SPECTRAL RIGIDITY OF A COMPACT NEGATIVELY CURVED MANIFOLD

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1. Introduction

In [1] V. Guillemin and D. Kazhdan introduced the following definition of spectral rigidity of a Riemannian manifold.

Let (M, g) be a compact boundaryless Riemannian manifold. A family g^τ of Riemannian metrics on M smoothly depending on the parameter $\tau \in [-\epsilon, \epsilon]$ is called the *deformation* of the metric g if $g^0 = g$. A deformation is called *trivial* if there exists a one-parameter family of diffeomorphisms $\varphi^\tau : M \to M$ such that $\varphi^0 = \mathrm{Id}$, and $g^\tau = (\varphi^\tau)^* g^0$. Given a deformation g^τ $(-\epsilon \le \tau \le \epsilon)$, let $\Delta^\tau : C^\infty(M) \to C^\infty(M)$ be the Laplace-Beltrami operator corresponding to the metric g^τ. The deformation is called *isospectral* if, for all $-\epsilon \le \tau \le \epsilon$ spectra of the operators Δ^τ and Δ^0 coincide (counting multiplicities). A Riemannian manifold (M, g) is called *spectrally rigid* if it does not admit non-trivial isospectral deformations.

The main result of the present article is the following

Theorem 1.1. *A compact negatively curved Riemannian manifold is spectrally rigid.*

For two-dimensional manifolds, this result was obtained by V. Guillemin and D. Kazhdan [1]. The same authors proved this fact for n-dimensional manifolds [2] under a pointwise curvature pinching assumption. That result was later extended by Min-Oo [3] to the case where the curvature operator is negative definite. On the other hand, since [1] was published, a number of examples of isospectral deformations of compact manifolds have been given. The first such example was due to Gordon and Wilson [4]. Hence to rule out isospectral deformations there must be some extra assumption, such as curvature above. The examples of Vignéras in [5] show that even nonisometric surfaces of constant negative curvature can be isospectral. Thus the best rigidity one can hope for is: for a compact manifold of negative curvature the space of isospectral manifolds is finite. Theorem 1.1 can be considered an important step in that direction.

A compact Riemannian manifold is said to have a simple length spectrum if there do not exist two different closed geodesics such that the ratio of their lengths is a rational number. This is a generic condition.

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Theorem 1.2. *Let (M, g) be a compact negatively curved Riemannian manifold with simple length spectrum, and $\Delta : C^\infty(M) \to C^\infty(M)$ be the corresponding Laplace-Beltrami operator. If real functions $q_1, q_2 \in C^\infty(M)$ are such that the operators $\Delta + q_1$ and $\Delta + q_2$ have coincident spectra, then $q_1 \equiv q_2$.*

In [1,2] (as we will point out) Guillemin and Kazhdan show that Theorems 1.1 and 1.2 follow from Theorem 1.3 below, which is a generalization of Theorem 4 of [2] to the case of negatively curved manifolds.

In the statement below $S^m\tau'_M$ represents the symmetric m tensors, and d is the operator that symmetrizes the covariant derivative (see the next section for definitions). For $f \in S^m\tau'_M$ and ξ a vector we let $\langle f, \xi^m \rangle$ represent the action of f on the m-th tensor product of ξ with itself.

Theorem 1.3. *Let (M, g) be a compact negatively curved Riemannian manifold. If a symmetric tensor field $f \in C^\infty(S^m\tau'_M)$ is such that the integral*

$$\int_0^1 \langle f(\gamma(t)), \dot\gamma^m(t) \rangle \, dt$$

is equal to zero for every closed geodesic $\gamma : [0, 1] \to M$, then there exists a field $v \in C^\infty(S^{m-1}\tau'_M)$ such that $dv = f$.

This result has three interesting special cases: namely for $m = 0, 1, 2$.

Corollary 1.4. *Let M be a compact negatively curved manifold and $f : M \to R$ a C^∞ function. If f integrates to zero along every closed geodesic then f must itself be zero. In particular a function is determined by its integral along closed geodesics.*

Theorem 1.2 follows from this corollary (as in [1,2]) since in our case the spectrum determines the integral of the potential along each closed geodesic.

Note that the corollary is not true in positive curvature (even if closed geodesics are dense in unit tangent bundle) as the round sphere shows.

