Application of geometric symbol calculus to computing heat invariants

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Abstract

The problem of evaluating heat invariants can be computerized. Geometric symbol calculus of pseudodifferential operators is the main tool of such computerization.

We are going to demonstrate that geometric symbol calculus can be used for evaluating heat invariants and that the calculations can be computerized. To this end we evaluate first three heat invariants. All calculations have been done manually in the paper since the author is not good personally with a computer. Nevertheless, we have used results of computer calculations that have been kindly done by Valery Djepko and Michal Skokan by author's request. We hope the paper will inspire a young mathematician for a further progress in computing heat invariants.

1   Heat invariants

We start with a short summary of Sections 1.6 and 1.7 of book [2] by Peter Gilkey.

Let (M, g) be a closed (= compact with no boundary) Riemannian manifold of dimension n and V be a Hermitian vector bundle over M. Let V_x denote the fiber of V over $x \in M$. Let $\mathrm{End}\, V$ be the vector bundle over M whose fiber over x consists of all linear operators $V_x \to V_x$. The cotangent bundle of M is denoted by T^*M and its points are denoted by pairs (x, ξ), where $x \in M$ and $\xi \in T_x^*M$.

We consider an elliptic self-adjoint differential operator $P : C^\infty(V) \to C^\infty(V)$ of order $\mu > 0$ with a positive definite principle symbol. The eigenvalue spectrum $\mathrm{Sp}(P) = \{\lambda_0 \le \lambda_1 \le \cdots \le \lambda_k \le \cdots \to +\infty\}$ is real and bounded from below. The initial value problem for the heat equation

$$(\partial/\partial t + P)f(t, x) = 0 \quad \text{for} \quad t \ge 0, \quad f(0, x) = f(x)$$

has a unique solution for every $f \in L^2(M)$ which can be written as

$$f(t, x) = e^{-tP} f(x) = \int_M K(t, x, y) f(y) \, dM(y),$$

where dM is the Riemannian volume form. The function $K(t, x, y)$ is the *fundamental solution* to the heat equation. It is a smooth function of $(t, x, y) \in \mathbb{R}^+ \times M \times M$ whose value is a linear operator from V_y to V_x. The function $\operatorname{tr} K(t, x, x)$ admits the asymptotic expansion

$$\operatorname{tr} K(t, x, x) \sim \frac{1}{(4\pi t)^{n/\mu}} \Big(a_0(x, P) + a_2(x, P) t^{2/\mu} + a_4(x, P) t^{4/\mu} + \dots \Big) \text{ as } t \to +0 \quad (1.1)$$

whose coefficients are *local heat invariants* of the operator P. Recall that $n = \dim M$ and μ is the order of P. The following locality constitutes the main property of heat invariants: $a_k(x, P)$ can be expressed in terms of the jet of P at x of some finite order. The integrated invariants $a_k(P) = \int_M a_k(x, P) \, dM(x)$ are of great importance in spectral geometry since they are determined by the eigenvalue spectrum of P:

$$\operatorname{tr}_{L^2} e^{-tP} = \sum_{k=0}^{\infty} e^{-t\lambda_k} \sim \frac{1}{(4\pi t)^{n/\mu}} \Big(a_0(P) + a_2(P) t^{2/\mu} + a_4(P) t^{4/\mu} + \dots \Big) \quad \text{as} \quad t \to +0.$$

The existence of the asymptotic expansion (1.1) is proved in Section 1.7 of [2]. Gilkey mentions also that the method of the proof can be used for computing invariants. But actually he uses another approach for computing $a_k(x, \Delta_\nu)$ in the case of the Laplacian on ν-forms. Namely, he proves that every $a_k(x, \Delta_\nu)$ is a homogeneous polynomial in partial derivatives of the metric tensor. Then, using Weyl's theorem on invariants, he finds all such polynomials invariant under the orthogonal group. This gives a formula for $a_k(x, \Delta_\nu)$ up to some constant coefficients. The coefficients are determined by considering some definite manifolds and using some functorial properties of invariants, see Section 4.8 of [2].

We will compute invariants $a_k(x, \Delta_\nu)$ more explicitly using the geometric symbol calculus developed in [3], see also Appendix below. Roughly speaking, our algorithm follows the proof of the existence theorem.

Let $I \in C^\infty(\operatorname{End} V)$ be the identity operator. For a complex $\lambda \notin \operatorname{Sp}(P)$, the operator $\lambda I - P$ has the bounded inverse $(\lambda I - P)^{-1} : L^2(V) \to L^2(V)$. Being considered as a function of λ, the *resolvent* $(\lambda I - P)^{-1}$ is a holomorphic function in $\mathbb{C} \backslash \operatorname{Sp}(P)$. In particular, the function is holomorphic in the cut plane $\mathbb{C}_{cut} = \mathbb{C} \backslash [C, \infty)$, where $C = \inf \operatorname{Sp}(P)$. Let γ be the oriented curve in $\mathbb{C}_{cut}$ which goes over the cutset from the point $\infty + ia$ (with some $a > 0$) to the point $C + ia$, then rounds the point C, and then goes under the cutset from $C - ia$ to $\infty - ia$. Then

$$e^{-tP} = \frac{1}{2\pi i} \int_\gamma e^{-t\lambda} (\lambda I - P)^{-1} d\lambda. \tag{1.2}$$

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The resolvent $(\lambda I - P)^{-1}$ is not a pseudodifferential operator. The main idea of the proof is to replace the factor $(\lambda I - P)^{-1}$ on (1.2) with some pseudodifferential operator $R(\lambda)$ that approximates the resolvent $(\lambda I - P)^{-1}$ in an appropriate sense. The main feature of the approximation is the right understanding the role of λ: we think of the parameter λ as being of the same order μ as the principal symbol of P. According to this idea, we introduce the following definition.

Let W be a vector bundle over M and $\mu > 0$. Fix a domain $\mathbb{C}_{cut} \subset \mathbb{C}$. For a real k, the space $S_\mu^k(T^*M, W, \lambda)$ of symbols of order $\leq k$ depending on the complex parameter $\lambda \in \mathbb{C}_{cut}$ consists of functions $q : T^*M \times \mathbb{C}_{cut} \to W$ satisfying
(a) $q(x, \xi, \lambda) \in W_x$ is smooth in $(x, \xi, \lambda) \in T^*M \times \mathbb{C}_{cut}$ and is holomorphic in λ;
(b) For all (α, β, γ) the estimate

$$|D_x^\alpha D_\xi^\beta D_\lambda^\gamma q(x, \xi, \lambda)| \leq C_{\alpha,\beta,\gamma}(1 + |\xi| + |\lambda|^{1/\mu})^{k-|\beta|-\mu|\gamma|}$$

holds with some constant $C_{\alpha,\beta,\gamma}$ uniformly in any compact belonging to the domain of a local coordinate system.

Compare this with the definition of $S^k(T^*M, W)$ in Appendix below. We say that $q(x, \xi, \lambda)$ is homogeneous of order k in (ξ, λ) if $q(x, t\xi, t^\mu\lambda) = t^k q(x, \xi, \lambda)$ for $t \geq 1$. We think of the parameter λ as being of order μ. It is clear if q is homogeneous in (ξ, λ), then it satisfies the decay condition (b).

Let V be a vector bundle over M furnished with a connection ∇^V. The latter, together with the Levi-Chivita connection ∇ of the Riemannian manifold M, allows us to define the covariant derivative $\nabla : C^\infty(V) \to C^\infty(V \otimes T^*M)$. More generally, if $\tau_s^r M$ is the bundle of (r, s)-tensors, the covariant derivative $\nabla : C^\infty(V \otimes \tau_s^r M) \to C^\infty(V \otimes \tau_{s+1}^r M)$ is well defined. Now, a differential operator $P : C^\infty(V) \to C^\infty(V)$ of order μ is uniquely written in the form $P = p(x, -i\nabla)$, where $p(x, \xi) \in \mathrm{End} V_x$ is a polynomial of order μ in ξ. The polynomial $p(x, \xi)$ is called the *full geometric symbol* of the differential operator P. We will write $p = \sigma P$. See Section 7 of [3] for details.

Next, we define the space $\Psi_\mu^k(M, \nabla, V, \lambda)$ of pseudodifferential operators with full geometric symbols in $S_\mu^k(T^*M, \mathrm{End} V, \lambda)$ in the complete analogy with definition (8.1) of [3], see also Appendix below. For $Q(\lambda) \in \Psi_\mu^k(M, \nabla, V, \lambda)$, we denote the full geometric symbol by $q(\lambda) = \sigma Q \in S_\mu^k(T^*M, \mathrm{End} V, \lambda)$. The new feature arises from the dependence on the parameter λ. All facts of the geometric symbol calculus are obviously generalized to the class $\Psi_\mu^k(M, \nabla, V, \lambda)$ of operators depending on λ, with the only one exception: given a sequence $q_j \in S_\mu^{k-j}(T^*M, \mathrm{End} V, \lambda)$, in the general case there is no operator $Q(\lambda) \in \Psi_\mu^k(M, \nabla, V, \lambda)$ with the symbol $\sigma Q(\lambda) = \sum_{j=0}^\infty q_j(\lambda)$ since the sum of the series can be not holomorphic in λ. Nevertheless, there is no problem with a finite sum $\sum_{j=0}^{j_0} q_j(\lambda)$. Thus, in constructing an approximation for the resolvent, we will always restrict to a finite sum rather than an infinite series.

We wish to solve the equation

$$\sigma(R(\lambda)(\lambda I - P)) \sim I. \tag{1.3}$$

Standard arguments of symbol calculus show that, for an arbitrary elliptic P of order μ, the equation has a unique solution $R(\lambda)$ with the geometric symbol $r = r_0 + r_1 + \dots$, where $r_k = r_k(x, \xi, \lambda) \in S_\mu^{-\mu - k}(T^*M, \operatorname{End} V, \lambda)$ is homogeneous of degree $-\mu - k$ in (ξ, λ). Then the asymptotic expansion (1.1) holds with

$$a_k(x, P) = \frac{\pi^{-n/2}}{2\pi i} \int\limits_{T_x^*M} \int\limits_\gamma e^{-\lambda} \operatorname{tr} r_k(x, \xi, \lambda)\, d\lambda d\xi, \tag{1.4}$$

as is proved in Section 1.7 of [2].

Our algorithm for computing the heat invariants $a_k(x, P)$ is as follows. First we have to find the full geometric symbol $r(x, \xi, \lambda) = r_0(x, \xi, \lambda) + r_1(x, \xi, \lambda) + \dots$ of the operator $R(\lambda)$ by solving equation (1.3). Then the invariants are computed by formula (1.4). We emphasize that the algorithm does not involve any ambiguity unlike the corresponding procedure of [2]. Indeed, since we use geometric symbol calculus, (1.3) is a coordinate free equation or, to be more precise, the equation does not change its form under a coordinate change. The difference between the geometric version of equation (1.3) and classical one can be illustrated in the case of $P = \Delta_\nu$, the Laplacian on ν-forms. In the latter case, the classical version of equation (1.3) strongly depends on higher order derivatives of the metric tensor while the geometric version of the equation involves only the curvature tensor and its covariant derivatives.

In the case of a general elliptic P, solution of equation (1.3) is a very hard business since the geometric symbol of the product is expressed by a rather complicated formula, see formula (A.3) in Appendix below. In the next section, we will solve the equation in the case of $P = -\nabla^p \nabla_p + A$ with an algebraic operator A. The Laplacian on forms is of this kind.

2 Recurrent formula for r_k

Let (V, ∇^V) be a vector bundle with connection over a Riemannian manifold (M, g). As we have mentioned in the previous section, the covariant derivative ∇ is well defined on V-valued tensor fields. In particular, the operator $\nabla^p \nabla_p = g^{ij} \nabla_i \nabla_j : C^\infty(V) \to C^\infty(V)$ is well defined. Hereafter, $\nabla^i = g^{ij} \nabla_j$ and (g^{ij}) is the inverse matrix of (g_{ij}). We use Einstein's rule: the summation from 1 to $n = \dim M$ is assumed over an index repeated in upper and low position in a monomial.

