Proof of Soul Theorem
(translated with some modifications frome [10])

Vladimir Sharafutdinov

January 2006, Seattle

1 Introduction

The paper is devoted to the proof of the following

Theorem 1.1 *Let M be an open (= complete, connected, noncompact, with no boundary) Riemannian manifold of nonnegative sectional curvature and S, a soul of M. Then M is diffeomorphic to the space $\nu(S)$ of the normal bundle of the soul.*

The theorem was first stated by Cheeger and Gromoll in [2]. The homeomorphism is proved in [3]. As far as I know, the proof of the diffeomorphism is not published yet. Some results on convex sets (Theorems 2.3, 4.6, 5.3) obtained in the proof are of some independent interest.

Throughout the paper, *smooth manifold* means a C^∞-manifold. A *Riemannian manifold* is a smooth manifold with a smooth Riemannian metric.

2 Convex sets in a Riemannian manifold

Let M be a Riemannian manifold. For $p, q \in M$, we denote the distance between p and q either by $\rho(p,q)$ or by pq. A minimal geodesic between p and q is sometimes also denoted by pq. For $A \subset M$, we denote by $\bar{A}$, ∂A, and $\operatorname{int} A$ the closure, boundary, and interior of A respectively. For $\varepsilon > 0$, the *ε-neighborhood* of A is the set

$$O(A,\varepsilon) = \{p \in M \mid \rho(p,A) < \varepsilon\}.$$

In particular, for $p \in M$, the open ball of radius $r > 0$ centered at p is $B_r(p) = O(\{p\}, r)$.

We start with a local estimate for small triangles. Locally, in a small neighborhood of a point $p \in M$, geometry of the manifold is approximated by the Euclidean geometry of the tangent space M_p. There are many specifications of the last nonrigourous statement. Probably, the following lemma is the most precise of such specifications.

Lemma 2.1 *Let C be a compact set in a Riemannian manifold M. There exist positive numbers δ and κ such that, for any geodesic triangle $p_0p_1p_2$ with vertices $p_i \in C$ $(i = 0,1,2)$ and side lengths $a_i = p_{i+1}p_{i+2}$ (indices are taken modulo 3) satisfying $a_i < \delta$, the equality*

$$a_0^2 = a_1^2 + a_2^2 - 2a_1a_2\cos\alpha_0 + Ka_1^2a_2^2$$

holds with some K satisfying $|K| < \kappa$, where α_0 is the angle of the triangle at the vertex p_0.

This statement belongs to E. Cartan [1]. But Cartan's proof is not quite rigorous from the modern viewpoint. A modern proof can be obtained as follows. We first prove the statement for the model spaces $\mathbf{R}^n$, $\mathbf{S}^n$, $\mathbf{H}^n$ of constant sectional curvature. There is the corresponding cosine law in each of the spaces which implies the statement easily. In the general case, let κ be the maximum of $|K(p,\sigma)|$ over all points $p \in C$ and all two-dimensional subspaces $\sigma \subset M_p$, where $K(p,\sigma)$ is the sectional curvature at the point p in the direction σ. For a small triangle $p_0p_1p_2$, we construct the triangle $p_0^-p_1^-p_2^-$ in the model space of the constant curvature $-\kappa$ and the triangle $p_0^+p_1^+p_2^+$ in the model space of the constant curvature κ such that

$$a_1 = p_2p_3 = p_2^-p_3^- = p_2^+p_3^+, \quad a_2 = p_3p_1 = p_3^-p_1^- = p_3^+p_1^+, \quad \alpha_0 = \alpha_0^- = \alpha_0^+.$$

Then by Rauch's comparison theorem

$$a_0^+ \leq a_0 \leq a_0^-.$$

Since the statement of the lemma holds for a_0^- and for a_0^+, the last inequalities imply the statement for a_0.

Definition. A set A in a Riemannian manifold is said to be *convex* (*totally convex*) if, for any $p, q \in A$, every minimal geodesics (every geodesic) joining p and q lies completely in A. A is said to be *locally convex* if every point $p \in A$ has a neighborhood U such that $U \cap A$ is convex.

The topological structure of closed convex sets is described by the following

Theorem 2.2 *Let C be a closed connected locally convex set in a Riemannian manifold M. Then C, considered with the induced from M topology, is a topological manifold with (nonsmooth, possibly empty) boundary bC. The set $C \setminus bC$ is a smooth totally geodesic submanifold of M.*

Let us remind that a submanifold N of a Riemannian manifold M is said to be *totally geodesic* if every geodesic in N (which is considered as a Riemannian manifold with the metric induced from M) is also the geodesic in the ambient M. This is equivalent to the following statement: the second fundamental form of N is identically equal to zero. Theorem 2.2 is proved in [3]. The proof is pretty easy but rather technical. One can prove Theorem 2.2 by him/herself starting with considering the case of $M = \mathbf{R}^n$ and expanding then the arguments to the general case.

The following observation plays the crucial role in our proof of Theorem 1.1.

Theorem 2.3 *Let C be a compact locally convex set in a Riemannian manifold M. There exists an open neighborhood U of C such that the function $f(p) = \rho(p, C)$ is a C^1-function in $U \setminus C$ and $\|grad\, f\| = 1$.*

We will first prove the following statement.

Lemma 2.4 *Let C be a compact locally convex set in a Riemannian manifold M. There exists an open neighborhood U of C such that, for every point $p \in C$, there exists a unique nearest to p point of C, i.e., such $q \in C$ that*

$$\rho(p, q) = \inf_{q' \in C} \rho(q', C),$$

and $p \mapsto q$ is the continuous map $U \to C$.

