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ON EMISSION TOMOGRAPHY OF INHOMOGENEOUS MEDIA

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Abstract. The problem of finding the source distribution for particles (or radiation) in a bounded domain D from the emitting flow through the boundary of D is considered. The particles are supposed to move with unit velocity along geodesics of a Riemannian metric and can be absorbed by the medium. The metric and the absorption are known. Uniqueness of the solution to this problem and a stability estimate are obtained under a certain assumption on the absorption and curvature of the metric.

Key words. tomography, X-ray transform, Riemannian manifolds

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1. Introduction. The attenuated X-ray transform of a compactly-supported function f is given on $\mathbf{R}^n$ by

$$I^{\varepsilon} f(x,\xi) = \int\limits_{-\infty}^{\infty} f(x+t\xi) \exp\left[-\int\limits_{t}^{\infty} \varepsilon(x+s\xi)\,ds\right] dt \tag{1.1}$$

where $x \in \mathbf{R}^n,\ 0 \neq \xi \in \mathbf{R}^n$. In this formula $\varepsilon(x) \geq 0$ represents the absorption and is assumed to be compactly-supported. The operator I^{ε} arises in the study of emission tomography. The basic problem in the mathematical theory of the attenuated X-ray transform is whether or not for a given absorption ε the operator I^{ε} has a non-trivial kernel. For non-constant ε all known results include some assumptions on smallness of ε. A review of such results is given in [1].

In (1.1) the integration is made along straight lines. Actually, in tomographic problems the integration is to be made along rays of sounding radiation. Strictly speaking, these rays are straight only in homogeneous media. Any inhomogeneity of the medium implies a ray refraction. It is very small and can be ignored in X-ray tomography. But in other fields such as acoustic, geophysic and optic tomography the ray refraction is to be taken into account. In many important cases the rays are geodesic lines of a Riemannian metric. We thus come to the definition of the attenuated X-ray transform along the geodesics of a Riemannian manifold which is formulated in the next section.

2. Formulation of the result. A compact Riemannian manifold (M, g) with a boundary ∂M is called a *compact dissipative Riemannian manifold* (CDRM briefly) if

1) the boundary is strictly convex, i.e., at every point $x \in \partial M$ the second quadratic form $\mathrm{II}(\xi,\xi) = \langle \nabla_{\xi}\nu, \xi\rangle$ is positive-definite on the tangent space $T_x(\partial M)$; where ν is the unit outer normal vector to $\partial M,\ \nabla$ is the covariant derivative in the metric g and $\langle , \rangle$ is the scalar product in the metric g;

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2) for every point $x \in M$ and every vector $0 \neq \xi \in T_xM$, the maximal geodesic $\gamma_{x\xi}(t)$ determined by the initial conditions $\gamma_{x\xi}(0) = x,\ \dot\gamma_{x\xi}(0) = \xi$ is defined on the finite segment $[\tau_-(x,\xi), \tau_+(x,\xi)]$.

We denote by $TM = \{(x,\xi) \mid x \in M,\ \xi \in T_xM\}$ the space of the tangent bundle. Let $T^0M = \{(x,\xi) \in TM \mid \xi \neq 0\}$ be the manifold of non-vanishing tangent vectors. Note that, together with the definition of CDRM, we have introduced two functions $\tau_+, \tau_- : T^0M \to \mathbf{R}$.

Let $\Omega M = \{(x,\xi) \in TM \mid |\xi|^2 = \langle \xi, \xi\rangle = g_{ij}(x)\xi^i\xi^j = 1\}$ be the manifold of unit tangent vectors. Its boundary $\partial\Omega M$ is the union of the two submanifolds

$$\partial_\pm M = \{(x,\xi) \in \Omega M \mid x \in \partial M,\ \pm\langle \xi, \nu(x)\rangle \geq 0\}$$

of inner and outer vectors.

We fix a smooth non-negative function ε on M that will be called the *absorption*. The linear operator

$$I^\varepsilon : C^\infty(M) \to C^\infty(\partial_+\Omega M) \tag{2.1}$$

defined by the equality

$$I^\varepsilon f(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} f(\gamma_{x\xi}(t))\, \exp\left[-\int\limits_{t}^{0} \varepsilon(\gamma_{x\xi}(s))\,ds\right] dt \tag{2.2}$$

is called the *attenuated X-ray transform on CDRM* (M,g) *corresponding to absorption* ε; here $\gamma_{x\xi} : [\tau_-(x,\xi), 0] \to M$ is the maximal geodesic satisfying the initial conditions $\gamma_{x\xi}(0) = x,\ \dot\gamma_{x\xi}(0) = \xi$.

