j_{m+1}\leqslant n$.

An elegant application of equations (1.5) to the problem of inversion of the x-ray transform from incomplete data was found by Denisjuk [2].

Theorem 1.1 is definitely false in the case of $n = 2$. Indeed, in the case of $(m, n) = (0, 2)$, operator (1.4) coincides, up to notation, with the Radon transform on the plane. Unlike (1.5), the corresponding consistency conditions for the Radon transform are of integral nature, see [3, chapter 1, theorem 2.4]. These conditions are well known under the name 'Helgason—Ludwig's conditions' although they have been first written down by Gelfand *et al* [1, section 1.6]. Helgason—Ludwig's conditions were generalized to the case of $n = 2$ and of arbitrary m by Pantjukhina [5].

How many linearly independent equations does system (1.5) contain? Of course, every pair (i_s, j_s) $(1 \leqslant s \leqslant m + 1)$ can be ordered so that $i_s < j_s$. There are $N = n(n - 1)/2$ ordered pairs. Since factors on the left-hand side of (1.5) commute with each other, the system contains $\binom{N+m}{m+1}$ linearly independent equations.

Now, we consider the three-dimensional case. System (1.5) contains $(m + 2)(m + 3)/2$ linearly independent equations. Nevertheless, each of these equations turns out equivalent to a single equation for the function χ expressed in appropriate coordinates.

For the open set $U = \{(x, \xi) \in T\mathbb{S}^2 | \xi_3 > 0\} \subset T\mathbb{S}^2$, we define the diffeomorphism

$$\Phi : U \to \mathbb{R}^4, \quad (x, \xi) = (x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) \mapsto (y, \alpha) = (y_1, y_2, \alpha_1, \alpha_2)$$

by

$$y_1 = x_1 - \frac{\xi_1}{\xi_3}x_3, \quad y_2 = x_2 - \frac{\xi_2}{\xi_3}x_3, \quad \alpha_1 = \frac{\xi_1}{\xi_3}, \quad \alpha_2 = \frac{\xi_2}{\xi_3}. \tag{1.6}$$

Then (U, Φ) is a coordinate patch on $T\mathbb{S}^2$. For a function $\chi \in C^\infty(U)$, we define $\varphi \in C^\infty(\mathbb{R}^4)$ by

$$\varphi = (\|\alpha\|^2 + 1)^{(m-1)/2}\chi \circ \Phi^{-1}.$$

These two functions are expressed through each other by the formulas

$$\chi(x, \xi) = \xi_3^{m-1}\varphi\left(x_1 - \frac{\xi_1}{\xi_3}x_3, x_2 - \frac{\xi_2}{\xi_3}x_3, \frac{\xi_1}{\xi_3}, \frac{\xi_2}{\xi_3}\right), \tag{1.7}$$

$$(\|\alpha\|^2 + 1)^{-(m-1)/2}\varphi(y, \alpha) = \chi\left(y_1 - \frac{\langle\alpha, y\rangle\alpha_1}{\|\alpha\|^2 + 1}, y_2 - \frac{\langle\alpha, y\rangle\alpha_2}{\|\alpha\|^2 + 1}, -\frac{\langle\alpha, y\rangle}{\|\alpha\|^2 + 1}, \right.$$
$$\left. \frac{\alpha_1}{\sqrt{\|\alpha\|^2 + 1}}, \frac{\alpha_2}{\sqrt{\|\alpha\|^2 + 1}}, \frac{1}{\sqrt{\|\alpha\|^2 + 1}}\right). \tag{1.8}$$

If, like in the statement of theorem 1.1, a function $\chi \in C^\infty(T\mathbb{S}^2)$ satisfies $\chi(x, -\xi) = (-1)^m\chi(x, \xi)$, then it is uniquely determined by the function

$$\varphi = (\|\alpha\|^2 + 1)^{(m-1)/2}\chi|_U \circ \Phi^{-1} \in C^\infty(\mathbb{R}^4).$$

For a tensor field $f \in \mathcal{S}(S^m\mathbb{R}^n)$, the function

$$\varphi = (\|\alpha\|^2 + 1)^{(m-1)/2}(If)|_U \circ \Phi^{-1} \in C^\infty(\mathbb{R}^4)$$

is expressed through f by the formula

$$\varphi(y,\alpha)=\sum_{i_1,\dots,i_m=1}^{3}\int_{-\infty}^{\infty}f_{i_1\dots i_m}(y_1+\alpha_1 t,\, y_2+\alpha_2 t,\, t)\alpha_{i_1}\dots\alpha_{i_m}\,\mathrm{d}t, \tag{1.9}$$

where $\alpha_3=1$. This easily follows from (1.1). Formula (1.9) has the obvious meaning: the family of non-horizontal lines in $\mathbb{R}^3$ is parameterized by the variable $(y,\alpha)\in\mathbb{R}^4$. Exactly this parametrization was used by John in [4].

We can now formulate the main result of the current paper.

Theorem 1.2. *In the case of* $n=3$, *a function* $\chi\in\mathcal{S}(T\mathbb{S}^2)$ *belongs to the range of operator (1.4) if and only if the following two conditions hold:*

(1) $\chi(x,-\xi)=(-1)^m\chi(x,\xi)$;
(2) being defined by (1.8), the function $\varphi\in C^\infty(\mathbb{R}^4)$ *solves the equation*

$$L^{m+1}\varphi=0, \tag{1.10}$$

where

$$L=\frac{\partial^2}{\partial y_1\partial\alpha_2}-\frac{\partial^2}{\partial y_2\partial\alpha_1}. \tag{1.11}$$

Theorem 1.2 follows from theorem 1.1 with the help of the following statement.