We fix a self-adjoint algebraic operator $A \in C^\infty(\operatorname{End} V)$ and consider the second order differential operator on the bundle V

$$P = P_A = -\nabla^p \nabla_p + A : C^\infty(V) \to C^\infty(V) \tag{2.1}$$

The full geometric symbol of the operator is

$$(\sigma P)(x, \xi) = |\xi|^2 I + A(x) = g^{ij}(x)\xi_i \xi_j I + A(x).$$

Therefore $\sigma(\lambda I - P) = (\lambda - |\xi|^2)I - A$.

We proceed to solving equation (1.3). Let $r = r(x, \xi, \lambda)$ be the full geometric symbol of $R(\lambda)$. By formula (A.3) for the symbol of a product, equation (1.3) is written as

$$\sum_{\alpha} \frac{1}{\alpha!} \overset{v}{\nabla}{}^{\alpha} r \sum_{\beta,\gamma} \frac{1}{\gamma!} \binom{\alpha}{\beta} (-i\overset{h}{\nabla})^{\beta} \overset{v}{\nabla}{}^{\gamma} (\lambda I - |\xi|^2 I - A) \cdot \rho_{\alpha-\beta,\gamma} \sim I.$$

We use the central dot in our formulas to avoid extra parentheses. For example the expression $(-i\overset{h}{\nabla})^{\beta} \overset{v}{\nabla}{}^{\gamma} A \cdot \rho_{\alpha-\beta,\gamma}$ means the same as $\left((-i\overset{h}{\nabla})^{\beta} \overset{v}{\nabla}{}^{\gamma} A\right) \rho_{\alpha-\beta,\gamma}$. See Appendix below for the definition of the vertical and horizontal derivatives $\overset{v}{\nabla}$ and $\overset{h}{\nabla}$. These operators commute. Since

$$\rho_{0,0} = I, \quad \rho_{\alpha,0} = \rho_{0,\alpha} = 0 \quad \text{for} \quad |\alpha| > 0, \tag{2.2}$$

the equation can be rewritten in the form

$$r(\lambda I - |\xi|^2 I - A) + \sum_{|\alpha|>0} \frac{1}{\alpha!} \overset{v}{\nabla}{}^{\alpha} r \sum_{\beta,\gamma} \frac{1}{\gamma!} \binom{\alpha}{\beta} (-i\overset{h}{\nabla})^{\beta} \overset{v}{\nabla}{}^{\gamma} (\lambda I - |\xi|^2 I - A) \cdot \rho_{\alpha-\beta,\gamma} \sim I.$$

We distinguish terms corresponding to $\beta = 0$ and rewrite the equation once more as

$$\begin{aligned}
r(\lambda I - |\xi|^2 I - A) &+ \sum_{|\alpha|>0} \frac{1}{\alpha!} \overset{v}{\nabla}{}^{\alpha} r \Big[\sum_{\gamma} \frac{1}{\gamma!} \overset{v}{\nabla}{}^{\gamma} (\lambda I - |\xi|^2 I - A) \cdot \rho_{\alpha,\gamma} \\
&+ \sum_{|\beta|>0} \sum_{\gamma} \frac{1}{\gamma!} \binom{\alpha}{\beta} \overset{v}{\nabla}{}^{\gamma} (-i\overset{h}{\nabla})^{\beta} (\lambda I - |\xi|^2 I - A) \cdot \rho_{\alpha-\beta,\gamma} \Big] \sim I.
\end{aligned} \tag{2.3}$$

Of course, the parameter λ is considered as a constant with respect to the both differentiations, i.e., $\overset{v}{\nabla}\lambda = \overset{h}{\nabla}\lambda = 0$. Therefore $(-i\overset{h}{\nabla})^{\beta}(\lambda I - |\xi|^2 I - A) = -(-i\nabla)^{\beta} A$ for $|\beta| > 0$. Observe also that $\overset{v}{\nabla} A = 0$ since A is independent of ξ. Therefore the summation over γ in the second line of (2.3) is reduced to $\gamma = 0$. Taking also (2.2) into account, we see that the summation over β in the second line of (2.3) is reduced to $\beta = \alpha$. Equation (2.3) is thus simplified to

$$r(\lambda I - |\xi|^2 I - A) + \sum_{|\alpha|>0} \frac{1}{\alpha!} \overset{v}{\nabla}{}^{\alpha} r \Big(\sum_{|\gamma|>0} \frac{1}{\gamma!} \overset{v}{\nabla}{}^{\gamma} (\lambda I - |\xi|^2 I - A) \cdot \rho_{\alpha,\gamma} - (-i\nabla)^{\alpha} A \Big) \sim I. \tag{2.4}$$

The summation over γ is restricted to $|\gamma| > 0$ in virtue of (2.2). Actually the summation can be restricted to $|\gamma| = 1$ and $|\gamma| = 2$ since $(\lambda - |\xi|^2)I - A$ is the second order polynomial in ξ. Namely,

$$\overset{v}{\nabla}{}^{i}(\lambda I - |\xi|^2 I - A) = -\overset{v}{\nabla}{}^{i}|\xi|^2 \cdot I = -2\xi^i I, \quad \overset{v}{\nabla}{}^{i}\overset{v}{\nabla}{}^{j}(\lambda I - |\xi|^2 I - A) = -2g^{ij}I.$$

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Substitute these values into (2.4) to obtain

$$r(\lambda I - |\xi|^2 I - A) - \sum_{|\alpha|>0} \frac{1}{\alpha!} \overset{v}{\nabla}{}^\alpha r \cdot (g^{ij}\rho_{\alpha,\langle ij\rangle} + 2\rho_{\alpha,\langle i\rangle}\xi^i + (-i\nabla)^\alpha A) \sim I.$$

See Appendix below for the notation $\langle j_1 \ldots j_m\rangle$ for multi-indices. Let us remind that the coordinates $(x^1, \ldots, x^n, \xi_1, \ldots, \xi_n)$ are used as independent variables on T^*M. Nevertheless, we use also contravariant coordinates $\xi^i = g^{ij}\xi_j$. After introducing the notation

$$\chi_\alpha = g^{ij}\rho_{\alpha,\langle ij\rangle} + 2\rho_{\alpha,\langle i\rangle}\xi^i, \tag{2.5}$$

the equation takes the form

$$r(\lambda I - |\xi|^2 I - A) - \sum_{|\alpha|>0} \frac{1}{\alpha!}(\overset{v}{\nabla}{}^\alpha r) \cdot (\chi_\alpha + (-i\nabla)^\alpha A) \sim I. \tag{2.6}$$

Every function $\rho_{\alpha,\beta}(x,\xi)$ is a polynomial of degree $\le |\beta|$ in ξ. Therefore χ_α is a second degree polynomial in ξ and can be written in the form

$$\chi_\alpha = \chi_\alpha^{(0)} + \chi_\alpha^{(1)} + \chi_\alpha^{(2)}, \tag{2.7}$$

where $\chi_\alpha^{(p)}$ is a homogeneous polynomial of degree p in ξ for $p = 0, 1, 2$. Introduce the similar notation $\rho_{\alpha,\beta}^{(p)}$ for homogeneous parts of $\rho_{\alpha,\beta}$. Formula (2.5) implies

$$\chi_\alpha^{(0)} = g^{ij}\rho_{\alpha,\langle ij\rangle}^{(0)}, \quad \chi_\alpha^{(1)} = g^{ij}\rho_{\alpha,\langle ij\rangle}^{(1)} + \rho_{\alpha,\langle i\rangle}^{(0)}\xi^i, \quad \chi_\alpha^{(2)} = g^{ij}\rho_{\alpha,\langle ij\rangle}^{(2)} + \rho_{\alpha,\langle i\rangle}^{(1)}\xi^i. \tag{2.8}$$

We are looking for the solution to equation (2.6) in the form $r = r_0 + r_1 + \ldots$, where $r_k = r_k(x, \xi, \lambda) \in S_2^{-2-k}(T^*M, \operatorname{End} V, \lambda)$. Substitute this expression and (2.7) into (2.6) to obtain

$$(r_0+r_1+\ldots)(\lambda I-|\xi|^2 I-A) - \sum_{|\alpha|>0} \frac{1}{\alpha!}\overset{v}{\nabla}{}^\alpha(r_0+r_1+\ldots)\cdot(\chi_\alpha^{(0)}+\chi_\alpha^{(1)}+\chi_\alpha^{(2)}+(-i\nabla)^\alpha A) \sim I. \tag{2.9}$$

Let us remind that λ is considered as a variable of the second degree of homogeneity. The derivative $\overset{v}{\nabla}{}^\alpha r_j$ is homogeneous of degree $-2 - j - |\alpha|$ in (λ, ξ). The operator A is of the zero degree of homogeneity while $\chi_\alpha^{(p)}$, of degree p. Equating the homogeneous terms of the zero degree on the left- and right-hand sides of (2.9), we obtain

$$r_0 = (\lambda - |\xi|^2)^{-1}I. \tag{2.10}$$

Equating to zero the sum of homogeneous terms of degree -1 on the left-hand side of (2.9), we obtain

$$r_1(\lambda - |\xi|^2) + \sum_{|\alpha|=1} \frac{1}{\alpha!}\overset{v}{\nabla}{}^\alpha r_0 \cdot \chi_\alpha^{(2)} = 0.$$

By formula (A.18) of the Appendix below, $\chi_\alpha^{(2)} = 0$ for $|\alpha| = 1$. Therefore the previous formula gives $r_1 = 0$. Finally, equating to zero the sum of homogeneous terms of degree $-k$ ($k \geq 2$), we obtain the recurrent relation

$$r_k = \frac{1}{\lambda - |\xi|^2}\Big(r_{k-2}A + \sum_{j=0}^{k-1}\sum_{|\alpha|=k-j}\frac{1}{\alpha!}\overset{v}{\nabla}^\alpha r_j \cdot \chi_\alpha^{(2)}$$
$$+ \sum_{j=0}^{k-2}\sum_{|\alpha|=k-j-1}\frac{1}{\alpha!}\overset{v}{\nabla}^\alpha r_j \cdot \chi_\alpha^{(1)} + \sum_{j=0}^{k-3}\sum_{|\alpha|=k-j-2}\frac{1}{\alpha!}\overset{v}{\nabla}^\alpha r_j \cdot (\chi_\alpha^{(0)} + (-i\nabla)^\alpha A)\Big).$$

The summation limits of the first sum can be restricted to $0 \leq j \leq k-2$ since $\chi_\alpha^{(2)} = 0$ for $|\alpha| = 1$, as we have already mentioned. Thus, the formula takes the form

$$r_k = \frac{1}{\lambda - |\xi|^2}\Big(r_{k-2}A + \sum_{j=0}^{k-3}\sum_{|\alpha|=k-j-2}\frac{1}{\alpha!}\overset{v}{\nabla}^\alpha r_j \cdot (-i\nabla)^\alpha A + \sum_{j=0}^{k-2}\sum_{|\alpha|=k-j}\frac{1}{\alpha!}\overset{v}{\nabla}^\alpha r_j \cdot \chi_\alpha^{(2)}$$
$$+ \sum_{j=0}^{k-2}\sum_{|\alpha|=k-j-1}\frac{1}{\alpha!}\overset{v}{\nabla}^\alpha r_j \cdot \chi_\alpha^{(1)} + \sum_{j=0}^{k-3}\sum_{|\alpha|=k-j-2}\frac{1}{\alpha!}\overset{v}{\nabla}^\alpha r_j \cdot \chi_\alpha^{(0)}\Big).$$

The term $r_{k-2}A$ coincides with the summand of the first sum for $j = k - 2$. Observe also that summation limits of the last sum can be changed to $0 \leq j \leq k - 2$ since $\chi_0^{(0)} = 0$, see (A.17). In such the way, the recurrent formula takes its final form: for $k \geq 2$,

$$r_k = \frac{1}{\lambda - |\xi|^2}\sum_{j=0}^{k-2}\Big(\sum_{|\alpha|=k-j-2}\frac{1}{\alpha!}\overset{v}{\nabla}^\alpha r_j \cdot (-i\nabla)^\alpha A + \sum_{p=0}^{2}\sum_{|\alpha|=k-j-p}\frac{1}{\alpha!}\overset{v}{\nabla}^\alpha r_j \cdot \chi_\alpha^{(2-p)}\Big). \quad (2.11)$$

The formula has two important specifics. First, there is no term with $j = 1$ on the right-hand side since $r_1 = 0$. Second, we need to know $r_2, \ldots, r_{k-2}$ for calculating r_k, but we do not need r_{k-1}.