Definition. The closure $\bar{U}$ of an open set $U \subset M$ is said to be *strongly convex* if, for any $p, q \in \bar{U}$, all inner points of a minimal geodesic between p and q belong to U. Let us remind [7] that, for a Riemannian manifold M, there exists a continuous function $r : M \to (0, \infty]$, the convexity radius, such that the closed ball $\overline{B_\varepsilon(p)}$ is strongly convex for any $p \in M$ and $0 < \varepsilon < r(p)$.

Proof of Lemma 2.4. Choose $\varepsilon > 0$ such that $\overline{O(C, \varepsilon)}$ is compact and $r(p) > \varepsilon$ for any $p \in \overline{O(C, \varepsilon)}$. Set $U = O(C, \varepsilon)$. Let $p \in U \setminus C$ and $q \in C$ be the nearest point of C to p. Then $0 < d = \rho(p, C) = \rho(p, q) < \varepsilon$. Assume that there exists another point $q \neq q' \in C$ nearest to p, i.e., $d = \rho(p, q')$. Any inner point q'' of the minimal geodesic qq' belongs to C because of the convexity of C and belongs to $B_d(p)$ because of the strong convexity of $\overline{B_d(p)}$. Therefore $\rho(p, C) \leq \rho(p, q'') < d$. This contradicts to $d = f(p)$.

So, for any $p \in U$, there exists a unique nearest to p point $n(p)$ in C. If the map $n : U \to C$ is discontinuous at some $p \in U$, there exists a sequence of points $p_k \in U$ ($k = 1, 2, \ldots$) converging to p and such that

$$\rho(n(p_k), n(p)) \geq \delta > 0.$$

The sequence $n(p_k)$ has a limit point q since C is compact. Passing to the limit in the last inequality, we see that $\rho(q, n(p)) \geq \delta$, i.e., the points $n(p)$ and q are different. On the other hand,

$$\rho(p, q) = \lim_{k \to \infty} \rho(p_k, C) = \rho(p, C).$$

Thus, q is the nearest point from C to p and does not coincide with $n(p)$. This contradicts to the first statement of the lemma.

Corollary 2.5 *Let C be as in Lemma 2.4. C is the strong deformation retract of $O(C, \varepsilon)$ for any sufficiently small ε.*

Proof of Theorem 2.3. We will use the following obvious statement which is sometimes called Lemma on two policemen:

Let f be a continuous function in a neighborhood W of a pont p_0 of a manifold. If there exist two smooth functions g_1 and g_2 in W satisfying

$$g_1 \leq f \leq g_2, \qquad g_1(p_0) = f(p_0) = g_2(p_0),$$

then f is differentiable at p_0.

We are going to prove that the function $f(p) = \rho(p, C)$ is differentiable at every point $p_0 \in U \setminus C$, where $U = O(C, \varepsilon)$ with sufficiently small $\varepsilon > 0$. Let $d = f(p_0)$, $q_0 \in C$ be the nearest point to p_0, and $W = B_d(p_0)$. The function $g_2(p) = \rho(p, q_0)$ is smooth in W and satisfies

$$f(p) \leq g_2(p) \quad \text{for} \quad p \in W, \qquad f(p_0) = g_2(p_0).$$

To construct the second policeman, we consider the hypersurface $N \subset M$ formed by all geodesics of length d starting from q_0 perpendicular to the radius p_0q_0. It is easy to see that N does not intersect the interior $C \setminus bC$ because otherwise there would exist a point $q' \in C$ satisfying $\rho(p, q') < d$; the letter inequality contradicts to the definition of d. Any short geodesic from $p \in W$ to C must intersect N. Therfore the smooth function $g_1 : W \to \mathbf{R}$, $g_1(p) = \rho(p, N)$, satisfies

$$g_1(p) \leq f(p) \quad \text{for} \quad p \in W, \qquad g_1(p_0) = f(p_0).$$

Applying Lemma on two policemen, we see that f is differentiable at every point $p_0 \in U$. The gradient of f at p_0 coincides with $-v$, where $v \in M_{p_0}$ is the unit vector tangent to the radius p_0q_0. The vector v depends continously on $p_0 \in U \setminus C$ by Lemma 2.2.

Definition. For a compact locally convex set in a Riemannian manifold M, we define the *regularity radius* $r(C)$ as the supremum of such $\varepsilon > 0$ that

(1) $\overline{O(C, \varepsilon)}$ is compact;

(2) $C \cap B(p, \varepsilon')$ is convex for any $p \in C$ and any $0 < \varepsilon' < \varepsilon$;

(3) for any $p \in O(C, \varepsilon)$ there exists a unique nearest point $q \in C$ and the map $p \mapsto q$ is continuous;

(4) $f(p) = \rho(p, C)$ is a C^1-function in $O(C, \varepsilon) \setminus C$.

Theorem 2.3 and Lemma 2.4 guarantee the positiveness of $r(C)$.

3 Continuous families of sets

Here we define continuous families of sets and list some properties of such families. Proofs are omitted because they are very elementary.

Definition. Let M be a Riemannian manifold; $\alpha, \beta \in \mathbf{R}$; $\alpha \leq \beta$. We say that an *increasing family* is defined on $[\alpha, \beta]$ if, for every $t \in [\alpha, \beta]$, a compact set $C(t) \subset M$ is defined such that $C(t_1) \subset C(t_2)$ for $t_1 \leq t_2$. The family is said to be *uniformly locally convex* if every point $p \in M$ has a neighborhood U such that $U \cap C(t)$ is convex for any t. In the latter case, the number $r(C|_{[\alpha,\beta]}) = \inf_{t \in [\alpha,\beta]} r(C(t))$ is positive as one can see by revising the proof of Theorem 2.3.