Proposition 1.3. *Let a function* $\chi\in C^\infty(T\mathbb{S}^2)$ *satisfy*

$$\chi(x,-\xi)=(-1)^m\chi(x,\xi) \tag{1.12}$$

for some non-negative integer m. *Define functions* $\psi\in C^\infty(\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\}))$ *and* $\varphi\in C^\infty(\mathbb{R}^4)$ *by formulas (1.3) and (1.8) respectively. The function* ψ *satisfies equations (1.5) for all* $1\leqslant i_1,j_1,\dots,i_{m+1},j_{m+1}\leqslant 3$ *if and only if the function* φ *solves equation (1.10).*

Observe that, formally speaking, proposition 1.3 does not relate to the x-ray transform. The proof of proposition 1.3 consists of changing independent variables in (1.5) and demonstrating that, after the change, each equation of system (1.5) becomes equivalent to (1.10). The proof is presented in the next section.

2. Proof of proposition 1.3

Given a function $\chi\in C^\infty(T\mathbb{S}^2)$ satisfying (1.12), define $\psi\in C^\infty(\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\}))$ and $\varphi\in C^\infty(\mathbb{R}^4)$ by formulas (1.3) and (1.8) respectively. Then the function ψ satisfies (1.2). It suffices to control the validity of equations (1.5) on the open set

$$\mathbb{R}^3\times\mathbb{R}^3_+=\{(x,\xi)\in\mathbb{R}^3\times\mathbb{R}^3|\xi_3>0\}\subset\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\}).$$

Indeed, if equations (1.5) hold for $\xi_3>0$, they also hold for $\xi_3<0$ in virtue of $\psi(x,-\xi)=(-1)^{m-1}\psi(x,\xi)$; and then, by continuity, equations (1.5) hold for all $\xi\neq 0$. Therefore we assume $\psi\in C^\infty(\mathbb{R}^3\times\mathbb{R}^3_+)$ in what follows.

Introduce the differential operators

$$J_1,\ J_2,\ J_3 : C^\infty(\mathbb{R}^3 \times \mathbb{R}^3_+) \to C^\infty(\mathbb{R}^3 \times \mathbb{R}^3_+)$$

by

$$J_1 = \frac{\partial^2}{\partial x_2 \partial \xi_3} - \frac{\partial^2}{\partial x_3 \partial \xi_2}, \quad J_2 = \frac{\partial^2}{\partial x_3 \partial \xi_1} - \frac{\partial^2}{\partial x_1 \partial \xi_3}, \quad J_3 = \frac{\partial^2}{\partial x_1 \partial \xi_2} - \frac{\partial^2}{\partial x_2 \partial \xi_1}. \tag{2.1}$$

Since factors on the left-hand side of (1.5) commute with each other, system (1.5) can be written as

$$J_1^r J_2^s J_3^t \psi = 0 \quad (r + s + t = m + 1). \tag{2.2}$$

Hereafter $(r,\ s,\ t)$ are non-negative integers.

Being defined by formulas (1.3) and (1.8) respectively, the functions $\psi \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3_+)$ and $\varphi \in C^\infty(\mathbb{R}^4)$ are related by

$$\psi(x,\ \xi) = \xi_3^{m-1} \varphi(y,\ \alpha), \tag{2.3}$$

where

$$y = (y_1,\ y_2) = \left(x_1 - \frac{\xi_1}{\xi_3} x_3,\ x_2 - \frac{\xi_2}{\xi_3} x_3\right), \quad \alpha = (\alpha_1,\ \alpha_2) = \left(\frac{\xi_1}{\xi_3},\ \frac{\xi_2}{\xi_3}\right). \tag{2.4}$$

In order to write (2.3) in a more invariant form, we introduce the smooth map

$$F : \mathbb{R}^3 \times \mathbb{R}^3_+ \to \mathbb{R}^4, \quad F : (x,\ \xi) \mapsto (y,\ \alpha) \tag{2.5}$$

by formulas (2.4) and let $F^* : C^\infty(\mathbb{R}^4) \to C^\infty(\mathbb{R}^3 \times \mathbb{R}^3_+)$ be the pull-back operator, i.e., $F^* u = u \circ F$ for $u \in C^\infty(\mathbb{R}^4)$. Then (2.3) is written as

$$\psi = \xi_3^{m-1} F^* \varphi. \tag{2.6}$$

Lemma 2.1. *On using the notation* $\mathbb{R}^3 \times \mathbb{R}^3_+ = \{(x_1,\ x_2,\ x_3,\ \xi_1,\ \xi_2,\ \xi_3) | \xi_3 > 0\}$ *and* $\mathbb{R}^4 = \{(y_1,\ y_2,\ \alpha_1,\ \alpha_2)\}$, *the identities*

$$J_1^r J_2^s J_3^t (\xi_3^{m-1} F^* \varphi) = \frac{1}{\xi_3^2} F^*(\alpha_1^r \alpha_2^s L^{m+1} \varphi) \quad (r + s + t = m + 1) \tag{2.7}$$

hold for every function $\varphi \in C^\infty(\mathbb{R}^4)$, *where the differential operator* $L : C^\infty(\mathbb{R}^4) \to C^\infty(\mathbb{R}^4)$ *is defined by (1.11).*

Proposition 1.3 follows from lemma 2.1. Indeed, (2.6) and (2.7) imply

$$J_1^r J_2^s J_3^t\ \psi = \frac{1}{\xi_3^2} F^*(\alpha_1^r \alpha_2^s L^{m+1} \varphi) \quad (r + s + t = m + 1).$$

Since F^* is an injective operator, each equation of system (2.2) is equivalent to $L^{m+1}\varphi = 0$.