Corollary 1.5. *Let M be a compact negatively curved manifold, ω a smooth one form (not a-priori closed) on M, and $[\alpha]$ a real 1-cohomology class. If for every closed geodesic γ the value of $[\alpha]$ on the homology class of γ is the same as the integral of ω around γ then ω is a closed form that represents $[\alpha]$.*

To see this let α be a closed 1-form representing $[\alpha]$. Then by Theorem 1.3 $\alpha - \omega$ is exact, so the corollary follows.

Corollary 1.6. *Let M be a compact negatively curved manifold and ω a smooth symmetric 2 form that integrates to 0 along every closed geodesic then $\omega = \frac{d}{dt}\phi_t^* g$ where ϕ_t is a smooth 1-parameter family of diffeomorphisms.*

Theorem 1.3 gives us a one form α such that $d\alpha = \omega$ (note d is not exterior derivative). Let V be the vector field dual to α, and ϕ_t be the 1-parameter group of diffeomorphisms corresponding to $V/2$. A standard computation yields the corollary. Theorem 1.1 follows from a similar argument as in [1,2].

The rest of the paper is devoted to the proof of Theorem 1.3. The proof is based on the ideas developed in [7] (also see [8]) where the corresponding theorem for convex domains with boundary and negative sectional curvature is proved. We pass from the convex with boundary case to the compact with no boundary case with the help of Livčic's theorem [9,p. 566] in a way similar to [1,2].

2. Symmetric tensor fields on a compact negatively curved manifold

Given a manifold M, by $\tau_M = (TM, p, M)$ and $\tau'_M = (T'M, p', M)$ we denote the tangent and cotangent vector bundles respectively. Points of the tangent space TM are denoted by pairs (x, ξ), where $x \in M,\ \xi \in T_xM$. Let $\tau^r_s M$ be the complex bundle of tensors over M which are r times contravariant and s times covariant. Recall that every permutation π of the set $\{1, \dots, m\}$ defines the corresponding automorphism ρ_π of the bundle $\tau^0_m M$ which is called the *transposition of indices.* Let $S^m\tau'_M$ be the subbundle of $\tau^0_m M$ consisting of tensors invariant with respect to all transpositions of indices. The *canonical projection* (*symmetrization*) $\sigma : \tau^0_m M \to S^m\tau'_M$ is defined by the equality $\sigma = \frac{1}{m!}\sum_\pi \rho_\pi$. The *symmetric product* is defined by the formula $uv = \sigma(u \otimes v)$. This product turns $S^*\tau'_M = \oplus_{m=0}^\infty S^m\tau'_M$ into a bundle of commutative graded algebras. Therefore $C^\infty(S^*\tau'_M)$ is a commutative graded $C^\infty(M)$-algebra whose sections are called *covariant symmetric tensor fields on* M.

Now let (M, g) be a Riemannian manifold. By

$$\Omega M = \{(x, \xi) \in TM \mid |\xi|^2 = \langle \xi, \xi\rangle = g_{ij}(x)\xi^i\xi^j = 1\}$$

we denote the manifold of unit tangent vectors. The metric g establishes a canonical isomorphism of the bundles $\tau^r_s M \cong \tau^{r+s}_0 M \cong \tau^0_{r+s} M$ which is expressed in coordinate form by the well-known operations of raising and lowering indices. Therefore in the case of a Riemannian manifold one does not need to distinguish co- and contravariant tensors, and can talk about co- and contravariant components of the same tensor. The scalar product is introduced in fibers of the bundle $\tau^0_m M$ by the formula $\langle u, v\rangle = u^{i_1\dots i_m}\bar v_{i_1\dots i_m}$; it is evidently independent of the choice of local coordinates. It turns $S^m\tau'_M$ into a Hermitian vector bundle and allows us to define the space $L_2(S^m\tau'_M)$ with the scalar product

$$(u, v)_{L_2(S^m\tau'_M)} = \int_M \langle u(x), v(x)\rangle\, dV^n(x), \tag{2.1}$$

where dV^n is the Riemannian volume form on M. The lower index in (2.1) will sometimes be omitted. For $u \in C^\infty(S^m\tau'_M)$, the equality

$$\langle u(x), \xi^m\rangle = u_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m}$$

is valid which we will make use of abbreviation various formulas.