An increasing family $C(t)$ on $[\alpha, \beta]$ is said to be *continuous* if, for every $t \in [\alpha, \beta]$ and for every $\varepsilon > 0$, there exists $\tau > 0$ such that

$$C(t + \tau) \setminus C(t) \subset O(\partial C(t), \varepsilon); \quad C(t) \setminus C(t - \tau) \subset O(\partial C(t), \varepsilon).$$

The family is said to be *uniformly right-hand continuous* if, for any $\varepsilon > 0$, there exists $\tau > 0$ such that

$$C(t') \subset O(C(t), \varepsilon)$$

for any $t, t' \in [\alpha, \beta]$ satisfying $0 < t' - t < \tau$.

Lemma 3.1 *Let a continuous increasing family $C(t)$ be defined on a finite segment $[\alpha, \beta]$. Assume $C(\alpha)$ to be nonempty and $C(t) = \overline{\text{int} C(t)}$ for any $t > \alpha$. Then the family is uniformly right-hand continuous on $[\alpha, \beta]$.*

Definition. For $C \subset M$ and $t \geq 0$, the closed set

$$\mathrm{int}\,(C,t) = \{p \in C \mid \rho(p, \partial C) \geq t\}$$

is called the *t-interior* of C.

Lemma 3.2 *$int(int(C,t),t') = int(C,t+t')$.*

Lemma 3.3 *For a compact set C in a Riemannian manifold, the map $t \mapsto int(C,-t)$ defines a continuous increasing family on $(-\infty, 0]$.*

4 Basic construction

In this section, we continue studying convex sets (Theorem 4.6) and prove Lemmas 4.8 and 4.9 which allow us to reduce the problem of constructing a diffeomorphism to the problem of finding continuous families of convex sets. We start with recalling some elementary facts of cobordism theory.

Definition. A *cobordism* $(W; V_0, V_1)$ is a smooth compact manifold W whose boundary $\partial W = V_1 \cup V_2$ is represented as the union of two disjoint manifolds. Two cobordisms $(W; V_0, V_1)$ and $(W'; V_0', V_1')$ are *equivalent* if there exists a diffeomorphism $h : W \to W'$ such that $h(V_0) = V_0'$ and $h(V_1) = V_1'$. A cobordism is *trivial* if it is equivalent to $(V \times [0,1]; V \times 0, V \times 1)$.

Let $(W; V_0, V_1)$ and $(W'; V_1', V_2')$ be two cobordisms. Given a diffeomorphism $h : V_1 \to V_1'$, let $W \cup_h W'$ be the space obtained from W and W' by the h-identification of V_1 and V_1'. The following statement is a partial case of Theorem 1.4 of [8].

Lemma 4.1 *There exists a smooth structure $\mathcal{L}$ on $W \cup_h W'$ which is agreed with the given smooth structures, i.e., such that the both embeddings $W \to W \cup_h W'$ and $W' \to W \cup_h W'$ are diffeomorphisms onto their ranges. The structure $\mathcal{L}$ is unique up to a diffeomorphism identical in some neighborhood of $V_0 \cup V_2'$.*

The following statement can be proved by some modification of the proof of Theorem 3.4 of [8].

Lemma 4.2 *Let $(W; V_0, V_1)$ be a cobordism. Assume that there exist a smooth Riemannian metric on W and a smooth function $g : W \to \mathbf{R}$ such that*
(1) $\mathrm{grad}\, g$ does not vanish in W;
(2) $\langle \nu, \mathrm{grad}\, g \rangle < 0$ in V_1;
(3) $\langle \nu, \mathrm{grad}\, g \rangle > 0$ in V_0,
where ν is the inner normal to the boundary. Then the cobordism is trivial.

Of course, the proof of this lemma consists in fibering the manifold W to integral curves of the vector field $\mathrm{grad}\, g$.

Definition. Let V be a manifold with no boundary. A compact set $D \subset V$ is said to be a *smooth compact* if D is a smooth submanifold (with boundary) of V and the dimension of D is the same as the dimension of V.

All cobordisms considered below are obtained in the following way. Let $D_0, D_1 \subset V$ be two smooth compacts such that $D_0 \subset \mathrm{int}\, D_1$. Then $(\overline{D_2 \setminus D_0}; \partial D_0, \partial D_1)$ is the cobordism which will be denoted by (D_0, D_1) for brevity. Lemma 4.1 implies

Lemma 4.3 *Let a smooth compact $D_i \subset V$ be defined for every $i = 0, 1, \dots$ such that*

$$D_i \subset \mathit{int}\, D_{i+1}, \quad \cup_i D_i = V.$$

Assume the cobordism (D_{i-1}, D_i) to be trivial for every $i = 1, 2, \dots$. Then there exists a diffeomorphism

$$F : V \setminus \mathit{int}\, D_0 \to \partial D_0 \times [0, 1).$$

Basic construction. Let C be a compact locally convex set in a Riemannian manifold M and $r(C)$ be the regularity radius of C. Choose $0 < \varepsilon' < \varepsilon < r(C)$ and set $K = \overline{O(C,\varepsilon) \setminus O(C,\varepsilon')}$. A function $g : K \to \mathbf{R}$ is said to be a C^1-function if it admits a C^1-extention to some open neighborhood of K. Let $C^1(K)$ be the linear space of all such functions endowed with the norm