Before proving lemma 2.1, we study some properties of the operators $J_1,\ J_2,\ J_3$. First of all, they commute with each other as any differential operators with constant coefficients. The operators J_1 and $-J_2$ are obtained from each other by the transposition $x_1 \leftrightarrow x_2$, this symmetry will be used in what follows. Since J_3 is independent of ξ_3, it commutes with the operator of multiplication by an arbitrary smooth function $g(\xi_3)$, i.e.

$$J_3 g(\xi_3) = g(\xi_3) J_3. \tag{2.8}$$

For J_1 and J_2, the corresponding commutator formulas look as follows:

$$J_1 g(\xi_3) = g(\xi_3) J_1 + g'(\xi_3)\frac{\partial}{\partial x_2}, \quad J_2 g(\xi_3) = g(\xi_3) J_2 - g'(\xi_3)\frac{\partial}{\partial x_1}. \tag{2.9}$$

From this, one obtains by induction the commutator formulas for powers of these operators

$$J_1^r g(\xi_3) = \sum_{\rho=0}^{r}\binom{r}{\rho} g^{(\rho)}(\xi_3) J_1^{r-\rho}\frac{\partial^\rho}{\partial x_2^\rho}, \quad J_2^s g(\xi_3) = \sum_{\sigma=0}^{s}(-1)^\sigma\binom{s}{\sigma} g^{(\sigma)}(\xi_3) J_2^{s-\sigma}\frac{\partial^\sigma}{\partial x_1^\sigma}. \tag{2.10}$$

The Jacobi matrix of map (2.4) can be easily computed. We write the result in the operator form

$$\frac{\partial}{\partial x_1}F^* = F^*\frac{\partial}{\partial y_1}, \quad \frac{\partial}{\partial x_2}F^* = F^*\frac{\partial}{\partial y_2}, \quad \frac{\partial}{\partial x_3}F^* = -F^*\left(\alpha_1\frac{\partial}{\partial y_1} + \alpha_2\frac{\partial}{\partial y_2}\right),$$
$$\frac{\partial}{\partial \xi_1}F^* = -\frac{x_3}{\xi_3}F^*\frac{\partial}{\partial y_1} + \frac{1}{\xi_3}F^*\frac{\partial}{\partial \alpha_1}, \quad \frac{\partial}{\partial \xi_2}F^* = -\frac{x_3}{\xi_3}F^*\frac{\partial}{\partial y_2} + \frac{1}{\xi_3}F^*\frac{\partial}{\partial \alpha_2},$$
$$\frac{\partial}{\partial \xi_3}F^* = \frac{x_3}{\xi_3}F^*\left(\alpha_1\frac{\partial}{\partial y_1} + \alpha_2\frac{\partial}{\partial y_2}\right) - \frac{1}{\xi_3}F^*\left(\alpha_1\frac{\partial}{\partial \alpha_1} + \alpha_2\frac{\partial}{\partial \alpha_2}\right). \tag{2.11}$$

On using these formulas and definitions (1.11) and (2.1), we derive the important formulas relating operators J_1, J_2, J_3 to L

$$J_1 F^* = \frac{1}{\xi_3}F^*\left(\alpha_1 L + \frac{\partial}{\partial y_2}\right), \quad J_2 F^* = \frac{1}{\xi_3}F^*\left(\alpha_2 L - \frac{\partial}{\partial y_1}\right), \quad J_3 F^* = \frac{1}{\xi_3}F^* L. \tag{2.12}$$

Finally, we will need a commutator formula for the operator L. As follows immediately from definition (1.11),

$$L\alpha_1 = \alpha_1 L - \frac{\partial}{\partial y_2}, \quad L\alpha_2 = \alpha_2 L + \frac{\partial}{\partial y_1}.$$

This implies with the help of induction in r and s

$$L\alpha_1^r\alpha_2^s = \alpha_1^r\alpha_2^s L + s\alpha_1^r\alpha_2^{s-1}\frac{\partial}{\partial y_1} - r\alpha_1^{r-1}\alpha_2^s\frac{\partial}{\partial y_2}. \tag{2.13}$$

Proof of lemma 2.1. In the case of $m = 0$, the lemma states that

$$J_1(\xi_3^{-1}F^*\varphi) = \frac{1}{\xi_3^2}F^*(\alpha_1 L\varphi), \quad J_2(\xi_3^{-1}F^*\varphi) = \frac{1}{\xi_3^2}F^*(\alpha_2 L\varphi),$$
$$J_3(\xi_3^{-1}F^*\varphi) = \frac{1}{\xi_3^2}F^*(L\varphi). \tag{2.14}$$

The last of these equalities is almost obvious. Indeed, operator J_3 commutes with the multiplication by ξ_3^{-1} in virtue of (2.8), i.e.

$$J_3(\xi_3^{-1}F^*\varphi) = \xi_3^{-1}(J_3F^*)\varphi.$$

By (2.12), $J_3F^* = \xi_3^{-1}F^*L$. Substituting this value into the previous formula, we obtain the last of equalities (2.14). First two equalities on (2.14) are very similar to each other because of the above-mentioned symmetry between J_1 and J_2. We present the proof of the first equality. First we permute J_1 and the multiplication by ξ_3^{-1} with the help of (2.9)

$$J_1(\xi_3^{-1}F^*\varphi) = (J_1\xi_3^{-1})F^*\varphi = \xi_3^{-1}(J_1F^*)\varphi - \xi_3^{-2}\left(\frac{\partial}{\partial x_2}F^*\right)\varphi.$$

Substituting the values $J_1F^* = \xi_3^{-1}F^*\left(\alpha_1 L + \frac{\partial}{\partial y_2}\right)$ from (2.12) and $\frac{\partial}{\partial x_2}F^* = F^*\frac{\partial}{\partial y_2}$ from (2.11), we arrive to the first of equalities (2.14). The lemma is thus proved in the case of $m = 0$.

We continue the proof by induction in m. On assuming (2.7) to be valid for some $m \geqslant 0$, we have to prove the equality

$$J_1^r J_2^s J_3^t(\xi_3^m F^*\varphi) = \frac{1}{\xi_3^2}F^*(\alpha_1^r\alpha_2^s L^{m+2}\varphi) \quad (r + s + t = m + 2). \tag{2.15}$$

Again, this equality is almost obvious and is proved similarly to the last equality of (2.14) in the case of $r + s = 0$. Therefore we assume $r + s > 0$. Moreover, we can assume $r > 0$ because of the above-mentioned symmetry between J_1 and J_2. We start with the identity

$$J_1^r J_2^s J_3^t(\xi_3^m F^*\varphi) = J_1^r J_2^s J_3^t(\xi_3\xi_3^{m-1}F^*\varphi) = J_1^r J_2^s(J_3^t\xi_3)\xi_3^{m-1}F^*\varphi$$

and try to move the factor ξ_3 to the first position. Since J_3 commutes with the multiplication by ξ_3, the formula can be written as

$$J_1^r J_2^s J_3^t(\xi_3^m F^*\varphi) = J_1^r(J_2^s\xi_3)J_3^t\xi_3^{m-1}F^*\varphi. \tag{2.16}$$

