Pogorelov — Klingenberg theorem for manifolds homeomorphic to $\mathbf{R}^n$ (translated from [4])

Vladimir Sharafutdinov

January 2006, Seattle

1 Introduction

The paper is devoted to the proof of the following

Theorem 1.1 *Let M be a complete Riemannian manifold homeomorphic to the Euclidean space. Let the sectional curvature of M satisfy $0 \le K \le 1$ at any point and any two-dimensional direction. Then every geodesic in M of length $\le \pi$ is minimal.*

This statement was proved by Toponogov in the case $0 < K \le 1$ [5]. In the same paper, Toponogov proves that, if $0 \le K \le 1$ (not assuming the manifold to be homeomorphic to $\mathbf{R}^n$), then the injectivity radius of M is positive, i.e., there exists $\delta > 0$ such that every geodesic of length $\le \delta$ is minimal. Actually we will prove a more general statement, see Theorem 3.1 below, which generalizes both Toponogov's results.

Theorem 1.1 was proved by Veiner [6] in the case when M is a hypersurface in $\mathbf{R}^{n+1}$. Veiner's proof is based on the Buzemann — Feller theorem. The latter theorem says that, given a convex set $C \subset \mathbf{R}^n$, the metric projection $\varphi : \mathbf{R}^n \to C$ sending a point p to the nearest point of C does not increase lengths of curves. For a convex set C in a Riemannian manifold, the metric projection $\varphi : U \to C$ is still defined in some neighborhood of C, but can increase distances as simple examples show. Nevertheless, if C is the level surface of some convex function defined in U, the metric projection can be slightly corrected in such a way that the corrected map satisfies the Buzemann — Feller theorem. This construction is presented in Section 2. Theorem 1.1 is proved in Section 3.

Pogorelov and Klingenberg are mentioned in the title of the paper because the corresponding theorem for compact manifolds is proved by Pogorelov in the two-dimensional case and by Klingenberg, in any dimension.

2 Integral curves of the gradient of a convex function

Let M be a Riemannian manifold. A set $C \subset M$ is said to be *convex* (*totally convex*) if, for any two points $p, q \in C$, every minimal geodesic (every geodesic) between p and q lies completely in C. The set C is *locally convex* if every $p \in C$ has a neighborhood U such that $U \cap C$ is convex. Let us remind [1] that a closed connected locally convex set $C \subset M$ is a topological manifold with boundary ∂C.

Let $C \subset M$ be a connected locally convex set. For $p, q \in C$, by $\rho_C(p,q)$ we denote the infimum of lengths of curves joining p and q in C. Then $\rho_C(p,q) \geq \rho_M(p,q)$ (where ρ_M is the metric of M) and $\rho_C(p,q) = \rho_M(p,q)$ for sufficiently close p and q. In particular, ρ_C and ρ_M define the same topology on C.

Lemma 2.1 *Let $C \subset M$ be a connected closed locally convex set. For any $p, q \in C$, there exists a geodesic of the length $\rho_C(p,q)$ joining p and q in C.*

The proof is obvious.

A function $f : C \to \mathbf{R}$ is said to be *convex* if, for any geodesic $\gamma : [0,1] \to C$, the function $f \circ \gamma$ is convex, i.e., $f(\gamma(\alpha_1 t_1 + \alpha_2 t_2)) \geq \alpha_1 f(\gamma(t_1)) + \alpha_2 f(\gamma(t_2))$ for $t_1, t_2 \in [0,1],\ \alpha_1, \alpha_2 \geq 0,\ \alpha_1 + \alpha_2 = 1$.

Let now $C \subset M$ be a compact connected locally convex set with nonemty boundary and $f : C \to \mathbf{R}$ be a function satisfying the following hypotheses:

(i) f is a convex function;

(ii) $f|_{\partial C} = 0$;

(iii) f is Lipschitz-continuous, i.e., there exists a constant K such that

$$|f(p) - f(q)| \leq K \rho_C(p,q)$$

for any $p, q \in C$;

(iv) $m = \max\{f(p) \mid p \in C\} > 0$.

The set C and function f are fixed in this Section. Let us agree to write $\rho(p,q)$ instead of $\rho_C(p,q)$. The latter will be sometimes abbreviated to pq. The minimal geodesic as in Lemma 2.1 will be sometimes also denoted by pq.

Given $p \in C$, denote $C_p = \{x \in M_p \mid \exp_p tx \in C$ for sufficiently small $t > 0\}$. The closure $\overline{C}_p$ is a convex cone in M_p. By ∂C_p we denote the boundary of C_p with respect to its linear span in M_p. If $p \in C \setminus \partial C$, then $\partial C_p = \emptyset$. Set $\mathring{C}_p = C_p \setminus \partial C_p$. Obviously, $\mathring{C}_p = \{x \in M_p \mid \exp_p tx \in C \setminus \partial C$ for sufficiently small $t > 0\}$. Set also $TC = \cup_{p \in C} C_p$, $\mathring{T}C = \cup_{p \in C} \mathring{C}_p$. TC is the subset of TM, and we endow TC with the induced topology. Then $\mathring{T}C$ is the open subset of TC.