Formula (2.11) implies in particular that $r_k(x, \xi, \lambda)$ depends on λ through factors $(\lambda - |\xi|^2)^{-m}$ with different values of m. More precisely, the representation

$$\operatorname{tr} r_k(x, \xi, \lambda) = \sum_{m=1}^{M_k}\frac{f_{k,m}(x, \xi)}{(\lambda - |\xi|^2)^m}$$

holds for every k where, moreover, $f_{k,m}(x, \xi)$ are polynomials in ξ. In virtue of this fact, both integrations of (1.4) become trivial procedures. Indeed, the integration over γ reduces to the formula

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{-\lambda}d\lambda}{(\lambda - |\xi|^2)^m} = \frac{(-1)^{m-1}}{(m-1)!}e^{-|\xi|^2}. \quad (2.12)$$

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The formula is obviously true since the left-hand side is just the residue of the integrand at the point $\lambda = |\xi|^2$. Now, the integration over $T_x^* M$ in (1.4) reduces with the help of an orthonormal basis to the evaluation of the integral

$$\pi^{-n/2} \int\limits_{\mathbb{R}^n} e^{-|\xi|^2} \xi^\alpha \, d\xi$$

for different values of the multi-index α. The integral is obviously equal to zero if α is not even. For an even multi-index,

$$\pi^{-n/2} \int\limits_{\mathbb{R}^n} e^{-|\xi|^2} \xi^{2\alpha} \, d\xi = \prod_{k=1}^n \pi^{-1/2} \int\limits_{-\infty}^\infty e^{-t^2} t^{\alpha_k} \, dt = 2^{-|\alpha|} \prod_{k=1}^n (2\alpha_k - 1)!!,$$

where $(2m - 1)!! = (2m - 1)(2m - 3)\ldots 1$ with the standard agreement $(-1)!! = 1$. In this paper, we will use this equality for $|\alpha| \leq 2$ only in the following tensor form:

Lemma 2.1 *If $C = (C_{ij})$ and $D = (D_{ijkl})$ are tensor fields on an n-dimensional Riemannian manifold (M, g), then*

$$\pi^{-n/2} \int\limits_{T_x^* M} e^{-|\xi|^2} \, d\xi = 1,$$

$$\pi^{-n/2} \int\limits_{T_x^* M} e^{-|\xi|^2} C_{ij} \xi^i \xi^j \, d\xi = \frac{1}{2} g^{ij} C_{ij},$$

$$\pi^{-n/2} \int\limits_{T_x^* M} e^{-|\xi|^2} D_{ijkl} \xi^i \xi^j \xi^k \xi^l \, d\xi = \frac{3}{4} (g^2)^{ijkl} D_{ijkl},$$

where

$$(g^2)^{ijkl} = \sigma(ijkl)(g^{ij} g^{kl}) = \frac{1}{3} (g^{ij} g^{kl} + g^{ik} g^{jl} + g^{il} g^{jk}).$$

3 Computing the invariants a_0 and a_2

For a Riemannian manifold (M, g), let $R = (R_{ijkl})$ be the curvature tensor of the Levi-Chivita connection ∇. The Ricci curvature tensor $Ric = (R_{ij})$ is defined by $R_{ij} = g^{pq} R_{ipjq}$ and the scalar curvature is $S = g^{ij} R_{ij}$. We normalize the curvature tensor such that the scalar curvature of the unit two-dimensional sphere is equal to $+1$. This differs by the sign from Gilkey's choice [2]. Let $\Delta S = -\nabla^p \nabla_p S$.

Now, let (V, ∇^V) be a Hermitian vector bundle with connection over M. We denote the curvature tensor of the connection ∇^V by $\mathcal{R} = (\mathcal{R}_{ij})$. Thus, $\mathcal{R}_{ij}(x) \in \operatorname{End} V_x$ for $x \in M$, $\mathcal{R}_{ij}$ is skew symmetric in (i, j) and behaves like an ordinary second rank tensor under a coordinate change.

Theorem 3.1 *Let (V, ∇^V) be a Hermitian vector bundle with connection over a closed Riemannian manifold (M, g). Denote the dimension of the fiber of V by d. Assume the curvature tensor of the connection ∇^V to satisfy*

$$\operatorname{tr} \mathcal{R}_{ij} = 0. \tag{3.1}$$

Fix a self-adjoint operator $A \in C^\infty(\operatorname{End} V)$ and define the second order differential operator $P = P_A$ on V by formula (2.1). Then first three heat invariants of P are as follows:

(a) $\quad a_0(x, P) = d,$

(b) $\quad a_2(x, P) = \dfrac{d}{6} S - \operatorname{tr} A,$

(c) $\quad a_4(x, P) = \dfrac{d}{360}(-12 \Delta S + 5S^2 - 2|Ric|^2 + 2|R|^2)$

$\qquad\qquad + \dfrac{1}{12} \operatorname{tr} (g^{ik} g^{jl} \mathcal{R}_{ij} \mathcal{R}_{kl} - 2\nabla^p \nabla_p A + 6A^2 - 2SA).$

Of course the result is not new, compare with Theorem 4.8.16 of [2]. The main news is about the proof. Our proof consists of explicit calculations strictly following the algorithm presented above, with no extra argument. In our opinion, this approach can be computerized in order to obtain similar formulas for $a_k(x, P)$ $(k = 6, 8, \ldots)$.

Let us give a couple of remarks about hypothesis (3.1). It definitely holds if the connection ∇^V is compatible with the Hermitian inner product on V. In particular, it holds in the case of the Laplacian on forms. The hypothesis is not used in our proof of statements (a) and (b) of the theorem. As far as the proof of statement (c) is concerned, we use the hypothesis to abbreviate some of our calculations. Namely, condition (3.1) allows us to ignore terms depending linearly on $\mathcal{R}$ in any formula if we are going to apply the operator tr to the formula. Most probably, hypothesis (3.1) can be removed from Theorem 3.1, but some of our calculations would become much longer. No such hypothesis is mentioned in the statement of Theorem 4.8.16 of [2].

We start with evaluating $a_0(x, P)$. By formulas (1.4) and (2.10),

$$a_0(x, P) = \frac{\pi^{-n/2}}{2\pi i} \int\limits_{T_x^* M} \int\limits_\gamma e^{-\lambda} \operatorname{tr} r_0(x, \xi, \lambda) \, d\lambda d\xi = d\pi^{-n/2} \int\limits_{T_x * M} \left(\frac{1}{2\pi i} \int\limits_\gamma \frac{e^{-\lambda} d\lambda}{\lambda - |\xi|^2} \right) d\xi.$$

Applying (2.12) and Lemma 2.1, we obtain the desired result

$$a_0(x, P) = d\pi^{-n/2} \int\limits_{T_x^* M} e^{-|\xi|^2} d\xi = d.$$

We use the abbreviated notation for higher order derivatives $\overset{v}{\nabla}{}^{i_1 \ldots i_k} = \overset{v}{\nabla}{}^{i_1} \ldots \overset{v}{\nabla}{}^{i_k}$. Find the derivatives of r_0 up to the fourth order by differentiating (2.10)

$$\overset{v}{\nabla}{}^i r_0 = \frac{2\xi^i}{(\lambda - |\xi|^2)^2}, \qquad \overset{v}{\nabla}{}^{ij} r_0 = \frac{2g^{ij}}{(\lambda - |\xi|^2)^2} + \frac{8\xi^i \xi^j}{(\lambda - |\xi|^2)^3}, \tag{3.2}$$

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$$\overset{v}{\nabla}{}^{ijk} r_0 = 8\frac{g^{ij}\xi^k + g^{ik}\xi^j + g^{jk}\xi^i}{(\lambda - |\xi|^2)^3} + 48\frac{\xi^i\xi^j\xi^k}{(\lambda - |\xi|^2)^4}, \tag{3.3}$$

$$\overset{v}{\nabla}{}^{ijkl} r_0 = \frac{8}{(\lambda - |\xi|^2)^3}(g^{ij}g^{kl} + g^{ik}g^{jl} + g^{il}g^{jk}) + \frac{384}{(\lambda - |\xi|^2)^5}\xi^i\xi^j\xi^k\xi^l$$
$$+ \frac{48}{(\lambda - |\xi|^2)^4}(g^{il}\xi^k\xi^l + g^{ik}\xi^j\xi^l + g^{il}\xi^j\xi^k + g^{jk}\xi^i\xi^l + g^{jl}\xi^i\xi^k + g^{kl}\xi^i\xi^j). \tag{3.4}$$

We have omitted the factor I on right-hand sides of (3.2)–(3.4) for brevity.

Now, we calculate $r_2(x, \xi, \lambda)$. By (2.11),

$$r_2 = \frac{1}{\lambda - |\xi|^2}\Big(r_0 A + \frac{1}{2}\overset{v}{\nabla}{}^{ij} r_0 \cdot \chi^{(2)}_{\langle ij\rangle} + \overset{v}{\nabla}{}^i r_0 \cdot \chi^{(1)}_{\langle i\rangle}\Big).$$

We substituting values (2.10) and (3.2) for r_0 and its derivatives. Then we substitute values (A.18) and (A.19) for $\chi^{(1)}_{\langle i\rangle}$ and $\chi^{(2)}_{\langle ij\rangle}$ (see Appendix below) to obtain

$$r_2 = \frac{A}{(\lambda - |\xi|^2)^2} + \frac{1}{3}\Big[\frac{2R_{ij}\xi^i\xi^j}{(\lambda - |\xi|^2)^3} - \frac{8R_{ijkl}\xi^i\xi^j\xi^k\xi^l}{(\lambda - |\xi|^2)^4}\Big]I + \frac{2\mathcal{R}_{ij}\xi^i\xi^j}{(\lambda - |\xi|^2)^3}. \tag{3.5}$$

The last term on the right-hand side is equal to zero since $\mathcal{R}_{ij}$ is skew-symmetric in (i, j) while the factor $\xi^i\xi^j$ is symmetric in these indices. The second term in brackets is equal to zero by the same reason. We thus obtain the final formula

$$r_2 = \frac{1}{(\lambda - |\xi|^2)^2}A + \frac{2}{3}\frac{R_{ij}\xi^i\xi^j}{(\lambda - |\xi|^2)^3}I. \tag{3.6}$$

Now, we evaluate $a_2(x, P)$. Take the trace of (3.6), multiply the result by $e^{-\lambda}$, and integrate over the curve γ with the help of (2.12)

$$\frac{1}{2\pi i}\int_\gamma e^{-\lambda}\operatorname{tr} r_2\, d\lambda = e^{-|\xi|^2}\Big(-\operatorname{tr} A + \frac{d}{3}R_{ij}\xi^i\xi^j\Big).$$

Integrate this equality over $T_x^* M$ with the help of Lemma 2.1

$$\frac{\pi^{-n/2}}{2\pi i}\int_{T_x^* M}\int_\gamma e^{-\lambda}\operatorname{tr} r_2\, d\lambda = -\operatorname{tr} A + \frac{d}{6}S.$$

In view of (1.4), this coincides with statement (b) of Theorem 3.1.