$$\|g\|_{C^1} = \sup_K |g| + \sup_K \|\mathrm{grad}\, g\|.$$

The space $C^\infty(K)$ of smooth functions is dense in $C^1(K)$. Define the function $f : K \to \mathbf{R}$ by

$$f(p) = \rho(p, C).$$

By Theorem 2.3, $f \in C^1(K)$ and

$$\|\mathrm{grad}\, f\| = 1. \tag{4.1}$$

Choose $g \in C^\infty(K)$ such that

$$\|f - g\|_{C^1} < (\varepsilon - \varepsilon')/2 \tag{4.2}$$

and set

$$D = O(C, \varepsilon') \cup g^{-1}([0, (\varepsilon + \varepsilon')/2]).$$

(4.1) and (4.2) imply that D is a smooth compact and

$$O(C, \varepsilon') \subset D \subset O(C, \varepsilon). \tag{4.3}$$

Definition. A smooth compact D obtained by the construction above is called the *smooth* $(\varepsilon', \varepsilon)$*-approximation* of the set C.

The nonuniqueness in the construction of D consists only in the choice of the approximation $g \in C^\infty(K)$ of the function f.

Lemma 4.4 *Let C be a compact locally convex set in a Riemannian manifold M and $r(C)$, the regularity radius of C. Choose $0 < \varepsilon_1' < \varepsilon_1 \le \varepsilon_2' < \varepsilon_2 < r(C)$. Let D_1 and D_2 be smooth $(\varepsilon_1', \varepsilon_1)$- and $(\varepsilon_2', \varepsilon_2)$-approximations of C respectively. Then $D_1 \subset \mathit{int}\, D_2$ and the cobordism (D_1, D_2) is trivial.*

Proof. The inclusion $D_1 \subset \mathrm{int}\, D_2$ follows from (4.3). Set

$$K_i = \overline{O(C, \varepsilon_i) \setminus O(C, \varepsilon_i')} \ \ (i = 1, 2); \quad K_0 = \overline{O(C, \varepsilon_2) \setminus O(C, \varepsilon_1')}.$$

Define the functions $f_i : K_i \to \mathbf{R}$ by

$$f_i = \rho(p, C) \quad (p \in K_i;\ i = 0, 1, 2).$$

Then $f_i \in C^1(K_i)$ and

$$\|\mathrm{grad}\, f_i\| = 1 \quad (i = 0, 1, 2). \tag{4.4}$$

By the definition of D_i $(i = 1, 2)$, there exist $g_i \in C^\infty(K_i)$ such that

$$\|f_i - g_i\|_{C^1} < (\varepsilon_i - \varepsilon_i')/2, \tag{4.5}$$

$$D_i = O(C, \varepsilon_i') \cup g_i^{-1}([0, (\varepsilon_i + \varepsilon_i')/2]). \tag{4.6}$$

Choose $g_0 \in C^\infty(K_0)$ satisfying

$$\|f_0 - g_0\|_{C^1} < \varepsilon_1/2. \tag{4.7}$$

To prove the triviality of (D_1, D_2), we will show that the restriction of g_0 to $\overline{D_2 \setminus D_1}$ satisfies all hypotheses of Lemma 4.2. Indeed, the first hypothesis, $\mathrm{grad}\, g_0 \neq 0$, follows from (4.4), (4.5), and (4.7). To check the second hypothesis, we observe that $\langle \nu, \mathrm{grad}\, g_2 \rangle = -\|\mathrm{grad}\, g_2\|$ on $\partial D_2 = g_2^{-1}((\varepsilon_2 + \varepsilon_2')/2)$. This implies, with the help of (4.4)–(4.7), that $\langle \nu, \mathrm{grad}\, g_0 \rangle < 0$ on ∂D_2. The third hypothesis of Lemma 4.2 is checked in the same way. The application of Lemma 4.2 finishes the proof.

Lemmas 4.3 and 4.4 imply

Lemma 4.5 *Let C be a compact locally convex set in a Riemannian manifold M and $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon < r(C)$. Let D be a smooth $(\varepsilon_0, \varepsilon_1)$-approximation of C. Then there exists a diffeomorphism*

$$F : O(C, \varepsilon) \setminus \mathit{int}\, D \to \partial D \times [0, 1).$$

Theorem 4.6 *Let C be a compact locally convex set in a Riemannian manifold. For any $\varepsilon, \varepsilon'$ satisfying $0 < \varepsilon, \varepsilon' < r(C)$, there exists a diffeomorphism*

$$F : O(C, \varepsilon) \to O(C, \varepsilon')$$

identical on C.

Proof. Choose numbers ε_i $(i = 0, 1, 2)$ satisfying $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \min(\varepsilon, \varepsilon')$. Let D_0 and D_1 be smooth $(\varepsilon_0, \varepsilon_1)$- and $(\varepsilon_1, \varepsilon_2)$-approximations of C respectively. By Lemma 4.5, both the manifolds $O(C, \varepsilon) \setminus \mathrm{int}\, D_0$ and $O(C, \varepsilon') \setminus \mathrm{int}\, D_0$ are diffeomorphic to $\overline{(D_1 \setminus D_0)} \cup_h (\partial D_1 \times [0, 1))$, where $h(x) = (x, 0)$ for $x \in \partial D_1$. By Lemma 4.1, there exists a diffeomorphism $F' : O(C, \varepsilon) \setminus \mathrm{int}\, D_0 \to O(C, \varepsilon') \setminus \mathrm{int}\, D_0$ identical in some neighborhood of ∂D_0. Extending F' to D_0 by the identical map, we obtain the desired diffeomorphism F.