Let $p \in C,\ x \in C_p$. Set $f_x(t) = f(\exp_p tx)$. The function f_x is defined, convex, and Lipschitz-continuous on some interval $[0, \varepsilon)$ and hence has a finite right-hand derivative at any point of the interval. Let $f'(p,x)$ denote the value of the right-hand derivative of f_x at $t = 0$. Extend the function $f'(p,\cdot)$ to the closure $\overline{C}_p$ of C_p by setting $f'(p,x) = 0$ for $x \in \overline{C}_p \setminus C_p$. The following lemma contains all properties of $f'(p,x)$ we need.

Lemma 2.2 *For any $p \in C$, the function $f'(p,\cdot) : \overline{C}_p \to \mathbf{R}$ is Lipschitz-continuous, convex, and positively homogeneous, i.e., $f'(p,tx) = tf'(p,x)$ for $t \geq 0$; $f'(p,x) = 0$ if $x \in \partial C_p$. If $f(p) < m$, then the function $f'(p,\cdot)$ reaches the positive maximal value on the set $C_p \cap S_p$ (S_p is the unit sphere in M_p) in a unique point of the set. The function $f' : \mathring{T}C \to \mathbf{R}$ is lower semicontinuous.*

All these statements are well known in the case $M = \mathbf{R}^n$, see [3]. In the general case, the proofs differ in minor details that are omitted. By the Lemma, given $p \in C$ such that

$f(p) < m$, there exists a unique vector $x \in C_p \cap S_p$ such that $0 < f'(p,x) = \max\{f'(p,x') \mid x' \in C_p \cap S_p\}$. The vector $f'(p,x)x \in M_p$ will be called the (generalized) *gradient* of the function f at the point p and will be denoted by $\nabla f(p)$.

Lemma 2.3 *For every $x \in C_p$, the inequality holds*

$$f'(p,x) \le \langle x, \nabla f(p)\rangle.$$

Proof. $\nabla f(p) + \lambda x \in C_p$ for any $\lambda > 0$. By the definition of $\nabla f(p)$,

$$f'(p, \nabla f(p) + \lambda x) = |\nabla f(p) + \lambda x| \cdot f'(p, (\nabla f(p) + \lambda x)/|\nabla f(p) + \lambda x|) \le$$
$$\le |\nabla f(p)| \cdot |\nabla f(p) + \lambda x|.$$

On the other hand, by the convexity of $f'(p,\cdot)$,

$$f'(p, \nabla f(p) + \lambda x) \ge f'(p, \nabla f(p)) + \lambda f'(p,x) = |\nabla f(p)|^2 + \lambda f'(p,x).$$

Comparing two last inequalities, we obtain

$$|\nabla f(p)|^2 + \lambda f'(p,x) \le |\nabla f(p)| \cdot |\nabla f(p) + \lambda x| = |\nabla f(p)|^2 + \lambda\langle x, \nabla f(p)\rangle + o(\lambda).$$

This implies the desired inequality. The lemma is proved.

Corollary 2.4 *Let $p, q \in C$ be such that $f(p) < m$ and $f(p) \le f(q)$. Then the angle at the point p between the vector $\nabla f(p)$ and the minimal geodesic pq is not greater than $\pi/2$.*

Proof. Let $\gamma : [0,1] \to C$ be the geodesic parameterization of the curve pq. Then $f \circ \gamma$ is the convex function on $[0,1]$ and $f(\gamma(0)) \le f(\gamma(1))$. Therefore $f'(p, \dot\gamma(0) \ge 0$.

Let $\varphi : [a,b] \to M$ be a continuous curve, $t \in [a,b)$, and $x \in M_{\varphi(t)}$. We say that x is the *right-hand tangent vector* to the curve φ at the point t and write this fact as $x = \dot\varphi_+(t)$ if, for any $g \in C^\infty(M)$, the function $g \circ \varphi$ has the right-hand derivative at t which is equal to xg.

Lemma 2.5 *Let $\varphi : [a,b] \to M$ be a continuous curve and $t \in [a,b)$. For $\tau > 0$, let $c_\tau : [0,1] \to M$ be the geodesic parameterization of the minimal geodesic $\varphi(t)\varphi(t+\tau)$. Set $x(\tau) = \dot c_\tau(0)$. A vector $x \in M_{\varphi(t)}$ is the right-hand tangent to φ at t if and only if*

$$\lim_{\tau \downarrow 0} x(\tau)/\tau = x.$$

The proof is obvious.

Let a point $p \in C$ and number T be such that $f(p) = t_0 < T \le m$. A continuous curve $\varphi_p : [t_0, T] \to C$ will be called the *integral curve of the field ∇f starting from the point p* if it satisfies the following three conditions:

(1) $\varphi_p(t_0) = p$;

(2) φ_p is locally Lipschitz-continuous on $[t_0, T)$, i.e., for any $t \in [t_0, T)$ there exist such K and $\varepsilon > 0$ that

$$\rho(\varphi_p(t'), \varphi_p(t'')) \le K|t' - t''| \quad \text{if} \quad |t' - t| < \varepsilon,\ |t'' - t| < \varepsilon;$$

(3) $f(\varphi_p(t)) < m$ and $\dot\varphi_{p+}(t) = \nabla f(\varphi_p(t))/|\nabla f(\varphi_p(t))|^2$ for every $t \in [t_0, T)$.

