IOP PUBLISHING INVERSE PROBLEMS

Inverse Problems **24** (2008) 035010 (21pp) doi:10.1088/0266-5611/24/3/035010

The problem of polarization tomography: II

Vladimir Sharafutdinov[1]

Sobolev Institute of Mathematics, 4 Koptjug Avenue, Novosibirsk, 630090, Russia

E-mail: sharaf@math.nsc.ru

Received 5 February 2008, in final form 14 March 2008
Published 10 April 2008
Online at stacks.iop.org/IP/24/035010

Abstract

Let f be a matrix function on a bounded domain $D \subset \mathbb{R}^n$ furnished with a Riemannian metric. For a unit speed geodesic $\gamma : [0, l] \to D$ between boundary points, let $\Phi[f](\gamma) = U(l)$, where $U(t)$ is the solution to the Cauchy problem $DU/dt = (Q_{\dot\gamma(t)} f(\gamma(t)))U, U(0) = E, E$ being the unit matrix. Here Q_ξ is an orthogonal projection onto the space $\{h \in gl(\mathbb{C}^n) | h\xi = h^*\xi = 0, \operatorname{tr} h = 0\}$. We consider the inverse problem of recovering the function f from the data $\Phi[f]$ known on the manifold of all unit speed geodesics between boundary points. The problem arises in optical tomography of weakly anisotropic media. The local uniqueness theorem is proved: a C^1-small function f can be recovered from the data uniquely up to a natural obstruction.

1. Introduction

This paper is a continuation of [4] that is referred to as [NS]. The reference (NS.1.1) stands for formula (1.1) of [NS]. Here, we consider the problem posed at the end of section 2 of [NS]. We start with the physical motivation of the problem.

Let us consider propagation of time-harmonic electromagnetic waves of frequency ω in a medium with the zero conductivity, unit magnetic permeability and the dielectric permittivity tensor of the form

$$\varepsilon_{ij} = n^2\delta_{ij} + \frac{1}{k}\chi_{ij}, \tag{1.1}$$

where $k = \omega/c$ is the wave number, c being the light velocity. Here $n > 0$ is a function of a point $x \in \mathbb{R}^3$, and the tensor $\chi_{ij} = \chi_{ij}(x)$ determines a small anisotropy of the medium. The smallness is emphasized by the factor $1/k$. The tensor χ is assumed to be Hermitian, $\chi_{ij} = \bar\chi_{ji}$.

In the scope of the zero approximation of geometric optics, propagation of electromagnetic waves in such media is described as follows. Exactly as in the background isotropic medium,

[1] Partially supported by CRDF grant KAM1-2851-AL-07.

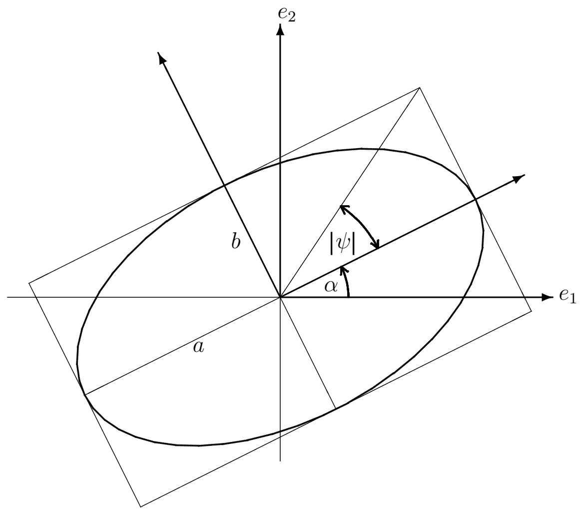

Figure 1. Polarization ellipse.

light rays are geodesics of the Riemannian metric

$$dt^2 = n^2(x)|dx|^2; \tag{1.2}$$

the electric vector $E(x)$ and magnetic vector $H(x)$ are orthogonal to each other as well as to the ray, and the polarization vector $\eta = n^{-1}|E|^{-1}E$ satisfies the equation (generalized Rytov's law)

$$\frac{D\eta}{dt} = \frac{i}{2n^2}\pi_{\dot\gamma}\chi\eta \tag{1.3}$$

along a geodesic ray $\gamma(t)$. Here $\pi_{\dot\gamma}$ is the orthogonal projection onto the plane $\dot\gamma^{\perp}$, and $D/dt = \dot\gamma^k\nabla_k$ is the covariant derivative along γ in metric (1.2).

For a fixed unit speed geodesic $\gamma(t)$, let $(e_1(t), e_2(t), e_3(t) = \dot\gamma(t))$ be an orthonormal basis parallel along γ in the sense of metric (1.2). Let $\eta(t) = \eta_1(t)e_1(t) + \eta_2(t)e_2(t)$ be the representation of the polarization vector in this basis, and χ_{ij} be the components of the tensor χ in this basis. Equation (1.3) is equivalent to the system

$$\begin{aligned} \frac{d\eta_1}{dt} &= \frac{i}{2n^2}(\chi_{11}\eta_1 + \chi_{12}\eta_2), \\ \frac{d\eta_2}{dt} &= \frac{i}{2n^2}(\chi_{21}\eta_1 + \chi_{22}\eta_2). \end{aligned} \tag{1.4}$$

The vectors η and E are complex. It is the real vector

$$\xi(t, T) = \mathrm{Re}[\eta(t)\,e^{i(kt-\omega T)}]$$

that has a physical meaning. We fix a point $t = t_0$ on the ray. With time T, the end of the vector

$$\xi(T) = \mathrm{Re}[(\eta_1 e_1 + \eta_2 e_2)\,e^{i(kt_0-\omega T)}]$$

runs an ellipse in the plane of vectors e_1, e_2; it is called the *polarization ellipse*. The shape and disposition of the ellipse are determined by the angles α and ψ shown in figure 1. The sign of ψ depends on whether the polarization is right or left. The angle α is not defined if $\psi = \pm\pi/4$ (the case of circular polarization).

Only the angles α and ψ are measured in practical polarimetry. Simple arguments presented in section 6.1 of [5] lead to the following conclusion: *the complex ratio* η_2/η_1 *of the components of the vector* η *is in one-to-one correspondence with the pair of the angles* (α, ψ) *that determine the shape and disposition of the polarization ellipse*. Note also that $|\eta_1|^2 + |\eta_2|^2 = \text{const.}$ on the ray since (1.4) is a system with a skew-Hermitian matrix.

Let us now consider the inverse problem. Assume a medium under investigation to be contained in a bounded domain $D \subset \mathbb{R}^3$ with a smooth boundary. The background isotropic medium is assumed to be known, i.e., metric (1.2) is given. The domain D is assumed to be convex with respect to the metric, i.e., for any two boundary points $x_0, x_1 \in \partial D$, there exists a unique unit speed geodesic $\gamma : [0, l] \to D$ such that $\gamma(0) = x_0, \gamma(l) = x_1$. We consider the inverse problem of determining the anisotropic part χ_{ij} of the dielectric permittivity tensor. The data for the inverse problem are the angles α and ψ that are measured for outcoming light along every unit speed geodesic $\gamma : [0, l] \to D$ with the endpoints on the boundary of D. We denote by $U(l)$ the fundamental matrix of system (1.4), i.e.,

$$\begin{pmatrix}\eta_1(l)\\ \eta_2(l)\\ 0\end{pmatrix} = \begin{pmatrix}u_{11} & u_{12} & 0\\ u_{21} & u_{22} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\eta_1(0)\\ \eta_2(0)\\ 0\end{pmatrix}, \qquad U(l) = \begin{pmatrix}u_{11} & u_{12} & 0\\ u_{21} & u_{22} & 0\\ 0 & 0 & 1\end{pmatrix}.$$

In [NS], studying the inverse problem, we assumed the matrix $U(l)$ to be completely known. Now, by the above conclusion, we assume that the ratio $\eta_2(l)/\eta_1(l)$ is known as a function of the ratio $\eta_2(0)/\eta_1(0)$, for all solutions to system (1.4). As one can easily see, this is equivalent to the fact that the matrix $U(l)$ is known up to a factor

$$\begin{pmatrix}e^{i\lambda} & 0 & 0\\ 0 & e^{i\lambda} & 0\\ 0 & 0 & 1\end{pmatrix} \tag{1.5}$$

with a real λ. If the tensor field χ/n^2 is sufficiently small, λ on (1.5) can be assumed to satisfy

$$|\lambda| < \pi/2 \tag{1.6}$$

since the fundamental matrix $U(l)$ is sufficiently close to the unit matrix. In other words, the results of the measurement do not change if a solution $(\eta_1(t), \eta_2(t))$ is multiplied by $e^{i\lambda(t)}$, where $\lambda(t)$ is a real function satisfying $\lambda(0) = 0$ and $|\lambda(t)| < \pi/2$.

Using the last observation, we change the variables in system (1.4) as follows:

$$f = \frac{i}{2n^2}\chi, \qquad \zeta = \exp\left[-\frac{i}{4n^2}\int_0^t (\chi_{11} + \chi_{22})\,dt\right]\eta.$$

Then the system is transformed to the following one:

$$\begin{aligned}\frac{d\zeta_1}{dt} &= \frac{1}{2}(f_{11} - f_{22})\zeta_1 + f_{12}\zeta_2,\\ \frac{d\zeta_2}{dt} &= f_{21}\zeta_1 + \frac{1}{2}(f_{22} - f_{11})\zeta_2.\end{aligned} \tag{1.7}$$

Let us observe that the structure of this system coincides with that of system (1.22) of [1].

As compared with (1.4), system (1.7) has the next advantage: the results of the measurements allow us to determine completely the fundamental matrix $U(l)$ of system (1.7). Indeed, note that the trace of the matrix of this system is equal to zero. Therefore the fundamental matrix of the system satisfies the condition

$$\det U(l) = 1. \tag{1.8}$$

Assume that, for every solution $\zeta(t)$ to system (1.7), the ratio $\zeta_2(l)/\zeta_1(l)$ is known as a function of $\zeta_2(0)/\zeta_1(0)$. As above, this allows us to determine the matrix $U(l)$ up to a factor (1.5). Under assumption (1.6), the factor is uniquely determined by condition (1.8).

Let $gl(\mathbb{C}^3)$ be the space of all linear operators on $\mathbb{C}^3$. Equations (1.7) are written in a basis $(e_1(t), e_2(t), e_3(t) = \dot\gamma(t))$ related to the ray γ. To find an invariant form of the equations, we note that the matrix of the system

$$Q_{\dot\gamma} f = \begin{pmatrix} \frac12(f_{11} - f_{22}) & f_{12} & 0 \\ f_{21} & \frac12(f_{22} - f_{11}) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

considered as the linear operator on $\mathbb{C}^3$ is the orthogonal projection of the tensor f onto the subspace

$$\{h \in gl(\mathbb{C}^3) | h\dot\gamma = h^*\dot\gamma = 0, \operatorname{tr} h = 0\}.$$

Thus system (1.7) takes the invariant form

$$\frac{D\zeta}{\mathrm{d}t} = (Q_{\dot\gamma} f)\zeta. \tag{1.9}$$

Instead of (1.9), we will consider the corresponding operator equation

$$\frac{DU}{\mathrm{d}t} = (Q_{\dot\gamma} f)U. \tag{1.10}$$

Equation (1.10) has a unique solution satisfying the initial condition $U(0) = E$, where E is the identity operator. Since $Q_{\dot\gamma} f$ is a trace-free skew-Hermitian operator satisfying $(Q_{\dot\gamma} f)\dot\gamma = 0$, the solution $U(t)$ belongs to $SU(3)$ and satisfies $U(t)\dot\gamma(t) = \dot\gamma(t)$. The final value of the solution

$$\Phi[f](\gamma) = U(l)$$

is the data for the inverse problem. Given the function $\Phi[f]$ on the set of unit speed geodesics between boundary points, we have to determine the tensor field $f = (f_{ij}(x))$ on the domain D.