For a compact manifold N and an integer $k \geq 0$, let $H^k(N)$ be the topological Hilbert space of functions that have generalized locally square integrable derivatives up to order k in any local coordinate system. We denote by $\|\cdot\|_k$ one of equivalent norms on this space. As in [3,4], one shows that operator (2.1) has some bounded continuation

$$I^\varepsilon : H^k(M) \to H^k(\partial_+\Omega M) \tag{2.3}$$

for any integer $k \geq 0$.

Let (R_{ijkl}) be the curvature tensor of the Riemannian manifold (M,g). For a point $x \in M$ and a two-dimensional subspace $\sigma \subset M$ we denote by

$$K(x,\sigma) = R_{ijkl}\xi^i\xi^k\eta^j\eta^k/|\xi\wedge\eta|^2 \tag{2.4}$$

the sectional curvature of M at the point x in the direction σ; here ξ, η is a basis for σ. For $(x,\xi) \in T^0M$, we define

$$K(x,\xi) = \sup_{\sigma \ni \xi} K(x,\sigma). \tag{2.5}$$

For a CDRM (M,g) and a non-negative function $\varepsilon \in C^\infty(M)$ we introduce the following characteristic

$$\kappa(M,g,\varepsilon) = \sup_{(x,\xi)\in\partial_-\Omega M} \int\limits_{0}^{\tau_+(x,\xi)} t\left[2\varepsilon^2(\gamma_{x\xi}(t)) + K(\gamma_{x\xi}(t), \dot\gamma_{x\xi}(t))\right]_+ dt. \tag{2.6}$$

Here $\gamma_{x\xi} : [0, \tau_+(x,\xi)] \to M$ is the maximal geodesic satisfying the initial conditions $\gamma_{x\xi}(0) = x,\ \dot\gamma_{x\xi}(0) = \xi$; and the notation

$$[a]_+ = \begin{cases} a, & \text{if} \quad a \ge 0, \\ 0, & \text{if} \quad a \le 0 \end{cases}$$

is used.

The main result of this paper is the following

THEOREM 2.1. *Let a CDRM (M, g) of dimension $n \ge 2$ and a non-negative function $\varepsilon \in C^\infty(M)$ satisfy the condition*

$$\kappa(M, g, \varepsilon) \le \frac{1}{2}. \tag{2.7}$$

Then any function $f \in H^1(M)$ is uniquely determined by its attenuated X-ray transform $I^\varepsilon f$ and the stability estimate

$$\|f\|_0 \le C\|I^\varepsilon f\|_1 \tag{2.8}$$

holds with a constant C independent of f.

Let us make some remarks on this theorem.

It follows from (2.6) that large values of the absorption can be compensated by negative values of the curvature so as restriction (2.7) holds.

Restriction (2.7) is of an integral nature. Roughly speaking, it means that the values of $[2\varepsilon^2(x) + K(x,\xi)]_+$ are not to be accumulated along with geodesics. At the same time the values of this quantity at some points can be very large.

Let us put

$$\begin{gathered} \varepsilon_0 = \sup_{x\in M} \varepsilon(x), \qquad k(M, g) = \sup_{(x,\xi)\in\Omega M} K(x, \xi), \\ \operatorname{diam}(M, g) = \sup_{(x,\xi)\in\partial_-\Omega M} \tau_+(x, \xi). \end{gathered} \tag{2.9}$$

Condition (2.7) is satisfied if

$$\big[2\varepsilon_0^2 + k(M, g)\big]_+ \operatorname{diam}^2(M, g) \le 1. \tag{2.10}$$

In particular, for a flat metric, i. e., such that the corresponding curvature vanishes identically, restriction (2.10) takes the form

$$\varepsilon_0 \operatorname{diam}(M, g) \le \sqrt{2}/2. \tag{2.11}$$

This corollary of our theorem (the metric is flat and restriction (2.7) is replaced by (2.11)) is stronger than the result obtained in [2]. At the same time it is weaker than the theorem of [1] in which the constant $\sqrt{2}/2$ on the right side of (2.11) is replaced with 5.37.

The proof of Theorem 2.1 is presented in Section 4. Now we only note that by the boundedness of (2.3) one can see that it is sufficient to prove the theorem for a real smooth f.

3. Semibasic tensor fields. Here some notions and results of tensor analysis are exposed that are needed for proving Theorem 2.1. We give the formulations of all definitions and statements but do not present the proofs. The latter can be found in [3] and [5].

The modern mathematical style presumes that invariant (independent of the choice of coordinates) notions are introduced by invariant definitions. Risking to look old-fashioned, here the author consciously chooses the opposite approach. The notions under considerations will be introduced with the help of local coordinates. We will pay the particular attention to the rules of transformation of the quantities under definition with respect to the change of coordinates. Invariant definitions are also possible but they require some more preliminary notions.