If additionally $T = m$, then φ_p is called the *maximal integral curve of the field ∇f starting from p.*

Theorem 2.6 *For every point $p \in C$ satisfying $f(p) < m$, there exists a unique maximal integral curve of the field ∇f starting from p.*

The proof of the theorem is presented below as a sequence of lemmas.

Lemma 2.7 *Let $p \in C$, $f(p) = t_0 < T \le m$, and $\varphi_p : [t_0, T] \to C$ be an integral curve of the field ∇f starting from p. Then $f(\varphi_p(t)) = t$ for all $t \in [t_0, T]$.*

Proof. The function $g(t) = f(\varphi_p(t))$ is locally Lipschitz-continuous on $[t_0, T)$ as the composition of two Lipschitz-continuous functions. Therefore g is differentiable almost at all t and

$$g(t_2) - g(t_1) = \int_{t_1}^{t_2} g'(t)dt \quad \text{for} \quad t_1, t_2 \in [t_0, T].$$

Thus, it suffices to prove that $g'(t) = 1$ under the assumption of the existence of $g'(t)$. If the derivative $g'(t)$ exists for some $t \in [t_0, T)$, it coincides with the right-hand derivative $g'_+(t)$. We will prove that the right-hand derivative exists and is equal to 1 for any $t \in [t_0, T)$.

Fix $t \in [t_0, T)$ and set $x = \nabla f(\varphi_p(t))/|\nabla f(\varphi_p(t))|^2$. By Lemma 2.5,

$$\rho(\varphi_p(t+\tau), \exp_{\varphi_p(t)} \tau x) = o(\tau) \quad \text{for} \quad \tau > 0.$$

Using the Lipschitz-continuity of f, this implies

$$g(t+\tau) = f(\varphi_p(t+\tau)) = f(\exp_{\varphi_p(t)} \tau x) + o(\tau).$$

Therefore $g'_+(t)$ exists and is equal to 1. The lemma is proved.

Lemma 2.8 *Let points $p_0, p_1 \in C$ and number T be such that $f(p_0) = f(p_1) = t_0 < T \le m$. Let $\varphi_{p_i} : [t_0, T] \to C$ $(i = 0, 1)$ be integral curves of the field ∇f starting from p_0 and p_1 respectively. Then $g(t) = \rho(\varphi_{p_0}(t), \varphi_{p_1}(t))$ is the nonincreasing function on $[t_0, T]$.*

Proof. By the same arguments as in the previous lemma, it suffices to prove that the right-hand derivative $g'_+(t)$ is nonpositive if it exists. By Lemma 2.1, there exists a geodesic $c : [0, 1] \to C$ of the length $g(t)$ such that $c(i) = \varphi_{p_i}(t)$ $(i = 0, 1)$. Let α_i $(i = 0, 1)$ be the angle at the point p_i between the geodesic c and vector $x_i = \nabla f(\varphi_{p_i}(t))/|\nabla f(\varphi_{p_i}(t))|^2$. By Lemma 2.6 and Corollary 2.4,

$$\alpha_i \le \pi/2 \quad (i = 0, 1). \tag{2.1}$$

Let $x(s) \in C_{c(s)}$ $(0 \le s \le 1)$ be a smooth vector field along c satisfying $x(i) = x_i$ $(i = 0, 1)$. For $\tau \ge 0$, let $l(\tau)$ be the length of the curve $c_\tau : [0, 1] \to C$, $c_\tau(s) = \exp_{c(s)} \tau x(s)$. The function $l(\tau)$ is smooth, $l(0) = g(t)$, and by the formula for the first variation of length, $l'(0) = -(|x_0| \cos \alpha_0 + |x_1| \cos \alpha_1)$. This implies with the help of (2.1)

$$l(\tau) \le g(t) + o(\tau). \tag{2.2}$$

By Lemma 2.5,

$$\rho(\varphi_{p_i}(t+\tau), c_\tau(i)) = o(\tau) \quad (i = 0, 1). \tag{2.3}$$

From (2.2) and (2.3)

$$g(t+\tau)-g(t)=\rho(\varphi_{p_0}(t+\tau),\varphi_{p_1}(t+\tau))-g(t)\le$$

$$\le\rho(\varphi_{p_0}(t+\tau),c_\tau(0))+\rho(c_\tau(0),c_\tau(1))+\rho(c_\tau(1),\varphi_{p_1}(t+\tau))-g(t)\le$$
$$\le l(\tau)-g(t)+o(\tau)\le o(\tau),$$

i.e., $g(t+\tau)-g(t)\le o(\tau)$. Hence $g'_+(t)\le 0$. The lemma is proved.

Applying this lemma for $p_0=p_1$, we obtain

Corollary 2.9 *There exists at most one maximal integral curve of the field ∇f starting from any point $p\in C$ such that $f(p)<m$.*

The following statement is proved along the same line as Lemma 2.8.