4 Computing the invariant a_4

Since $r_1 = 0$, formula (2.11) for $k = 4$ gives

$$r_4 = \frac{1}{\lambda - |\xi|^2}\Big(r_2 A - \frac{1}{2}\overset{v}{\nabla}{}^{ij} r_0 \cdot \nabla_{ij} A + \overset{v}{\nabla}{}^i r_2 \cdot \chi^{(1)}_{\langle i\rangle} + \frac{1}{2}\overset{v}{\nabla}{}^{ij} r_2 \cdot \chi^{(2)}_{\langle ij\rangle}$$
$$+ \frac{1}{2}\overset{v}{\nabla}{}^{ij} r_0 \cdot \chi^{(0)}_{\langle ij\rangle} + \frac{1}{6}\overset{v}{\nabla}{}^{ijk} r_0 \cdot \chi^{(1)}_{\langle ijk\rangle} + \frac{1}{24}\overset{v}{\nabla}{}^{ijkl} r_0 \cdot \chi^{(2)}_{\langle ijkl\rangle}\Big). \tag{4.1}$$

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First of all we will eliminate r_2 from this formula. Differentiate (3.6) with respect to ξ to get

$$\overset{v}{\nabla}{}^i r_2 = \frac{1}{(\lambda - |\xi|^2)^3} A + \frac{4}{3} \frac{R_p^i \xi^p}{(\lambda - |\xi|^2)^3} I + 4 \frac{R_{pq} \xi^p \xi^q \xi^i}{(\lambda - |\xi|^2)^4} I, \tag{4.2}$$

$$\overset{v}{\nabla}{}^{ij} r_2 = \frac{4 g^{ij}}{(\lambda - |\xi|^2)^3} A + \frac{24 \xi^i \xi^j}{(\lambda - |\xi|^2)^4} A + \frac{4}{3} \frac{R^{ij}}{(\lambda - |\xi|^2)^3} I$$
$$+ 4 \frac{2 R_p^i \xi^p \xi^j + 2 R_p^j \xi^p \xi^i + g^{ij} R_{pq} \xi^p \xi^q}{(\lambda - |\xi|^2)^4} I + 32 \frac{R_{pq} \xi^p \xi^q \xi^i \xi^j}{(\lambda - |\xi|^2)^5} I. \tag{4.3}$$

We substitute (3.6) and (4.2)–(4.3) into (4.1) and then group all terms on the right-hand side of the resulting formula into three clusters so that the first cluster contains terms dependent on A, the second cluster contains terms independent of A but dependent on r_0, and the last cluster consists of all other terms. Thus, $r_4 = r_4^1 + r_4^2 + r_4^3$, where

$$r_4^1 = -\frac{1}{2} \frac{1}{\lambda - |\xi|^2} \overset{v}{\nabla}{}^{ij} r_0 \cdot \nabla_{ij} A + \frac{1}{(\lambda - |\xi|^2)^3} A^2 + \frac{2}{3} \frac{R_{ij} \xi^i \xi^j}{(\lambda - |\xi|^2)^4} A$$
$$+ \frac{4 \xi^i}{(\lambda - |\xi|^2)^4} A \chi_{\langle i \rangle}^{(1)} + \frac{2 g^{ij}}{(\lambda - |\xi|^2)^4} A \chi_{\langle ij \rangle}^{(2)} + \frac{12 \xi^i \xi^j}{(\lambda - |\xi|^2)^5} A \chi_{\langle ij \rangle}^{(2)}, \tag{4.4}$$

$$r_4^2 = \frac{1}{\lambda - |\xi|^2} \left(\frac{1}{2} \overset{v}{\nabla}{}^{ij} r_0 \cdot \chi_{\langle ij \rangle}^{(0)} + \frac{1}{6} \overset{v}{\nabla}{}^{ijk} r_0 \cdot \chi_{\langle ijk \rangle}^{(1)} + \frac{1}{24} \overset{v}{\nabla}{}^{ijkl} r_0 \cdot \chi_{\langle ijkl \rangle}^{(2)} \right), \tag{4.5}$$

$$r_4^3 = \frac{4 R_{pq} \xi^p \xi^q \xi^i}{(\lambda - |\xi|^2)^5} \chi_{\langle i \rangle}^{(1)} + \frac{4}{3} \frac{R_p^i \xi^p}{(\lambda - |\xi|^2)^4} \chi_{\langle i \rangle}^{(1)} + \frac{2}{3} \frac{R^{ij}}{(\lambda - |\xi|^2)^4} \chi_{\langle ij \rangle}^{(2)}$$
$$+ \frac{8 R_p^i \xi^p \xi^j + 2 g^{ij} R_{pq} \xi^p \xi^q}{(\lambda - |\xi|^2)^5} \chi_{\langle ij \rangle}^{(2)} + \frac{16 R_{pq} \xi^p \xi^q \xi^i \xi^j}{(\lambda - |\xi|^2)^6} \chi_{\langle ij \rangle}^{(2)}. \tag{4.6}$$

We first evaluate the term r_4^1. Substitute value (3.2) for $\overset{v}{\nabla}{}^{ij} r_0$ into (4.4)

$$r_4^1 = -\frac{g^{ij}}{(\lambda - |\xi|^2)^3} \nabla_{ij} A - \frac{4 \xi^i \xi^j}{(\lambda - |\xi|^2)^4} \nabla_{ij} A + \frac{1}{(\lambda - |\xi|^2)^3} A^2 + \frac{2}{3} \frac{R_{ij} \xi^i \xi^j}{(\lambda - |\xi|^2)^4} A$$
$$+ \frac{4 \xi^i}{(\lambda - |\xi|^2)^4} A \chi_{\langle i \rangle}^{(1)} + \frac{2 g^{ij}}{(\lambda - |\xi|^2)^4} A \chi_{\langle ij \rangle}^{(2)} + \frac{12 \xi^i \xi^j}{(\lambda - |\xi|^2)^5} A \chi_{\langle ij \rangle}^{(2)}.$$

The dependence on λ is now explicitly designated in this formula. We multiply the formula by $e^{-\lambda}$ and integrate over γ with the help of (2.12)

$$\frac{1}{2\pi i} \int_\gamma e^{-\lambda} r_4^1 \, d\lambda = e^{-|\xi|^2} \left(-\frac{1}{2} \nabla^p \nabla_p A + \frac{2}{3} \nabla_{ij} A \xi^i \xi^j + \frac{1}{2} A^2 - \frac{1}{9} A R_{ij} \xi^i \xi^j \right.$$

$$\left. - \frac{2}{3} A \chi_{\langle i \rangle}^{(1)} \xi^i - \frac{1}{3} A g^{ij} \chi_{\langle ij \rangle}^{(2)} + \frac{1}{2} A \chi_{\langle ij \rangle}^{(2)} \xi^i \xi^j \right).$$

Next, we substitute values (A.18) and (A.19) for $\chi_{\langle i \rangle}^{(1)}$ and $\chi_{\langle ij \rangle}^{(2)}$ to obtain

$$\frac{1}{2\pi i} \int_{\gamma} e^{-\lambda} r_4^1 \, d\lambda = e^{-|\xi|^2} \Big(-\frac{1}{2} \nabla^p \nabla_p A + \frac{2}{3} \nabla_{ij} A \xi^i \xi^j + \frac{1}{2} A^2 - \frac{1}{9} A R_{ij} \xi^i \xi^j$$

$$+ \frac{2}{3} A \mathcal{R}_{ij} \xi^i \xi^j - \frac{4}{9} A R_{ij} \xi^i \xi^j + \frac{2}{9} A R_{ij} \xi^i \xi^j - \frac{1}{3} A R_{ikjl} \xi^i \xi^j \xi^k \xi^l \Big).$$

The fifth and last terms in parentheses are equal to zero and the formula takes the form

$$\frac{1}{2\pi i} \int_{\gamma} e^{-\lambda} r_4^1 \, d\lambda = e^{-|\xi|^2} \Big(-\frac{1}{2} \nabla^p \nabla_p A + \frac{2}{3} \nabla_{ij} A \xi^i \xi^j + \frac{1}{2} A^2 - \frac{1}{3} A R_{ij} \xi^i \xi^j \Big).$$

We integrate this relation over $T_x^* M$ with the help of Lemma 2.1 and obtain the final formula for r_4^1

$$\frac{\pi^{-n/2}}{2\pi i} \int_{T_x^* M} \int_{\gamma} e^{-\lambda} \operatorname{tr} r_4^1 \, d\lambda d\xi = \operatorname{tr} \Big(-\frac{1}{6} \nabla^p \nabla_p A + \frac{1}{2} A^2 - \frac{1}{6} S A \Big). \qquad (4.7)$$

Next, we evaluate r_4^3. The dependence on λ is explicitly designated in formula (4.6) since $\chi_\alpha^{(p)}$ are independent of λ. We multiply (4.6) by $e^{-\lambda}$ and integrate over γ with the help of (2.12)

$$\frac{1}{2\pi i} \int_{\gamma} e^{-\lambda} r_4^3 \, d\lambda = e^{-|\xi|^2} \Big(\frac{1}{6} R_{pq} \xi^p \xi^q \xi^i \chi_{\langle i \rangle}^{(1)} - \frac{2}{9} R_p^i \xi^p \chi_{\langle i \rangle}^{(1)} - \frac{1}{9} R^{ij} \chi_{\langle ij \rangle}^{(2)}$$

$$+ \frac{1}{12} (4 R_p^i \xi^p \xi^j + g^{ij} R_{pq} \xi^p \xi^q) \chi_{\langle ij \rangle}^{(2)} - \frac{2}{15} R_{pq} \xi^p \xi^q \xi^i \xi^j \chi_{\langle ij \rangle}^{(2)} \Big).$$

Substitute values (A.18) and (A.19) for $\chi_{\langle i \rangle}^{(1)}$ and $\chi_{\langle ij \rangle}^{(2)}$. In the resulting formula, some terms will be equal to zero because of the skew-symmetry of curvature tensors. After canceling such terms and grouping similar terms, the formula takes the form

$$\frac{1}{2\pi i} \int_{\gamma} e^{-\lambda} r_4^3 \, d\lambda = e^{-|\xi|^2} \Big(\frac{2}{27} (3 R_i^p \mathcal{R}_{pj} - 2 R_p^i R_{pj} I + R^{pq} R_{ipjq} I) \xi^i \xi^j + \frac{1}{18} R_{ij} R_{kl} \xi^i \xi^j \xi^k \xi^l I \Big).$$

We apply the operator tr to this equality and integrate over $T_x^* M$ with the help of Lemma 2.1. The first term on the right-hand side will give the zero contribution to the integral since R^{ij} is symmetric in (i, j) while $\mathcal{R}_{ij}$ is skew-symmetric. We thus obtain

$$\frac{\pi^{-n/2}}{2\pi i} \int_{T_x^* M} \int_{\gamma} e^{-\lambda} \operatorname{tr} r_4^3 \, d\lambda d\xi = d\Big(\frac{1}{27} g^{ij} (-2 R_p^i R_{pj} + R^{pq} R_{ipjq}) + \frac{1}{24} (g^2)^{ijkl} R_{ij} R_{kl} \Big).$$

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Since

$$g^{ij}(-2R_p^i R_{pj} + R^{pq} R_{ipjq}) = -|Ric|^2, \quad (g^2)^{ijkl} R_{ij} R_{kl} = \frac{1}{3}S^2 + \frac{2}{3}|Ric|^2,$$

the formula takes its final form

$$\frac{\pi^{-n/2}}{2\pi i} \int\limits_{T_x^* M} \int\limits_{\gamma} e^{-\lambda} \operatorname{tr} r_4^3 \, d\lambda d\xi = \frac{d}{216}(3S^2 - 2|Ric|^2). \qquad (4.8)$$