Let M be a manifold of dimension n and $\tau_M = (TM, p, M)$ be its tangent bundle. Points of the manifolds TM are designated by the pairs (x, ξ) where $x \in M,\ \xi \in T_xM$. If $(x^1, \dots, x^n)$ is a local coordinate system defined in a domain $U \subset M$, then by $\partial_i = \partial/\partial x^i \in C^\infty(\tau_M; U)$ we mean the coordinate vector fields and by $dx^i \in C^\infty(\tau'_M; U)$ we mean the coordinate covector fields. Recall that the coordinates of a vector $\xi \in T_xM$ are the coefficients of the expansion $\xi = \xi^i \partial/\partial x^i$. On the domain $p^{-1}(U) \subset TM$ the family of the functions $(x^1, \dots, x^n, \xi^1, \dots, \xi^n)$ constitutes a local coordinate system (strictly speaking, we have to write $x^i \circ p$; nevertheless we will use a more brief notation x^i, hoping that it will not lead to misunderstanding) which is called *associated* with the system $(x^1, \dots, x^n)$. From now on we will use only such coordinate systems on TM. If $({x'}^1, \dots, {x'}^n)$ is another coordinate system defined in a domain $U' \subset M$, then in $p^{-1}(U \cap U')$ the associated coordinates are connected by the transformation formulas

$$ {x'}^i = {x'}^i(x^1, \dots, x^n); \qquad {\xi'}^i = \frac{\partial {x'}^i}{\partial x^j}\xi^j. \tag{3.1} $$

Unlike the case of general coordinates, these formulas have the next peculiarity: the first n transformation functions are independent of ξ^i while the last n functions depend linearly on these variables. This peculiarity is the base of all further constructions in the current section.

The algebra of tensor fields of the manifold TM is generated locally by the coordinate fields $\partial/\partial x^i,\ \partial/\partial \xi^i,\ dx^i,\ d\xi^i$. Differentiating (3.1), we obtain the next rules for transforming the fields with respect to change of associated coordinates:

$$ \frac{\partial}{\partial \xi^i} = \frac{\partial {x'}^j}{\partial x^i}\frac{\partial}{\partial {\xi'}^j}, \qquad d{x'}^i = \frac{\partial {x'}^i}{\partial x^j}dx^j, \tag{3.2} $$

$$ \frac{\partial}{\partial x^i} = \frac{\partial {x'}^j}{\partial x^i}\frac{\partial}{\partial {x'}^j} + \frac{\partial^2 {x'}^j}{\partial x^i \partial x^k}\xi^k \frac{\partial}{\partial {\xi'}^j}, \quad d{\xi'}^i = \frac{\partial^2 {x'}^i}{\partial x^j \partial x^k}\xi^k dx^j + \frac{\partial {x'}^i}{\partial x^j}d\xi^j. \tag{3.3} $$

We note that formulas (3.2) contain only the first-order derivatives of the transformation functions and take the observation as the basis for the next definition.

A tensor $u \in T^r_{s,(x,\xi)}(TM)$ of degree (r, s) at a point (x, ξ) of the manifold TM is called *semibasic* if in some (and, consequently, in any) associated coordinate system it can be represented as:

$$ u = u^{i_1 \dots i_r}_{j_1 \dots j_s} \frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}. \tag{3.4} $$

The numbers $u^{i_1 \dots i_r}_{j_1 \dots j_s}$ are called the *coordinates* (or *components*) of the tensor u. Assuming the choice of the associated coordinate system to be clear from the context (or arbitrary), we will abbreviate equality (3.4) to the next one:

$$u = \left(u^{i_1\dots i_r}_{j_1\dots j_s}\right). \tag{3.5}$$

It follows from (3.2) that, under change of associated coordinates, the components of a semibasic tensor are transformed by the formulas

$$u'^{i_1\dots i_r}_{j_1\dots j_s} = \frac{\partial x'^{i_1}}{\partial x^{k_1}} \dots \frac{\partial x'^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial x'^{j_1}} \dots \frac{\partial x^{l_s}}{\partial x'^{j_s}} u^{k_1\dots k_r}_{l_1\dots l_s} \tag{3.6}$$

which are identical in form with formulas for transforming components of an ordinary tensor on M. The set of all semibasic tensors of degree (r,s) constitutes the subbundle in $\tau^r_s(TM)$. We shall denote the subbundle by $\beta^r_s M = (B^r_s M, p^r_s, TM)$. Sections of this bundle are called *semibasic tensor fields of degree* (r,s). For such a field $u \in C^\infty(\beta^r_s M)$, equalities (3.4) and (3.5) are valid in the domain $p^{-1}(U)$ in which an associated coordinate system acts; here $u^{i_1\dots i_r}_{j_1\dots j_s} \in C^\infty(p^{-1}(U))$. Note that $C^\infty(\beta^0_0 M) = C^\infty(TM)$, i.e., semibasic tensor fields of degree $(0,0)$ are just smooth functions on TM. The elements of $C^\infty(\beta^1_0 M)$ are called the *semibasic vector fields*, and the elements of $C^\infty(\beta^0_1 M)$ are called *semibasic covector fields.*