We consider the inverse problem in a more general setting. Instead of a domain $D \subset \mathbb{R}^3$ with metric (1.2), we will consider a convex non-trapping manifold (CNTM in brief, see the definition in [NS]) (M, g) of an arbitrary dimension $n \geqslant 3$ and an arbitrary complex tensor field $f = (f_{ij})$ on M. In such a setting, equation (1.10) makes sense along a geodesic γ.

Quite similarly to section 2 of [NS], the inverse problem can be equivalently posed as follows. Given a tensor field $f \in C^\infty(\tau_1^1 M)$ on a CNTM (M, g), let us consider the boundary value problem

$$HU = (Q_\xi f)U \quad \text{on} \quad \Omega M, \qquad U|_{\partial_-\Omega M} = E. \tag{1.11}$$

The inverse problem is now posed as follows: one has to recover the tensor field f given the trace

$$\Phi[f] = U|_{\partial_+\Omega M}$$

of the solution to (1.11). In the same way as in [NS], this nonlinear inverse problem is reduced to the linear problem of recovering f from the data

$$F[f] = u|_{\partial_+\Omega M},$$

where u is the solution to the boundary value problem

$$Hu = p(Q_\xi f)q, \qquad u|_{\partial_-\Omega M} = 0$$

with some weights p and q that are semibasic tensor fields. The linear problem is studied in sections 2 and 3. In section 4, we present our main result on the local uniqueness in the nonlinear inverse problem.

The problem under consideration is of some applied interest for photoelasticity [2, 6] and for other kinds of optical tomography [3]. As the authors of [2] insist, the nonlinear problem is important for photoelasticity in the case of testing solid objects with big point loads.

2. The linear problem in dimensions greater than 3

First we are going to correct some inaccuracy made in [NS]. Let us recall that a number $k(M,g)$ has been defined by formula (NS.3.6) for a CNTM (M,g). Unfortunately, the definition given in [NS] is wrong and must be replaced with the following one:

$$k(M,g) = \sup_{(x,\xi)\in\partial_-\Omega M} \int_0^{\tau_+(x,\xi)} tK(\gamma_{x,\xi}(t))\,\mathrm{d}t, \tag{2.1}$$

where $K(x)$ is the supremum of the absolute values of sectional curvatures at the point x over all two-dimensional subspaces of T_xM. This coincides with definition (5.2.8) of [5].

Let us recall that, for a point x of a Riemannian manifold (M,g) and a vector $0 \neq \xi \in T_xM$, we have introduced the linear operator $Q_\xi : gl\big(T_x^{\mathbb{C}}M\big) \to gl\big(T_x^{\mathbb{C}}M\big)$ as the orthogonal projection onto the subspace $\big\{f \in gl\big(T_x^{\mathbb{C}}M\big)\,\big|\, f\xi = f^*\xi = 0, \operatorname{tr} f = 0\big\}$. Obviously, $Q_\xi g = 0$, g being the metric tensor. If f is a scalar multiple of the metric tensor, i.e., $f = \lambda g$ with $\lambda \in C^\infty(M)$, equation (1.11) gives no information on f. Therefore we will consider the inverse problem on the subspace of $C^\infty\big(\tau_1^1 M\big)$ consisting of trace-free tensor fields, i.e., f will always be assumed to satisfy

$$\operatorname{tr} f = f_i^i = 0. \tag{2.2}$$

The quadratic form $\int_{\Omega_x M} |Q_\xi f|^2\,\mathrm{d}\omega_x(\xi)$ is positive definite on the space of second-rank tensors at x satisfying (2.2). Therefore the estimate

$$|f(x)|^2 \leqslant C \int_{\Omega_x M} |Q_\xi f(x)|^2\,\mathrm{d}\omega_x(\xi) \tag{2.3}$$

holds for a trace-free $f \in C^\infty\big(\tau_1^1 M\big)$ with constant C depending only on $n = \dim M$.

We start with studying the corresponding linear inverse problem.

Let (M,g) be a CNTM. Choose two semibasic tensor fields $p, q \in C^\infty\big(\beta_1^1 M; \Omega M\big)$ satisfying

$$p^*(x,\xi)\xi = \xi, \qquad q(x,\xi)\xi = \xi. \tag{2.4}$$

We can also assume these fields to satisfy

$$\det p = \det q = 1, \tag{2.5}$$

but this assumption has not been used so far. For a trace-free tensor field $f \in C^\infty\big(\tau_1^1 M\big)$, consider the boundary value problem on ΩM

$$Hu = p(Q_\xi f)q, \qquad u|_{\partial_-\Omega M} = 0. \tag{2.6}$$

The problem has a unique solution $u \in C\big(\beta_1^1 M; \Omega M\big)$ and, by virtue of (2.4), the solution satisfies

$$u(x,\xi)\xi = u^*(x,\xi)\xi = 0. \tag{2.7}$$

In this section, we consider the inverse problem of recovering the tensor field f from the data

$$F[f] = u|_{\partial_+\Omega M}. \tag{2.8}$$

In the case of a real symmetric f and unit weights, the problem was considered in chapter 6 of [5], see theorem 6.2.2 of [5]. Compared with [5], the main difficulty of our case relates to the trace $\mathrm{tr}\, u$ of the solution u to (2.6). Indeed, $\mathrm{tr}\, u = 0$ in the case of $p = q = E$, and this fact plays a crucial role in the proof of theorem 6.2.2 of [5]. Therefore we start with estimating $\mathrm{tr}\, u$.

The factors p and q on (2.6) are considered as weights. We will assume the weights to be close to the unit weight E in the following sense: the inequalities

$$|p - E| < \varepsilon, \qquad |q - E| < \varepsilon, \qquad |\overset{v}{\nabla} p| < \varepsilon, \qquad |\overset{v}{\nabla} q| < \varepsilon \tag{2.9}$$

hold uniformly on ΩM. The value of ε will be specified later.

Equation (2.6) is initially considered on ΩM. To get some freedom in treating the equation, we extend it to the manifold $T^0M = \{(x, \xi) \in TM | \xi \neq 0\}$ of nonzero vectors. The weights are assumed to be positively homogeneous of zero degree in ξ

$$p(x, t\xi) = p(x, \xi), \qquad q(x, t\xi) = q(x, \xi) \qquad \text{for} \quad t > 0.$$

Then the right-hand side of (2.6) is positively homogeneous in ξ of zero degree because f is independent of ξ. The solution u must be extended to T^0M as a homogeneous function of degree -1

$$u(x, t\xi) = t^{-1}u(x, \xi) \qquad \text{for} \quad t > 0$$

because the operator H increases the degree of homogeneity by 1.

Exactly as in [NS], the solution u to (2.6) is continuous on T^0M and C^∞ smooth on $T^0M \setminus T(\partial M)$.

We rewrite the boundary value problem (2.6) in the form

$$Hu = Q_\xi f + r \quad \text{on} \quad T^0M, \qquad u|_{\partial_+\Omega M} = 0, \tag{2.10}$$

where

$$r = (p - E)Q_\xi f + pQ_\xi f(q - E). \tag{2.11}$$

By (2.9), the remainder r satisfies

$$|r| \leqslant C\varepsilon|f|, \qquad |\overset{v}{\nabla} r| \leqslant \frac{C\varepsilon}{|\xi|}|f| \tag{2.12}$$

with constant C dependent only on $n = \dim M$. In what follows in this section, we denote different constants dependent only on n by the same letter C.

For sufficiently small ε, equation (2.10) and inequalities (2.12) imply the estimate

$$|f(x)|^2 \leqslant C \int_{\Omega_x M} |Hu|^2 \, \mathrm{d}\omega_x(\xi) \leqslant C \int_{\Omega_x M} |\overset{h}{\nabla} u|^2 \, \mathrm{d}\omega_x(\xi). \tag{2.13}$$

Indeed, from (2.3) and (2.10)

$$\begin{aligned} |f(x)|^2 &\leqslant C \int_{\Omega_x M} |Q_\xi f(x)|^2 \, \mathrm{d}\omega_x(\xi) = C \int_{\Omega_x M} |Hu - r|^2 \, \mathrm{d}\omega_x(\xi) \\ &\leqslant C \int_{\Omega_x M} |Hu|^2 \, \mathrm{d}\omega_x(\xi) + C \int_{\Omega_x M} |r|^2 \, \mathrm{d}\omega_x(\xi). \end{aligned}$$

Estimating the last integral on the right-hand side with the help of (2.12), we obtain

$$(1 - C\varepsilon^2)|f(x)|^2 \leqslant C \int_{\Omega_x M} |Hu|^2 \, \mathrm{d}\omega_x(\xi) \leqslant C \int_{\Omega_x M} |\overset{h}{\nabla} u|^2 \, \mathrm{d}\omega_x(\xi).$$

This implies (2.13) under the assumption $C\varepsilon^2 < 1/2$.

Lemma 2.1. *Let a CNTM* (M, g) *satisfy*

$$k(M, g) < \varepsilon < 1/8 \tag{2.14}$$

For every tensor field $f \in C^\infty\big(\tau_1^1 M\big)$ *and for every semibasic tensor field* $r \in C^\infty\big(\beta_1^1 M; T^0 M\big)$*, if the solution* $u \in C\big(\beta_1^1 M; T^0 M\big)$ *to the boundary value problem (2.10) is positively homogeneous of degree* -1 *in* ξ*, then the estimate*

$$\int_{\Omega M} |\stackrel{h}{\nabla}(tr\, u)|^2\, \mathrm{d}\Sigma \leqslant C \int_{\Omega M} |\stackrel{v}{\nabla} r|^2\, \mathrm{d}\Sigma + D' \|u|_{\partial_+\Omega M}\|^2_{H^1} \tag{2.15}$$

holds with constant C *dependent only on* $n = \dim M$ *and constant* D' *dependent on* (M, g) *but not on* f *and* r*.*

Let us emphasize that no estimate for the remainder r is assumed in the lemma. If the remainder satisfies (2.12), then (2.15) implies

$$\int_{\Omega M} |\stackrel{h}{\nabla}(\mathrm{tr}\, u)|^2\, \mathrm{d}\Sigma \leqslant C\varepsilon^2 \|f\|^2_{L^2} + D' \|u|_{\partial_+\Omega M}\|^2_{H^1}. \tag{2.16}$$

Before proving the lemma, let us give a remark. Estimates for $|\mathrm{tr}\, u|$ and $|\stackrel{v}{\nabla}(\mathrm{tr}\, u)|$ in terms of $|f|$ can be easily derived from (2.9)–(2.11). But the corresponding estimate for $|\stackrel{h}{\nabla}(\mathrm{tr}\, u)|$ will involve $|\nabla f|$. Such an estimate does not fit our approach since the norm $|\nabla f|$ does not participate in the Pestov identity. The more tricky estimate (2.16) involves $|f|$ but does not involve $|\nabla f|$.