Given a Riemannian manifold (M, g), let

$$\nabla : C^\infty(\tau^r_s M) \to C^\infty(\tau^r_{s+1} M), \qquad \nabla : \left(u^{i_i\dots i_r}_{j_1\dots j_s}\right) \mapsto \left(\nabla_k u^{i_i\dots i_r}_{j_1\dots j_s}\right)$$

be the Levi-Civita covariant derivative. *Inner differentiation*

$$d : C^\infty(S^m\tau'_M) \to C^\infty(S^{m+1}\tau'_M)$$

is defined by the equality $d = \sigma\nabla$. The *divergence*

$$\delta : C^\infty(S^{m+1}\tau'_M) \to C^\infty(S^m\tau'_M)$$

is defined in coordinate form by the formula $(\delta u)_{i_1\dots i_m} = g^{jk}\nabla_j u_{ki_1\dots i_m}$. The operators d and $-\delta$ are adjoints with respect to the scalar product (2.1).

Given a complete Riemannian manifold (M, g), we will let $G^t : TM \to TM$ denote the geodesic flow, and H denote the vector field, on the manifold TM, generating the flow G^t. Like every vector field, H can be considered as a first order differential operator

$$H : C^\infty(TM) \to C^\infty(TM).$$

Since the manifold ΩM is invariant under the geodesic flow, H can also be considered as a differential operator

$$H : C^\infty(\Omega M) \to C^\infty(\Omega M).$$

Given a local coordinate system $(x^1, \dots, x^n)$ in M with a domain $U \subset M$, the functions $(x^1, \dots, x^n, \xi^1, \dots, \xi^n)$ constitute a local coordinate system in TM with domain $p^{-1}(U)$, where p is the projection of the tangent bundle. (Strictly speaking, we should write $x^i \circ p$ above; nevertheless, we will use the abbreviated notation x^i instead of $x^i \circ p$, hoping that it will not lead to misunderstanding.) These are the only coordinate systems on TM that are used in the present article. The vector field H in coordinate form is:

$$H = \xi^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk}(x)\xi^j\xi^k \frac{\partial}{\partial \xi^i},$$

where Γ^i_{jk} are the Christoffel symbols. A direct calculation in coordinates shows that for a symmetric tensor field $v \in C^\infty(S^m\tau'_M)$,

$$H\left(\langle v(x), \xi^m\rangle\right) = \langle (dv)(x), \xi^{m+1}\rangle. \tag{2.2}$$

This shows how the differential operator H is related to the inner differentiation d.

Lemma 2.1. *Let a complete Riemannian manifold (M, g) be such that there exists an orbit of the geodesic flow which is dense in ΩM. If a symmetric tensor field $v \in C^\infty(S^m\tau'_M)$ satisfies the equation*

$$dv = 0, \tag{2.3}$$

then

(i) if m is odd,
v is identically zero;
(ii) if $m = 2l$ is even, v is of the form $v = cg^l$, where c is a constant.

It is well-known [6] that in the case of a compact negatively curved Riemannian manifold almost all orbits of the geodesic flow on ΩM are dense in ΩM. Therefore the conclusion of Lemma 2.1 holds for such manifolds.

Proof of Lemma 2.1. Define the function $\varphi \in C^\infty(TM)$ by the equality $\varphi(x, \xi) = \langle v(x), \xi^m\rangle$. It follows from (2.2) and (2.3) that φ is constant on every orbit of the geodesic flow. Therefore, the restriction of φ to ΩM is constant. From this, taking the homogeneity of $\varphi(x, \xi)$ in its second argument into account, we obtain

$$\langle v(x), \xi^m\rangle = c|\xi|^m.$$

This equality clearly implies the claim of the lemma.

Given a compact manifold M and an integer $k \geq 0$, by $H^k(S^m\tau'_M)$ we denote the topological Hilbert space of sections of $S^m\tau'_M$ whose components have locally quadratically integrable derivatives up to order k with respect to any local coordinates. Let $\|\cdot\|_k$ denote one of equivalent norms of the space.

Theorem 2.2. *Let a compact Riemannian manifold (M, g) be such that there exists an orbit of the geodesic flow which is dense in ΩM, and let $k \geq 1$ be an integer.*

1. *For even m, every symmetric tensor field $f \in H^k(S^m\tau'_M)$ can be uniquely represented in the form*

$$f = dv + \tilde{f}, \tag{2.4}$$

where $v \in H^{k+1}(S^{m-1}\tau'_M)$, and the field $\tilde{f} \in H^k(S^m\tau'_M)$ is solenoidal, i.e., satisfies the equation

$$\delta\tilde{f} = 0. \tag{2.5}$$

These fields satisfy the estimates

$$\|v\|_{k+1} \leq C\|\delta f\|_{k-1}, \qquad \|\tilde{f}\|_k \leq C\|f\|_k \tag{2.6}$$

with a constant C independent of f.