The set $C^\infty(\beta^r_s M)$ is a $C^\infty(TM)$-module, i.e., the semibasic tensor fields of the same degree can be summed and multiplied by functions $\varphi(x,\xi)$ depending smoothly on $(x,\xi) \in TM$. Using the formal analogy between ordinary tensors and semibasic ones that is established by formula (3.6), one introduces the usual algebraic operations on semibasic tensors: the tensor product, transpositions of indices and convolutions with respect to two indices. Note also that ordinary tensor fields on M can be identified with semibasic tensor fields whose components do not depend on ξ.

Let now (M,g) be a Riemannian manifold. In this case $\beta^r_s M$ is furnished by structure of a Riemannian bundle. Thus for $u, v \in C^\infty(\beta^r_s M)$, the scalar product

$$\langle u, v\rangle = g_{i_1k_1}\dots g_{i_rk_r} g^{j_1l_1}\dots g^{j_sl_s} u^{i_1\dots i_r}_{j_1\dots j_s} \overline{v}^{k_1\dots k_r}_{l_1\dots l_s}$$

is a smooth function on TM.

A Riemannian metric establishes the canonical isomorphism $\beta^r_s M \cong \beta^{r+s} M \cong \beta_{r+s} M$. For this reason we do not distinguish co- and contravariant tensors and say about co- and contravariant components of the same tensor. In coordinate form this fact is expressed by the well-known rule of raising and lowering of indices, and we will use it.

For $u \in C^\infty(\beta^r_s M)$, we define two semibasic tensor fields $\overset{v}{\nabla} u = \left(\overset{v}{\nabla}_k u^{i_1\dots i_r}_{j_1\dots j_s}\right)$ and $\overset{h}{\nabla} u = \left(\overset{h}{\nabla}_k u^{i_1\dots i_r}_{j_1\dots j_s}\right)$ by the formulas

$$\overset{v}{\nabla}_k u^{i_1\dots i_r}_{j_1\dots j_s} = \frac{\partial}{\partial \xi^k} u^{i_1\dots i_r}_{j_1\dots j_s}, \tag{3.7}$$

$$\begin{aligned}\overset{h}{\nabla}_k u^{i_1\dots i_r}_{j_1\dots j_s} = {} & \frac{\partial}{\partial x^k} u^{i_1\dots i_r}_{j_1\dots j_s} - \Gamma^p_{kq}\xi^q \frac{\partial}{\partial \xi^p} u^{i_1\dots i_r}_{j_1\dots j_s} + \\ & + \sum_{m=1}^{r} \Gamma^{i_m}_{kp} u^{i_1\dots i_{m-1} p i_{m+1}\dots i_r}_{j_1\dots j_s} - \sum_{m=1}^{s} \Gamma^p_{kj_m} u^{i_1\dots i_r}_{j_1\dots j_{m-1} p j_{m+1}\dots j_s}\end{aligned} \tag{3.8}$$

where Γ^i_{jk} are the Christoffel symbols. Pay attention to a formal analogy between (3.8) and the formula for the covariant derivative of an ordinary tensor field: comparing with the latter, the right-hand side of (3.8) contains one additional summand related to dependence of the field u on the coordinates ξ^i. One can verify that $\overset{v}{\nabla} u$ and $\overset{h}{\nabla} u$ defined by (3.8) and (3.9) are really semibasic tensor fields, i.e., that their components are transformed according to (3.6) under change of associated coordinates. We thus obtain two well-defined differential operators $\overset{v}{\nabla}, \overset{h}{\nabla} : C^\infty(\beta^r_s M) \to C^\infty(\beta^r_{s+1} M)$ that are called the *vertical* and *horizontal covariant derivatives* respectively. One can also show that they are derivatives with respect to tensor product, commute with convolutions and satisfy the next commutation formulas

$$\overset{v}{\nabla}_k \overset{v}{\nabla}_l - \overset{v}{\nabla}_l \overset{v}{\nabla}_k = 0, \tag{3.9}$$

$$\overset{v}{\nabla}_k \overset{h}{\nabla}_l - \overset{h}{\nabla}_l \overset{v}{\nabla}_k = 0, \tag{3.10}$$

$$\begin{gathered}\left(\overset{h}{\nabla}_k \overset{h}{\nabla}_l - \overset{h}{\nabla}_l \overset{h}{\nabla}_k\right) u^{i_1\dots i_r}_{j_1\dots j_s} = -R^p{}_{qkl}\xi^q \overset{v}{\nabla}_p u^{i_1\dots i_r}_{j_1\dots j_s} + \\ + \sum_{m=1}^{r} R^{i_m}{}_{pkl} u^{i_1\dots i_{m-1} p i_{m+1}\dots i_r}_{j_1\dots j_s} - \sum_{m=1}^{s} R^p{}_{j_m kl} u^{i_1\dots i_r}_{j_1\dots j_{m-1} p j_{m+1}\dots j_s}.\end{gathered} \tag{3.11}$$