Proof of lemma 2.1. Let us denote $\varphi = \mathrm{tr}\, u$. Since operators tr and H commute, equation (2.10) implies

$$H\varphi = \mathrm{tr}\, r. \tag{2.17}$$

We write the Pestov identity for the function φ

$$2\,\mathrm{Re}\langle \stackrel{h}{\nabla}\varphi, \stackrel{v}{\nabla} H\varphi\rangle = |\stackrel{h}{\nabla}\varphi|^2 + \stackrel{h}{\nabla}_i v^i + \stackrel{v}{\nabla}_i w^i - R_{ikjl}\xi^k\xi^l \stackrel{v}{\nabla}{}^i\varphi \cdot \stackrel{v}{\nabla}{}^j\bar\varphi, \tag{2.18}$$

where

$$v^i = \mathrm{Re}(\xi^i \stackrel{h}{\nabla}{}^j\varphi \cdot \stackrel{v}{\nabla}_j\bar\varphi - \xi^j \stackrel{v}{\nabla}{}^i\varphi \cdot \stackrel{h}{\nabla}_j\bar\varphi), \tag{2.19}$$

$$w^i = \mathrm{Re}(\xi^j \stackrel{h}{\nabla}{}^i\varphi \cdot \stackrel{h}{\nabla}_j\bar\varphi). \tag{2.20}$$

By (2.17), the left-hand side of the Pestov identity admits the estimate

$$2\,\mathrm{Re}\langle \stackrel{h}{\nabla}\varphi, \stackrel{v}{\nabla} H\varphi\rangle = 2\,\mathrm{Re}\langle \stackrel{h}{\nabla}\varphi, \stackrel{v}{\nabla}(\mathrm{tr}\, r)\rangle \leqslant \tfrac12 |\stackrel{h}{\nabla}\varphi|^2 + 2|\stackrel{v}{\nabla}(\mathrm{tr}\, r)|^2 \leqslant \tfrac12 |\stackrel{h}{\nabla}\varphi|^2 + C|\stackrel{v}{\nabla} r|^2.$$

Therefore, the Pestov identity implies

$$\tfrac12 |\stackrel{h}{\nabla}\varphi|^2 + \stackrel{v}{\nabla}_i w^i \leqslant C|\stackrel{v}{\nabla} r|^2 - \stackrel{h}{\nabla}_i v^i + R_{ikjl}\xi^k\xi^l \stackrel{v}{\nabla}{}^i\varphi \cdot \stackrel{v}{\nabla}{}^j\bar\varphi$$

for $|\xi| = 1$. We multiply the inequality by the volume form $\mathrm{d}\Sigma$, integrate over ΩM, and transform the integrals of divergent terms by Gauss–Ostrogradskii

$$\int_{\Omega M} \left[\frac12 |\stackrel{h}{\nabla}\varphi|^2 + (n-2)|H\varphi|^2\right] \mathrm{d}\Sigma \leqslant C \int_{\Omega M} |\stackrel{v}{\nabla} r|^2\, \mathrm{d}\Sigma - \int_{\partial\Omega M} \langle v, \nu\rangle\, \mathrm{d}\Sigma^{2n-2} + \int_{\Omega M} R_{ikjl}\xi^k\xi^l \stackrel{v}{\nabla}{}^i\varphi \cdot \stackrel{v}{\nabla}{}^j\bar\varphi\, \mathrm{d}\Sigma. \tag{2.21}$$

The integrand $\langle v, \nu\rangle$ of the boundary integral on (2.21) is equal to zero on $\partial_-\Omega M$ as is seen from (2.19) and boundary condition $u|_{\partial_-\Omega M} = 0$. On $\partial_+\Omega M$, the integrand is the value on φ of some quadratic first-order differential operator, as is shown at the end of section 4.6 of [5]. Therefore the boundary integral admits the estimate

$$\left|\int_{\partial\Omega M}\langle v, \nu\rangle\,\mathrm{d}\Sigma^{2n-2}\right| \leqslant D_1\|\varphi|_{\partial_+\Omega M}\|^2_{H^1} \leqslant D_2\|u|_{\partial_+\Omega M}\|^2_{H^1}$$

with some constant D_2 dependent on (M, g). Inequality (2.21) takes the form

$$\int_{\Omega M}\left[\frac{1}{2}|\overset{h}{\nabla}\varphi|^2 + (n-2)|H\varphi|^2\right]\mathrm{d}\Sigma \leqslant \int_{\Omega M} R_{ikjl}\xi^k\xi^l\overset{v}{\nabla}{}^i\varphi\cdot\overset{v}{\nabla}{}^j\bar{\varphi}\,\mathrm{d}\Sigma + C\int_{\Omega M}|\overset{v}{\nabla}r|^2\,\mathrm{d}\Sigma + D_2\|u|_{\partial_+\Omega M}\|^2_{H^1}. \tag{2.22}$$

The first integral on the right-hand side of (2.22) is estimated with the help of the Poincaré inequality (see section 4.5 of [5]) like in section 4.7 of [5]. Namely, the integrand admits the estimate

$$R_{ikjl}\xi^k\xi^l\overset{v}{\nabla}{}^i\varphi\cdot\overset{v}{\nabla}{}^j\bar{\varphi} \leqslant K^+(x,\xi)|\overset{v}{\nabla}\varphi(x,\xi)|^2, \tag{2.23}$$

where $K^+(x,\xi)$ is defined by formula (4.3.2) of [5]. The vector field $\overset{v}{\nabla}\varphi$ vanishes on $\partial_-\Omega M$, so the Poincaré inequality can be applied and gives

$$\int_{\Omega M} R_{ikjl}\xi^k\xi^l\overset{v}{\nabla}{}^i\varphi\cdot\overset{v}{\nabla}{}^j\bar{\varphi}\,\mathrm{d}\Sigma \leqslant k^+(M,g)\int_{\Omega M}|H\overset{v}{\nabla}\varphi|^2\,\mathrm{d}\Sigma,$$

where $k^+(M,g)$ is defined by (4.3.3) of [5]. This gives with the help of (2.14)

$$\int_{\Omega M} R_{ikjl}\xi^k\xi^l\overset{v}{\nabla}{}^i\varphi\cdot\overset{v}{\nabla}{}^j\bar{\varphi}\,\mathrm{d}\Sigma \leqslant \varepsilon\int_{\Omega M}|H\overset{v}{\nabla}\varphi|^2\,\mathrm{d}\Sigma. \tag{2.24}$$

We have thus to estimate $|H\overset{v}{\nabla}\varphi|$. To this end, we apply the operator $\overset{v}{\nabla}$ to equation (2.17) and use the commutator formula $\overset{v}{\nabla}H = H\overset{v}{\nabla} + \overset{h}{\nabla}$

$$H\overset{v}{\nabla}\varphi = -\overset{h}{\nabla}\varphi + \overset{v}{\nabla}(\operatorname{tr} r).$$

This implies the estimate

$$|H\overset{v}{\nabla}\varphi|^2 \leqslant 2|\overset{h}{\nabla}\varphi|^2 + C|\overset{v}{\nabla}r|^2$$

for $|\xi| = 1$. Combining this inequality with (2.24), we obtain

$$\int_{\Omega M} R_{ikjl}\xi^k\xi^l\overset{v}{\nabla}{}^i\varphi\cdot\overset{v}{\nabla}{}^j\bar{\varphi}\,\mathrm{d}\Sigma \leqslant 2\varepsilon\int_{\Omega M}|\overset{h}{\nabla}\varphi|^2\,\mathrm{d}\Sigma + C\varepsilon\int_{\Omega M}|\overset{v}{\nabla}r|^2\,\mathrm{d}\Sigma.$$

Estimating the first integral on the right-hand side of (2.22) with the help of the last inequality, we arrive at the final estimate

$$\left(\frac{1}{2}-2\varepsilon\right)\int_{\Omega M}|\overset{h}{\nabla}\varphi|^2\,\mathrm{d}\Sigma + (n-2)\int_{\Omega M}|H\varphi|^2\,\mathrm{d}\Sigma \leqslant C\int_{\Omega M}|\overset{v}{\nabla}r|^2\,\mathrm{d}\Sigma + D_2\|u|_{\partial_+\Omega M}\|^2_{H^1}.$$

This gives statement of the lemma assuming $\varepsilon < 1/8$. □

The following statement is an analog of formula (NS.3.17).

Lemma 2.2. *A trace-free tensor field $f \in C^\infty(\tau_1^1 M)$ is uniquely represented as*

$$f_{ij}(x) = (Q_\xi f)_{ij}(x,\xi) + \xi_j a_i(x,\xi) + \xi_i\bar{b}_j(x,\xi) + \xi_i\xi_j c(x,\xi) + d(x,\xi)g_{ij}(x) \tag{2.25}$$

with the semibasic covector fields a and b orthogonal to ξ

$$a_i\xi^i = b_i\xi^i = 0 \tag{2.26}$$

and scalar functions $c(x,\xi)$ and $d(x,\xi)$. The (vector versions of the) fields a and b are expressed through f by the formulae

$$a = \frac{1}{|\xi|^2}\pi_\xi f\xi, \qquad b = \frac{1}{|\xi|^2}\pi_\xi f^*\xi \tag{2.27}$$

and the functions c and d by

$$c = \frac{n}{n-1}\frac{\langle f\xi,\xi\rangle}{|\xi|^4}, \qquad d = -\frac{1}{n-1}\frac{\langle f\xi,\xi\rangle}{|\xi|^2}, \tag{2.28}$$

where $n = \dim M$.

Proof. Compare with lemma 6.2.1 of [5]. We first prove the uniqueness statement. Assume (2.25) and (2.26) to be valid. Take the contraction of (2.25) with ξ^j (multiply by ξ^j and take the sum over j). Taking (2.26) into account, we obtain

$$f\xi = |\xi|^2 a + (|\xi|^2 c + d)\xi, \qquad \langle a,\xi\rangle = 0.$$

This means that $a = \pi_\xi f\xi/|\xi|^2$ and

$$|\xi|^2 c + d = \frac{\langle f\xi,\xi\rangle}{|\xi|^2}. \tag{2.29}$$

In the same way we obtain $b = \pi_\xi f^*\xi/|\xi|^2$. On the other hand, applying the operator tr to equation (2.25), we see that

$$|\xi|^2 c + nd = 0. \tag{2.30}$$

Equations (2.29) and (2.30) imply (2.28). This proves the uniqueness statement.

The existence is proved by reverse arguments. Define a, b, c and d by (2.27)–(2.28) and then define $Q_\xi f$ by (2.25). Check that $Q_\xi f$ belongs to the subspace $A_\xi = \{h | h\xi = h^*\xi = 0, \operatorname{tr} h = 0\}$ and the difference $f - Q_\xi f$ belongs to $A_\xi^\perp$. □

The main result of the current section is the following

Theorem 2.3. *For any $n \geqslant 4$, there exists a positive number $\varepsilon = \varepsilon(n)$ such that, for any n-dimensional CNTM (M,g) satisfying (2.14) and for any weights $p, q \in C^\infty(\beta_1^1 M; \Omega M)$ satisfying (2.4) and (2.9), every trace-free tensor field $f \in C^\infty(\tau_1^1 M)$ can be uniquely recovered from the trace (2.8) of the solution to the boundary value problem (2.6) and the stability estimate*

$$\|f\|_{L^2} \leqslant D\|F[f]\|_{H^1} \tag{2.31}$$

holds with constant D dependent on (M,g) but not on f, p, q.