2. *For odd $m = 2l + 1$, the previous claim is also valid under the additional assumption that v satisfies the relation*

$$(v, g^l)_{L_2(S^{2l}\tau'_M)} = 0. \tag{2.7}$$

In particular, in both 1 and 2 above if f is smooth then $\tilde{f}$ and v are also smooth.

The terms of decomposition (2.4) are called the *potential* and *solenoidal parts* of the symmetric tensor field f respectively.

Proof. Assume existence of symmetric tensor fields v and $\tilde{f}$ satisfying (2.4) and (2.5), and apply the operator δ to the first of the equalities to obtain

$$\delta dv = \delta f. \tag{2.8}$$

Conversely, if equation (2.8) has a solution satisfying the first of estimates (2.6), then, putting $\tilde{f} = f - dv$, we would arrive at the claim of the theorem.

As can be easily shown [8, p. 89], the operator

$$\delta d : H^{k+1}(S^{m-1}\tau'_M) \to H^{k-1}(S^{m-1}\tau'_M) \tag{2.9}$$

is elliptic. Therefore, its kernel $\mathrm{Ker}(\delta d)$ is a finite-dimensional vector space consisting of smooth fields; the image $\mathrm{Im}(\delta d)$ is a closed subspace in $H^{k-1}(S^{m-1}\tau'_M)$; the orthogonal complement $(\mathrm{Im}(\delta d))^\perp$ is a finite-dimensional vector space consisting of smooth fields; and operator (2.9) induces an isomorphism of the topological Hilbert spaces

$$H^{k+1}(S^{m-1}\tau'_M)/\mathrm{Ker}(\delta d) \to \mathrm{Im}(\delta d). \tag{2.10}$$

Let us show that

$$\mathrm{Ker}(\delta d) = (\mathrm{Im}(\delta d))^\perp = \{v \in C^\infty(S^{m-1}\tau'_M) \mid dv = 0\}. \tag{2.11}$$

Indeed, if $v \in \mathrm{Ker}(\delta d)$, then

$$(dv, dv) = -(v, \delta dv) = 0.$$

If $v \in (\mathrm{Im}(\delta d))^\perp$, then for every $u \in C^\infty(S^{m-1}\tau'_M)$

$$(\delta dv, u) = (v, \delta du) = 0.$$

Therefore, $\delta dv = 0$, i.e., $v \in \mathrm{Ker}(\delta d)$.

Observe that the right-hand side of equation (2.8) belongs to $\mathrm{Im}(\delta d)$ since $(\delta f, v) = -(f, dv) = 0$ if $dv = 0$. Therefore, equation (2.8) has a solution for every $f \in H^k(S^m\tau'_M)$.

In the case of even m, equalities (2.11) with the help of Lemma 2.1 imply that $\mathrm{Ker}(\delta d) = 0$. Thus, equation (2.8) has a unique solution for every $f \in H^k(S^m\tau'_M)$. Since (2.9) is an isomorphism, the first of estimates (2.6) holds.

In the case of odd $m = 2l + 1$, equalities (2.11) with the help of Lemma 2.1 imply that $\mathrm{Ker}(\delta d)$ consists of the fields cg^l. Therefore, equation (2.8) has a unique solution satisfying condition (2.7). The first of the estimates (2.6) also holds for the solution. Thus the theorem is proved.

Theorem 2.3. *Let (M, g) be a compact non-positively curved Riemannian manifold such that there exists an orbit of the geodesic flow which is dense in ΩM. If a function $u \in C^\infty(\Omega M)$ and a symmetric tensor field $f \in C^\infty(S^m\tau'_M)$ satisfy the equation*

$$Hu(x, \xi) = \langle f(x), \xi^m\rangle \tag{2.12}$$

on ΩM, then the field f is potential, i.e., there exists a symmetric tensor field $v \in C^\infty(S^{m-1}\tau'_M)$ such that $dv = f$.

Remark. In [2] equation (2.12) was investigated by expanding the function $u(x, \xi)$ into the Fourier series of spherical harmonics in the argument ξ and considering the corresponding infinite system of differential equations on M. We use quite different approach in investigating equation (2.12).

We first show that Theorem 2.3 follows from the following special case.

Lemma 2.4. *Let (M, g) be as in Theorem 2.3. If a symmetric tensor field $f \in C^\infty(S^m\tau'_M)$ is solenoidal, i.e., satisfies the equation*

$$\delta f = 0, \tag{2.13}$$

and if there exists a function $u \in C^\infty(\Omega M)$ satisfying (2.12), then $f \equiv 0$.