where $(R^i{}_{jkl})$ is the curvature tensor. We again pay attention to an analogy between (3.11) and the corresponding formula for ordinary tensor fields. The next relations are valid:

$$\overset{v}{\nabla}_k g_{ij} = \overset{h}{\nabla}_k g_{ij} = 0, \quad \overset{v}{\nabla}_k \delta^i_j = \overset{h}{\nabla}_k \delta^i_j = 0, \quad \overset{h}{\nabla}_k \xi^i = 0, \quad \overset{v}{\nabla}_k \xi^i = \delta^i_k.$$

In what follows we will also use the notations: $\overset{v}{\nabla}{}^i = g^{ij}\overset{v}{\nabla}_j$, $\overset{h}{\nabla}{}^i = g^{ij}\overset{h}{\nabla}_j$.

The operator $H : C^\infty(\beta^r_s M) \to C^\infty(\beta^r_s M)$ is defined by the equality $H = \xi^i \overset{h}{\nabla}_i$. In the case $r = s = 0$ this operator $H : C^\infty(TM) \to C^\infty(TM) = C^\infty(\beta^0_0 M)$ is expressed in coordinate form as

$$H = \xi^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk}\xi^j\xi^k \frac{\partial}{\partial \xi^i}$$

and is called the *geodesic vector field* on TM. This vector field generates the one-parameter group, of diffeomorphisms of TM, which is called the *geodesic flow.*

With the help of commutation formulas (3.9)–(3.11), the next claim can be obtained that plays a crucial role in treating integral geometry problems on Riemannian manifolds.

Lemma 3.1 (Pestov's identity). *For a real function $u \in C^\infty(TM)$, the identity*

$$2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = |\overset{h}{\nabla} u|^2 + \overset{h}{\nabla}_i v^i + \overset{v}{\nabla}_i w^i - R_{ijkl}\xi^i\xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u \tag{3.12}$$

holds where semibasic vector fields v and w are defined by the equalities

$$v^i = \xi^i \overset{h}{\nabla}{}^j u \cdot \overset{v}{\nabla}_j u - \xi^j \overset{v}{\nabla}{}^i u \cdot \overset{h}{\nabla}_j u, \tag{3.13}$$

$$w^i = \xi^j \overset{h}{\nabla}{}^i u \cdot \overset{h}{\nabla}_j u. \tag{3.14}$$

Note that the second and third terms on the right-hand side of (3.12) are of divergence form.

The last tools we need for proving Theorem 2.1 are the Gauss-Ostrogradskiĭ formulas for the vertical and horizontal divergences. To formulate them, we note that the Riemannian metric induces an Euclidean structure on the tangent space T_xM which in turn induces a metric on the unit sphere $\Omega_xM = \Omega M \cap T_xM$. By $d\omega_x$ we denote the corresponding volume form on Ω_xM. We introduce also the volume forms $d\Sigma^{2n-1}$ and $d\Sigma^{2n-2}$ on the manifolds ΩM and $\partial\Omega M$ by the formulas

$$d\Sigma^{2n-1}(x,\xi) = d\omega_x(\xi)\, dV^n(x), \qquad d\Sigma^{2n-2}(x,\xi) = (-1)^n d\omega_x(\xi)\, dV^{n-1}(x)$$

where dV^n and dV^{n-1} are the Riemannian volumes on M and ∂M respectively.

Lemma 3.2. *Let (M,g) be a compact n-dimensional Riemannian manifold, $u = (u^i(x,\xi))$ be a semibasic vector field on TM smooth for $\xi \neq 0$ and positively homogeneous in its second argument*

$$u(x,t\xi) = t^\lambda u(x,\xi) \qquad (t>0).$$

Then the next Gauss-Ostrogradskiĭ formulas are valid:

$$\int\limits_{\Omega M} \overset{v}{\nabla}_i u^i \, d\Sigma^{2n-1} = (\lambda + n - 1) \int\limits_{\Omega M} \langle u, \xi\rangle \, d\Sigma^{2n-1}, \tag{3.15}$$

$$\int\limits_{\Omega M} \overset{h}{\nabla}_i u^i \, d\Sigma^{2n-1} = \int\limits_{\partial\Omega M} \langle u, \nu\rangle \, d\Sigma^{2n-2} \tag{3.16}$$

where ν is the unit outer normal vector to ∂M.