Proof. It follows approximately the same line as the proof of theorem 3.1 of [NS] by making use of lemma 2.1 at a crucial point. We start by writing the Pestov identity for u

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla} Hu\rangle = |\overset{h}{\nabla} u|^2 + \overset{h}{\nabla}_i v^i + \overset{v}{\nabla}_i w^i - \mathcal{R}_1[u], \tag{2.32}$$

where

$$v^i = \mathrm{Re}\big(\xi^i \overset{h}{\nabla}{}^j u^{i_1 i_2}\cdot \overset{v}{\nabla}_j \bar u_{i_1 i_2} - \xi^j \overset{v}{\nabla}{}^i u^{i_1 i_2}\cdot \overset{h}{\nabla}_j \bar u_{i_1 i_2}\big), \tag{2.33}$$

$$w^i = \mathrm{Re}\big(\xi^j \overset{h}{\nabla}{}^i u^{i_1 i_2}\cdot \overset{h}{\nabla}_j \bar u_{i_1 i_2}\big), \tag{2.34}$$

$$\mathcal{R}_1[u] = R_{kplq}\xi^p\xi^q \overset{v}{\nabla}{}^k u^{i_1i_2} \cdot \overset{v}{\nabla}{}^l \bar{u}_{i_1i_2} + \mathrm{Re}\big((R^{i_1}_{pqj}u^{pi_2} + R^{i_2}_{pqj}u^{i_1p})\xi^q \overset{v}{\nabla}{}^j \bar{u}_{i_1i_2}\big). \tag{2.35}$$

By (2.10), the left-hand side of the Pestov identity can be represented as

$$\langle \overset{h}{\nabla} u, \overset{v}{\nabla} Hu\rangle = \langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Q_\xi f)\rangle + \langle \overset{h}{\nabla} u, \overset{v}{\nabla} r\rangle. \tag{2.36}$$

We will first investigate the first term on the right-hand side of (2.36). By (2.25),

$$(Q_\xi f)_{ij} = f_{ij} - a_i\xi_j - \bar{b}_j\xi_i - c\xi_i\xi_j - dg_{ij}. \tag{2.37}$$

Differentiating the last equality with respect to ξ and using the fact that f is independent of ξ, we obtain

$$\overset{v}{\nabla}_k(Q_\xi f)_{ij} = -\xi_j \overset{v}{\nabla}_k a_i - \xi_i \overset{v}{\nabla}_k \bar{b}_j - \xi_i\xi_j \overset{v}{\nabla}_k c - g_{ij}\overset{v}{\nabla}_k d - g_{jk}a_i - g_{ik}\bar{b}_j - (g_{ik}\xi_j + g_{jk}\xi_i)c.$$

Therefore

$$\begin{aligned}\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Q_\xi f)\rangle &= \overset{h}{\nabla}{}^k u^{ij} \cdot \overset{v}{\nabla}_k(Q_\xi \bar{f})_{ij} \\ &= \overset{h}{\nabla}{}^k u^{ij}\big(-\xi_j \overset{v}{\nabla}_k \bar{a}_i - \xi_i \overset{v}{\nabla}_k b_j - \xi_i\xi_j \overset{v}{\nabla}_k \bar{c} \\ &\quad - g_{ij}\overset{v}{\nabla}_k \bar{d} - g_{jk}\bar{a}_i - g_{ik}b_j - (g_{ik}\xi_j + g_{jk}\xi_i)\bar{c}\big).\end{aligned}$$

The tensor $\overset{h}{\nabla}{}^k u^{ij}$ is orthogonal to ξ in the indices i and j as follows from (2.7). Therefore the last formula is simplified to the following one:

$$\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Q_\xi f)\rangle = -\overset{h}{\nabla}{}^p u_{ip} \cdot \bar{a}^i - \overset{h}{\nabla}{}^p u_{pi} \cdot b^i - \overset{h}{\nabla}{}^k(g_{ij}u^{ij}) \cdot \overset{v}{\nabla}_k \bar{d}.$$

Introducing the semibasic covector fields $\overset{h}{\delta}_1 u$ and $\overset{h}{\delta}_2 u$ by the equalities

$$(\overset{h}{\delta}_1 u)_i = \overset{h}{\nabla}{}^p u_{ip}, \qquad (\overset{h}{\delta}_2 u)_i = \overset{h}{\nabla}{}^p u_{pi}, \tag{2.38}$$

we write the result in the form

$$\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Q_\xi f)\rangle = -\langle \overset{h}{\delta}_1 u, a\rangle - \langle \overset{h}{\delta}_2 u, \bar{b}\rangle - \langle \overset{h}{\nabla}(\mathrm{tr}\, u), \overset{v}{\nabla} d\rangle. \tag{2.39}$$

This implies the estimate

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Q_\xi f)\rangle \leqslant \frac{\beta}{2}(|\overset{h}{\delta}_1 u|^2 + |\overset{h}{\delta}_2 u|^2) + \frac{2}{\beta}(|a|^2 + |b|^2) + \frac{1}{\varepsilon}|\overset{h}{\nabla}(\mathrm{tr}\, u)|^2 + \varepsilon|\overset{v}{\nabla} d|^2, \tag{2.40}$$

where β is an arbitrary positive number.

The first term on the right-hand side of (2.40) is estimated exactly as in [NS], we just reproduce (NS.3.35)

$$|\overset{h}{\delta}_1 u|^2 + |\overset{h}{\delta}_2 u|^2 \leqslant 2|z|^2 + \overset{h}{\nabla}_i \tilde{v}^i + \mathcal{R}_4[u], \tag{2.41}$$

where terms on the right-hand side are defined in [NS].

Let us estimate the last term on the right-hand side of (2.40). Differentiating the second of equalities (2.28) with respect to ξ and taking the independence f of ξ into account, we obtain

$$\overset{v}{\nabla} d = -\frac{1}{n-1}\left(\frac{1}{|\xi|^2} f\xi + \frac{1}{|\xi|^2}\bar{f}^*\xi - \frac{2\langle f\xi, \xi\rangle}{|\xi|^4}\xi\right).$$

This implies the estimate

$$|\overset{v}{\nabla} d|^2 \leqslant \frac{9}{(n-1)^2}\frac{|f|^2}{|\xi|^2}. \tag{2.42}$$

We substitute values (2.27) for a and b into (2.40) and then use (2.41) and (2.42) to obtain the following analog of (NS.3.36)

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Q_\xi f)\rangle \leqslant \beta|z|^2 + \frac{2}{\beta|\xi|^4}(|\pi_\xi f\xi|^2 + |\pi_\xi f^*\xi|^2) + \frac{9\varepsilon}{(n-1)^2|\xi|^2}|f|^2 + \frac{1}{\varepsilon}|\overset{h}{\nabla}(\mathrm{tr}\,u)|^2 + \frac{\beta}{2}\overset{h}{\nabla}_i \tilde{v}^i + \frac{\beta}{2}\mathcal{R}_4[u]. \tag{2.43}$$

The second term on the right-hand side of (2.36) is estimated as in [NS], we just reproduce (NS.3.37)

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla} r\rangle \leqslant C\varepsilon\left(|\overset{h}{\nabla} u|^2 + \frac{1}{|\xi|^2}|f|^2\right). \tag{2.44}$$

Let us recall that we denote different constants dependent only on $n = \dim M$ by the same letter C. Combining (2.43) and (2.44), we obtain from (2.36)

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla} Hu\rangle \leqslant \beta|z|^2 + \frac{2}{\beta|\xi|^4}(|\pi_\xi f\xi|^2 + |\pi_\xi f^*\xi|^2) + \frac{\beta}{2}\overset{h}{\nabla}_i \tilde{v}^i + C\varepsilon\left(|\overset{h}{\nabla} u|^2 + \frac{1}{|\xi|^2}|f|^2\right) + \frac{\beta}{2}\mathcal{R}_4[u] + \frac{1}{\varepsilon}|\overset{h}{\nabla}(\mathrm{tr}\,u)|^2. \tag{2.45}$$

Estimating the left-hand side of the Pestov identity (2.32) by (2.45), we obtain for $|\xi| = 1$

$$|\overset{h}{\nabla} u|^2 + \overset{v}{\nabla}_i w^i - \beta|z|^2 - \frac{2}{\beta}(|\pi_\xi f\xi|^2 + |\pi_\xi f^*\xi|^2) - C\varepsilon(|\overset{h}{\nabla} u|^2 + |f|^2) \leqslant \overset{h}{\nabla}_i\left(\frac{\beta}{2}\tilde{v}^i - v^i\right) + \mathcal{R}[u] + \frac{1}{\varepsilon}|\overset{h}{\nabla}(\mathrm{tr}\,u)|^2, \tag{2.46}$$

where $\mathcal{R}[u] = \mathcal{R}_1[u] + \frac{\beta}{2}\mathcal{R}_4[u]$. We multiply inequality (2.46) by the volume form $\mathrm{d}\Sigma$, integrate over ΩM, and transform the integrals of divergent terms by Gauss–Ostrogradskii formulae. In such a way we obtain the following analog of (NS.3.40):

$$\int_{\Omega M}\left[|\overset{h}{\nabla} u|^2 + (n-2)|Hu|^2 - \beta|z|^2 - \frac{2}{\beta}(|\pi_\xi f\xi|^2 + |\pi_\xi f^*\xi|^2) - C\varepsilon(|\overset{h}{\nabla} u|^2 + |f|^2)\right]\mathrm{d}\Sigma \leqslant \int_{\partial\Omega M}\left\langle \frac{\beta}{2}\tilde{v} - v, \nu\right\rangle \mathrm{d}\Sigma^{2n-2} + \int_{\Omega M}\mathcal{R}[u]\,\mathrm{d}\Sigma + \frac{1}{\varepsilon}\int_{\partial\Omega M}|\overset{h}{\nabla}(\mathrm{tr}\,u)|^2\,\mathrm{d}\Sigma. \tag{2.47}$$

The first integral on the right-hand side of (2.47) can be estimated as follows,

$$\left|\int_{\partial\Omega M}\left\langle \frac{\beta}{2}\tilde{v} - v, \nu\right\rangle \mathrm{d}\Sigma^{2n-2}\right| \leqslant D''\|u|_{\partial_+\Omega M}\|^2_{H^1} \tag{2.48}$$

with constant D'' in (2.48) depending on (M, g). Indeed, analyzing the integrand by the same arguments as that used for proving (5.5.14) of [5], we show that the integrand is the value on $u|_{\partial\Omega M}$ of some first-order quadratic differential operator.

The second integral on the right-hand side of (2.47) is estimated as follows:

$$\int_{\Omega M}\mathcal{R}[u]\,\mathrm{d}\Sigma \leqslant C\varepsilon\int_{\Omega M}|\overset{h}{\nabla} u|^2\,\mathrm{d}\Sigma. \tag{2.49}$$

In [NS], we have written the corresponding estimate (NS.3.42) with no proof just saying that the estimate can be proved exactly as in [5]. The letter statement is not quite right. So, let us discuss the proof of (2.49) in more detail.