Next, we evaluate r_4^2. Substitute values (3.2)–(3.4) for derivatives of r_0 into (4.5)

$$r_4^2 = \frac{g^{ij}}{(\lambda - |\xi|^2)^3}\chi_{\langle ij\rangle}^{(0)} + \frac{4\xi^i\xi^j}{(\lambda - |\xi|^2)^4}\chi_{\langle ij\rangle}^{(0)} + \frac{4g^{ij}\xi^k}{(\lambda - |\xi|^2)^4}\chi_{\langle ijk\rangle}^{(1)} + \frac{8\xi^i\xi^j\xi^k}{(\lambda - |\xi|^2)^5}\chi_{\langle ijk\rangle}^{(1)}$$
$$+ \frac{g^{ij}g^{kl}}{(\lambda - |\xi|^2)^4}\chi_{\langle ijkl\rangle}^{(2)} + \frac{12g^{ij}\xi^k\xi^l}{(\lambda - |\xi|^2)^5}\chi_{\langle ijkl\rangle}^{(2)} + \frac{16\xi^i\xi^j\xi^k\xi^l}{(\lambda - |\xi|^2)^6}\chi_{\langle ijkl\rangle}^{(2)}.$$

We multiply this by $e^{-\lambda}$ and integrate over γ with the help of (2.12)

$$\frac{1}{2\pi i} \int\limits_{\gamma} e^{-\lambda} r_4^2 \, d\lambda = e^{-|\xi|^2}\left(\frac{1}{2}g^{ij}\chi_{\langle ij\rangle}^{(0)} - \frac{2}{3}\chi_{\langle ij\rangle}^{(0)}\xi^i\xi^j - \frac{2}{3}g^{ij}\chi_{\langle ijk\rangle}^{(1)}\xi^k \right.$$
$$\left. + \frac{1}{3}\chi_{\langle ijk\rangle}^{(1)}\xi^i\xi^j\xi^k - \frac{1}{6}g^{ij}g^{kl}\chi_{\langle ijkl\rangle}^{(2)} + \frac{1}{2}g^{ij}\chi_{\langle ijkl\rangle}^{(2)}\xi^k\xi^l - \frac{2}{15}\chi_{\langle ijkl\rangle}^{(2)}\xi^i\xi^j\xi^k\xi^l\right). \qquad (4.9)$$

Let us calculate separately each term on the right-hand side. Using formula (A.19) for $\chi_{\langle ij\rangle}^{(0)}$, we obtain

$$\frac{1}{2}g^{ij}\chi_{\langle ij\rangle}^{(0)} = \frac{1}{4}g^{ik}g^{jl}\mathcal{R}_{ij}\mathcal{R}_{kl} + \ldots, \qquad (4.10)$$

$$-\frac{2}{3}\chi_{\langle ij\rangle}^{(0)}\xi^i\xi^j = -\frac{1}{3}g^{pq}\mathcal{R}_{ip}\mathcal{R}_{jq}\xi^i\xi^j + \ldots, \qquad (4.11)$$

where dots mean some operator depending linearly on $\mathcal{R}$. It is a trace free operator by hypothesis (3.1).

Using formula (A.20) for $\chi_{\langle ijk\rangle}^{(1)}$, we obtain

$$-\frac{2}{3}g^{ij}\chi_{\langle ijk\rangle}^{(1)}\xi^k = \frac{1}{135}\left(27\nabla_i\nabla_j R_{kp} + 7\nabla_i\nabla^q R_{pjkq} + 2\nabla^q\nabla_i R_{pjkq} - 4R_{\cdot ij\cdot}^{q\cdot\cdot r} R_{pqkr} \right.$$
$$\left. - 12R_{\cdot ij\cdot}^{q\cdot\cdot r} R_{prkq} - 16R_i^q R_{pjkq}\right)(g^{ij}\xi^k\xi^p + g^{ik}\xi^j\xi^p + g^{jk}\xi^i\xi^p) + \ldots$$

For brevity, we do not write the factor I in this and several further formulas. After

opening the parentheses

$$-\frac{2}{3}g^{ij}\chi^{(1)}_{\langle ijk\rangle}\xi^k = \frac{1}{135}\Big(27\nabla^i\nabla_i R_{kp}\xi^k\xi^p + 27\nabla^k\nabla_j R_{kp}\xi^j\xi^p + 27\nabla_i\nabla^k R_{kp}\xi^i\xi^p$$
$$+ 7\nabla^j\nabla^q R_{pjkq}\xi^k\xi^p + 7\nabla^k\nabla^q R_{pjkq}\xi^j\xi^p + 7\nabla_i\nabla^q(g^{jk}R_{pjkq})\xi^i\xi^p$$
$$+ 2\nabla^q\nabla^j R_{pjkq}\xi^k\xi^p + 2\nabla^q\nabla^k R_{pjkq}\xi^j\xi^p + 2\nabla^q\nabla_i(g^{jk}R_{pjkq})\xi^i\xi^p$$
$$- 4(g^{ij}R^{q\cdot\cdot r}_{\cdot ij\cdot})R_{pqkr}\xi^k\xi^p - 4R^{qk\cdot r}_{\cdot\cdot j\cdot}R_{pqkr}\xi^j\xi^p - 4R^{q\cdot kr}_{\cdot i\cdot\cdot}R_{pqkr}\xi^i\xi^p$$
$$- 12(g^{ij}R^{q\cdot\cdot r}_{\cdot ij\cdot})R_{prkq}\xi^k\xi^p - 12R^{qk\cdot r}_{\cdot\cdot j\cdot}R_{prkq}\xi^j\xi^p - 12R^{q\cdot kr}_{\cdot i\cdot\cdot}R_{prkq}\xi^i\xi^p$$
$$- 16R^{qj}R_{pjkq}\xi^k\xi^p - 16R^{qk}R_{pjkq}\xi^j\xi^p - 16R^q_i(g^{jk}R_{pjkq})\xi^i\xi^p\Big) + \ldots.$$

The fifth term on the right-hand side is equal to zero since R_{pjkq} is skew-symmetric in (p, j) while $\xi^j\xi^p$ is symmetric in these indices. By the same reason 8th and 17th terms on the right-hand side are equal to zero too. Deleting that terms and changing summation indices, we transform the formula to the form

$$-\frac{2}{3}g^{ij}\chi^{(1)}_{\langle ijk\rangle}\xi^k = \frac{1}{135}\Big(27\nabla^p\nabla_p R_{ij} + 27\nabla^p\nabla_i R_{jp} + 27\nabla_i\nabla^p R_{jp} + 7\nabla^p\nabla^q R_{ipjq}$$
$$- 7\nabla_i\nabla^p R_{jp} + 2\nabla^p\nabla^q R_{iqjp} - 2\nabla^p\nabla_i R_{jp} + 4R^{pq}R_{ipjq}$$
$$- 4R_{pqir}R^{\cdot pqr}_j - 4R_{piqr}R^{\cdot pqr}_j + 12R^{pq}R_{ipjq} - 12R_{pqir}R^{\cdot rqp}_j$$
$$- 12R_{piqr}R^{\cdot rqp}_j - 16R^{pq}R_{ipjq} + 16R^p_i R_{jp}\Big)\xi^i\xi^j + \ldots$$

After grouping similar terms

$$-\frac{2}{3}g^{ij}\chi^{(1)}_{\langle ijk\rangle}\xi^k = \frac{1}{135}\Big(27\nabla^p\nabla_p R_{ij} + 25\nabla^p\nabla_i R_{jp} + 20\nabla_i\nabla^p R_{jp} + 9\nabla^p\nabla^q R_{ipjq}$$
$$+ 16R^p_i R_{jp} + 4R_{ipqr}(4R^{\cdot pqr}_j + R^{\cdot qpr}_j - 3R^{\cdot rpq}_j)\Big)\xi^i\xi^j + \ldots$$

The last term on the right-hand side can be simplified a little bit with the help of the Ricci identity $R^{\cdot rpq}_j = -R^{\cdot pqr}_j - R^{\cdot qrp}_j = -R^{\cdot pqr}_j + R^{\cdot qpr}_j$. The formula becomes

$$-\frac{2}{3}g^{ij}\chi^{(1)}_{\langle ijk\rangle}\xi^k = \frac{1}{135}\Big(27\nabla^p\nabla_p R_{ij} + 25\nabla^p\nabla_i R_{jp} + 20\nabla_i\nabla^p R_{jp} + 9\nabla^p\nabla^q R_{ipjq}$$
$$+ 16R^p_i R_{jp} + 4R_{ipqr}(7R^{\cdot pqr}_j - 2R^{\cdot qpr}_j)\Big)\xi^i\xi^j I + \ldots \tag{4.12}$$

The fourth term on the right-hand side of (4.9) is treated similarly. The result is as follows:

$$\frac{1}{3}\chi^{(1)}_{\langle ijk\rangle}\xi^i\xi^j\xi^k = -\frac{1}{90}(27\nabla_{ij}R_{kl} + 16R^p_{ijq}R^q_{klp})\xi^i\xi^j\xi^k\xi^l I + \ldots \tag{4.13}$$

Formula (A.21) for $\chi^{(2)}_{\langle ijkl\rangle}$ can be written as

$$\chi^{(2)}_{\langle ijkl\rangle} = \frac{1}{15}\sigma(ij)\sigma(kl)\Big(3\nabla_{ij}R_{kplq} + 4R_{pijr}R^r_{klq} + 3\nabla_{ik}R_{jplq} + 4R_{pikr}R^r_{jlq}$$
$$+ 3\nabla_{ik}R_{lpjq} + 4R_{pilr}R^r_{kjq} + 3\nabla_{ki}R_{lpjq} + 4R_{pklr}R^r_{ijq}$$
$$+ 3\nabla_{ki}R_{jplq} + 4R_{pkjr}R^r_{ilq} + 3\nabla_{kl}R_{jpiq} + 4R_{pkjr}R^r_{liq}\Big)\xi^p\xi^q I + \ldots$$

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On using this representation, one easily derives

$$-\frac{1}{6}g^{ij}g^{kl}\chi^{(2)}_{\langle ijkl\rangle} = -\frac{1}{45}\Big(3\nabla^p\nabla_p R_{ij} + 6\nabla^{pq}R_{ipjq}$$

$$+ 4R_{ip}R^p_j + 4R_{ipqr}(R^{\cdot pqr}_j + R^{\cdot qpr}_j)\Big)\xi^i\xi^j I + \dots, \tag{4.14}$$

$$\frac{1}{2}g^{ij}\chi^{(2)}_{\langle ijkl\rangle}\xi^k\xi^l = \frac{1}{30}(3\nabla_i\nabla_j R_{kl} + 4R^p_{ijq}R^q_{klp})\xi^i\xi^j\xi^k\xi^l I + \dots, \tag{4.15}$$

$$\chi^{(2)}_{\langle ijkl\rangle}\xi^i\xi^j\xi^k\xi^l = 0 + \dots \tag{4.16}$$

We substitute expressions (4.10)–(4.16) into (4.9) and write the result in the form

$$\frac{1}{2\pi i}\int_\gamma e^{-\lambda}r^2_4\,d\lambda = e^{-|\xi|^2}\Big(\frac{1}{4}g^{ij}B_{ij} + (-\frac{1}{3}B_{ij} + C_{ij}I)\xi^i\xi^j + D_{ijkl}I\xi^i\xi^j\xi^k\xi^l\Big) + \dots, \tag{4.17}$$

where

$$B_{ij} = g^{pq}\mathcal{R}_{ip}\mathcal{R}_{jq}, \tag{4.18}$$

$$C_{ij} = \frac{1}{135}\Big(18\nabla^p\nabla_p R_{ij} + 25\nabla^p\nabla_i R_{jp} + 20\nabla_i\nabla^p R_{jp} - 9\nabla^p\nabla^q R_{ipjq}$$

$$+ 4R^p_i R_{jp} + 4R_{ipqr}(4R^{\cdot pqr}_j - 5R^{\cdot qpr}_j)\Big), \tag{4.19}$$

$$D_{ijkl} = -\frac{1}{45}(9\nabla_i\nabla_j R_{kl} + 2R^p_{ijq}R^q_{klp}). \tag{4.20}$$