By (2.10)

$$J_2^s\xi_3 = \xi_3 J_2^s - sJ_2^{s-1}\frac{\partial}{\partial x_1}.$$

Substitute this value into (2.16)

$$J_1^r J_2^s J_3^t(\xi_3^m F^*\varphi) = (J_1^r\xi_3)J_2^s J_3^t\xi_3^{m-1}F^*\varphi - sJ_1^r J_2^{s-1}\frac{\partial}{\partial x_1}J_3^t\xi_3^{m-1}F^*\varphi. \tag{2.17}$$

Quite similarly, on using the commutator formula

$$J_1^r\xi_3 = \xi_3 J_1^r + rJ_1^{r-1}\frac{\partial}{\partial x_2}$$

we transform (2.17) to the form

$$\begin{aligned} J_1^r J_2^s J_3^t(\xi_3^m F^*\varphi) &= \xi_3 J_1^r J_2^s J_3^t\xi_3^{m-1}F^*\varphi + rJ_1^{r-1}\frac{\partial}{\partial x_2}J_2^s J_3^t\xi_3^{m-1}F^*\varphi \\ &\quad - sJ_1^r J_2^{s-1}\frac{\partial}{\partial x_1}J_3^t\xi_3^{m-1}F^*\varphi. \end{aligned} \tag{2.18}$$

Of course, the partial derivative $\frac{\partial}{\partial x_i}$ commutes with J_j. Therefore (2.18) can be written as (recall that $r > 0$)

$$J_1^r J_2^s J_3^t (\xi_3^m F^*\varphi) = \xi_3 J_1 (J_1^{r-1} J_2^s J_3^t \xi_3^{m-1} F^*\varphi) + r\frac{\partial}{\partial x_2}(J_1^{r-1} J_2^s J_3^t \xi_3^{m-1} F^*\varphi) - s\frac{\partial}{\partial x_1}(J_1^r J_2^{s-1} J_3^t \xi_3^{m-1} F^*\varphi). \tag{2.19}$$

By the induction hypothesis

$$J_1^{r-1} J_2^s J_3^t \xi_3^{m-1} F^*\varphi = \frac{1}{\xi_3^2} F^*(\alpha_1^{r-1}\alpha_2^s L^{m+1}\varphi), \quad J_1^r J_2^{s-1} J_3^t \xi_3^{m-1} F^*\varphi = \frac{1}{\xi_3^2} F^*(\alpha_1^r \alpha_2^{s-1} L^{m+1}\varphi)$$

Substitute these values into (2.19) and use the permutability of partial derivatives $\frac{\partial}{\partial x_i}$ with the multiplication by ξ_3^{-2} to get

$$J_1^r J_2^s J_3^t (\xi_3^m F^*\varphi) = \xi_3 (J_1 \xi_3^{-2}) F^*(\alpha_1^{r-1}\alpha_2^s L^{m+1}\varphi) + \frac{r}{\xi_3^2}\frac{\partial}{\partial x_2} F^*(\alpha_1^{r-1}\alpha_2^s L^{m+1}\varphi) - \frac{s}{\xi_3^2}\frac{\partial}{\partial x_1} F^*(\alpha_1^r \alpha_2^{s-1} L^{m+1}\varphi). \tag{2.20}$$

In the first term on the right-hand side, we permute J_1 and ξ_3^{-2} with the help of the commutator formula

$$J_1 \xi_3^{-2} = \xi_3^{-2} J_1 - 2\xi_3^{-3}\frac{\partial}{\partial x_2}$$

that follows from (2.9). In this way (2.20) takes the form

$$J_1^r J_2^s J_3^t (\xi_3^m F^*\varphi) = \frac{1}{\xi_3}(J_1 F^*)(\alpha_1^{r-1}\alpha_2^s L^{m+1}\varphi) + \frac{r-2}{\xi_3^2}\frac{\partial}{\partial x_2} F^*(\alpha_1^{r-1}\alpha_2^s L^{m+1}\varphi) - \frac{s}{\xi_3^2}\frac{\partial}{\partial x_1} F^*(\alpha_1^r \alpha_2^{s-1} L^{m+1}\varphi).$$

Substituting the values $\frac{\partial}{\partial x_i} F^* = F^* \frac{\partial}{\partial y_i}$ from (2.11), we obtain

$$J_1^r J_2^s J_3^t (\xi_3^m F^*\varphi) = \frac{1}{\xi_3}(J_1 F^*)(\alpha_1^{r-1}\alpha_2^s L^{m+1}\varphi) + \frac{r-2}{\xi_3^2} F^*\frac{\partial}{\partial y_2}(\alpha_1^{r-1}\alpha_2^s L^{m+1}\varphi) - \frac{s}{\xi_3^2} F^*\frac{\partial}{\partial y_1}(\alpha_1^r \alpha_2^{s-1} L^{m+1}\varphi).$$

Then we insert value (2.12) for $J_1 F^*$

$$J_1^r J_2^s J_3^t (\xi_3^m F^*\varphi) = \frac{1}{\xi_3^2} F^*\left(\alpha_1 L + \frac{\partial}{\partial y_2}\right)(\alpha_1^{r-1}\alpha_2^s L^{m+1}\varphi) + \frac{r-2}{\xi_3^2} F^*\frac{\partial}{\partial y_2}(\alpha_1^{r-1}\alpha_2^s L^{m+1}\varphi) - \frac{s}{\xi_3^2} F^*\frac{\partial}{\partial y_1}(\alpha_1^r \alpha_2^{s-1} L^{m+1}\varphi).$$