Repeating arguments in the beginning of section 5.5 of [5], we prove the estimate

$$\int_{\Omega M}\mathcal{R}[u]\,\mathrm{d}\Sigma \leqslant Ck(M, g)\int_{\Omega M}(|Hu|^2 + |\overset{v}{\nabla} Hu|^2)\,\mathrm{d}\Sigma, \tag{2.50}$$

where $k(M,g) < \varepsilon$ is defined by (2.1). The inequality $|Hu|^2 \leqslant |\overset{h}{\nabla} u|^2$ for $|\xi| = 1$ holds since $Hu = \xi^i \overset{h}{\nabla}_i u$. To derive (2.49) from (2.50), we have thus to estimate $\int_{\Omega M} |\overset{v}{\nabla} Hu|^2\,\mathrm{d}\Sigma$ through $\int_{\Omega M} |\overset{h}{\nabla} u|^2\,\mathrm{d}\Sigma$. To this end we apply the operator $\overset{v}{\nabla}$ to equation (2.10)

$$\overset{v}{\nabla} Hu = \overset{v}{\nabla}(Q_\xi f) + \overset{v}{\nabla} r.$$

Therefore

$$|\overset{v}{\nabla} Hu|^2 \leqslant 2|\overset{v}{\nabla}(Q_\xi f)|^2 + 2|\overset{v}{\nabla} r|^2 \leqslant C|f|^2 + 2|\overset{v}{\nabla} r|^2. \tag{2.51}$$

The second inequality holds since f is independent of ξ. Estimating the second term with the help of (2.12), we obtain from (2.51)

$$|\overset{v}{\nabla} Hu|^2 \leqslant C|f|^2$$

Together with (2.13), this gives

$$\int_{\Omega M} |\overset{v}{\nabla} Hu|^2\,\mathrm{d}\Sigma \leqslant C \int_{\Omega M} |\overset{h}{\nabla} u|^2\,\mathrm{d}\Sigma.$$

We emphasize that (2.49) has been proved with the help of estimates (2.12). This remark will be important in the next section where the remainder r will be arbitrary.

Estimating integrals on the right-hand side of (2.47) by (2.48), (2.49) and (2.16) respectively, we obtain

$$\int_{\Omega M} \left[(1 - C\varepsilon)|\overset{h}{\nabla} u|^2 + (n-2)|Hu|^2 - \beta|z|^2 - \frac{2}{\beta}(|\pi_\xi f\xi|^2 + |\pi_\xi f^*\xi|^2) - C\varepsilon|f|^2\right]\mathrm{d}\Sigma$$
$$\leqslant C\varepsilon\|f\|^2_{L^2} + (D'' + D'/\varepsilon)\|u|_{\partial_+\Omega M}\|^2_{H^1}.$$

Using the relation $|\overset{h}{\nabla} u|^2 = |z|^2 + |Hu|^2$, the last inequality is transformed to the following analog of (NS.3.44)

$$\int_{\Omega M} \left[(1 - \beta - C\varepsilon)|z|^2 + (n - 1 - C\varepsilon)|Hu|^2 - \frac{2}{\beta}(|\pi_\xi f\xi|^2 + |\pi_\xi f^*\xi|^2) - C\varepsilon|f|^2\right]\mathrm{d}\Sigma$$
$$\leqslant D\|u|_{\partial_+\Omega M}\|^2_{H^1},$$

where

$$D = D'' + D'/\varepsilon. \tag{2.52}$$

Then, using the inequality $|Hu|^2 \geqslant |Q_\xi f|^2 - C\varepsilon|f|^2$ we transform our estimate to the final form

$$\int_M \int_{\Omega_x M} \Big[(1 - \beta - C\varepsilon)|z|^2 + (n - 1 - C\varepsilon)|Q_\xi f|^2$$
$$- \frac{2}{\beta}(|\pi_\xi f\xi|^2 + |\pi_\xi f^*\xi|^2) - C\varepsilon|f|^2\Big]\mathrm{d}\omega_x(\xi) dV^n(x) \leqslant D\|u|_{\partial_+\Omega M}\|^2_{H^1}. \tag{2.53}$$

Let us recall that β is an arbitrary number satisfying $0 < \beta \leqslant 1$.

Lemma 2.4. *For every Riemannian manifold (M, g) of dimension $n \geqslant 4$ and every point $x \in M$, the Hermitian form*

$$A(f, f) = \int_{\Omega_x M} [(n-1)|Q_\xi f|^2 - 2(|\pi_\xi f\xi|^2 + |\pi_\xi f^*\xi|^2)]\,\mathrm{d}\omega_x(\xi)$$

is positive definite on the space of trace-free second-rank tensors at x*. Moreover, the estimate*

$$A(f, f) \geqslant c|f|^2$$

holds with positive constant c *depending only on* n*. In the case of* $n = 3$*, the form* A *is identically equal to zero on the space of trace-free symmetric tensors at* x*.*

With the help of the lemma, the proof of theorem 2.3 is finished as in [NS]. As far as the proof of the lemma is concerned, we first observe that it suffices to prove it for a real tensor f. Again, as in [NS], symmetric and skew-symmetric tensors are orthogonal to each other with respect to the quadratic form A. Therefore, it suffices to prove the positiveness of A on the spaces of real symmetric and skew-symmetric tensors separately.

The positiveness of A on the space of real trace-free symmetric tensors in the case of $n \geqslant 4$ is proved in lemma 6.3.1 of [5], as well as $A = 0$ on real trace-free symmetric tensors in the case of $n = 3$. On skew-symmetric tensors, projections P_ξ and Q_ξ coincide and therefore the quadratic form A coincides with the form B of lemma NS.3.2. The form B is positive in the case of $n \geqslant 4$.

Remark. The dependence of the coefficient D on ε is a little bit strange. According to (2.52), D grows to infinity as ε approaches zero. On the other hand, formula (2.52) is replaced with $D = D''$ in the case of $\varepsilon = 0$. Indeed, $\varepsilon = 0$ means that weights p and q coincide with E, the remainder r is identically equal to zero, and $\operatorname{tr} u = 0$.

3. The linear problem in the 3D case

Since the projections P_ξ and Q_ξ coincide on skew-symmetric tensors, the same counterexamples as in section 4 of [NS] are valid in our case for both linear and nonlinear problems. Here, we will prove the uniqueness for the linear problem under the same closeness condition on f as in section 4 of [NS].

Theorem 3.1. *There exists* $\varepsilon > 0$ *such that, for any three-dimensional CNTM* (M, g) *satisfying (2.14) and for any weights* $p, q \in C^\infty(\beta_1^1 M; \Omega M)$ *satisfying (2.4) and (2.9), every closed trace-free tensor field* $f \in C^\infty(\tau_1^1 M)$ *can be uniquely recovered from the trace (2.8) of the solution to the boundary value problem (2.6) and the stability estimate*

$$\|f\|_{L^2}^2 \leqslant D\big(\|F[f]\|_{H^1}^2 + \|f|_{\partial M}\|_{L^2}\|F[f]\|_{L^2}^2\big) \tag{3.1}$$

holds with constant D *independent of* f, p, q*.*

While proving this theorem, we would like to separate the cases of a symmetric and skew-symmetric f. Such a separation was already used implicitly in the proof of theorem NS.4.2, see formula (NS.4.29). The complete separation is impossible since the weights p and q in equation (2.6) mix the symmetric and skew-symmetric parts of f. Therefore we will apply the separation to equation (2.10) with a remainder. In the separated equation, we cannot use estimates of the remainder like (2.44).

For a semibasic tensor field h, we will use the notation

$$\|h\|^2 = \int_{\Omega M} |h(x, \xi)|^2 \, \mathrm{d}\Sigma(x, \xi).$$

Actually, this formula defines a norm on the subspace of $C^\infty(\beta_s^r M; T^0 M)$ consisting of fields h satisfying the homogeneity condition $h(x, t\xi) = t^\lambda h(x, \xi) (t > 0)$ for a fixed λ. All semibasic tensor fields under consideration will be of this kind but for different values of λ. In particular, for $f \in C^\infty(\tau_1^1 M)(\dim M = 3)$,

$$\|f\|^2 = \int_{\Omega M} |f(x)|^2 \, d\Sigma(x,\xi) = C\|f\|^2_{L^2},$$

where C is the volume of the unit sphere in $\mathbb{R}^3$. In the current section, we denote different universal constants by the same letter C, while D denotes different constants depending on (M, g) and probably on ε.

First of all we will demonstrate that theorem 3.1 can be derived from the following two lemmas.

Lemma 3.2. *There exists $\varepsilon_0 > 0$ such that the following statement is valid for any positive $\varepsilon < \varepsilon_0$. Let a three-dimensional CNTM (M, g) satisfy (2.14). For every trace-free symmetric tensor field $f \in C^\infty(\tau_1^1 M)$ and every symmetric semibasic tensor field $r \in C^\infty(\beta_1^1 M; T^0 M)$ satisfying*

$$r(x,\xi)\xi = 0, \tag{3.2}$$

$$r(x,t\xi) = r(x,\xi)(t > 0), \tag{3.3}$$

if the solution $u \in C(\beta_1^1 M; T^0 M)$ to the boundary value problem

$$Hu = Q_\xi f + r, \qquad u|_{\partial_-\Omega M} = 0 \tag{3.4}$$

is positively homogeneous in ξ of degree -1

$$u(x,t\xi) = t^{-1}u(x,\xi) \qquad \text{for} \quad t > 0, \tag{3.5}$$

then the estimate

$$\|\overset{h}{\nabla} u\|^2 \leqslant C\varepsilon^{-3/2}(\|r\|^2 + \|\overset{v}{\nabla} r\|^2) + D\|u|_{\partial_+\Omega M}\|^2_{H^1} \tag{3.6}$$

holds with some universal constant C and some constant D depending on (M, g) but not on f and r.

Lemma 3.3. *There exists $\varepsilon_0 > 0$ such that the following statement is valid for any positive $\varepsilon < \varepsilon_0$. Let a three-dimensional CNTM (M, g) satisfy (2.14). For every closed skew-symmetric tensor field $f \in C^\infty(\tau_1^1 M)$ and every skew-symmetric semibasic tensor field $r \in C^\infty(\beta_1^1 M; T^0 M)$ satisfying (3.2)–(3.3), if the solution $u \in C(\beta_1^1 M; T^0 M)$ to the boundary value problem (3.4) is positively homogeneous in ξ of degree -1, then the estimate*

$$\|\overset{h}{\nabla} u\|^2 \leqslant C\varepsilon^{-3/2}(\|r\|^2 + \|\overset{v}{\nabla} r\|^2) + D\big(\|u|_{\partial_+\Omega M}\|^2_{H^1} + \|f|_{\partial M}\|_{L^2}\|u|_{\partial_+\Omega M}\|_{L^2}\big) \tag{3.7}$$

holds with some universal constant C and some constant D depending on (M, g) but not on f and r.

Proof of theorem 3.1. We write equation (2.6) in form (2.10) with the remainder r defined by (2.11). The remainder satisfies $r(x,\xi)\xi = r^*(x,\xi)\xi = 0$ and estimates (2.12).