We apply the operator tr to (4.17) and integrate over $T^*_x M$ with the help of Lemma 2.1

$$\frac{\pi^{-n/2}}{2\pi i}\int_{T^*_x M}\int_\gamma e^{-\lambda}\,\mathrm{tr}\,r^2_4\,d\lambda d\xi = \frac{1}{12}\,\mathrm{tr}\,(g^{ij}B_{ij}) + d\Big(\frac{1}{2}g^{ij}D_{ij} + \frac{3}{4}(g^2)^{ijkl}E_{ijkl}\Big). \tag{4.21}$$

From (4.19) and (4.20) we obtain

$$\frac{1}{2}g^{ij}C_{ij} + \frac{3}{4}(g^2)^{ijkl}D_{ijkl} = \frac{1}{540}\Big(9\nabla^i\nabla_i S + 18\nabla^{ij}R_{ij} + 2|Ric|^2 + 2R_{ijkl}(13R^{ijkl} - 23R^{ikjl})\Big). \tag{4.22}$$

This formula can be simplified with the help of the identities

$$\nabla^{ij}R_{ij} = \frac{1}{2}\nabla^i\nabla_i S = -\Delta S \tag{4.23}$$

and

$$R_{ijkl}R^{ikjl} = \frac{1}{2}R_{ijkl}R^{ikjl} = \frac{1}{2}|R|^2 \tag{4.24}$$

that will be proved at the end of the section. (4.22) takes now the form

$$\frac{1}{2}g^{ij}C_{ij} + \frac{3}{4}(g^2)^{ijkl}D_{ijkl} = \frac{1}{540}(-18\Delta S + 2|Ric|^2 + 3|R|^2). \tag{4.25}$$

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Substitute (4.18) and (4.25) into (4.21) to obtain

$$\frac{\pi^{-n/2}}{2\pi i} \int\limits_{T_x^*M} \int\limits_{\gamma} e^{-\lambda} \operatorname{tr} r_4^2 \, d\lambda d\xi = \frac{d}{540}(-18\Delta S + 2|Ric|^2 + 3|R|^2) + \frac{1}{12} \operatorname{tr}(g^{ik}g^{jl}\mathcal{R}_{ij}\mathcal{R}_{kl}).$$

(4.26)

Let us recall that $r_4 = r_4^1 + r_4^2 + r_4^3$. Take the sum of (4.7), (4.8), and (4.26) to obtain

$$\frac{\pi^{-n/2}}{2\pi i} \int\limits_{T_x^*M} \int\limits_{\gamma} e^{-\lambda} \operatorname{tr} r_4 \, d\lambda d\xi = \frac{d}{360}(-12\Delta S + 5S^2 - 2|Ric|^2 + 2|R|^2)$$

$$+ \operatorname{tr}\left(\frac{1}{12}g^{ik}g^{jl}\mathcal{R}_{ij}\mathcal{R}_{kl} - \frac{1}{6}\nabla^i\nabla_i A + \frac{1}{2}A^2 - \frac{1}{6}SA\right).$$

In virtue of (1.4), this coincides with statement (c) of Theorem 3.1.

Let us prove (4.23). By the Bianchi identity,

$$\nabla_i R_{jkpm} + \nabla_j R_{kipm} + \nabla_k R_{ijpm} = 0.$$

Contracting this equality with g^{km} (i.,e., multiplying by g^{km} and taking the sum over k and m), we obtain

$$\nabla_i R_{jp} - \nabla_j R_{ip} + \nabla^q R_{ijpq} = 0.$$

Transpose the indices j and p on this equality

$$\nabla_i R_{jp} = \nabla_p R_{ij} - \nabla^q R_{ipjq}.$$

Applying the operator ∇^p to the last equality and summing over p, we obtain

$$\nabla^p\nabla_i R_{jp} = \nabla^p\nabla_p R_{ij} - \nabla^p\nabla^q R_{ipjq}.$$

Contracting this equality with g^{ij}, we arrive to (4.23).

Finally, we prove (4.24). To this end we first transform the second factor in the product $R_{ijkl}R^{ikjl}$ with the help of the Ricci identity

$$R_{ijkl}R^{ikjl} = -R_{ijkl}(R^{ijlk} + R^{ilkj}) = R_{ijkl}R^{ijkl} + R_{ijkl}R^{iljk}.$$

Transpose the summation indices k and l in the last term on the right-hand side

$$R_{ijkl}R^{ikjl} = R_{ijkl}R^{ijkl} + R_{ijlk}R^{ikjl}$$

and then use the skew-symmetry of R_{ijlk} in two last indices to obtain

$$R_{ijkl}R^{ikjl} = R_{ijkl}R^{ijkl} - R_{ijkl}R^{ikjl}.$$

This is equivalent to (4.24).

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5 Laplacian on forms

For a Riemannian manifold (M, g), we denote the Hodge Laplacian on ν-forms by

$$\Delta_\nu = d\delta + \delta d : C^\infty(\Lambda^\nu(T^*M)) \to C^\infty(\Lambda^\nu(T^*M)).$$

Theorem 5.1 *For a closed n-dimensional Riemannian manifold (M, g), the first three heat invariants of Δ_ν are expressed by the formulas*

$$a_0(x, \Delta_\nu) = \binom{n}{\nu},$$

$$a_2(x, \Delta_\nu) = \frac{1}{6}\Big[\binom{n}{\nu} - 6\binom{n-2}{\nu-1}\Big]S,$$

$$a_4(x, \Delta_\nu) = \frac{1}{360}\Big(c_1(n, \nu)\Delta S + c_2(n, \nu)S^2 + c_3(n, \nu)|Ric|^2 + c_4(n, \nu)|R|^2\Big),$$

where

$$c_1(n, \nu) = -12\Big[\binom{n}{\nu} - 5\binom{n-2}{\nu-1}\Big],$$

$$c_2(n, \nu) = 5\Big[\binom{n}{\nu} - 12\binom{n-2}{\nu-1} + 36\binom{n-4}{\nu-2}\Big],$$

$$c_3(n, \nu) = -2\Big[\binom{n}{\nu} - 90\binom{n-2}{\nu-1} + 360\binom{n-4}{\nu-2}\Big],$$

$$c_4(n, \nu) = 2\Big[\binom{n}{\nu} - 15\binom{n-2}{\nu-1} + 90\binom{n-4}{\nu-2}\Big].$$

The binomial coefficients $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ are assumed to be defined for all integers m and k under the agreement: $\binom{m}{k} = 0$ if either $m < 0$ or $k < 0$ or $m < k$. The curvature tensor is normalized so that the scalar curvature S is equal to $+1$ for the two-dimensional unit sphere.

The result actually belongs to Patodi and is reproduced in Theorem 4.8.18 of [2]. We emphasize that our formulas for $c_i(n, \nu)$ are valid for all n and ν while the corresponding formulas in Theorem 4.8.18 of [2] make sense for $n \geq 4$ only. By the way, it is a good exercise to check the agreement of our formulas with that of [2].

The Laplacian Δ_ν can be written in form (2.1) with the algebraic operator $A = A_\nu \in C^\infty(\mathrm{End}\,(\Lambda^\nu(T^*M)))$ that is expressed in local coordinates as follows. If a ν-form is written as $\omega = \omega_{i_1...i_\nu}dx^{i_1} \wedge \cdots \wedge dx^{i_\nu}$ with skew-symmetric $\omega_{i_1...i_\nu}$, then

$$A_\nu\omega = \Big(\nu R^p_{i_1}\omega_{pi_2...i_\nu} - \nu(\nu - 1)R^{p\cdot q\cdot}_{\cdot i_1 \cdot i_2}\,\omega_{pqi_3...i_\nu}\Big)dx^{i_1} \wedge \cdots \wedge dx^{i_\nu}. \qquad (5.1)$$

Another useful representation of A_ν is

$$A_\nu = \sum_{a=1}^\nu A_\nu^a - 2\sum_{1\leq a<b\leq\nu} A_\nu^{ab}, \qquad (5.2)$$

where

$$A_\nu^a(dx^{i_1} \wedge \cdots \wedge dx^{i_\nu}) = R_p^{i_a} dx^{i_1} \wedge \cdots \wedge dx^{i_{a-1}} \wedge dx^p \wedge dx^{i_{a+1}} \wedge \cdots \wedge dx^{i_\nu} \qquad (5.3)$$

and

$$A_\nu^{ab}(dx^{i_1} \wedge \cdots \wedge dx^{i_\nu}) = R_{.\,p\,.\,q}^{i_a\,.\,i_b\,.} dx^{i_1} \wedge \cdots \wedge dx^{i_{a-1}} \wedge dx^p \wedge dx^{i_{a+1}} \wedge \cdots$$
$$\wedge\, dx^{i_{b-1}} \wedge dx^q \wedge dx^{i_{b+1}} \wedge \cdots \wedge dx^{i_\nu}. \qquad (5.4)$$

The Levi-Chivita connection of (M, g) induces the connection on $\Lambda^\nu(T^*M))$ whose curvature tensor $\mathcal{R}^\nu = (\mathcal{R}_{ij}^\nu)$ can be written as

$$\mathcal{R}_{ij}^\nu = -\sum_{a=1}^\nu \mathcal{R}_{ij}^{\nu,a}, \qquad (5.5)$$

where

$$\mathcal{R}_{ij}^{\nu,a}(dx^{i_1} \wedge \cdots \wedge dx^{i_\nu}) = R_{pij}^{i_a} dx^{i_1} \wedge \cdots \wedge dx^{i_{a-1}} \wedge dx^p \wedge dx^{i_{a+1}} \wedge \cdots \wedge dx^{i_\nu}. \qquad (5.6)$$

Lemma 5.2 *For all n and ν,*

$$\operatorname{tr} A_\nu = \binom{n-2}{\nu-1} S, \qquad (5.7)$$

$$\operatorname{tr} A_\nu^2 = \binom{n-4}{\nu-2} S^2 + \left[\binom{n-2}{\nu-1} - 4\binom{n-4}{\nu-2}\right] |Ric|^2 + \binom{n-4}{\nu-2} |R|^2, \qquad (5.8)$$

$$\operatorname{tr}(g^{ik} g^{jl} \mathcal{R}_{ij}^\nu \mathcal{R}_{kl}^\nu) = -\binom{n-2}{\nu-1} |R|^2. \qquad (5.9)$$

The proof of Theorem 5.1 consists of substituting values (5.7)–(5.9) into the statement of Theorem 3.1. So, it remains to prove Lemma 5.2.

Lemma 5.2 is of a pure algebraic nature. It can be proved in different ways. Probably, the shortest proof is as follows. The idea of the proof is taken from [2], see arguments presented before Theorem 4.8.18 of [2].

Obviously, $\operatorname{tr} A_\nu$ must be a linear scalar function of the curvature tensor which, moreover, must be invariant under action of the orthogonal group. As well known, every such linear invariant is a multiple of the scalar curvature. Thus $\operatorname{tr} A_\nu = a(n, \nu)S$. To find the coefficient $a(n, \nu)$, it suffices to consider the case of a metric of the constant sectional curvature K. In the latter case $R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk})$, $R_{ij} = (n-1)g_{ij}$, $S = n(n-1)K$, and formula (5.7) is easily derived from definition (5.1).