After opening big parenthesis and grouping similar terms, this takes the form

$$J_1^r J_2^s J_3^t (\xi_3^m F^*\varphi) = \frac{1}{\xi_3^2} F^*(\alpha_1 (L\alpha_1^{r-1}\alpha_2^s) L^{m+1}\varphi) + \frac{r-1}{\xi_3^2} F^*\frac{\partial}{\partial y_2}(\alpha_1^{r-1}\alpha_2^s L^{m+1}\varphi) - \frac{s}{\xi_3^2} F^*\frac{\partial}{\partial y_1}(\alpha_1^r \alpha_2^{s-1} L^{m+1}\varphi). \tag{2.21}$$

By (2.13)

$$L\alpha_1^{r-1}\alpha_2^s = \alpha_1^{r-1}\alpha_2^s L + s\alpha_1^{r-1}\alpha_2^{s-1}\frac{\partial}{\partial y_1} - (r-1)\alpha_1^{r-2}\alpha_2^s\frac{\partial}{\partial y_2}.$$

Substituting this value into (2.21) and grouping similar terms, we arrive to (2.15). This finishes the induction step. □

3. Higher dimensions

Here, we discuss a generalization of theorem 1.2 to the higher-dimensional case. The manifold of all non-horizontal lines in $\mathbb{R}^n$ is parameterized similarly to the parametrization in section 1. Formulas (1.8) and (1.9) have the obvious generalizations:

$$(\|\alpha\|^2+1)^{-(m-1)/2}\varphi(y,\alpha) = \chi\Bigg(y_1 - \frac{\langle\alpha, y\rangle\alpha_1}{\|\alpha\|^2+1}, \dots, y_{n-1} - \frac{\langle\alpha, y\rangle\alpha_{n-1}}{\|\alpha\|^2+1}, \frac{-\langle\alpha, y\rangle}{\|\alpha\|^2+1}, \frac{\alpha_1}{\sqrt{\|\alpha\|^2+1}}, \dots, \frac{\alpha_{n-1}}{\sqrt{\|\alpha\|^2+1}}, \frac{1}{\sqrt{\|\alpha\|^2+1}}\Bigg), \tag{3.1}$$

$$\varphi(y,\alpha) = \sum_{i_1,\dots,i_m=1}^{n}\int_{-\infty}^{\infty} f_{i_1\dots i_m}(y_1+\alpha_1 t, \dots, y_{n-1}+\alpha_{n-1}t, t)\alpha_{i_1}\dots\alpha_{i_m}\,dt, \tag{3.2}$$

where $(y, \alpha) \in \mathbb{R}^{n-1}\times\mathbb{R}^{n-1} = \mathbb{R}^{2(n-1)}$ and $\alpha_n = 1$.

Theorem 3.1. *A function $\chi \in \mathcal{S}(T\mathbb{S}^n)$ $(n \geqslant 3)$ belongs to the range of operator (1.4) if and only if the following two conditions hold:*

(1) $\chi(x, -\xi) = (-1)^m\chi(x, \xi)$;
(2) being defined by (3.1), the function $\varphi \in C^\infty(\mathbb{R}^{2(n-1)})$ satisfies the equations

$$\left(\frac{\partial^2}{\partial y_{p_1}\partial\alpha_{q_1}} - \frac{\partial^2}{\partial y_{q_1}\partial\alpha_{p_1}}\right)\dots\left(\frac{\partial^2}{\partial y_{p_{m+1}}\partial\alpha_{q_{m+1}}} - \frac{\partial^2}{\partial y_{q_{m+1}}\partial\alpha_{p_{m+1}}}\right)\varphi = 0 \tag{3.3}$$

that are written for all indices $1 \leqslant p_1, q_1, \dots, p_{m+1}, q_{m+1} \leqslant n-1$.

System (3.3) has $\binom{N'+m}{m+1}$ linearly independent equations, where $N' = (n-1)(n-2)/2$. This is substantially less than $\binom{N+m}{m+1}$ with $N = n(n-1)/2$.

The following statement is the higher-dimensional generalization of proposition 1.3. It immediately implies theorem 3.1.

Proposition 3.2. *Given a function $\psi \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ $(n \geqslant 3)$ satisfying (1.2), let the function $\varphi \in C^\infty(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ be related to ψ by*

$$\varphi(y,\alpha) = \psi(y_1,\dots,y_{n-1}, 0, \alpha_1, \dots, \alpha_{n-1}, 1),$$

$$\psi(x, \xi) = \xi_n^{m-1}\mathrm{sgn}(\xi_n)\varphi\left(x_1 - \frac{\xi_1}{\xi_n}x_n, \dots, x_{n-1} - \frac{\xi_{n-1}}{\xi_n}x_n, \frac{\xi_1}{\xi_n}, \dots, \frac{\xi_{n-1}}{\xi_n}\right).$$

Then $\psi(x, \xi)$ satisfies equations (1.5) for all $1 \leqslant i_1, j_1, \dots, i_{m+1}, j_{m+1} \leqslant n$ if and only if $\varphi(y, \alpha)$ satisfies equations (3.3) for all $1 \leqslant p_1, q_1, \dots, p_{m+1}, q_{m+1} \leqslant n - 1$.

Similarly to (2.1), we introduce the operators

$$J_p = \frac{\partial^2}{\partial x_p \partial \xi_n} - \frac{\partial^2}{\partial x_n \partial \xi_p} \quad (1 \leqslant p \leqslant n - 1),$$
$$J_{pq} = \frac{\partial^2}{\partial x_p \partial \xi_q} - \frac{\partial^2}{\partial x_q \partial \xi_p} \quad (1 \leqslant p, q \leqslant n - 1).$$

In this notation, (1.5) is written as the system of equations

$$J_1^{r_1} \dots J_{n-1}^{r_{n-1}} J_{p_1 q_1}^{s_1} \dots J_{p_k q_k}^{s_k} \psi = 0 \quad (1 \leqslant p_1, q_1, \dots, p_k, q_k \leqslant n - 1) \tag{3.4}$$

that should be valid for $r_1 + \dots + r_{n-1} + s_1 + \dots + s_k = m + 1$.