Lemma 2.10 *Let points $p_0,p_1\in C$ be such that $f(p_0)=t_0<m,\ t_0\le f(p_1)=t_1$. Let $\varphi_{p_0}:[t_0,t_1]\to C$ be the integral curve of the field ∇f starting from p_0. Then the function $g(t)=\rho(\varphi_{p_0}(t),p_1)$ does not increase on $[t_0,t_1]$.*

Let $p\in C$ and $t>0$ be such that $f(p)+t\le m$. By p^t we denote the nearest to p point of the set $\{q\in C\mid f(q)\ge f(p)+t\}$. If there are several such points, let p^t denote one of them.

Lemma 2.11 *Let $p_0\in C,\ f(p_0)<m$. For any $\varepsilon>0$, there exist a neighborhood U of p_0 and $t_0>0$ such that the inequality $t/pp^t\ge|\nabla f(p_0)|-\varepsilon$ holds for any $p\in U\cap C$ and $0<t<t_0$.*

Proof. The vector $x_0=\nabla f(p_0)/|\nabla f(p_0)|$ belongs to $\mathring{C}_{p_0}$, $|x_0|=1$, and $f'(p_0,x_0)=|\nabla f(p_0)|$. Choose $\varepsilon'>0$ such that $(|\nabla f(p_0)|-\varepsilon')/(1+\varepsilon')>|\nabla f(p_0)|-\varepsilon$.

Since f' is lower semicontinuous on $\mathring{T}C$, we can find an open neighborhood W of the point (p_0,x_0) in $\mathring{T}C$ such that

$$f'(p,x)\ge|\nabla f(p_0)|-\varepsilon' \tag{2.4}$$

for $(p,x)\in W$. Let us now choose a compact neighborhood U of the point p_0 in C and a continuous section $x(p)$ of the bundle $\mathring{T}C$ over U whose value at p_0 coincides with x_0. By decreasing U, we can assume that $x(p)\in W$ and

$$|x(p)|<1+\varepsilon'\quad(p\in U). \tag{2.5}$$

For $p\in W$, we set

$$h(p)=\sup\{s\mid d(\exp_p s'x(p))/ds'(s')\in W\text{ for }0\le s'\le s\}.$$

The function h is positive and lower semicontinuous on the compact set U as one can easily show. Therefore

$$t'_0=\inf\{h(p)\mid p\in U\}>0.$$

Let now $p \in U$ and $0 < t' < t'_0$. The function $g(s) = f(\exp_p sx(p))$ is convex on $[0, t']$ and, by (2.4),

$$g'_+(s) = f'(\exp_p sx(p), d(\exp_p sx(p))/ds(s)) \geq |\nabla f(p_0)| - \varepsilon'.$$

Hence

$$f(\exp_p t'x(p)) - f(p) = \int_0^{t'} g'_+(s)ds \geq t'(|\nabla f(p_0)| - \varepsilon'). \tag{2.6}$$

On the other hand, (2.5) implies

$$\rho(p, \exp_p t'x(p)) = t'|x(p)| \leq t'(1 + \varepsilon'). \tag{2.7}$$

Comparing (2.6) and (2.7), we see that

$$pp^{t'(|\nabla f(p_0)| - \varepsilon')} \leq t'(1 + \varepsilon').$$

The last inequality holds for any t' satisfying $0 < t' < t'_0$. Setting $t'(|\nabla f(p_0)| - \varepsilon') = t$, we obtain

$$t/pp^t \geq (|\nabla f(p_0)| - \varepsilon')/(1 + \varepsilon') \geq |\nabla f(p_0)| - \varepsilon.$$

The latter inequality holds for any t satisfying $0 < t < t'_0/(|\nabla f(p_0)| - \varepsilon') = t_0$. The lemma is proved.

Let a point $p \in C$ and number T be such that $f(p) = t_0 < T < m$. For a partition $\alpha = \{t_0 < t_1 < \ldots < t_n = T\}$ of the segment $[t_0, T]$, let $L(p, \alpha)$ denote the broken geodesic $p_0 p_1 \ldots p_n$, where $p_0 = p$, $p_i = p_{i-1}^{t_i - t_{i-1}}$ $(i = 1, \ldots, n)$. Introduce the parameterization $L(p, \alpha) : [t_0, T] \to C$ of the broken geodesic by setting $L(p, \alpha)(t) = c_i(t)$ for $t \in [t_{i-1}, t_i]$ $(i = 1, \ldots, n)$, where $c_i : [t_{i-1}, t_i] :\to C$ is the geodesic parameterization of the minimal geodesic $p_{i-1}p_i$.

The *rank* of a partition $\alpha = \{t_0 < t_1 < \ldots < t_n = T\}$ is the maximum of $t_i - t_{i-1}$ $(i = 1, \ldots, n)$.

Lemma 2.12 *Let $p \in C$ and T be such that $f(p) = t_0 < T < m$. The family $L(p, \alpha)$ of parameterized curves, where α runs over all partitions of the segment $[t_0, T]$, is Lipschitz equicontinuous, i.e., there exists K such that*

$$\rho(L(p, \alpha)(t), L(p, \alpha)(t')) \leq K|t - t'| \tag{2.8}$$

for any α and $t, t' \in [t_0, T]$.