4. Proof of Theorem 2.1. We shall consider only real tensor fields in this section. For $u, v \in C^\infty(\beta^0_m M)$, we denote

$$\langle u, v\rangle = u_{i_1\dots i_m} v^{i_1\dots i_m}, \qquad |u|^2 = \langle u, u\rangle, \qquad \|u\|^2 = \int\limits_{\Omega M} |u|^2 \, d\Sigma;$$

here $d\Sigma = d\Sigma^{2n-1}$ is the volume form on ΩM introduced above.

Lemma 4.1. *Let (M,g) be a CDRM; $\lambda \geq 0$ and $\mu > 0$ be two continuous functions on ΩM. If a semibasic tensor field $u \in C^\infty(\beta^0_m M)$ satisfies the boundary condition*

$$u|_{\partial_-\Omega M} = 0, \tag{4.1}$$

then the inequality

$$\int\limits_{\Omega M} \mu |u|^2 \, d\Sigma \leq C_{\lambda,\mu} \int\limits_{\Omega M} \mu |Hu|^2 \, d\Sigma \tag{4.2}$$

is valid; here

$$C_{\lambda,\mu} = \sup_{(x,\xi)\in\partial_-\Omega M} \int\limits_0^{\tau_+(x,\xi)} \lambda(\gamma_{x\xi}(t), \dot\gamma_{x\xi}(t)) \left[\int\limits_0^t \frac{ds}{\mu(\gamma_{x\xi}(s), \dot\gamma_{x\xi}(s))}\, ds \right] dt. \tag{4.3}$$

For $\mu \equiv 1$ this lemma was proved in [4]. In the general case the proof is similar and, thus, omitted.

COROLLARY. *Let (M,g) be CDRM, ε and φ be two non-negative continuous functions on ΩM. If a semibasic tensor field $u \in C^\infty(\beta^0_m M)$ satisfies boundary condition* (4.1), *then the estimate*

$$\int\limits_{\Omega M} \varphi |u|^2 \, d\Sigma \le D_\varphi \|(H+\varepsilon)u\|^2 \tag{4.4}$$

is valid; here

$$D_\varphi = \sup_{(x,\xi)\in\partial_-\Omega M} \int\limits_0^{\tau_+(x,\xi)} t\varphi(\gamma_{x\xi}(t), \dot\gamma_{x\xi}(t)). \tag{4.5}$$

Proof. Let us define the semibasic tensor field v by the equality $Hu + \varepsilon u = v$. We note that it can be written in the form

$$H(e^E u) = e^E v; \tag{4.6}$$

here E is a function on ΩM satisfying the equation

$$HE = \varepsilon. \tag{4.7}$$

Applying the lemma with $\lambda = \varphi e^{-2E}$ and $\mu = e^{-2E}$ and using (4.6), we derive

$$\int\limits_{\Omega M} \varphi|u|^2\, d\Sigma = \int\limits_{\Omega M} \lambda |e^E u|^2 \, d\Sigma \le C_{\lambda,\mu} \int\limits_{\Omega M} \mu |H(e^E u)|^2 \, d\Sigma =$$

$$= C_{\lambda,\mu} \int\limits_{\Omega M} \mu |e^E v|^2 \, d\Sigma = C_{\lambda,\mu}\|v\|^2 = C_{\lambda,\mu}\|(H+\varepsilon)u\|^2.$$

We thus came to the inequality

$$\int\limits_{\Omega M} \varphi|u|^2\, d\Sigma \le C_{\lambda,\mu}\|(H+\varepsilon)u\|^2, \tag{4.8}$$

where, according to (4.3),

$$C_{\lambda,\mu} = \sup_{(x,\xi)\in\partial_-\Omega M} \int\limits_0^{\tau_+(x,\xi)} \varphi(\gamma_{x\xi}(t), \dot\gamma_{x\xi}(t)) \int\limits_0^t \exp\left[-\mathcal{E}_{x\xi}(t) + \mathcal{E}_{x\xi}(s)\right] ds\, dt \tag{4.9}$$

and the notation $\mathcal{E}_{x\xi}(t) = 2E(\gamma_{x\xi}(t), \dot\gamma_{x\xi}(t))$ is used for brevity. By (4.7), $d\mathcal{E}_{x\xi}(t)/dt = 2\varepsilon \ge 0$; so the function in the brackets in (4.9) is non-positive. Consequently, quantities (4.5) and (4.9) satisfy the inequality $C_{\lambda,\mu} \le D_\varphi$. The last inequality together with (4.8) implies (4.4). □

Proof of Theorem 2.1. Let $f \in C^\infty(M)$ be a real function. We define the function $u : T^0M \to \mathbf{R}$ as

$$u(x,\xi) = \int\limits_{\tau_-(x,\xi)}^0 f(\gamma_{x\xi}(t)) \exp\left[-|\xi| \int\limits_t^0 \varepsilon(\gamma_{x\xi}(s))\, ds\right] dt, \tag{4.10}$$