We also introduce the operators

$$L_{pq} = \frac{\partial^2}{\partial y_p \partial \alpha_q} - \frac{\partial^2}{\partial y_q \partial \alpha_p} \quad (1 \leqslant p, q \leqslant n - 1).$$

Then (3.3) becomes

$$L_{p_1 q_1} \dots L_{p_{m+1} q_{m+1}} \varphi = 0 \quad (1 \leqslant p_1, q_1, \dots, p_{m+1}, q_{m+1} \leqslant n - 1). \tag{3.5}$$

To formulate a generalization of lemma 2.1 for any $n \geqslant 3$, we introduce the operator $L^{(r)}$ for an $(n - 1)$-uple $r = (r_1, \dots, r_{n-1})$ of non-negative integers by the recursive formulas

$$L^{(0,\dots,0)} = \mathrm{Id}; \quad L^{(r+1_p)} = \sum_{q=1}^{n-1} \alpha_q L^{(r)} L_{qp} \quad (1 \leqslant p \leqslant n - 1), \tag{3.6}$$

where $1_p = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 on the pth position. In particular

$$L^{(1_p)} = \sum_{q=1}^{n-1} \alpha_q L_{qp}. \tag{3.7}$$

The definition is correct. Indeed, $L^{(r+1_p+1_q)}$ can be written in two ways:

$$L^{(r+1_p+1_q)} = \sum_{s=1}^{n-1} \alpha_s L^{(r+1_q)} L_{sp} = \sum_{s,t=1}^{n-1} \alpha_s \alpha_t L^{(r)} L_{tq} L_{sp},$$
$$L^{(r+1_p+1_q)} = \sum_{t=1}^{n-1} \alpha_t L^{(r+1_p)} L_{tq} = \sum_{s,t=1}^{n-1} \alpha_s \alpha_t L^{(r)} L_{sp} L_{tq}.$$

The right-hand sides coincide since operators L_{tq} and L_{sp} commute (as any differential operators with constant coefficients).

Definition (3.6) implies the important statement: Every operator $L^{(r)}$ can be represented as a homogeneous polynomial of degree $|r| = r_1 + \dots + r_{n-1}$ in the variables L_{pq} $(1 \leqslant p, q \leqslant n - 1)$ with coefficients polynomially depending on α.

As above in section 2, we introduce the smooth map

$$F:\mathbb{R}^n\times\mathbb{R}^n_+\to\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}=\mathbb{R}^{2n-2},\quad F:(x,\xi)\mapsto(y,\alpha)$$

by

$$y_p=x_p-\frac{\xi_p}{\xi_n}x_n,\quad \alpha_p=\frac{\xi_p}{\xi_n}\quad(1\leqslant p\leqslant n-1).$$

The Jacobi matrix of F is easily computed similar to (2.11) which leads to the following analogues of (2.12):

$$\begin{aligned} J_pF^*&=\frac{1}{\xi_n}F^*\left(L^{(1_p)}+\frac{\partial}{\partial y_p}\right)(1\leqslant p\leqslant n-1),\\ J_{pq}F^*&=\frac{1}{\xi_n}F^*L_{pq}\ (1\leqslant p,q\leqslant n-1). \end{aligned}\tag{3.8}$$

The following statement serves as a base of the proof of proposition 3.2.

Lemma 3.3. *Let $k\geqslant 0$ and $n\geqslant 3$ be integers. Given two sequences $r=(r_1,\cdots,r_{n-1})$ and $(s_1,\cdots,s_k)$ of non-negative integers, set*

$$m=r_1+\ldots+r_{n-1}+s_1+\ldots+s_k-1.$$

For every function $\varphi\in C^\infty(\mathbb{R}^{2n-2})$, the identities

$$J_1^{r_1}\ldots J_{n-1}^{r_{n-1}}J_{p_1q_1}^{s_1}\ldots J_{p_kq_k}^{s_k}(\xi_n^{m-1}F^*\varphi)=\frac{1}{\xi_n^2}F^*(L^{(r)}L_{p_1q_1}^{s_1}\ldots L_{p_kq_k}^{s_k}\varphi)\tag{3.9}$$

hold for all indices $1\leqslant p_1,q_1,\ \ldots p_k,q_k\leqslant n-1$.

Let us demonstrate how proposition 3.2 follows from lemma 3.3. If a function $\psi=\xi_n^{m-1}F^*(\varphi)$ satisfies (3.4), then the validity of (3.5) follows from (3.9) with $r=(0,\ldots,0)$ (recall F^* is an injective operator). Conversely, if φ satisfies (3.5), then the right-hand side of (3.9) is equal to zero since $L^{(r)}L_{p_1q_1}^{s_1}\ldots L_{p_kq_k}^{s_k}$ is a homogeneous polynomial of degree $m+1$ in L_{pq}. Equating the left-hand side of (3.9) to zero, we obtain (3.4) for $\psi=\xi_n^{m-1}F^*(\varphi)$.

Before proving lemma 3.3, we derive some commutator formulas for operators participating in the lemma.

The commutator formula for J_p $(1\leqslant p\leqslant n-1)$ and the operator of multiplication by ξ_n appear as follows:

$$J_p\xi_n=\xi_nJ_p+\frac{\partial}{\partial x_p}.\tag{3.10}$$

For a sequence $r=(r_1,\ldots,r_{n-1})$ of non-negative integers, set $J^r=J_1^{r_1}\ldots J_{n-1}^{r_{n-1}}$. The commutator formula

$$J^r\xi_n=\xi_nJ^r+\sum_{p=1}^{n-1}r_pJ^{r-1_p}\frac{\partial}{\partial x_p}\tag{3.11}$$

is proved on the base of (3.10) by induction in $|r|=r_1+\cdots+r_{n-1}$. In (3.11) as well as in some formulas below, the following agreement is used: $r_pJ^{r-1_p}=0$ in the case of $r_p=0$ although the operator J^{r-1_p} is not defined in the latter case.

The commutator formula

$$L^{(1_p)}\alpha_q = \alpha_q L^{(1_p)} - \alpha_q \frac{\partial}{\partial y_p} + \delta_q^p \sum_{s=1}^{n-1} \alpha_s \frac{\partial}{\partial y_s}, \tag{3.12}$$

where δ_q^p is the Kronecker symbol, follows immediately from (3.7).