Proof. Set $K = d/(m - T)$, where d is the diameter of C. Let $\alpha = \{t_0 < t_1 < \ldots < t_n = T\}$ be a partition of $[t_0, T]$ and $p_i = L(p, \alpha)(t_i)$ $(i = 0, \ldots, n)$. Since $L(p, \alpha)$ coincides with the geodesic parameterization of the minimal geodesic $p_{i-1}p_i$ on $[t_{i-1}, t_i]$, it suffices to prove the inequalities

$$p_{i-1}p_i \leq K(t_i - t_{i-1}) \quad (i = 1, \ldots, n). \tag{2.9}$$

Fix some $i = 1, \ldots, n$. Choose a point q satisfying $f(q) = m$. Let $c : [0, 1] \to C$ be the geodesic parameterization of a minimal geodesic $p_{i-1}q$. Set $q' = c((t_i - t_{i-1})/(m - t_{i-1}))$. Since f is convex along c,

$$f(q') \geq f(c(0))(m - t_i)/(m - t_{i-1}) + f(c(1))(t_i - t_{i-1})/(m - t_{i-1}).$$

Taking the equalities $f(c(0)) = f(p_{i-1}) = t_{i-1}$ and $f(c(1)) = m$ into account, we obtain $f(q') \geq t_i$. Since p_i is the closest point to p_{i-1} among all points q' satisfying the last inequality, we deduce $p_{i-1}p_i \leq p_{i-1}q'$. On the other hand,

$$p_{i-1}q' = p_{i-1}q \cdot (t_i - t_{i-1})/(m - t_{i-1}) \leq d(t_i - t_{i-1})/(m - T) = K(t_i - t_{i-1}).$$

Comparing two last inequalities, we obtain (2.9). The lemma is proved.

Remark. Lemma 2.12 is the only point in our arguments where the compactness of C is used. I do not know whether Theorem 2.6 is true for a noncompact C. It is true in the case $M = \mathbf{R}^n$.

Lemma 2.13 *Let $p_0 \in C$ and T be such that $f(p_0) = t_0 < T < m$. Let α_k be a sequence of partitions of the segment $[t_0, T]$ such that rank $\alpha_k \to 0$ as $k \to \infty$. Assume the sequence $L(p_0, \alpha_k)$ to converge uniformly to some parameterized curve $\varphi : [t_0, T] \to C$. Then*

(1) $\varphi(t_0) = p_0$;

(2) φ is Lipschitz-continuous;

(3) $f(\varphi(t)) = t$ for $t \in [t_0, T]$;

(4) for any $t \in [t_0, T)$,

$$\overline{\lim_{\tau \downarrow 0}} \rho(\varphi(t + \tau), \varphi(t))/\tau \leq 1/|\nabla f(\varphi(t))|.$$

Proof. The first statement is obvious, the second statement follows from Lemma 2.12. The third statement follows from the equalities $f(L(p_0, \alpha)(t_i)) = t_i$ $(i = 0, \ldots, n)$ which hold for any partition $\alpha = \{t_0 < t_1 < \ldots < t_n = T\}$.

Let us prove the last statement. Let $t \in [t_0, T)$ and $\varepsilon > 0$. By Lemma 2.11, there exist a neighborhood U of the point $\varphi(t)$ and $\tau_0 > 0$ such that

$$\tau/pp^\tau \geq |\nabla f(\varphi(t))| - \varepsilon \tag{2.10}$$

for any $p \in U \cap C$ and $0 < \tau < \tau_0$. By decreasing τ_0, we can assume that

$$\varphi([t, t + \tau_0]) \subset U.$$

Since $L(p_0, \alpha_k)$ converges uniformly to φ, we can find such N that rank $\alpha_k < \tau_0$ and $L(p_0, \alpha_k)([t, t + \tau_0]) \subset U$ for $k > N$.

Let $0 < \tau < \tau_0$. Fix $k > N$ and consider the partition $\alpha_k = \{t_0 < t_1 < \ldots < t_n = T\}$. Set

$$r = r(k) = \min\{i \mid t \leq t_i\}; \quad s = s(k) = \max\{i \mid t_i \leq t + \tau\}.$$

Let $r < i \leq s$, $p_i = L(p_0, \alpha_k)(t_i)$, $p_{i-1} = L(p_0, \alpha_k)(t_{i-1})$. Since $p_i = p_{i-1}^{t_1 - t_{i-1}}$, (2.10) implies that

$$p_{i-1}p_i \leq (t_i - t_{i-1})/(|\nabla f(\varphi(t)) - \varepsilon) \quad (i = r + 1, \ldots, s).$$

Taking the sum of these inequalities, we obtain

$$\rho(L(p, \alpha_k)(t_{r(k)}), L(p, \alpha_k)(t_{s(k)})) \leq (t_{s(k)} - t_{r(k)})/(|\nabla f(\varphi(t)) - \varepsilon).$$

Passing to the limit as $k \to \infty$, we arrive to the inequality

$$\rho(\varphi(t), \varphi(t + \tau)) \leq \tau/(|\nabla f(\varphi(t)) - \varepsilon)$$

which proves the desired statement because ε is arbitrary. The lemma is proved.