We decompose each of the fields f, u, r into the sum of symmetric and skew-symmetric tensors

$$f = f^+ + f^-, \qquad u = u^+ + u^-, \qquad r = r^+ + r^-.$$

The field f^+ is trace-free, and f^- is a closed field. The fields $u^\pm$ are solutions to the boundary value problems

$$Hu^\pm = Q_\xi f^\pm + r^\pm, \qquad u^\pm|_{\partial_-\Omega M} = 0.$$

Thus, the triple (f^+, r^+, u^+) satisfies all hypotheses of lemma 3.2, and (f^-, r^-, u^-) satisfies hypotheses of lemma 3.3. Let us emphasize that we have estimates (2.12) for $\|r\|^2 = \|r^+\|^2 + \|r^-\|^2$ through $\|f\|^2 = \|f^+\|^2 + \|f^-\|^2$ but we have no estimate for $\|r^\pm\|$ through $\|f^\pm\|$. Applying lemmas 3.2 and 3.3, we obtain the estimates

$$\|\overset{h}{\nabla}u^+\|^2 \leqslant C\varepsilon^{-3/2}(\|r^+\|^2 + \|\overset{v}{\nabla}r^+\|^2) + D\|u^+|_{\partial_+\Omega M}\|^2_{H^1},$$
$$\|\overset{h}{\nabla}u^-\|^2 \leqslant C\varepsilon^{-3/2}(\|r^-\|^2 + \|\overset{v}{\nabla}r^-\|^2) + D\big(\|u^-|_{\partial_+\Omega M}\|^2_{H^1} + \|f^-|_{\partial M}\|_{L^2}\|u^-|_{\partial_+\Omega M}\|_{L^2}\big).$$

Taking the sum of these inequalities, we arrive at the estimate

$$\|\overset{h}{\nabla}u\|^2 \leqslant C\varepsilon^{-3/2}(\|r\|^2 + \|\overset{v}{\nabla}r\|^2) + D\big(\|u|_{\partial_+\Omega M}\|^2_{H^1} + \|f|_{\partial M}\|_{L^2}\|u|_{\partial_+\Omega M}\|_{L^2}\big)$$

which gives together with (2.12)

$$\|\overset{h}{\nabla}u\|^2 \leqslant C\varepsilon^{1/2}\|f\|^2 + D\big(\|F[f]\|^2_{H^1} + \|f|_{\partial M}\|_{L^2}\|F[f]\|_{L^2}\big). \tag{3.8}$$

By (2.13), $\|\overset{h}{\nabla}u\|^2 \geqslant \|f\|^2_{L^2}/C$. Estimating the left-hand side of (3.8) with the help of the last inequality, we obtain

$$(1/C - C\varepsilon^{1/2})\|f\|^2_{L^2} \leqslant D\big(\|F[f]\|^2_{H^1} + \|f|_{\partial M}\|_{L^2}\|F[f]\|_{L^2}\big).$$

This gives the statement (3.1) of the theorem under the assumption $C^2\varepsilon^{1/2} < 1/2$. □

Proof of lemma 3.2. The field u is symmetric and orthogonal to ξ, i.e., $u_{ij}\xi^j = 0$ as follows from (3.2) and (3.4). We write the Pestov identity (2.32) for u with terms defined by (2.33)–(2.35). We represent the left-hand side of the Pestov identity in form (2.36). For a symmetric f, formula (2.37) takes the form

$$(Q_\xi f)_{ij} = f_{ij} - a_i\xi_j - a_j\xi_i - c\xi_i\xi_j - dg_{ij}$$

since $b = \bar a$ as is seen from (2.27). The terms a, c and d are defined by formulae (2.27) and (2.28). Treating the last formula as before, we obtain the following analog of (2.39)

$$\langle\overset{h}{\nabla}u, \overset{v}{\nabla}(Q_\xi f)\rangle = -2\langle\overset{h}{\delta}u, a\rangle - \langle\overset{h}{\nabla}(\mathrm{tr}\,u), \overset{v}{\nabla}d\rangle, \tag{3.9}$$

where

$$(\overset{h}{\delta}u)_i = \overset{h}{\nabla}{}^p u_{ip} \tag{3.10}$$

This implies the estimate

$$2\,\mathrm{Re}\langle\overset{h}{\nabla}u, \overset{v}{\nabla}(Q_\xi f)\rangle \leqslant |\overset{h}{\delta}u|^2 + 4|a|^2 + \frac{1}{\varepsilon}|\overset{h}{\nabla}(\mathrm{tr}\,u)|^2 + \varepsilon|\overset{v}{\nabla}d|^2. \tag{3.11}$$

Like in the proof of theorem NS.3.1, we distinguish a divergent term from $|\overset{h}{\delta}u|^2$ and obtain the following analog of formula (NS.3.31),

$$|\overset{h}{\delta}u|^2 = \overset{h}{\nabla}{}^i u^{jk}\cdot\overset{h}{\nabla}_j\bar u_{ik} + \overset{h}{\nabla}_i\tilde v^i + \mathcal{R}_4[u], \tag{3.12}$$

where

$$\tilde v^i = \mathrm{Re}(u^{ij}\overset{h}{\nabla}{}^k\bar u_{jk} - u_{jk}\overset{h}{\nabla}{}^k\bar u^{ij}) \tag{3.13}$$

and

$$\mathcal{R}_4[u] = \mathrm{Re}\big(u_i^p\big(R^k_{jpq}\xi^j\overset{v}{\nabla}_k\bar u^{qi} - R^q_{jpq}\bar u^{ji} - R^i_{jpq}\bar u^{qi}\big)\big). \tag{3.14}$$

We introduce the semibasic tensor field z by the formula

$$\overset{h}{\nabla}_i u_{jk} = \frac{\xi_i}{|\xi|^2}(Hu)_{jk} + z_{ijk}. \tag{3.15}$$

The summands on the right-hand side of (3.15) are orthogonal to each other, therefore

$$|\overset{h}{\nabla} u|^2 = \frac{1}{|\xi|^2}|Hu|^2 + |z|^2. \tag{3.16}$$

Then we represent z in the form

$$z = z^+ + z^-, \tag{3.17}$$

where

$$z^+_{ijk} = \tfrac{1}{2}(z_{ijk} + z_{jik}), \qquad z^-_{ijk} = \tfrac{1}{2}(z_{ijk} - z_{jik}).$$

The summands on the right-hand side of (3.17) are orthogonal to each other, therefore

$$|z|^2 = |z^+|^2 + |z^-|^2.$$

It follows from (3.15) and (3.17) that

$$\overset{h}{\nabla}{}^i u^{jk} \cdot \overset{h}{\nabla}_j \bar{u}_{ik} = z^{ijk}\bar{z}_{jik} = |z^+|^2 - |z^-|^2.$$

Therefore formulae (3.12) and (3.16) take the form

$$|\overset{h}{\delta} u|^2 = |z^+|^2 - |z^-|^2 + \overset{h}{\nabla}_i \tilde{v}^i + \mathcal{R}_4[u], \tag{3.18}$$

$$|\overset{h}{\nabla} u|^2 = \frac{1}{|\xi|^2}|Hu|^2 + |z^+|^2 + |z^-|^2. \tag{3.19}$$

Substituting expressions (3.18) and (2.27) into (3.11) and estimating the last term of (3.11) by (2.42), we arrive at the inequality

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Q_\xi f)\rangle \leqslant |z^+|^2 - |z^-|^2 + \frac{4}{|\xi|^4}|\pi_\xi f\xi|^2 + \frac{9\varepsilon}{4|\xi|^2}|f|^2 + \frac{1}{\varepsilon}|\overset{h}{\nabla}(\mathrm{tr}\,u)|^2 + \overset{h}{\nabla}_i \tilde{v}^i + \mathcal{R}_4[u].$$

The second term on the right-hand side of (2.36) can be estimated as

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla} r\rangle \leqslant \varepsilon|\overset{h}{\nabla} u|^2 + \frac{1}{\varepsilon}|\overset{v}{\nabla} r|^2.$$

Taking the sum of two last inequalities and using (2.36), we obtain

$$\begin{aligned} 2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla} Hu\rangle \leqslant{}& |z^+|^2 - |z^-|^2 + \frac{4}{|\xi|^4}|\pi_\xi f\xi|^2 + \frac{9\varepsilon}{4|\xi|^2}|f|^2 + \frac{1}{\varepsilon}|\overset{h}{\nabla}(\mathrm{tr}\,u)|^2 \\ &+ \varepsilon|\overset{h}{\nabla} u|^2 + \frac{1}{\varepsilon}|\overset{v}{\nabla} r|^2 + \overset{h}{\nabla}_i \tilde{v}^i + \mathcal{R}_4[u]. \end{aligned} \tag{3.20}$$

Estimating the left-hand side of the Pestov identity (2.32) by (3.20), we arrive at the inequality for $|\xi| = 1$

$$\begin{aligned} |\overset{h}{\nabla} u|^2 + \overset{v}{\nabla}_i w^i - |z^+|^2 + |z^-|^2 - 4|\pi_\xi f\xi|^2 \leqslant{}& \varepsilon|\overset{h}{\nabla} u|^2 + \overset{h}{\nabla}_i(\tilde{v}^i - v^i) + \mathcal{R}[u] \\ &+ \frac{9\varepsilon}{4}|f|^2 + \frac{1}{\varepsilon}|\overset{h}{\nabla}(\mathrm{tr}\,u)|^2 + \frac{1}{\varepsilon}|\overset{v}{\nabla} r|^2, \end{aligned} \tag{3.21}$$

where

$$\mathcal{R}[u] = \mathcal{R}_1[u] + \mathcal{R}_4[u].$$

We multiply (3.21) by the volume form $\mathrm{d}\Sigma$, integrate the result over ΩM, and transform the integrals of divergent terms by Gauss–Ostrogradskii formulae. In such a way, we obtain

$$\|\overset{h}{\nabla}u\|^2 + \|Hu\|^2 - \|z^+\|^2 + \|z^-\|^2 - 4\|\pi_\xi f\xi\|^2 \\ \leqslant \varepsilon\|\overset{h}{\nabla}u\|^2 + \int_{\partial\Omega M}\langle\nu, \tilde{v} - v\rangle\,\mathrm{d}\Sigma' + \int_{\Omega M}\mathcal{R}[u]\,\mathrm{d}\Sigma + \frac{9\varepsilon}{4}\|f\|^2 \\ + \frac{1}{\varepsilon}\|\overset{h}{\nabla}(\mathrm{tr}\,u)\|^2 + \frac{1}{\varepsilon}\|\overset{v}{\nabla}r\|^2. \qquad (3.22)$$

The first integral on the right-hand side of (3.22) can be estimated as

$$\left|\int_{\partial\Omega M}\langle\nu, \tilde{v} - v\rangle\,\mathrm{d}\Sigma'\right| \leqslant D\|u|_{\partial_+\Omega M}\|^2_{H^1} \qquad (3.23)$$

since the integrand is the value on $u|_{\partial\Omega M}$ of some quadratic first-order differential operator. Let us demonstrate that the curvature-dependent integral on (3.22) admits the estimate

$$\int_{\Omega M}\mathcal{R}[u]\,\mathrm{d}\Sigma \leqslant C\varepsilon(\|\overset{h}{\nabla}u\|^2 + \|f\|^2 + \|\overset{v}{\nabla}r\|^2). \qquad (3.24)$$

Indeed, repeating the corresponding arguments from the proof of theorem 2.3, we derive estimates (2.50) and (2.51) which imply (3.24).