Lemma 3.4. *For a sequence $r = (r_1, \dots, r_{n-1})$, the product formula holds*

$$L^{(1_p)}L^{(r)} = L^{(r+1_p)} - \sum_{q=1}^{n-1} r_q L^{(r+1_p-1_q)} \frac{\partial}{\partial y_q} \quad (1 \leqslant p \leqslant n-1). \tag{3.13}$$

Proof. We argue by induction in $|r|$. There is nothing to prove if $|r| = 0$. Otherwise $r_t > 0$ for some t and $L^{(r)} = \sum_{q=1}^{n-1} \alpha_q L^{(r-1_t)} L_{qt}$. Therefore

$$L^{(1_p)}L^{(r)} = \sum_{q=1}^{n-1} L^{(1_p)} \alpha_q L^{(r-1_t)} L_{qt}.$$

This gives with the help of (3.12)

$$\begin{aligned} L^{(1_p)}L^{(r)} &= \sum_{q=1}^{n-1} \left(\alpha_q L^{(1_p)} - \alpha_q \frac{\partial}{\partial y_p} + \delta_q^p \sum_{s=1}^{n-1} \alpha_s \frac{\partial}{\partial y_s} \right) L^{(r-1_t)} L_{qt} \\ &= \sum_{q=1}^{n-1} \alpha_q L^{(1_p)} L^{(r-1_t)} L_{qt} + \sum_{q=1}^{n-1} \alpha_q L^{(r-1_t)} \left(\frac{\partial}{\partial y_q} L_{pt} - \frac{\partial}{\partial y_p} L_{qt} \right). \end{aligned} \tag{3.14}$$

On using the obvious relation

$$\frac{\partial}{\partial y_q} L_{pt} - \frac{\partial}{\partial y_p} L_{qt} = -L_{qp} \frac{\partial}{\partial y_t},$$

the second sum on the right-hand side of (3.14) is transformed as follows:

$$\sum_{q=1}^{n-1} \alpha_q L^{(r-1_t)} \left(\frac{\partial}{\partial y_q} L_{pt} - \frac{\partial}{\partial y_p} L_{qt} \right) = -\sum_{q=1}^{n-1} \alpha_q L^{(r-1_t)} L_{qp} \frac{\partial}{\partial y_t} = -L^{(r+1_p-1_t)} \frac{\partial}{\partial y_t}.$$

Substitute this value into (3.14)

$$L^{(1_p)}L^{(r)} = \sum_{q=1}^{n-1} \alpha_q L^{(1_p)} L^{(r-1_t)} L_{qt} - L^{(r+1_p-1_t)} \frac{\partial}{\partial y_t}. \tag{3.15}$$

By the induction hypothesis

$$L^{(1_p)}L^{(r-1_t)} = L^{(r+1_p-1_t)} - \sum_{s=1}^{n-1} (r_s - \delta_s^t) L^{(r+1_p-1_t-1_s)} \frac{\partial}{\partial y_s}.$$

Substitute this value into (3.15)

$$L^{(1_p)}L^{(r)} = \sum_{q=1}^{n-1}\alpha_q L^{(r+1_p-1_t)}L_{qt} - \sum_{s=1}^{n-1}(r_s - \delta_s^t)\sum_{q=1}^{n-1}\alpha_q L^{(r+1_p-1_t-1_s)}L_{qt}\frac{\partial}{\partial y_s} - L^{(r+1_p-1_t)}\frac{\partial}{\partial y_t}.$$

In view of (3.6) this can be written as

$$L^{(1_p)}L^{(r)} = L^{(r+1_p)} - \sum_{s=1}^{n-1}(r_s - \delta_s^t)L^{(r+1_p-1_s)}\frac{\partial}{\partial y_s} - L^{(r+1_p-1_t)}\frac{\partial}{\partial y_t}.$$

This coincides with (3.13).

Proof of lemma 3.3 Goes by induction in $|r|$. In the case of $r = 0$, (3.9) is proved by induction in m exactly as in the proof of lemma 2.1.

Since operators J_{pq} $(1 \leqslant p,\, q \leqslant n-1)$ commute with the operator of multiplication by ξ_n, (3.9) can be written as

$$J_1^{r_1}\dots J_{n-1}^{r_{n-1}}(\xi_n J_{p_1q_1}^{s_1}\dots J_{p_kq_k}^{s_k}(\xi_n^{m-2}F^*\varphi)) = \frac{1}{\xi_n^2}F^*(L^{(r)}L_{p_1q_1}^{s_1}\dots L_{p_kq_k}^{s_k}\varphi). \tag{3.16}$$

The sequence $(s_1,\dots,s_k)$ and indices $p_1,\, q_1,\dots,p_k,\, q_k$ are fixed in the proof. Introducing the functions

$$\Psi = J_{p_1q_1}^{s_1}\dots J_{p_kq_k}^{s_k}(\xi_n^{m-2}F^*\varphi), \qquad \Phi = L_{p_1q_1}^{s_1}\dots L_{p_kq_k}^{s_k}\varphi,$$

we write (3.16) in the short form

$$J^r(\xi_n\Psi) = \frac{1}{\xi_n^2}F^*(L^{(r)}\Phi), \tag{3.17}$$

where $J^r = J_1^{r_1}\dots J_{n-1}^{r_{n-1}}$. We have to prove (3.17) for $|r| > 0$ under the induction hypothesis

$$J^{r'}\Psi = \frac{1}{\xi_n^2}F^*(L^{(r')}\Phi) \quad \text{for} \quad |r'| = |r| - 1. \tag{3.18}$$

With the help of (3.11), equation (3.17) takes the form

$$\xi_n J^r\Psi + \sum_{q=1}^{n-1}r_q J^{r-1_q}\frac{\partial\Psi}{\partial x_q} = \frac{1}{\xi_n^2}F^*(L^{(r)}\Phi). \tag{3.19}$$

If $|r| > 0$, then $r_p > 0$ for some p. Therefore (3.19) can be written as

$$\xi_n J_p J^{r-1_p}\Psi + \sum_{q=1}^{n-1}r_q J^{r-1_q}\frac{\partial\Psi}{\partial x_q} = \frac{1}{\xi_n^2}F^*(L^{(r)}\Phi). \tag{3.20}$$