By the same arguments

$$\operatorname{tr} A_\nu^2 = b_1(n, \nu)S^2 + b_2(n, \nu)|Ric|^2 + b_3(n, \nu)|R|^2. \qquad (5.10)$$

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It is easy to see that the coefficients satisfy the recurrent relation (Pascal's formula)

$$b_i(n+1, \nu) = b_i(n, \nu) + b_i(n, \nu - 1) \quad (i = 1, 2, 3). \tag{5.11}$$

Indeed, given an n-dimensional Riemannian manifold (M, g), let $M \times \mathbb{R}$ be the metric product. Then M and $M \times \mathbb{R}$ have the same values of S^2, $|Ric|^2$, and $|R|^2$ in the obvious sense. For a point $x \in M$, we have the natural isomorphism

$$\Lambda^\nu(T^*_{(x,0)}(M \times \mathbb{R})) \cong \Lambda^\nu(T^*_x M) \oplus \Lambda^{\nu-1}(T^*_x M). \tag{5.12}$$

Let $A_{\nu, n+1}$ be the operator A_ν for $M \times \mathbb{R}$ and $A_{\nu, n}$ be the same for M. As follows from (5.1), both summands on the right-hand side of (5.12) are invariant subspaces of $A_{\nu, n+1}$, the restriction of $A_{\nu, n+1}$ to the first summand coincides with $A_{\nu, n}$, and the restriction of $A_{\nu, n+1}$ to the second summand coincides with $A_{\nu-1, n}$. In other words, $A_{\nu, n+1} = A_{\nu, n} \oplus A_{\nu-1, n}$. Therefore $A^2_{\nu, n+1} = A^2_{\nu, n} \oplus A^2_{\nu-1, n}$ and $\operatorname{tr} A^2_{\nu, n+1} = \operatorname{tr} A^2_{\nu, n} + \operatorname{tr} A^2_{\nu-1, n}$. This implies (5.11).

The recurrent relation (5.11) makes sense for $n \geq 4$ since S^2, $|Ric|^2$, and $|R|^2$ become linearly dependent for $n = 3$. It is easy to check that the coefficients of formula (5.8)

$$b_1(n, \nu) = \binom{n-4}{\nu-2}, \quad b_2(n, \nu) = \binom{n-2}{\nu-1} - 4\binom{n-4}{\nu-2}, \quad b_3(n, \nu) = \binom{n-4}{\nu-2}$$

satisfy (5.11). Thus, to finish the proof of formula (5.8), we need to check the validity of the formula for $n \leq 4$. Obviously $A_0 = 0$. Beside this, $\operatorname{tr} A^2_\nu = \operatorname{tr} A^2_{n-\nu}$ since A_ν is agreed with the Hodge star. So, we need to consider four values of (n, ν)

$$(n, \nu) \in \{(2, 1); (3, 1); (4, 1); (4, 2)\}.$$

We will present the consideration of the last case only. Other three cases are much easier.

Fix a point x in a four-dimensional M and choose local coordinates in a neighborhood of x such that $g_{ij}(x) = \delta_{ij}$. The six 2-forms

$$dx^1 \wedge dx^2, \quad dx^1 \wedge dx^3, \quad dx^1 \wedge dx^4, \quad dx^2 \wedge dx^3, \quad dx^2 \wedge dx^4, \quad dx^3 \wedge dx^4$$

constitute the orthonormal basis of $\Lambda^2(T^*_x M)$. We find the matrix of A_2 in this basis by explicit calculations according formulas (5.2)–(5.4)

$$A_2 = B - 2C,$$

$$B = \begin{pmatrix} R_{11} + R_{22} & R_{23} & R_{24} & -R_{13} & -R_{14} & 0 \\ R_{23} & R_{11} + R_{33} & R_{34} & R_{12} & 0 & -R_{14} \\ R_{24} & R_{34} & R_{11} + R_{44} & 0 & R_{12} & R_{13} \\ -R_{13} & R_{12} & 0 & R_{22} + R_{33} & R_{34} & -R_{24} \\ -R_{14} & 0 & R_{12} & R_{34} & R_{22} + R_{44} & R_{23} \\ 0 & -R_{14} & R_{13} & -R_{24} & R_{23} & R_{33} + R_{44} \end{pmatrix},$$

$$C = \begin{pmatrix} R_{1212} & R_{1213} & R_{1214} & R_{1223} & R_{1224} & R_{1234} \\ R_{1213} & R_{1313} & R_{1314} & R_{1323} & R_{1324} & R_{1334} \\ R_{1214} & R_{1314} & R_{1414} & R_{1423} & R_{1424} & R_{1434} \\ R_{1223} & R_{1323} & R_{1423} & R_{2323} & R_{2324} & R_{2334} \\ R_{1224} & R_{1324} & R_{1424} & R_{2324} & R_{2424} & R_{2434} \\ R_{1234} & R_{1334} & R_{1434} & R_{2334} & R_{2434} & R_{3434} \end{pmatrix}.$$

Since the matrices are symmetric,

$$\operatorname{tr} A_2^2 = |B|^2 + 4|C|^2 - 4\operatorname{tr}(BC). \tag{5.13}$$

We evaluate
$$|B|^2 = (R_{11} + R_{22})^2 + (R_{11} + R_{33})^2 + (R_{11} + R_{44})^2$$
$$+ (R_{22} + R_{33})^2 + (R_{22} + R_{44})^2 + (R_{33} + R_{44})^2$$
$$+ 2(R_{12}^2 + R_{13}^2 + R_{14}^2 + R_{23}^2 + R_{24}^2 + R_{34}^2).$$

On using the equalities $S = R_{11} + R_{22} + R_{33} + R_{44}$ and

$$|Ric|^2 = R_{11}^2 + R_{22}^2 + R_{33}^2 + R_{44}^2 + 2(R_{12}^2 + R_{13}^2 + R_{14}^2 + R_{23}^2 + R_{24}^2 + R_{34}^2),$$

we transform the previous formula to the form

$$|B|^2 = 2|Ric|^2 + S^2. \tag{5.14}$$

Next,

$$\operatorname{tr}(BC) = R_{11}(R_{1212} + R_{1313} + R_{1414}) + R_{22}(R_{1212} + R_{2323} + R_{2424})$$
$$+ R_{33}(R_{1313} + R_{2323} + R_{3434}) + R_{44}(R_{1414} + R_{2424} + R_{3434})$$
$$+ 2\Big(R_{12}(R_{1323} + R_{1424}) + R_{13}(R_{1232} + R_{1434}) + R_{14}(R_{1242} + R_{1343})$$
$$+ \quad R_{23}(R_{1213} + R_{2434}) + R_{24}(R_{1214} + R_{2434}) + R_{34}(R_{1314} + R_{2324})\Big).$$

With the help of the relation $R_{ij} = R_{i1j1} + R_{i2j2} + R_{i3j3} + R_{i4j4}$, this gives

$$\operatorname{tr}(BC) = |Ric|^2. \tag{5.15}$$

Finally,

$$|C|^2 = R_{1212}^2 + R_{1313}^2 + R_{1414}^2 + R_{2323}^2 + R_{2424}^2 + R_{3434}^2$$
$$+ 2\Big(R_{1213}^2 + R_{1214}^2 + R_{1223}^2 + R_{1224}^2 + R_{1234}^2 + R_{1314}^2 + R_{1323}^2 \tag{5.16}$$
$$+ \quad R_{1324}^2 + R_{1334}^2 + R_{1423}^2 + R_{1424}^2 + R_{1434}^2 + R_{2324}^2 + R_{2434}^2\Big) = \frac{1}{4}|R|^2.$$

Substitute (5.14)–(5.16) into (5.13) to obtain $\operatorname{tr} A_2^2 = S^2 - 2|Ric|^2 + |R|^2$. This coincides with (5.8) in the case of $(n, \nu) = (4, 2)$.

We have thus finished the proof of (5.8). Formula (5.9) is proved in the same way.

In conclusion, we give a couple of remarks about the possibility of computerizing our calculations in order to evaluate heat invariants $a_k(x, \Delta_\nu)$ for $k = 6, 8, \ldots$.

First of all, the evaluation of the coefficients $\chi_\alpha^{(p)}$ definitely can be computerized and there is some experience of doing this. See our comments after formula (A.15) in Appendix below.

There is no problem with computer differentiation, i.e., with deriving higher order versions of formulas (3.2)–(3.4) and (4.2)–(4.3). Of course a computer can substitute a polynomial into another one and group similar terms. A little bit more problematic is the computer canceling of terms caused by the skew-symmetry of curvature tensors. We first have done such a canceling in formula (3.5) and then in a number of formulas of Section 4.

Probably, the main problem of the computerization relates to higher order analogies of (4.23) and (4.24). Recall that these relations are proved with the help of the Bianchi identity. There is an infinite sequence of Bianchi identities for higher order covariant derivatives of the curvature tensor which imply many relations between higher order curvature invariants. In author's opinion, this subject needs some theoretical investigation before the computerization.

Finally, our arguments based on Pascal's recurrent formula (5.11) are general enough to compute higher order algebraic invariants of A_ν and $\mathcal{R}^\nu$.

Appendix. Geometric symbol calculus

For reader's convenience, we summarize here main definitions and facts of geometric symbol calculus which are used in this paper. See [3] for proofs.

For a vector bundle V over a manifold M and for $m \in \mathbf{R}$, the space of symbols $S^m(T^*M, V)$ of order $\leq m$ consists of all smooth functions $a : T^*M \to V$ such that $a(x, \xi) \in V_x$ for $(x, \xi) \in T^*M$ and the estimate

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{K,\alpha,\beta}(1 + |\xi|)^{m-|\beta|} \quad (x \in K)$$

holds in any local coordinate system for any multi-indices α, β and for any compact $K \subset X$ contained in the domain of the system.

Let now (M, ∇) be a manifold with a fixed symmetric connection, (V, ∇^V) be a vector bundle with connection over M, and W be a second vector bundle over M. Given a symbol $a \in S^m(T^*M, \mathrm{Hom}(V, W))$, we say that a linear continuous operator

$$A = a(x, -i\nabla) : C_0^\infty(V) \to \mathcal{D}'(W)$$

belongs to $\Psi^m(M, \nabla; V, W)$ and has the geometric symbol a if the Schwartz kernel of A is smooth outside the diagonal and, for every point $x \in X$, there exists a

neighborhood U of x such that

$$Au(x) = (2\pi)^{-n} \int\limits_{T_x^*M} \int\limits_{T_xM} e^{-i\langle v,\xi\rangle} a(x,\xi) J_x^{\exp_x v} u(\exp_x v) \, dv d\xi \qquad (A.1)$$

for any section $u \in C^\infty(V)$ with supp $u \subset U$. Here $\langle v, \xi \rangle$ means the canonical pairing $T_x X \times T_x^* X \to \mathbf{R}$, dv and $d\xi$ are dual densities on $T_x X$ and $T_x^* X$ respectively, and $J_x^{\exp_x v} : V_{\exp_x v} \to V_x$ is the parallel transport along the geodesic $t \mapsto \exp_x v$ which is determined by the connection ∇^V. Note that the integrand $a(x, \xi) J_x^{\exp_x v} u(\exp_x v)$ belongs to the vector space W_x, so the integral is well defined. There is no ambiguity in (A.1) since the product $dv d\xi$ is uniquely determined. Observe that no coordinate system participates in the definition.

If the symbol depends polynomially on ξ, $a(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha$, then $a(x, -i\nabla)$ $= \sum_{|\alpha| \le m} a_\alpha(x)(-i\nabla)^\alpha$, where $(-i\nabla)^\alpha$ is the symmetrized covariant derivative on V.

We are going to present the formula that expresses the geometric symbol of the product of two pseudodifferential operators through symbols of the factors. To this end, given a bundle (V, ∇^V) with a connection, we introduce polynomials $R^{\alpha,\beta}(x, \xi) \in C^\infty(T^*M, \operatorname{End}(V))$ by the equalities

$$(-i\nabla)^\alpha (-i\nabla)^\beta = R^{\alpha,\beta}(x, -i\nabla). \qquad (A.2)$$

Let $\pi : T^*M \to M$ be the cotangent bundle and $\tau_s^r X$ be the bundle of (r, s)-tensors. The pull-back $\beta_s^r(M, V) = \pi^*(V \otimes \tau_s^r M)$ is a vector bundle over T^*M which is called the bundle of E-valued semibasic (r, s)-tensors. A connection ∇^V on V allows us to define the horizontal derivative $\overset{h}{\nabla} : C^\infty(\beta_s^r(M, V)) \to C^\infty(\beta_{s+1}^r(M, V))$. that commutes with the vertical derivative $\overset{v}{\nabla} = \partial_\xi : C^\infty(\beta_s^r(M, V)) \to C^\infty(\beta_s^{r+1}(M, V))$.