Estimating integrals on the right-hand side of (3.22) by (3.23) and (3.24), we obtain

$$\|\overset{h}{\nabla}u\|^2 + \|Hu\|^2 - \|z^+\|^2 + \|z^-\|^2 - 4\|\pi_\xi f\xi\|^2 \\ \leqslant C\varepsilon\|\overset{h}{\nabla}u\|^2 + D\|u|_{\partial_+\Omega M}\|^2_{H^1} + C\varepsilon\|f\|^2 + \frac{1}{\varepsilon}\|\overset{h}{\nabla}(\mathrm{tr}\,u)\|^2 + \frac{C}{\varepsilon}\|\overset{v}{\nabla}r\|^2.$$

Substituting the value $\|\overset{h}{\nabla}u\|^2 = \|Hu\|^2 + \|z^+\|^2 + \|z^-\|^2$ which follows from (3.16) into the left-hand side of the last inequality, we obtain

$$2\|Hu\|^2 + 2\|z^-\|^2 - 4\|\pi_\xi f\xi\|^2 \leqslant C\varepsilon\|\overset{h}{\nabla}u\|^2 + D\|u|_{\partial_+\Omega M}\|^2_{H^1} \\ + C\varepsilon\|f\|^2 + \frac{1}{\varepsilon}\|\overset{h}{\nabla}(\mathrm{tr}\,u)\|^2 + \frac{C}{\varepsilon}\|\overset{v}{\nabla}r\|^2.$$

Combining this estimate with the inequality

$$\|Hu\|^2 = \|Q_\xi f + r\|^2 \geqslant \|Q_\xi f\|^2 - 2\|Q_\xi f\|\cdot\|r\| \geqslant \|Q_\xi f\|^2 - 2\|f\|\cdot\|r\| \\ \geqslant \|Q_\xi f\|^2 - \varepsilon\|f\|^2 - \frac{1}{\varepsilon}\|r\|^2$$

which follows from (3.4), we obtain

$$2\|Q_\xi f\|^2 - 4\|\pi_\xi f\xi\|^2 + 2\|z^-\|^2 \leqslant C\varepsilon\|\overset{h}{\nabla}u\|^2 + D\|u|_{\partial_+\Omega M}\|^2_{H^1} \\ + C\varepsilon\|f\|^2 + \frac{1}{\varepsilon}\|\overset{h}{\nabla}(\mathrm{tr}\,u)\|^2 + \frac{1}{\varepsilon}(\|r\|^2 + \|\overset{v}{\nabla}r\|^2).$$

By lemma 2.4, the sum of the first two terms on the left-hand side is equal to zero, and the inequality takes the form

$$2\|z^-\|^2 \leqslant C\varepsilon\|\overset{h}{\nabla}u\|^2 + C\varepsilon\|f\|^2 + D\|u|_{\partial_+\Omega M}\|^2_{H^1} + \frac{1}{\varepsilon}(\|r\|^2 + \|\overset{v}{\nabla}r\|^2) + \frac{1}{\varepsilon}\|\overset{h}{\nabla}(\mathrm{tr}\,u)\|^2.$$

Estimating the last term on the right-hand side with the help of lemma 2.1, we arrive at the inequality

$$2\|z^-\|^2 \leqslant C\varepsilon\|\overset{h}{\nabla}u\|^2 + C\varepsilon\|f\|^2 + \frac{C}{\varepsilon}(\|r\|^2 + \|\overset{v}{\nabla}r\|^2) + D\|u|_{\partial_+\Omega M}\|^2_{H^1}. \qquad (3.25)$$

Now, following arguments presented at the end of section 6.3 of [5], we estimate $\|\overset{h}{\nabla}u\|^2$ through $\|f\|\cdot\|z^-\|$. To this end we write the equality

$$z^-_{ijk} = \frac{1}{2}\left(\overset{h}{\nabla}_i u_{jk} - \frac{1}{|\xi|^2}\xi_i(Hu)_{jk} - \overset{h}{\nabla}_j u_{ik} + \frac{1}{|\xi|^2}\xi_j(Hu)_{ik}\right)$$

which follows from (3.15). Multiplying this equality by g^{jk} and performing the summation over j and k

$$z^-_{ijk}g^{jk} = \frac{1}{2}\left(\overset{h}{\nabla}_i(\operatorname{tr}u) - \frac{1}{|\xi|^2}\xi_i\operatorname{tr}(Hu) - \overset{h}{\nabla}{}^k u_{ik} + \frac{1}{|\xi|^2}\xi^k(Hu)_{ik}\right).$$

The last term on the right-hand side is equal to zero since Hu is orthogonal to ξ, the third term is equal to $(\overset{h}{\delta}u)_i$, and the second term coincides with $\xi_i\operatorname{tr}r/|\xi|^2$ as is seen from (3.4). We thus obtain

$$(\overset{h}{\delta}u)_i = -2z^-_{ijk}g^{jk} + \overset{h}{\nabla}_i(\operatorname{tr}u) - \frac{1}{|\xi|^2}\xi_i\operatorname{tr}r.$$

Substituting this expression into (3.9) and using $\langle a,\xi\rangle = 0$, we obtain

$$\langle\overset{h}{\nabla}u, \overset{v}{\nabla}(Q_\xi f)\rangle = 4z^-_{ijk}\bar a^i g^{jk} - 2\bar a^i\overset{h}{\nabla}_i(\operatorname{tr}u) - \langle\overset{h}{\nabla}(\operatorname{tr}u), \overset{v}{\nabla}d\rangle.$$

Together with (2.36), this gives

$$\langle\overset{h}{\nabla}u, \overset{v}{\nabla}Hu\rangle = 4z^-_{ijk}\bar a^i g^{jk} - 2\bar a^i\overset{h}{\nabla}_i(\operatorname{tr}u) - \langle\overset{h}{\nabla}(\operatorname{tr}u), \overset{v}{\nabla}d\rangle + \langle\overset{h}{\nabla}(\operatorname{tr}u), \overset{v}{\nabla}r\rangle.$$

This implies the estimate

$$2\operatorname{Re}\langle\overset{h}{\nabla}u, \overset{v}{\nabla}Hu\rangle \leqslant C|z^-|\cdot|a| + \frac{3}{\varepsilon}|\overset{h}{\nabla}(\operatorname{tr}u)|^2 + \varepsilon|\overset{v}{\nabla}d|^2 + \frac{1}{2}|\overset{h}{\nabla}u|^2 + 2|\overset{v}{\nabla}r|^2.$$

Since $|a| \leqslant |f|$ and $|\overset{v}{\nabla}d| \leqslant \frac{3}{2}|f|$ for $|\xi| = 1$ as follows from (2.27) and (2.42), the estimate can be written as

$$2\operatorname{Re}\langle\overset{h}{\nabla}u, \overset{v}{\nabla}Hu\rangle \leqslant C|f|\cdot|z^-| + \frac{9\varepsilon}{4}|f|^2 + \frac{3}{\varepsilon}|\overset{h}{\nabla}(\operatorname{tr}u)|^2 + \frac{1}{2}|\overset{h}{\nabla}u|^2 + 2|\overset{v}{\nabla}r|^2.$$

Estimating the left-hand side of the Pestov identity (2.32) with the help of the last inequality, we obtain for $|\xi| = 1$

$$\frac{1}{2}|\overset{h}{\nabla}u|^2 + \overset{v}{\nabla}_i w^i \leqslant C|f|\cdot|z^-| + C\varepsilon|f|^2 + \frac{C}{\varepsilon}|\overset{h}{\nabla}(\operatorname{tr}u)|^2 + 2|\overset{v}{\nabla}r|^2 - \overset{h}{\nabla}_i v^i + \mathcal{R}_1[u].$$

We multiply this inequality by the volume form $\mathrm{d}\Sigma$, integrate the result over ΩM, and transform the integrals of divergent terms by Gauss–Ostrogradskii formulae. In such a way, we obtain

$$\frac{1}{2}\|\overset{h}{\nabla}u\|^2 + \|Hu\|^2 \leqslant C\|f\|\cdot\|z^-\| + C\varepsilon\|f\|^2 + \frac{C}{\varepsilon}\|\overset{h}{\nabla}(\operatorname{tr}u)\|^2 + 2\|\overset{v}{\nabla}r\|^2 - \int_{\partial\Omega M}\langle\nu, v\rangle\,\mathrm{d}\Sigma' + \int_{\Omega M}\mathcal{R}_1[u]\,\mathrm{d}\Sigma. \tag{3.26}$$

Integrals participating on (3.26) are estimated similarly to (3.23) and (3.24). Estimating also $\|\overset{h}{\nabla}(\operatorname{tr}u)\|^2$ with the help of lemma 2.1, we obtain from (3.26)

$$(1/2 - C\varepsilon)\|\overset{h}{\nabla}u\|^2 \leqslant C\|f\|\cdot\|z^-\| + C\varepsilon\|f\|^2 + D\|u|_{\partial_+\Omega M}\|^2_{H^1} + C\|\overset{v}{\nabla}r\|^2.$$

Assuming ε to be so small as $C\varepsilon < 1/4$, this implies

$$\|\overset{h}{\nabla}u\|^2 \leqslant 2C\|f\|\cdot\|z^-\| + C\varepsilon\|f\|^2 + C\|\overset{v}{\nabla}r\|^2 + D\|u|_{\partial_+\Omega M}\|^2_{H^1}. \tag{3.27}$$

The final part of the proof consists of comparing estimates (3.25) and (3.27). Without lost of generality, we can assume constants denoted by C in these estimates to coincide. The same is assumed for constants denoted by D. Inequality (3.27) implies

$$\|\overset{h}{\nabla}u\|^2 \leqslant \frac{C}{\sqrt{\varepsilon}}\|z^-\|^2 + C(\sqrt{\varepsilon}+\varepsilon)\|f\|^2 + C\|\overset{v}{\nabla}r\|^2 + D\|u|_{\partial_+\Omega M}\|^2_{H^1}.$$

Estimating the first term on the right-hand side of this inequality with the help of (3.25), we obtain

$$(1 - C^2\sqrt{\varepsilon})\|\overset{h}{\nabla}u\|^2 \leqslant C(C\sqrt{\varepsilon}+\varepsilon)\|f\|^2 + \frac{C^2}{\varepsilon^{3/2}}\|r\|^2 + C\left(\frac{C}{\varepsilon^{3/2}}+1\right)\|\overset{v}{\nabla}r\|^2 + D\left(\frac{C}{\varepsilon^{1/2}}+1\right)\|u|_{\partial_+\Omega M}\|^2_{H^1}. \tag{3.28}$$

Relations (2.3) and (3.4) imply the estimate

$$\|f\|^2 \leqslant C\|Q_\xi f\|^2 = C\|Hu - r\|^2 \leqslant C(\|Hu\|^2 + \|r\|^2) \leqslant C(\|\overset{h}{\nabla}u\|^2 + \|r\|^2). \tag{3.29}$$

Estimating the first term on the right-hand side of (3.28) with the help of the last inequality, we obtain

$$[1 - C^2(C\sqrt{\varepsilon}+2\sqrt{\varepsilon}+\varepsilon)]\|\overset{h}{\nabla}u\|^2 \leqslant (C^2\varepsilon^{-3/2}+C)\|r\|^2 + C(C\varepsilon^{-3/2}+1)\|\overset{v}{\nabla}r\|^2 + D(C\varepsilon^{-1/2}+1)\|u|_{\partial_+\Omega M}\|^2_{H^1}.$$

Assuming ε to be so small as the number in the brackets is $\geqslant 1/2$, this inequality implies the statement of lemma 3.2. □

Proof of lemma 3.3. We will actually repeat the proof of theorem NS.4.2 in the case of a skew-symmetric f but without estimating terms related to the remainder r.

The closeness condition for a skew-symmetric f is as follows:

$$\overset{h}{\nabla}_i f_{jk} + \overset{h}{\nabla}_j f_{ki} + \overset{h}{\nabla}_k f_{ij} = 0. \tag{3.30}$$

Formula (2.25) for a skew-symmetric f is simplified to the following one:

$$(Q_\xi f)_{ij} = f_{ij} + \xi_i a_j - \xi_j a_i, \tag{3.31}$$

where $a = f\xi/|\xi|^2$. The corresponding analog of the formula (2.39) is now

$$\langle\overset{h}{\nabla}u, \overset{v}{\nabla}(Q_\xi f)\rangle = -2\langle\overset{h}{\delta}u, a\rangle = -\frac{2}{|\xi|^2}\langle\overset{h}{\delta}u, f\xi\rangle, \tag{3.32}$$

where $\overset{h}{\delta}u$ is defined by formula (3.10).