By induction hypothesis (3.18),

$$J^{r-1_p}\Psi = \frac{1}{\xi_n^2}F^*(L^{(r-1_p)}\Phi) \tag{3.21}$$

and

$$J^{r-1_q}\frac{\partial\Psi}{\partial x_q}=\frac{\partial}{\partial x_q}J^{r-1_q}\Psi=\frac{1}{\xi_n^2}\frac{\partial}{\partial x_q}F^*(L^{(r-1_q)}\Phi)=\frac{1}{\xi_n^2}F^*\left(L^{(r-1_q)}\frac{\partial\Phi}{\partial y_q}\right).\tag{3.22}$$

We have used $\frac{\partial}{\partial x_q}F^*=F^*\frac{\partial}{\partial y_q}$, the permutability of $\frac{\partial}{\partial x_q}$ with J^{r-1_q}, and permutability of $\frac{\partial}{\partial y_q}$ with $L^{(r-1_q)}$. Substitute values (3.21) and (3.22) into (3.20)

$$\xi_n(J_p\,\xi_n^{-2})F^*(L^{(r-1_p)}\Phi)+\frac{1}{\xi_n^2}\sum_{q=1}^{n-1}r_qF^*\left(L^{(r-1_q)}\frac{\partial\Phi}{\partial y_q}\right)=\frac{1}{\xi_n^2}F^*(L^{(r)}\Phi).\tag{3.23}$$

With the help of the commutator formula

$$J_p\,\xi_n^{-2}=\xi_n^{-2}J_p-2\xi_n^{-3}\frac{\partial}{\partial x_p}$$

that is an analogous of (2.9) and (3.23) takes the form

$$\frac{1}{\xi_n}J_pF^*(L^{(r-1_p)}\Phi)-\frac{2}{\xi_n^2}F^*\left(L^{(r-1_p)}\frac{\partial\Phi}{\partial y_p}\right)+\frac{1}{\xi_n^2}\sum_{q=1}^{n-1}r_qF^*\left(L^{(r-1_q)}\frac{\partial\Phi}{\partial y_q}\right)=\frac{1}{\xi_n^2}F^*(L^{(r)}\Phi).\tag{3.24}$$

The first term on the left-hand side of (3.24) can be transformed with the help of (3.8) as follows:

$$\frac{1}{\xi_n}J_pF^*(L^{(r-1_p)}\Phi)=\frac{1}{\xi_n^2}F^*(L^{(1_p)}L^{(r-1_p)}\Phi)+\frac{1}{\xi_n^2}F^*\left(L^{(r-1_p)}\frac{\partial\Phi}{\partial y_p}\right).$$

With the help of this, (3.24) takes the form

$$F^*(L^{(1_p)}L^{(r-1_p)}\Phi)+\sum_{q=1}^{n-1}(r_q-\delta_q^p)F^*\left(L^{(r-1_q)}\frac{\partial\Phi}{\partial y_q}\right)=F^*(L^{(r)}\Phi).\tag{3.25}$$

By lemma 3.4

$$L^{(1_p)}L^{(r-1_p)}\Phi=L^{(r)}\Phi-\sum_{q=1}^{n-1}(r_q-\delta_q^p)L^{(r-1_q)}\frac{\partial\Phi}{\partial y_q}.$$

Substituting this value into the first term on the left-hand side of (3.25), we arrive to the identity. This completes the induction step. □

Remark. The proof of lemma 3.3 can be considerably simplified if we use, instead of (3.13), the dual product formula

$$L^{(r)}L^{(1_p)}=L^{(r+1_p)}-|r|L^{(r)}\frac{\partial}{\partial y_p}.\tag{3.26}$$

This coincides with (3.13) in the case of $r=1_q$.

Indeed, on using the induction hypothesis and (3.26), we can write for $|r| = m + 1$

$$\begin{aligned}\xi_n^2 J^r J_p(\xi_n^m F^*(\varphi)) &= \xi_n^2 J^r \left(\xi_n^m J_p + m\xi_n^{m-1}\frac{\partial}{\partial x_p}\right)F^*(\varphi)\\ &= \xi_n^2 J^r \xi_n^{m-1}(\xi_n J_p F^*(\varphi)) + m\xi_n^2 J^r \xi_n^{m-1} F^*\left(\frac{\partial\varphi}{\partial y_p}\right)\\ &= \xi_n^2 J^r \xi_n^{m-1} F^*\left(\left(L^{(1_p)} + \frac{\partial}{\partial y_p}\right)\varphi\right) + mF^*\left(L^{(r)}\frac{\partial\varphi}{\partial y_p}\right)\\ &= F^*\left(L^{(r)}\left(L^{(1_p)} + \frac{\partial}{\partial y_p}\right)\varphi\right) + mF^*\left(L^{(r)}\frac{\partial\varphi}{\partial y_p}\right)\\ &= F^*(L^{(r+1_p)}\varphi).\end{aligned}$$

Unfortunately, the proof of (3.26) is rather cumbersome. Actually, we have proved formula (3.26) implicitly since it is essentially equivalent to (3.9).

References

[1] Gelfand I M, Graev M I and Vilenkin N Y 1966 *Generalized Functions. vol 5: Integral Geometry and Representation Theory* (New York: Academic)

[2] Denisjuk A 2006 Inversion of the x-ray transform for 3D symmetric tensor fields with sourses on a curve *Inverse Provlems* **22** 399–412

[3] Helgason S 1984 *Groups and Geometric Analysis* (New York: Academic)

[4] John F 1938 The ultrahyperbolic differential equation with four independent variables *Duke Math. J.* **4** 300–22

[5] Pantyukhina E Yu 1990 Description of the image of a ray transformation in the two-dimensional case *Methods for Solving Inverse Problems* vol 144 (Novosibirsk: Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat.) pp 80–9 (in Russian)

[6] Sharafutdinov V A 1994 *Integral Geometry of Tensor Fields* (Utrecht: VSP)