Lemma 4.7 *Let M be a Riemannian manifold and $C(t) \subset M$, $t \in [\alpha, \beta]$, be an increasing uniformly locally convex and uniformly right-hand continuous family of compact sets. For any $\varepsilon > 0$, there exist numbers $\varepsilon_0, \varepsilon_0', \varepsilon_0''$ satisfying $0 < \varepsilon_0' < \varepsilon_0 < \varepsilon_0'' < 2\varepsilon_0 < \varepsilon$, smooth $(\varepsilon_0', \varepsilon_0)$-approximation $D(\alpha)$ of the set $C(\alpha)$, and smooth $(\varepsilon_0'', 2\varepsilon_0)$-approximation $D(\beta)$ of $C(\beta)$ such that $D(\alpha) \subset \mathit{int}\, D(\beta)$ and the cobordism $(D(\alpha), D(\beta))$ is trivial.*

Proof. Set $r_0 = r(C|_{[}\alpha, \beta])$ (see the definition in Section 3). For the compact set $\overline{O(C(\beta), r_0/2)}$, find positive δ and κ as in Lemma 2.1. Choose such small positive ε_0 that

$$4\varepsilon_0 < r_0; \quad 6\varepsilon_0 < \delta; \quad \kappa\varepsilon_0^2 < 10^{-4}; \quad 1 - 10^{-3} < (1-\varepsilon_0)^2. \tag{4.8}$$

Since the family $C(t)$ is uniformly right-hand continuous, we can find a sequence $\alpha = t_0 < t_1 < \ldots < t_k = \beta$ such that

$$C_i = C(t_i) \subset O(C_{i-1}, \varepsilon_0/8). \tag{4.9}$$

This implies that

$$O(C_i, \varepsilon_0/2) \subset O(C_{i-1}, 3\varepsilon_0/4). \tag{4.10}$$

Choose a sequence

$$3\varepsilon_0/4 < \varepsilon_0' < \varepsilon_0 < \varepsilon_1' < \varepsilon_1 < \ldots < \varepsilon_k' < \varepsilon_k = 2\varepsilon_0 \tag{4.11}$$

and set

$$K_0 = \overline{O(C_0, \varepsilon_0) \setminus O(C_0, 3\varepsilon_0/4)}; \quad K_i = \overline{O(C_i, \varepsilon_i) \setminus O(C_{i-1}, \varepsilon_{i-1}')} \quad (i = 1, \ldots, k). \tag{4.12}$$

Define the functions $f_i, f_i' : K_i \to \mathbf{R}$ by

$$f_i(p) = \rho(p, C_i) \ (i = 0, \ldots, k); \quad f_i'(p) = \rho(p, C_{i-1}) \ (i = 1, \ldots, k).$$

(4.10)–(4.12) implies that

$$K_i \subset O(C_i, 3\varepsilon_0) \backslash O(C_i, \frac{\varepsilon_0}{2}) \ (i = 0, \ldots, k); \ K_i \subset O(C_{i-1}, 3\varepsilon_0) \backslash O(C_{i-1}, \frac{3\varepsilon_0}{4}) \ (i = 1, \ldots, k). \tag{4.13}$$

This inclusions and Theorem 2.3 imply that $f_i, f_i' \in C^1(K_i)$. Note that f_{i-1} coincides with f_i' on $K_{i-1} \cap K_i$.

Let us show that the inequality

$$\|\mathrm{grad}\,(f_i - f_i')\| < \sqrt{2}(1 - \varepsilon_0) \tag{4.14}$$

holds on K_i. Since

$$\|\mathrm{grad}\, f_i\| = \|\mathrm{grad}\, f_i'\| = 1, \tag{4.15}$$

the difference of the gradients satisfies

$$\|\mathrm{grad}\,(f_i - f_i')\|^2 = 2(1 - \cos\alpha),$$

where α is the angle between the vectors $\mathrm{grad}\, f_i$ and $\mathrm{grad}\, f_i'$. Thus (4.14) is equivalent to

$$1 - \cos\alpha < (1 - \varepsilon_0)^2. \tag{4.16}$$

Let $p \in K_i$, q be the nearest to p point of C_i, and q' be the nearest to p point of C_{i-1}. The vector $\mathrm{grad}\, f_i$ ($\mathrm{grad}\, f_i'$) is the unit vector tangent to the minimal geodesic pq (pq'). The angle of the triangle pqq' at the vertex p is equal to α. Introduce the notations: $r = pq, \ r' = pq', \ d = qq', \ \beta = \angle pqq'$. From (4.13), we obtain

$$\varepsilon_0/2 \le r \le 3\varepsilon_0; \quad 3\varepsilon_0/4 \le r' \le 3\varepsilon_0; \quad 0 \le d \le 6\varepsilon_0; \tag{4.17}$$

and from (4.9),

$$r \le r' \le r + \varepsilon_0/8. \tag{4.18}$$

Finally, from the convexity of $C_i \cap B_{4\varepsilon_0}(p)$,

$$\beta \ge \pi/2. \tag{4.19}$$

Applying Lemma 2.1, we have

$$r'^2 = r^2 + d^2 - 2rd\cos\beta + Kr^2d^2$$

with $|K| \le \kappa$. This implies, with the help of (4.17)–(4.19), that

$$d^2 \le 3\varepsilon_0^2/4 + 324\kappa\varepsilon_0^4. \tag{4.20}$$

Applying Lemma 2.1 again, we have

$$d^2 = r^2 + r'^2 - 2rr'\cos\alpha + K'r^2r'^2$$

with $|K'| \le \kappa$. This implies, with the help of (4.17) and (4.20), that

$$\cos\alpha > 1/300 - 18\kappa\varepsilon_0^2. \tag{4.21}$$

Inequality (4.16) follows from (4.8) and (4.21). We have thus proved (4.14).