Proof. Indeed, let the assumptions of Theorem 2.3 be fulfilled, and let (2.4) be the decomposition of the field f into potential and solenoidal parts. Putting

$$\tilde{u}(x, \xi) = u(x, \xi) - \langle v(x), \xi^{m-1}\rangle$$

from (2.2) and (2.12) we derive

$$H\tilde{u}(x, \xi) = \langle \tilde{f}(x), \xi^m\rangle.$$

Assuming Lemma 2.4 to be valid, the last equality implies $\tilde{f} = 0$. Now formula (2.4) yields $dv = f$.

Before proving Lemma 2.4, we recall some more notions of tensor analysis.

Given a manifold M, the bundle $\beta^r_s M = p^*(\tau^r_s M)$ over TM, where $p : TM \to M$ is the projection of the tangent bundle, is called the *bundle of semibasic tensors of* *degree* (r,s). Sections of the bundle are called *semibasic tensor fields*. For such a field $u \in C^\infty(\beta^r_s M)$, the coordinate representation $u = \big(u^{i_1\dots i_r}_{j_i\dots j_s}(x,\xi)\big)$ holds in the domain of a local coordinate system on TM. Under a change of local coordinates, the components of a semibasic tensor field are transformed according to the same formula as for an ordinary tensor field. In particular, $C^\infty(\beta^0_0 M) = C^\infty(TM)$.

The *vertical covariant derivative*

$$\overset{v}{\nabla} : C^\infty(\beta^r_s M) \to C^\infty(\beta^r_{s+1} M)$$

is defined in coordinate form by the formula

$$\overset{v}{\nabla}_k u^{i_1\dots i_r}_{j_i\dots j_s} = \frac{\partial}{\partial \xi^k} u^{i_1\dots i_r}_{j_i\dots j_s}.$$

Thus for a tangent vectors $X,\xi \in T_xM$, $(\overset{v}{\nabla}_X u)(\xi)$ is just $\frac{d}{dt}u(\xi + tX)$.

For a Riemannian manifold (M,g), the *horizontal covariant derivative*

$$\overset{h}{\nabla} : C^\infty(\beta^r_s M) \to C^\infty(\beta^r_{s+1} M)$$

is defined in coordinate form by the equality

$$\overset{h}{\nabla}_k u^{i_1\dots i_r}_{j_i\dots j_s} = \frac{\partial}{\partial x^k} u^{i_1\dots i_r}_{j_i\dots j_s} - \Gamma^p_{kq}\xi^q \frac{\partial}{\partial \xi^p} u^{i_1\dots i_r}_{j_i\dots j_s} + \\ + \sum_{m=1}^{r} \Gamma^{i_m}_{kp} u^{i_1\dots i_{m-1} p i_{m+1}\dots i_r}_{j_i\dots j_s} - \sum_{m=1}^{s} \Gamma^p_{kj_m} u^{i_1\dots i_r}_{j_i\dots j_{m-1} p j_{m+1}\dots j_s}.$$

For tangent vectors $X,\xi \in T_xM$ let γ be a curve in M with $\gamma'(0) = X$, and $\tilde\gamma(t)$ be the horizontal lift of γ to TM such that $\tilde\gamma(0) = \xi$. Then $u(\tilde\gamma(t))$ is a tensor field along $\gamma(t)$ starting with $u(\xi)$, and $(\overset{h}{\nabla}_X u)(\xi)$ is the covariant derivative of this field.

As can be easily shown, $\overset{v}{\nabla}$ and $\overset{h}{\nabla}$ are well-defined differential operators, i.e., are independent of the choice of local coordinates. For a semibasic vector fields v we define the horizontal divergence, $\overset{h}{\operatorname{div}}\, v$, by

$$\overset{h}{\operatorname{div}}\, v = \overset{h}{\nabla}_i v^i.$$

Similarly define the vertical divergence $\overset{v}{\operatorname{div}}\, v$.