Using the closeness condition (3.30), we transform the expression $\langle\overset{h}{\delta}u, f\xi\rangle$ as follows:

$$\begin{aligned}
\langle\overset{h}{\delta}u, f\xi\rangle &= \overset{h}{\nabla}_p u^{ip}\cdot\bar{f}_{ik}\xi^k = \overset{h}{\nabla}_p(\xi^k u^{ip}\bar{f}_{ik}) - \xi^k u^{ip}\overset{h}{\nabla}_p\bar{f}_{ik}\\
&= \overset{h}{\nabla}_p(\xi^k u^{ip}\bar{f}_{ik}) - \tfrac{1}{2}\xi^k(u^{ip}\overset{h}{\nabla}_p\bar{f}_{ik} - u^{pi}\overset{h}{\nabla}_p\bar{f}_{ik})\\
&= \overset{h}{\nabla}_p(\xi^k u^{ip}\bar{f}_{ik}) - \tfrac{1}{2}\xi^k(u^{ip}\overset{h}{\nabla}_p\bar{f}_{ik} - u^{ip}\overset{h}{\nabla}_i\bar{f}_{pk})\\
&= \overset{h}{\nabla}_p(\xi^k u^{ip}\bar{f}_{ik}) - \tfrac{1}{2}\xi^k u^{ip}(\overset{h}{\nabla}_p\bar{f}_{ik} + \overset{h}{\nabla}_i\bar{f}_{kp})\\
&= \overset{h}{\nabla}_p(\xi^k u^{ip}\bar{f}_{ik}) - \tfrac{1}{2}\xi^k u^{ip}\overset{h}{\nabla}_k\bar{f}_{pi}\\
&= \overset{h}{\nabla}_p(\xi^k u^{ip}\bar{f}_{ik}) - \overset{h}{\nabla}_k\big(\tfrac{1}{2}\xi^k u^{ip}\bar{f}_{pi}\big) - \tfrac{1}{2}\xi^k\overset{h}{\nabla}_k u^{ip}\cdot\bar{f}_{pi}.
\end{aligned}$$

The result can be written as

$$2\langle \overset{h}{\delta} u, f\xi\rangle = \langle Hu, f\rangle + \overset{h}{\nabla}_i(\xi^i u^{jk}\bar f_{kj} + 2\xi^k u^{ji}\bar f_{jk}). \tag{3.33}$$

Using (3.6) and (3.4), we transform the first term on the right-hand side of (3.33) as follows:

$$\begin{aligned}\langle Hu, f\rangle &= \langle Q_\xi f + r, f\rangle = (Q_\xi f)^{ij}\bar f_{ij} + \langle r, f\rangle\\ &= (f^{ij} - a^i\xi^j + a^j\xi^i)\bar f_{ij} + \langle r, f\rangle = |f|^2 - 2a^i\bar f_{ij}\xi^j + \langle r, f\rangle\\ &= |f|^2 - \frac{2}{|\xi|^2}|f\xi|^2 + \langle r, f\rangle.\end{aligned}$$

Formula (3.33) takes now the form

$$2\langle \overset{h}{\delta} u, f\xi\rangle = |f|^2 - \frac{2}{|\xi|^2}|f\xi|^2 + \langle r, f\rangle + \overset{h}{\nabla}_i(\xi^i u^{jk}\bar f_{kj} + 2\xi^k u^{ji}\bar f_{jk}).$$

Substituting this expression into (3.32), we obtain

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Q_\xi f)\rangle = -\frac{2}{|\xi|^2}|f|^2 + \frac{4}{|\xi|^4}|f\xi|^2 - \frac{2}{|\xi|^2}\langle r, f\rangle + \overset{h}{\nabla}_i\tilde v^i, \tag{3.34}$$

where

$$\tilde v^i = -\frac{2}{|\xi|^2}\,\mathrm{Re}(\xi^i u^{jk}\bar f_{kj} + 2\xi^k u^{ji}\bar f_{jk}). \tag{3.35}$$

From (2.36) and (3.34),

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla} Hu\rangle = 2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla} r\rangle - \frac{2}{|\xi|^2}|f|^2 + \frac{4}{|\xi|^4}|f\xi|^2 - \frac{2}{|\xi|^2}\langle r, f\rangle + \overset{h}{\nabla}_i\tilde v^i.$$

This admits the estimate for $|\xi| = 1$

$$2\,\mathrm{Re}\langle \overset{h}{\nabla} u, \overset{v}{\nabla} Hu\rangle \leqslant \tfrac12|\overset{h}{\nabla} u|^2 - 2|f|^2 + 4|f\xi|^2 + \overset{h}{\nabla}_i\tilde v^i + 2|r|\cdot|f| + 2|\overset{v}{\nabla} r|^2.$$

Estimating the left-hand side of the Pestov identity (2.32) with the help of the last inequality, we obtain for $|\xi| = 1$

$$\tfrac12|\overset{h}{\nabla} u|^2 + \overset{v}{\nabla}_i w^i + 2|f|^2 - 4|f\xi|^2 \leqslant \overset{h}{\nabla}_i(\tilde v^i - v^i) + \mathcal{R}_1[u] + 2|r|\cdot|f| + 2|\overset{v}{\nabla} r|^2.$$

Integrating this inequality over ΩM and transforming integrals of divergent terms by Gauss–Ostrogradskii, we obtain

$$\begin{aligned}\frac12\|\overset{h}{\nabla} u\|^2 + \|Hu\|^2 + 2\int_{\Omega M}(|f|^2 - 2|f\xi|^2)\,\mathrm{d}\Sigma &\leqslant 2\|r\|\cdot\|f\| + 2\|\overset{v}{\nabla} r\|^2\\ &\quad + \int_{\partial\Omega M}\langle \nu, \tilde v - v\rangle\,\mathrm{d}\Sigma' + \int_{\Omega M}\mathcal{R}_1[u]\,\mathrm{d}\Sigma.\end{aligned} \tag{3.36}$$

The integral on the left-hand side of (3.36) is non-negative as is shown at the end of the proof of theorem NS.4.2. Integrals on the right-hand side admit the estimates

$$\left|\int_{\partial\Omega M}\langle \nu, \tilde v - v\rangle\,\mathrm{d}\Sigma'\right| \leqslant DN(u, f) := D\big(\|u|_{\partial_+\Omega M}\|^2_{H^1} + \|f|_{\partial M}\|_{L^2}\|u|_{\partial_+\Omega M}\|_{L^2}\big), \tag{3.37}$$

$$\int_{\Omega M}\mathcal{R}_1[u]\,\mathrm{d}\Sigma \leqslant C\varepsilon(\|\overset{h}{\nabla} u\|^2 + \|f\|^2 + \|\overset{v}{\nabla} r\|^2). \tag{3.38}$$

The second term on the right-hand side of (3.37) has appeared because of the dependence of the integrand on f which comes from (3.35). Estimate (3.38) is derived in the same way as (3.24).

With the help of (3.37) and (3.38), (3.36) implies

$$(1/2 - C\varepsilon)\|\overset{h}{\nabla}u\|^2 \leqslant 2\|r\| \cdot \|f\| + C\varepsilon\|f\|^2 + C\|\overset{v}{\nabla}r\|^2 + DN(u, f).$$

Assuming ε to be so small as $C\varepsilon < 1/4$, this gives

$$\|\overset{h}{\nabla}u\|^2 \leqslant 2C\|r\| \cdot \|f\| + C\varepsilon\|f\|^2 + C\|\overset{v}{\nabla}r\|^2 + DN(u, f). \tag{3.39}$$

Inequality (3.39) implies

$$\|\overset{h}{\nabla}u\|^2 \leqslant C\varepsilon\|f\|^2 + \frac{C}{\varepsilon}\|r\|^2 + C\|\overset{v}{\nabla}r\|^2 + DN(u, f).$$

Estimating the first term on the right-hand side with the help of (3.29), we obtain

$$(1 - C^2\varepsilon)\|\overset{h}{\nabla}u\|^2 \leqslant \left(\frac{C}{\varepsilon} + C^2\varepsilon\right)\|r\|^2 + C\|\overset{v}{\nabla}r\|^2 + DN(u, f).$$

If ε is chosen so small as $C^2\varepsilon < 1/2$, this inequality implies the statement of lemma 3.3. □

4. The nonlinear problem

Here, we consider the inverse problem of recovering a tensor field $f \in C^\infty(\tau_1^1 M)$ on a CNTM (M, g) from the data $\Phi[f] = U|_{\partial_+\Omega M}$, where $U \in C(\beta_1^1 M; \Omega M)$ is the solution to the boundary value problem (1.11). We will prove the uniqueness under the following smallness assumptions on f:

$$|f(x)| < \varepsilon \qquad \text{for} \quad x \in \partial M, \tag{4.1}$$

$$\int_{\tau_-(x,\xi)}^0 |f(\gamma_{x,\xi}(t))|\,\mathrm{d}t < \varepsilon, \qquad \int_{\tau_-(x,\xi)}^0 |\nabla f(\gamma_{x,\xi}(t))|\,\mathrm{d}t < \varepsilon \quad \text{for} \quad (x,\xi) \in \partial_+\Omega M. \tag{4.2}$$

The following theorem is proved on the basis of theorems 2.3 and 3.1 in full analogy with section 5 of [NS].

Theorem 4.1. *It is possible to choose a positive number $\varepsilon = \varepsilon(n)$ for $n \geqslant 4$ such that the following statement is true for an n-dimensional CNTM (M, g) satisfying the curvature condition (2.14). Let two trace-free tensor fields $f_i \in C^\infty(\tau_1^1 M)$ $(i = 1, 2)$ satisfy (4.1) and (4.2) and $\Phi_i = \Phi[f_i]$ be the corresponding data. Then the estimate*

$$\|f_2 - f_1\|_{L^2} \leqslant C\left\|\Phi_1^{-1}\Phi_2 - E\right\|_{H^1}$$

holds with constant C independent of f_i. In particular, $f_1 = f_2$ if $\Phi_1 = \Phi_2$. In the case of $n = 3$, the same statement is true under the additional assumption that $f_2 - f_1$ is a closed tensor field.

References

[1] Aben H K 1979 *Integrated Photoelasticity* (New York: McGraw-Hill)

[2] Hammer H B and Lionheart W R 2005 Reconstruction of spatially inhomogeneous dielectric tensors through optical tomography *J. Opt. Soc. Am.* **22** 250–5

[3] Lionheart W R B and Newton C J P 2007 Analysis of the inverse problem for determining nematic liquid crystal director profiles from optical measurements using singular value decomposition *New J. Phys.* **9** 63–73

[4] Novikov R and Sharafutdinov V 2007 On the problem of polarization tomography: I *Inverse Problems* **23** 1229–57

[5] Sharafutdinov V 1994 *Integral Geometry of Tensor Fields* (VSP, Utrecht: The Netherlands)

[6] Wijerathne M L L, Oguni K and Hori M 2002 Tensor field tomography based on 3D photoelasticity *Mech. Mater.* **34** 533–45