Theorem A.1. *Let (M, ∇) be a manifold with a symmetric connection, (V, ∇^V) and (W, ∇^W) be two vector bundles with connections, and Z be a third vector bundle over M. Let one of two operators $A = a(x, -i\nabla) \in \Psi^{m_1}(M, \nabla; W, Z)$ and $B = b(x, -i\nabla) \in \Psi^{m_2}(M, \nabla; V, W)$ be properly supported. Then the product $C = AB$ belongs to $\Psi^{m_1+m_2}(M, \nabla; V, Z)$ and the full geometric symbol $c(x, \xi)$ of C is expressed through $a(x, \xi)$ and $b(x, \xi)$ by the asymptotic series*

$$c(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \overset{v}{\nabla}^\alpha a \sum_{\beta,\gamma} \frac{1}{\gamma!} \binom{\alpha}{\beta} (-i\overset{h}{\nabla})^\beta \overset{v}{\nabla}^\gamma b \cdot \rho_{\alpha-\beta,\gamma}, \qquad (A.3)$$

where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$ are the binomial coefficients with $\binom{\alpha}{\beta} \ne 0$ only for $\beta \le \alpha$; and $\rho_{\alpha,\beta}(x, \xi)$ are polynomials expressed through polynomials (A.2) by the formula

$$\rho_{\alpha,\beta} = (-1)^{|\alpha|+|\beta|} \sum_{\lambda,\mu} (-1)^{|\lambda|+|\mu|} \binom{\alpha}{\lambda}\binom{\beta}{\mu} \xi^{\alpha+\beta-\lambda-\mu} R^{\lambda,\mu}.$$

Compared with [3], we have slightly changed the notation for the coefficients that are denoted by $\rho^{\alpha,\beta}$ in [3]. Let R be the curvature tensor of ∇ and $\mathcal{R}$ be the curvature tensor of ∇^V. Every function $\rho_{\alpha,\beta}$ is a homogeneous polynomial of degree $|\alpha| + |\beta|$ in the variables R, $\mathcal{R}, \nabla$, and ξ if the degree of homogeneity of R and $\mathcal{R}$ is equal to two and the degree of homogeneity of ∇ and ξ is equal to one. The degree of $\rho_{\alpha,\beta}$ in ξ satisfy the estimate

$$\deg_\xi \rho_{\alpha,\beta} \leq \min\{|\alpha|, |\beta|, (|\alpha| + |\beta|)/3\}. \tag{A.4}$$

There exists an efficient procedure for evaluating these polynomials based on the commutator formula for covariant derivatives, but the volume of calculations grows rapidly with $|\alpha|+|\beta|$. To write down some of these polynomials, we need the following correspondence between multi-indices and tensor indices. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and a sequence $(j_1, \ldots, j_m)$ with $m = |\alpha|$, $1 \leq j_a \leq n$ for $1 \leq a \leq m$, we write $\alpha = \langle j_1 \ldots j_m \rangle$ if the sequence $(j_1, \ldots, j_m)$ coincides with the sequence $(\underbrace{1, \ldots, 1}_{\alpha_1}, \underbrace{2, \ldots, 2}_{\alpha_2}, \ldots, \underbrace{n, \ldots, n}_{\alpha_n})$ up to the order of elements. Let also $\sigma(ij \ldots k)$ stand for the symmetrization in $(ij \ldots k)$.

Several first polynomials $\rho_{\alpha,\beta}$ are as follows (I is the identity operator):

$$\rho_{0,0} = I, \quad \rho_{\alpha,0} = \rho_{0,\alpha} = 0 \quad \text{for} \quad |\alpha| > 0, \tag{A.5}$$

$$\rho_{\langle j \rangle, \langle k \rangle} = -\frac{1}{2}\mathcal{R}_{jk}, \tag{A.6}$$

$$\rho_{\langle j \rangle, \langle kl \rangle} = -\frac{1}{3}(R^p_{klj} + R^p_{lkj})\xi_p I - \frac{1}{6}\left((-i\nabla)_k \mathcal{R}_{jl} + (-i\nabla)_l \mathcal{R}_{jk}\right), \tag{A.7}$$

$$\rho_{\langle jk \rangle, \langle l \rangle} = -\frac{1}{6}(R^p_{jlk} + R^p_{klj})\xi_p I - \frac{1}{3}\left((-i\nabla)_j \mathcal{R}_{kl} + (-i\nabla)_k \mathcal{R}_{jl}\right), \tag{A.8}$$

$$\rho_{\langle j \rangle, \langle klm \rangle} = \frac{1}{4}\sigma(klm)\left(2(-i\nabla)_k R^p_{ljm}\xi_p I - (-i\nabla)_k(-i\nabla)_l \mathcal{R}_{jm} + R^p_{klj}\mathcal{E}_{mp}\right), \tag{A.9}$$

$$\rho_{\langle jkl \rangle, \langle m \rangle} = \frac{1}{4}\sigma(jkl)\left(2(-i\nabla)_j R^p_{klm}\xi_p I - 3(-i\nabla)_j(-i\nabla)_k \mathcal{R}_{lm} - R^p_{jkm}\mathcal{R}_{lp}\right), \tag{A.10}$$

$$\rho_{\langle jk \rangle, \langle lm \rangle} = \frac{1}{6}\sigma(jk)\sigma(lm)\left(5(-i\nabla)_j R^p_{lkm}\xi_p I + (-i\nabla)_l R^p_{jkm}\xi_p I\right.$$
$$\left. -3(-i\nabla)_j(-i\nabla)_l \mathcal{R}_{km} + 2R^p_{lmj}\mathcal{R}_{kp} + R^p_{jkl}\mathcal{R}_{pm} + 3\mathcal{R}_{jl}\mathcal{R}_{km}\right). \tag{A.11}$$

The author derived these formulas by manual calculations. Later V. Djepko [1] computed $\rho_{\alpha,\beta}$ for $|\alpha| + |\beta| = 5$ but only in the scalar case, i.e., when $\mathcal{R} = 0$. He used MAPLE in his calculations. We will need the following two of his results:

$$\rho_{\langle ijk \rangle, \langle lm \rangle} = -\frac{1}{30}\sigma(ijk)\sigma(lm)\left(27\nabla_{ij} R^p_{lkm} + 7\nabla_{il} R^p_{jkm} + 2\nabla_{li} R^p_{jkm}\right.$$
$$\left. - 4R^q_{ijl} R^p_{qkm} - 12R^q_{ijl} R^p_{mkq} - 16R^q_{lim} R^p_{jkq}\right)\xi_p, \tag{A.12}$$

$$\rho_{\langle ijkl\rangle,\langle m\rangle} = \frac{1}{15}\sigma(ijkl)\Big(-9\nabla_{ij}R^p_{klm} + 7R^q_{ijm}R^p_{klq}\Big)\xi_p. \qquad (A.13)$$

Let $\rho^{(p)}_{\alpha,\beta}$ be the homogeneous in ξ part of degree p of the polynomial $\rho_{\alpha,\beta}$. Being valid in the scalar case, (A.12) and (A.13) imply the validity of formulas

$$\rho^{(1)}_{\langle ijk\rangle,\langle lm\rangle} = -\frac{1}{30}\sigma(ijk)\sigma(lm)\Big(27\nabla_{ij}R^p_{lkm} + 7\nabla_{il}R^p_{jkm} + 2\nabla_{li}R^p_{jkm}$$
$$\qquad\qquad\qquad - 4R^q_{ijl}R^p_{qkm} - 12R^q_{ijl}R^p_{mkq} - 16R^q_{lim}R^p_{jkq}\Big)\xi_p I + \dots, \qquad (A.14)$$

$$\rho^{(1)}_{\langle ijkl\rangle,\langle m\rangle} = \frac{1}{15}\sigma(ijkl)\Big(-9\nabla_{ij}R^p_{klm} + 7R^q_{ijm}R^p_{klq}\Big)\xi_p I + \dots \qquad (A.15)$$

in the general case, where dots stand for some terms linearly depending on $\mathcal{R}$. Indeed, observe that ξ_p always comes to $\rho_{\alpha,\beta}$ together with R, i.e., as a product $R^p_{qrs}\xi_p$. Therefore extra terms on (A.14) and (A.15) consist of monomials of the form $a^{qrs}(R,\mathcal{R},\nabla)R^p_{qrs}\xi_p$, where $a^{qrs}(R,\mathcal{R},\nabla)$ has the second degree in $(R,\mathcal{R},\nabla)$. So, it must be linear in $\mathcal{R}$.

As Djepko states in his PhD thesis, no modern computer is powerful enough to compute $\rho_{\alpha,\beta}$ for $|\alpha| + |\beta| = 6$. We are more optimistic. Probably, some progress can be achieved either by improving the algorithm or creating some special soft wear. Indeed, any universal soft wear like MAPLE is far of the optimal usage of computer resources. Observe that, to evaluate $\chi^{(p)}_\alpha$, we need to know $\rho_{\alpha,\beta}$ for $|\beta| \le 2$ only. Most probably, a fast algorithm can be found for computing $\rho_{\alpha,\beta}$ ($|\beta| \le 2$) which does not refer to $\rho_{\alpha,\beta}$ with $|\beta| > 2$.

M. Skokan [4] computed leading terms of $\rho_{\alpha,\beta}$ for $|\alpha| + |\beta| = 6$. From his results, we need the formula

$$\rho^{(2)}_{\langle ijkl\rangle,\langle pq\rangle} = \frac{2}{3}\sigma(ijkl)\sigma(pq)(R^r_{ijp}R^s_{klq}\xi_r\xi_s)I. \qquad (A.16)$$

Again, Skokan derived this formula in the scalar case only. But the same arguments as above show the validity of the formula in the general case.

Finally, we write down some of polynomials $\chi^{(p)}_\alpha$ ($p = 0, 1, 2$) that participate in the recurrent formula (2.11). The following formulas are obtained by substituting values (A.5)–(A.16) into the definition (2.8) of $\chi^{(p)}_\alpha$. Dots stand for some extra terms depending linearly on $\mathcal{R}$.

$$\chi^{(0)}_0 = 0, \quad \chi^{(1)}_0 = 0, \quad \chi^{(2)}_0 = 0, \qquad (A.17)$$

$$\chi^{(0)}_{\langle i\rangle} = 0 + \dots, \quad \chi^{(1)}_{\langle i\rangle} = \frac{2}{3}R_{ip}\xi^p I - \frac{1}{2}\mathcal{R}_{ip}\xi^p, \quad \chi^{(2)}_{\langle i\rangle} = 0, \qquad (A.18)$$

$$\chi^{(0)}_{\langle ij\rangle} = \frac{1}{2}g^{pq}\mathcal{R}_{ip}\mathcal{R}_{jq} + \dots, \quad \chi^{(2)}_{\langle ij\rangle} = -\frac{1}{3}R_{ipjq}\xi^p\xi^q I, \qquad (A.19)$$

$$\chi^{(1)}_{\langle ijk \rangle} = -\frac{1}{30}\sigma(ijk)\Big(27\nabla_{ij}R_{kp} + 7\nabla_i\nabla^q R_{pjkq} + 2\nabla^q\nabla_i R_{pjkq}$$
$$- 4R^{q\cdot\cdot r}_{\cdot ij\cdot}R_{pqkr} - 12R^{q\cdot\cdot r}_{\cdot ij\cdot}R_{prkq} - 16R^q_i R_{pjkq}\Big)\xi^p I + \dots, \qquad (A.20)$$

$$\chi^{(2)}_{\langle ijkl \rangle} = \frac{2}{5}\sigma(ijkl)(3\nabla_{ij}R_{kplq} + 4R_{pijr}R^r_{klq})\xi^p\xi^q I + \dots. \qquad (A.21)$$

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[4] M. Skokan. Rho's. Unpublished notes (2003).