Choose $g_i \in C^\infty(K_i)$ such that

$$\|f_i - g_i\|_{C^1} < (\varepsilon_i - \varepsilon_i')/2 \tag{4.22}$$

and set

$$D_i = O(C_i, \varepsilon_i') \cup g_i^{-1}([0, (\varepsilon_i + \varepsilon_i')/2]) \quad (i = 0, \dots, k).$$

Then D_i is a smooth compact, $D_{i-1} \subset \operatorname{int} D_i$, and $\overline{D_i \setminus D_{i-1}} \subset K_i$.

We will prove that the cobordism (D_{i-1}, D_i) is trivial for any $i = 1, \dots, k$. To this end we will show that the restriction of g_1 to $\overline{D_i \setminus D_{i-1}}$ satisfies the hypotheses of Lemma 4.2. The first hypothesis, $\operatorname{grad} g_i \ne 0$, follows from (4.15) and (4.22). The second hypothesis, $\langle \nu, \operatorname{grad} g_i \rangle < 0$ on ∂D_i, follows from the equality $\partial D_i = g_i^{-1}(\varepsilon_i + \varepsilon_i')/2)$. To check the third hypothesis, we note that $\langle \nu, \operatorname{grad} g_{i-1} \rangle = \|\operatorname{grad} g_{i-1}\|$ on ∂D_{i-1} since $\partial D_{i-1} = g_{i-1}^{-1}((\varepsilon_{i-1} + \varepsilon_{i-1}')/2)$. From this we obtain with the help of (4.14), (4.15), and (4.22) that $\langle \nu, \operatorname{grad} g_i \rangle > 0$ on ∂D_{i-1}.

Thus, the cobordism (D_{i-1}, D_i) is trivial for any $i = 1, \dots, k$. Applying Lemma 4.1, we see that (D_0, D_k) is a trivial cobordism. To finish the proof, it remains to set $\varepsilon_0'' = \varepsilon_k'$, $D(\alpha) = D_0$, $D(\beta) = D_k$.

Lemma 4.8 *Let hypotheses of Lemma 4.7 be satisfied. For any sufficiently small $\varepsilon > 0$, there exists a diffeomorphism*

$$F : O(C(\alpha), \varepsilon) \to O(C(\beta), \varepsilon)$$

identical on $C(\alpha)$.

The proof is the same as the proof of Theorem 4.6 but we have to use Lemma 4.7 instead of using Lemma 4.4.

Lemma 4.9 *Let M be a Riemannian manifold and $C(t) \subset M$ be an increasing family of compact sets defined for $t \in [0, \infty)$ and satisfying the following conditions:*

(1) $C(0)$ is nonempty;

(2) the family $C(t)$ is uniformly locally convex on any finite segment $[0, t_0]$;

(3) $C(t)$ is connected for all $t \geq 0$;

(4) $\partial C(t_1) \cap \partial C(t_2) = \emptyset$ for $t_1 \neq t_2$;

(5) $\cup_t C(t) = M$.

Then, for any sufficiently small $\varepsilon > 0$, there exists a diffeomorphism

$$F : M \to O(C(0), \varepsilon)$$

identical on $C(0)$.

Proof. First of all we note that condition (4) implies

$$C(t_1) \subset \operatorname{int} C(t_2) \quad \text{for} \quad t_1 < t_2 \tag{4.23}$$

and therefore

$$\operatorname{int} C(t) \neq \emptyset \quad \text{for} \quad t > 0. \tag{4.24}$$

Since $C(t)$ is connected an locally compact, it is a topological manifold (see Theorem 2.2) whose boundary is denoted by $bC(t)$. As follows from (4.24), $\dim C(t) = \dim M$ for $t > 0$ and therefore $bC(t) = \partial C(t)$. This implies that

$$C(t) = \overline{\operatorname{int} C(t)} \quad \text{for} \quad t > 0. \tag{4.25}$$

Applying Lemma 3.1, we see that the family $C(t)$ is uniformly right-hand continuous on any finite segment $[0, t_0]$.

Choose a sequence $0 = t_0 < t_1 < \ldots$ converging to infinity and set $C_i = C(t_i)$. By (4.23), we can choose ε_i for any $i = 0, 1, \ldots$ such that

$$O(C_i, \varepsilon_i) \subset C_{i+1}; \quad 0 < \varepsilon_i < r(C_i). \tag{4.26}$$

Let D_i be a smooth $(\varepsilon_i/2, \varepsilon_i)$-approximation of C_i. By (4.26), $D_i \subset \operatorname{int} D_{i+1}$. Let us prove that the cobordism (D_{i-1}, D_i) is trivial for any $i = 1, 2, \ldots$. By Lemma 4.7, there exist numbers $\delta_1, \delta_i', \delta_i''$ satisfying $0 < \delta_i' < \delta_i < \delta_i'' < 2\delta_i' < \varepsilon_i/2$, $2\delta_i < \varepsilon_{i-1}/2$, a smooth (δ_i', δ_i)-approximation D_i' of C_{i-1}, and a smooth $(\delta_i'', 2\delta_i)$-approximation D_i'' of C_i such that $D_i' \subset \operatorname{int} D_i''$ and the cobordism (D_i', D_i'') is trivial. By Lemma 4.4, $D_i' \subset \operatorname{int} D_{i-1}$, $D_i'' \subset \operatorname{int} D_i$ and the cobordisms (D_i', D_{i-1}), (D_i'', D_i) are trivial. It is easy to show that the triviality of (D_i', D_i) and of (D_i', D_{i-1}) implies the triviality of (D_{i-1}, D_i).