Lemma 2.14 *Let a point $p \in C$ be such that $f(p) = t_0 < m$. Given any T satisfying $t_0 < T < m$, there exists an integral curve $\varphi_p : [t_0, T] \to C$ of the field ∇f starting from p.*

Proof. Lemma 2.12 implies that the family $\{L(p, \alpha)\}$, where α runs over all partitions of the segment $[t_0, T]$, is equicontinuous and uniformly bounded. Therefore any sequence of the family contains a uniformly converging subsequence. Let α_k be such a sequence that $L(p, \alpha_k)$ converges uniformly to a curve $\varphi_p : [t_0, T] \to C$ and $\operatorname{rank} \alpha_k \to 0$.

By Lemma 2.13, the curve φ_p satisfies two first conditions of the definition of an integral curve. Let us check the third condition. To this end, by Lemma 2.5, it suffices to show that

$$\lim_{\tau \downarrow 0} x(\tau)/\tau = \nabla f(\varphi_p(t))/|\nabla f(\varphi_p(t))|^2 \tag{2.11}$$

for any $t \in [t_0, T)$, where $x(\tau) = \dot{c}_\tau(0)$ and $c_\tau : [0, 1] \to C$ is the geodesic parameterization of the minimal geodesic $\varphi_p(t)\varphi_p(t + \tau)$.

By Lemma 2.13,

$$\overline{\lim_{\tau \downarrow 0}} |x(\tau)|/\tau = \overline{\lim_{\tau \downarrow 0}} \rho(\varphi_p(t), \varphi_p(t + \tau))/\tau \leq 1/|\nabla f(\varphi_p(t))|. \tag{2.12}$$

In particular, the function $x(\tau)/\tau$ is bounded. Let x_0 be a partial limit of the function as $\tau \downarrow 0$, and $\tau_n > 0$ be a sequence converging to zero such that $x(\tau_n)/\tau_n$ converges to x_0. Then

$$|x_0| \leq 1/|\nabla f(\varphi_p(t))|. \tag{2.13}$$

The condition $x(\tau_n)/\tau_n \to x_0$ can be rewritten as:

$$x(\tau_n) = \tau_n x_0 + o(\tau_n).$$

This implies that

$$\rho(\varphi_p(t + \tau_n), \exp_{\varphi_p(t)} \tau_n x_0) = \rho(\exp_{\varphi_p(t)} x(\tau_n), \exp_{\varphi_p(t)} \tau_n x_0) = o(\tau_n).$$

On using the Lipschitz continuity of f, we obtain

$$f(\varphi_p(t + \tau_n)) - f(\exp_{\varphi_p(t)} \tau_n x_0) = o(\tau_n).$$

Since $f(\varphi_p(t + \tau_n)) = t + \tau_n = f(\varphi_p(t)) + \tau_n$ by Lemma 2.13, the previous formula can be rewritten as

$$f(\exp_{\varphi_p(t)} \tau_n x_0) - f(\varphi_p(t)) = \tau_n + o(\tau_n).$$

Dividing this equality by τ_n and passing to the limit as $\tau_n \downarrow 0$, we obtain $f'(\varphi_p(t), x_0) = 1$. This implies with the help of (2.13) and the homogeneity of $f'(\varphi_p(t), \cdot)$ that

$$f'(\varphi_p(t), x_0/|x_0|) = 1/|x_0| \geq |\nabla f(\varphi_p(t))|.$$

By Lemma 2.2, this implies that

$$1/|x_0| = |\nabla f(\varphi_p(t))| \quad \text{and} \quad x_0/|x_0| = \nabla f(\varphi_p(t))/|\nabla f(\varphi_p(t))|,$$

i.e., that $x_0 = \nabla f(\varphi_p(t))/|\nabla f(\varphi_p(t))|^2$. We have thus proved that every partial limit of the left-hand side of (2.11) coincides with the right-hand side. This proves (2.11) and the lemma.

Proof of Theorem 2.6. Let $p \in C,\ f(p) = t_0 < m$. The uniqueness of the maximal integral curve of the field ∇f starting from p is stated in Corollary 2.9. Let us prove the existence.

Let $t_0 < t_1 < t_2 < \dots$ be a sequence converging to m. Using Lemma 2.14, we construct by induction on n a curve $\varphi^n : [t_{n-1}, t_n] \to C$ for every $n = 1, 2, \dots$ such that φ^1 is the integral curve of ∇f starting from p and φ^n is the integral curve starting from $\varphi^{n-1}(t_{n-1})$ $(n = 2, 3, \dots)$. Define the curve $\varphi_p : [t_0, m) \to C$ by setting $\varphi_p(t) = \varphi^n(t)$ for $t \in [t_{n-1}, t_n]$. The restriction of φ_p to any segment $[t_0, T]$ with $t_0 < T < m$ is the integral curve of ∇f. To finish the proof, we have to show that $\varphi_p(t)$ converges to some point as $t \to m$.