The difference between (2.2) and (4.10) is that equality (2.2) is considered for $(x, \xi) \in \partial_+\Omega M$ and (4.10) is considered for $(x, \xi) \in T^0M$. The function u is smooth on $T^0M \setminus T(\partial M)$, satisfies the equation

$$Hu + \varepsilon|\xi|u = f(x), \tag{4.11}$$

the boundary conditions

$$u|_{\partial_-\Omega M} = 0, \tag{4.12}$$

$$u|_{\partial_+\Omega M} = I^\varepsilon f \tag{4.13}$$

and is homogeneous with respect to its second argument

$$u(x, \lambda\xi) = \lambda^{-1}u(x, \xi) \qquad (\lambda > 0). \tag{4.14}$$

Let us define semibasic vector fields $y = (y_i(x, \xi))$ and $z = (z_i(x, \xi))$ by the equalities

$$\overset{v}{\nabla} u = -\frac{u}{|\xi|^2}\xi + y, \tag{4.15}$$

$$\overset{h}{\nabla} u = \frac{Hu}{|\xi|^2}\xi + z. \tag{4.16}$$

The summands on the right-hand sides of (4.15) and (4.16) are orthogonal to each other. Indeed, being multiplied by ξ these equalities imply

$$\langle \xi, \overset{v}{\nabla} u\rangle = -u + \langle \xi, y\rangle, \tag{4.17}$$

$$Hu = \langle \xi, \overset{h}{\nabla} u\rangle = Hu + \langle \xi, z\rangle. \tag{4.18}$$

It follows from (4.14) by the Euler equation for homogeneous functions that the left hand side of (4.17) is equal to $-u$. Thus, (4.17) and (4.18) give $\langle \xi, y\rangle = \langle \xi, z\rangle = 0$. For $|\xi| = 1$, (4.16) implies

$$|\overset{h}{\nabla} u|^2 = |Hu|^2 + |z|^2. \tag{4.19}$$

Taking the scalar product of (4.15) and (4.16), we obtain

$$\langle \overset{h}{\nabla} u, \overset{v}{\nabla} u\rangle = -\frac{1}{|\xi|^2}u\,Hu + \langle y, z\rangle. \tag{4.20}$$

We also note that (4.12) and (4.15) imply the boundary condition

$$y|_{\partial_-\Omega M} = 0. \tag{4.21}$$

Applying the operator $\overset{v}{\nabla}$ to equation (4.11), we conclude

$$\overset{v}{\nabla} Hu + \varepsilon|\xi|\overset{v}{\nabla} u + \frac{\varepsilon u}{|\xi|}\xi = 0.$$

By the obvious relation $\overset{v}{\nabla} H - H\overset{v}{\nabla} = \overset{h}{\nabla}$, we transform the last equation to the form

$$H\overset{v}{\nabla} u + \varepsilon|\xi|\overset{v}{\nabla} u = -\overset{h}{\nabla} u - \frac{\varepsilon u}{|\xi|}\xi.$$

Substituting expressions (4.15), (4.16) for $\overset{v}{\nabla} u$ and $\overset{h}{\nabla} u$ to the last equation we infer

$$H\left(-\frac{u}{|\xi|^2}\xi + y\right) + \varepsilon|\xi|\left(-\frac{u}{|\xi|^2}\xi + y\right) = -\frac{Hu}{|\xi|^2}\xi - z - \frac{\varepsilon u}{|\xi|}\xi.$$

Since $H\xi = 0$, the previous relation gives

$$(H + \varepsilon|\xi|)y = -z. \tag{4.22}$$

For the function u, we write the Pestov identity (3.12). Using (4.11) and the independence of f and ε of ξ, we obtain

$$\overset{v}{\nabla}(Hu) = \overset{v}{\nabla}(-\varepsilon|\xi|u + f) = -\overset{v}{\nabla}(\varepsilon|\xi|u) = -\frac{\varepsilon u}{|\xi|}\xi - \varepsilon|\xi|\overset{v}{\nabla} u.$$

Consequently, the left-hand side of (3.12) can be transformed as follows:

$$2\langle\overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = -2\varepsilon\left(\frac{1}{|\xi|}u\,Hu + |\xi|\langle\overset{h}{\nabla} u, \overset{v}{\nabla} u\rangle\right).$$

Substituting expression (4.20) into the right-hand side of the last equality, we obtain

$$2\langle\overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = -2\varepsilon|\xi|\langle y, z\rangle.$$

By the inequality between the arithmetic and geometric means, the last formula gives the estimate

$$2\left|\langle\overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle\right| \le 2(\sqrt{2}\varepsilon|\xi|\,|y|)\,(\frac{\sqrt{2}}{2}|z|) \le 2\varepsilon^2|\xi|^2|y|^2 + \frac{1}{2}|z|^2. \tag{4.23}$$