Thus, the sequence D_i $(i = 0, 1, \ldots)$ satisfies all hypotheses of Lemma 4.3. Applying the lemma, we obtain the diffeomorphism

$$F : M \setminus \operatorname{int} D_0 \to \partial D_0 \times [0, 1).$$

Now the proof is completed as in Theorem 4.6.

5 Manifolds of nonnegative curvature

Let us recall (Theorem 2.2) that a closed connected locally convex set C of a Riemannian manifold M is a topological manifold whose boundary is denoted by bC (do not mix with ∂C!). For $p, q \in C$, we denote by $\rho_C(p,q)$ the infimum of lengths of curves running in C from p to q. Since C is locally convex, $\rho_C(p,q) = \rho(p,q)$ for sufficiently close p and q. In particular, the metric ρ_C determines the same topology on C as the topology induced from M.

Theorem 5.1 *Let M be a Riemannian manifold of nonnegative curvature and C be a compact connected locally convex set in M. For $t \in [0,\infty)$, denote $C(t) = \{p \in C \mid \rho_C(p, bC) \geq t\}$. Then $t \mapsto C(-t)$ is an increasing uniformly locally convex family on $(-\infty, 0]$. If C is totally convex, then $C(t)$ is also totally convex for any $t \in [0,\infty)$.*

The second statement is proved in [3]. The first statement can be proved in a similar way.

Corollary 5.2 *Under hypotheses of Theorem 5.1, $C(t)$ is connected for any $t \in [0,\infty)$.*

Proof. Denote by t_0 the supremum of such t that $C(t')$ is connected for $0 \leq t' \leq t$. We have to prove that $t_0 = \infty$. Assume that $t_0 < \infty$. Then, first of all, $C(t_0)$ is connected as the intersection of a decreasing sequence of connected compact sets. Applying Theorems 5.1 and 2.2, we see that $C(t)$ is a topological manifold with the boundary $bC(t)$ for any $t \in [0, t_0]$. The inequality $t_0 < \infty$ implies the existence of such $t_1 > t_0$ that $C(t_1)$ is nonempty. From this, $\dim C(t) = \dim C$ for $t \in [0, t_0]$. Therefore $bC(t) = \partial_C C(t)$ for $t \in (0, t_0]$, where $\partial_C C(t)$ means the boundary of $C(t)$ considered as a subset of the topological manifold C. The latter fact implies in the same way as in Lemma 3.2 that, for $t \in [0, t_0]$ and $t' \geq 0$,

$$C(t + t') = \{p \in C(t) \mid \rho_C(p, bC(t) \geq t'\}. \tag{5.1}$$

By Theorem 5.1, the family $t \mapsto C(-t)$ is uniformly locally convex. Set $\varepsilon = r(C|_{[t_0,t_0+1]})$. Since $C(t_0)$ is compact, there exists a finite $\varepsilon/2$-net in $C(t_0)$, i.e., a finite set $\{p_1, \ldots, p_k\}$ of points of $C(t_0)$ such that, for every $p \in C(t_0)$, there exists p_i such that $\rho(p, p_i) < \varepsilon/2$. For every p_i choose $q_1 \in C(t_0) \setminus bC(t_0)$ such that $\rho(p_i, q_i) < \varepsilon/2$. Then $\{q_1, \ldots, q_k\}$ is an ε-net in $C(t_0)$. Set $\varepsilon' = \min_i \rho(q_i, bC(t_0)$ and $\tau = \min(\varepsilon, \varepsilon')$. Then $C(t_0) \subset O(C(t_0 + \tau), \varepsilon)$ as follows from (5.1). The latter inclusion implies that, for any $t \in [t_0, t_0 + \tau]$ and for every point $p \in C(t_0)$, there exists a unique nearest to p point q in the set $C(t)$ and the formula $p \mapsto q$ defines the continuous map $\varphi_t : C(t_0) \to C(t)$. The range of φ_t coincides with $C(t)$ because φ_t is the identity on $C(t)$. Therefore $C(t)$ is connected for any $t \in [t_0, t_0 + \tau]$ as the range of the connected set $C(t_0)$ under the continuous map φ_t. This contradicts to the definition of t_0.

Theorem 5.3 *Let M be a Riemannian manifold of nonnegative curvature and C be a compact connected locally convex set in M. There exists a compact boundaryless totally geodesic submanifold S of M (the soul of C) such that (1) $S \subset C$ and (2) for any sufficiently small $\varepsilon > 0$, there exists a diffeomorphism*

$$F : O(C, \varepsilon) \to \nu(S)$$

identical on S, where $\nu(S)$ is the space of the normal bundle of the submanifold S. If C is totally convex, then S is totally convex too.

Proof. For a boundaryless compact submanifold $S \subset M$, $\nu(S)$ is diffeomorphic to $O(S, \varepsilon)$ with a sufficiently small $\varepsilon > 0$.