Let $q' \in C$ be such that $f(q') = m$. By Lemma 2.10, $\rho(\varphi_p(t), q')$ is a nonincreasing function. Hence, the curve φ_p is bounded and $\varphi_p(t)$ has at least one limit point q as $t \to m$. By the same Lemma 2.10, $\rho(\varphi_p(t), q)$ is a nonincreasing function. Hence $\varphi_p(t) \to q$ as $t \to m$, and we can extend φ_p to $[t_0, m]$ by setting $\varphi_p(m) = q$. The theorem is proved.

Now, we present the following analogue of the Buzemann — Feller theorem.

Theorem 2.15 *Let $C_t = \{p \in C \mid f(p) \geq t\}$ for $0 \leq t \leq m$. Define the map $R_t : C \to C_t$ as follows: $R_t(p) = p$ if $f(p) \geq t$ and $R_t(p) = \varphi_p(t)$ if $f(p) = t_0 < t$, where $\varphi_p : [t_0, m] \to C$ is the maximal integral curve of the field ∇f starting from p. Then*

(1) the map R_t does not increase lengths of curves, i.e.,

$$L(\gamma) \geq L(R_t \circ \gamma)$$

for any curve $\gamma : [0, 1] \to C$;

(2) the map $R : C \times [0, m] \to C,\ R(p, t) = R_t(p)$ is continuous.

Thus, $R_t : C \to C_t$ is a deformation retraction.

Proof. The first statement is equivalent to the following one: R_t does not increase distances, i.e.,

$$\rho(R_t(p_0), R_t(p_1)) \leq \rho(p_0, p_1) \quad (p_0, p_1 \in C). \tag{2.14}$$

For definiteness, let $f(p_0) = t_0 \leq t_1 = f(p_1)$. Consider three possible cases.

(a) $t \leq t_0$. (2.14) holds since $R_t(p_0) = p_0$ and $R_t(p_1) = p_1$.

(b) $t_0 \leq t \leq t_1$. In this case $R_t(p_1) = p_1$ and, by Lemma 2.10,

$$\rho(R_t(p_0), R_t(p_1)) = \rho(\varphi_{p_0}(t), p_1) \leq \rho(\varphi_{p_0}(t_0), p_1) = \rho(p_0, p_1).$$

(c) $t_1 \leq t$. By Lemma 2.9,

$$\rho(R_t(p_0), R_t(p_1)) = \rho(\varphi_{\varphi_{p_0}(t_1)}(t), \varphi_{p_1}(t)) \leq \rho(\varphi_{p_0}(t_1), p_1)$$

and

$$\rho(\varphi_{p_0}(t_1), p_1) \leq \rho(p_0, p_1)$$

by Lemma 2.10. Comparing two last inequality, we obtain (2.14).

Let us prove the second statement of the theorem. Let $(p_0, t_0) \in C \times [0, m]$. For $t \in [0, m]$, $R(p_0, t) = p_0$ if $t \leq f(p_0)$, and $R(p_0, t) = \varphi_{p_0}(t)$ if $f(p_0) \leq t$. So, the map $R(p_0, \cdot) : [0, m] \to C$ is continuous. Given $\varepsilon > 0$, we can find $\delta > 0$ such that $\rho(R(p_0, t_0), R(p_0, t)) < \varepsilon/2$ if $|t - t_0| < \delta$. Let now (p, t) be such that $\rho(p, p_0) < \varepsilon/2$ and $|t - t_0| < \delta$. Then

$$\rho(R(p_0, t_0), R(p, t)) \leq \rho(R(p_0, t_0), R(p_0, t)) + \rho(R_t(p_0), R_t(p)) < \varepsilon/2 + \rho(p_0, p_1) < \varepsilon.$$

The theorem is proved.

3 Proof of Theorem 1.1

We will prove a more general statement.

Let C be a compact convex set in a Riemannian manifold M. If ∂C is not empty, we set $C_t = \{p \in C \mid \rho(p, \partial C) \geq t\}$ for any $t \geq 0$. Let $t_{\max} = \max\{t \mid C_t \neq \emptyset\}$. The set $C_{t_{\max}}$ will be denoted by $C_{\max}$.

Let now M be an open (= complete, connected, noncompact, with no boundary) manifold of nonnegative curvature. Recall [1] that there exists a finite sequence

$$C^1 \supset C^2 \supset \ldots \supset C^{n+1} = S \tag{3.1}$$

of compact totally convex sets such that $C^{i+1} = C^i_{\max}$ $(i = 1, \ldots, n)$ and S is a totally geodesic submanifold (soul) of M with no boundary. Additionally, given a compact set $A \subset M$, the set C^1 can be chosen such that $A \subset C^1$ without changing other sets C^i of the sequence. M is homeomorphic to the Euclidean space if and only if the soul S is a one-point set.

Recall also that the injectivity radius $r(M)$ of a Riemannian manifold M is the supremum of such $s > 0$ that every geodesic of length s is minimizing. In the case of $\dim M = 0$, the injectivity radius of M is assumed to be equal to ∞.