Using the known symmetries of the curvature tensor and expression (4.15) for $\overset{v}{\nabla} u$, we transform the last summand on the right hand side of (3.12) as follows

$$R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u\cdot\overset{v}{\nabla}{}^l u = R_{ijkl}(\xi\wedge\overset{v}{\nabla} u)^{ij}(\xi\wedge\overset{v}{\nabla} u)^{kl} = R_{ijkl}(\xi\wedge y)^{ij}(\xi\wedge y)^{kl} = R_{ijkl}\xi^i\xi^k y^j y^l.$$

The last equality and (2.4), (2.5) imply the estimate

$$R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u\cdot\overset{v}{\nabla}{}^l u \le K(x,\xi)|\xi|^2\,|y|^2. \tag{4.24}$$

For $|\xi| = 1$, using estimates (4.23), (4.24) and expression (4.19) for $|\overset{h}{\nabla} u|^2$, from (3.12) we obtain

$$|Hu|^2 + \frac{1}{2}|z|^2 \le (2\varepsilon^2 + K)|y|^2 - \overset{h}{\nabla}_i v^i - \overset{w}{\nabla}_i w^i. \tag{4.25}$$

We multiply (4.25) by the volume element $d\Sigma = d\Sigma^{2n-1}$, integrate it over ΩM and transform the last two integrals on the right hand side of the so-obtained inequality by

Gauss-Ostrogradskiĭ formulas (3.15) and (3.16). In such a way we obtain the inequality

$$\begin{aligned}\|Hu\|^2 + \frac{1}{2}\|z\|^2 \le \int\limits_{\Omega M} (2\varepsilon^2 + K)|y|^2\, d\Sigma - \int\limits_{\partial\Omega M} \langle v, \nu\rangle\, d\Sigma^{2n-2} - \\ - (n-2)\int\limits_{\Omega M} \langle w, \xi\rangle\, d\Sigma.\end{aligned} \tag{4.26}$$

where ν is the unit outer normal to ∂M. The coefficient $n-2$ at the last integral in (4.26) is written with the homogeneity of w that follows from (4.14) and (3.14) taken into account.

The second integral from (4.26) can be estimated as follows

$$\left|\int\limits_{\partial\Omega M} \langle v, \nu\rangle\, d\Sigma^{2n-2}\right| \le C\|I^\varepsilon f\|_1^2, \tag{4.27}$$

where C is a constant depending on (M, g). This estimate can be obtained from (3.13) and boundary conditions (4.12), (4.13) in complete analogy with [3] and thus we omit its proof. We also note that $\langle w, \xi\rangle = |Hu|^2$ according to (3.14). Thus, (4.26) and (4.27) imply the inequality

$$(n-1)\|Hu\|^2 + \frac{1}{2}\|z\|^2 \le \int\limits_{\Omega M} (2\varepsilon^2 + K)|y|^2\, d\Sigma + C\|I^\varepsilon f\|_1^2. \tag{4.28}$$

The field y satisfies the equation (4.22) and boundary condition (4.21). Applying the corollary of Lemma 4.1 with $\varphi = [2\varepsilon^2 + K]_+$, we obtain the estimate

$$\int\limits_{\Omega M} (2\varepsilon^2 + K)|y|^2\, d\Sigma \le \kappa\|z\|^2 \tag{4.29}$$

where $\kappa = \kappa(M, g)$ is defined by (2.6). It follows from (4.28) and (4.29) that

$$(n-1)\|Hu\|^2 + \left(\frac{1}{2} - \kappa\right)\|z\|^2 \le C\|I^\varepsilon f\|_1^2.$$

If condition (2.7) holds, the last inequality gives

$$\|Hu\|^2 \le C_1\|I^\varepsilon f\|_1^2. \tag{4.30}$$

To finish the proof we have to estimate the norm $\|f\|$ from above by $\|Hu\|$. To this end, we note that equation (4.11) implies the inequality

$$\|f\| \le \|Hu\| + \varepsilon_0\|u\|, \tag{4.31}$$

where the constant ε_0 is defined by (2.9). By (4.12), the function u satisfies the conditions of Lemma 4.1. Applying the lemma with $\lambda \equiv \mu \equiv 1$, we obtain the estimate

$$\|u\| \le \frac{\sqrt{2}}{2}\,\mathrm{diam}\,(M, g)\|Hu\| \tag{4.32}$$

where the constant diam (M, g) is defined by (2.9). By combining (4.31) and (4.32), we conclude

$$\|f\| \le \left(1 + \frac{\sqrt{2}}{2}\varepsilon_0 \operatorname{diam}(M, g)\right) \|Hu\|. \tag{4.33}$$

Finally, (4.30) and (4.33) imply

$$\|f\|^2 \le C_2 \|I^\varepsilon f\|_1^2$$

and the theorem is proved. □

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