We prove the statement by induction on $\dim C$ (see Theorem 2.2). If bC is empty, we can take $S = C$. So, we assume bC to be nonempty. Let $C(t)$, $t \in [0, \infty)$, means the same as in Theorem 5.1. Let t_0 be the supremum of such t that $C(t)$ is nonempty. Then $C(t_0)$ is nonempty as the intersection of an increasing sequence of nonempty compact sets. By Theorem 5.1 and Corollary 5.2, $C(t_0)$ is locally (totally) convex and connected. Obviously, $\dim C(t_0) < \dim C$. By the induction hypothesis, there exist a compact boundaryless totally geodesic (totally convex) submanifold $S \subset C(t_0)$ of M and the diffeomorphism

$$F : O(C(t_0), \varepsilon) \to \nu(S) \tag{5.2}$$

identical on S. We will prove that all hypotheses of Lemma 4.7 are fulfilled for the family $t \mapsto C(-t)$ $t \in [-t_0, 0]$. Then the statement will follow from Lemma 4.7 and the induction hypothesis (5.2). Indeed, the family is uniformly locally convex by Theorem 5.1. It remains to check that the family is uniformly right-hand continuous.

Let ε be an arbitrary positive number. We have to find such $\tau > 0$ that, for any $t, t' \in [0, t_0]$ satisfying $0 < t - t' < \tau$, the inclusion holds

$$C(t') \subset O(C(t), \varepsilon). \tag{5.3}$$

To this end, choose a finite $\varepsilon/4$-net $\{p_1, \ldots, p_k\}$ in C. For every p_i, choose such $q_i \in C \setminus bC$ that $\rho_C(p_i, q_i) < \varepsilon/4$. Then $\{q_1, \ldots, q_k\}$ is the $\varepsilon/2$-net in C. Set $t_1 = \min_i \rho(q_i, bC)$. Then $t_1 > 0$ and (see the definition of $C(t)$)

$$C \subset O(C(t_1), \varepsilon/2). \tag{5.4}$$

Now, consider $t \mapsto C(-t)$ for $t \in [-t_0, -t_1]$ as a family of subsets of the Riemannian manifold $C \setminus bC$. Let us show that the family satisfies all hypotheses of Lemma 3.1. Indeed, $C(t)$ is connected and locally convex for any $t \in [t_1, t_0]$ by Theorem 5.1 and Corollary 5.2. Therefore $C(t)$ is a topological manifold. From the latter fact and the obvious statement $C(t_0) \subset \operatorname{int} C(t)$ (the interior is taken with respect to $C \setminus bC$), we see that $C(t) = \overline{\operatorname{int} C(t)}$. Thus, all hypotheses of Lemma 3.1 are fulfilled. Applying the lemma, we obtain that the family $t \mapsto C(-t)$ $(t \in [-t_0, t_1])$, considered as a family of subsets of $C \setminus bC$, is uniformly right-hand continuous from. Moreover this is true for the family considered as a family of subsets of M. The latter statement and (5.4) imply (5.3). This finishes the proof.

Theorem 5.4 *Let M be an open Riemannian manifold of nonnegative curvature. There exists an increasing family of compact totally geodesic sets $C(t) \subset M$ defined for $t \in [0, \infty)$ such that*

(1) $C(0)$ is nonempty;

(2) $C(t_1) = \{p \in C(t_2) \mid \rho(p, \partial C(t_2) \geq t_2 - t_1$ for $t_1 \leq t_2$;

(3) $\cup_t C(t) = M$.

See the proof in [3].

Together with Lemmas 3.3 and 4.9, Theorem 5.4 implies

Lemma 5.5 *Let M be an open Riemannian manifold of nonnegative curvature. There exists a compact totally geodesic set $C \subset M$ such that, for any sufficiently small positive ε, there exists a diffeomorphism $F : M \to O(C, \varepsilon)$ identical on C.*

Theorem 1.1 follows obviously from Theorem 5.3 and Lemma 5.5.

Final Remarks. Our approach is based on the following observation (Theorem 2.4): the distance function $f(p) = \rho(p, C)$ to a locally convex set C belongs to $C^1(U \setminus C)$ for a sufficiently small neighborhood U and has no critical point. We approximate f by a smooth function g with no critical point and then apply the standard techniques of noncritical Morse theory to g. independently of [10], this approach was realized by Poor [9] approximately at the same time. After the papers [10, 9] were published, a more general version of noncritical Morse theory for Lipschitz continuous functions was developed in [5, 6, 4].

References

[1] Cartan E. *Geometry of Riemannian Spaces.* Brookline, Mass.: Math. Sci Press, c1983.

[2] Cheeger J., Gromoll D. The structure of complete manifolds of nonnegative curvature. *Bull. Amer. Math. Soc.*, **74** (1968), 1147–1150.

[3] Cheeger J., Gromoll D. The structure of complete manifolds of nonnegative curvature. *Ann. Math.*, **96** (1972), 413–443.

[4] Greene R. A genealogy of noncompact manifolds of nonnegative curvature: history and logic. *Comparison Geometry.* MSRI Publications, **30** (1997), 99–134.

[5] Greene R., Shiohama K. Convex functions on complete noncompact manifolds. *Ann. scient. Ëc. Norm. Sup.* (4), **14** (1981), 357–367.

[6] Greene R., Wu H. Integrals of subharmonic functions on manifolds of nonnegative curvature. *Invent. Math.*, **27** (1974), 265–298.

[7] Klingenberg W. *Riemannian Geometry.* Berlin – New York, W. de Gruyter, 1995.

[8] Milnor J. *Lectures on the H-Cobordism Theorem.* Princeton N.J., Princeton University Press, 1965.

[9] Poor W. Some results on nonnegatively curved manifolds. *J. Diff. Geom.*, **9** (1974), 583–600.

[10] Sharafutdinov V. Complete open manifolds of nonnegative curvature. *Siberian Math. J.*, **15** (1974), no. 1, 177–191 [in Russian].