Theorem 3.1 *Let M be an open Riemannian manifold whose sectional curvature satisfies $0 \leq K \leq 1$ at any point and any two-dimensional direction. Let S be a soul of M. The injectivity radii of M and S satisfy the relation*

$$r(M) \geq \min\{\pi, r(S)\}.$$

Proof. Assume the statement to be wrong, and let c_0 be a geodesic in M of the length $s_0 < \pi$, $s_0 < r(S)$ which is not minimizing. Join the ends of c_0 by a minimal geodesic $\bar{c}_0$, and denote by γ_0 the geodesic biangle formed by c_0 and $\bar{c}_0$. The biangle is nondegenerate and its length is less then $2s_0$.

Let (3.1) be such a sequence of compact totally convex sets that $\gamma_0 \subset C^1$. Consider the family of all nondegenerate geodesic biangles in C^1, and let s be the infimum of lengths of such biangles. Then $s < 2s_0$ and $s > 0$ because of the compactness of C^1.

Choose a sequence γ_k of nondegenerate geodesic biangles in C^1 whose lengths converge to s. Using the total convexity of C^1 and the inequality $s < \pi$, one can show [2] that there exists a limit biangle γ for some subsequence of the sequence and that γ is a closed geodesic of length s. Let $\gamma : [0,1] \to C^1$ be a parameterization of the geodesic.

For every $i = 1, \ldots, n$, define the function $f^i : C^i \to \mathbf{R}$ by $f^i(p) = \rho(p, \partial C^i)$. As is shown in [1], f^i is a convex function. Besides this, $f^i|_{\partial C^i} = 0$ and f^i is Lipschitz-continuous

$$|f^i(p) - f^i(q)| \leq \rho(p,q) \quad (p, q \in C^i).$$

Thus, for every $i = 1, \ldots, n$, the set C^i and functions f^i satisfy conditions (i)–(iv) listed at the beginning of Section 2, and our results from Section 2 can be applied.

Let us set

$$m_i = \max\{f^i(p) \mid p \in C^i\} \ (i = 1, \ldots, n); \quad m = m_1 + \ldots + m_n,$$

So, $C^{i+1} = C^i_{m_i}$. For every $i = 1, \ldots, n$ and for every $t \in [0, m_i]$, let $R^i_t : C^i \to C^i_t$ be the retraction constructed in Theorem 2.15 with the help of the function f^i.

Now, for every $t \in [0, m]$, define the curve $\gamma_t : [0, 1] \to C^1$ by

$$\gamma_t = R^1_t \circ \gamma \quad \text{for} \quad t \in [0, m_1];$$

$$\gamma_{m_1+\ldots+m_{i-1}+t} = R^i_t \circ \gamma_{m_1+\ldots+m_{i-1}} \quad \text{for} \quad t \in [0, m_i] \ (i = 1, \ldots, n).$$

By Theorem 2.15, γ_t is a closed curve for any $t \in [0, m]$, $\gamma_0 = \gamma$, the length of γ_t is not greater than s, and the map $(t, \tau) \mapsto \gamma_t(\tau)$ is continuous on $[0, m] \times [0, 1]$. The curve γ_m lies in S.

Let t_0 be the supremum of those $t \in [0, m]$ for which γ_t is a nondegenerate closed geodesic. Then the curve $c = \gamma_{t_0}$ is a closed geodesic of the length s.

Observe that $t_0 < m$ since $s < 2r(S)$. Choose a sequence $t_k \in (t_0, m)$ converging to t_0. The sequence γ_{t_k} converges uniformly to c.

Since γ_{t_k} is not a geodesic, its length can be decreased by a small deformation. More precisely, there exists a closed curve $c_k : [0, 1] \to C^1$ whose length is less than the length of γ_{t_k} and such that

$$\sup\{\rho(c_k(\tau), \gamma_{t_k}(\tau)) \mid 0 \le \tau \le 1\} < 1/k \quad (k = 1, 2, \ldots).$$

The sequence c_k converges uniformly to c. We have thus found a closed geodesic c of length s and a uniformly converging to c sequence of closed curves c_k whose lengths are less than s. Now, the proof is finished by repeating the arguments of [2].

The author is grateful to V. Toponogov who has read the paper and made some remarks.

References

[1] Cheeger J., Gromoll D. The structure of complete manifolds of nonnegative curvature. *Ann. Math.*, **96** (1972), 413–443.

[2] Klingenberg W. Contributions to differential geometry in large. *Ann. Math.*, **69** (1959), no. 3, 654–666.

[3] Rockafellar R. *Convex Analysis.* Princeton N.J., Princeton University Press, 1970.

[4] Sharafutdinov V. Pogorelov — Klingenberg theorem for manifolds homeomorphic to $\mathbf{R}^n$. *Siberian Math. J.*, **18** (1977), no. 4, 915–925.

[5] Toponogov V. Theorems on minimal geodesics in noncompact Riemannian spaces of positive curvature. *Reports of Soviet Acad.*, **191** (1970), no. 3, 537–539.

[6] Veiner E. Pogorelov — Klingenberg theorem for convex hypersurfaces. *Siberian Math. J.*, **11** (1970), no. 6, 1396–1399.