A major role in our proof of Lemma 2.4 is played by *Pestov's identity* [8, p. 122] asserting that, for every real function $u \in C^\infty(TM)$, the equality

$$2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = |\overset{h}{\nabla} u|^2 + \overset{h}{\operatorname{div}}\, v + \overset{v}{\operatorname{div}}\, w - \langle R(\xi, \overset{v}{\nabla} u)\xi, \overset{v}{\nabla} u\rangle \tag{2.14}$$

holds on TM, where R is the curvature tensor of the Riemannian manifold (M,g), and the semibasic vector fields v and w are given by the formulas

$$v = \langle \overset{h}{\nabla} u, \overset{v}{\nabla} u\rangle \xi - \langle \xi, \overset{h}{\nabla} u\rangle \overset{v}{\nabla} u, \tag{2.15}$$

$$w = \langle \xi, \overset{h}{\nabla} u\rangle \overset{h}{\nabla} u. \tag{2.16}$$

We point out for future reference that $\langle w, \xi\rangle = |Hu|^2$.

Observe that the last term in the right-hand side of (2.14) is the sectional curvature multiplied by $|\xi \wedge \overset{v}{\nabla} u|^2$. Therefore, since the curvature is nonpositive, (2.14) implies the inequality

$$|\overset{h}{\nabla} u|^2 \le 2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle - \overset{h}{\operatorname{div}}\, v - \overset{v}{\operatorname{div}}\, w. \tag{2.17}$$

Proof of Lemma 2.4. Let f and u satisfy the lemma hypothesis. We extend the function $u(x, \xi)$ onto $TM \setminus 0$ in such a way that the function becomes positively homogeneous of degree $m-1$ in its second argument. Then $u \in C^\infty(TM \setminus 0)$, and equation (2.12) holds on $TM \setminus 0$. Using (2.12), we transform the first term in the right-hand side of inequality (2.17) as follows (hereafter the notation $\overset{v}{\nabla}{}^i = g^{ij}\overset{v}{\nabla}_j$ and $\overset{h}{\nabla}{}^i = g^{ij}\overset{h}{\nabla}_j$ are used):

$$2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = 2\overset{h}{\nabla}{}^j u \cdot \frac{\partial}{\partial \xi^j}(f_{i_1\dots i_m}\xi^{i_1}\dots\xi^{i_m}) = 2m\overset{h}{\nabla}{}^j u \cdot f_{ji_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} =$$
$$= \overset{h}{\nabla}{}^j (2mu f_{ji_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) - 2mu(\delta f)_{i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}.$$

Using condition (2.13), we obtain

$$2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = \overset{h}{\nabla}_i \tilde v^i, \tag{2.18}$$

where

$$\tilde v^i = 2mu g^{ij} f_{ji_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}.$$

That is $\tilde v$ is the vector field dual to the one form $2muf(\cdot, \xi, \dots, \xi)$. Replacing the first term in the right-hand side of (2.17) by its value (2.18), we obtain

$$|\overset{h}{\nabla} u|^2 \le \overset{h}{\operatorname{div}}(\tilde v - v) - \overset{v}{\operatorname{div}}\, w. \tag{2.19}$$

We multiply inequality (2.19) by the symplectic volume form $d\Sigma = d\Sigma^{2n-1}$ and integrate the result over ΩM. Transforming the integrals on the right-hand side of the so-obtained inequality by the Gauss-Ostrogradskiĭ formulas for vertical and horizontal divergences [8, p. 110], we arrive at the relation

$$\int_{\Omega M} |\overset{h}{\nabla} u|^2\, d\Sigma \le -(n + 2m - 2)\int_{\Omega M} \langle w, \xi\rangle\, d\Sigma. \tag{2.20}$$

The constant $n + 2m - 2$ above comes from the fact that the field $w(x, \xi)$ is homogeneous of degree $2m-1$ in its second argument, as can be seen from (2.16). Further, since $\langle w, \xi\rangle = |Hu|^2$, inequality (2.20) takes the form

$$\int_{\Omega M} |\overset{h}{\nabla} u|^2\, d\Sigma + (n + 2m - 2)\int_{\Omega M} |Hu|^2\, d\Sigma \le 0.$$

Consequently, $Hu \equiv 0$. Now (2.12) implies that $f \equiv 0$. The lemma is thus proved.

Proof of Theorem 1.3:. The condition of the theorem means that the integral of the function

$$F(x,\xi) = \langle f(x), \xi^m \rangle$$

over every closed orbit of the geodesic flow is equal to zero. Recall that the geodesic flow of a compact negatively curved manifold has Anosov's type [6]. By the smooth version of Livčic's theorem [9, p. 566], there exists a function $u \in C^\infty(\Omega M)$ satisfying equation (2.12). Applying Theorem 2.3, we arrive at the claim of Theorem 2.5.

As was mentioned above, Theorems 1.1 and 1.2 can be proved using Theorem 1.3 in the same way as in [1,2].

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