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COMPUTABLE FUNCTIONALS OF FINITE TYPES IN MONTAGUE SEMANTICS

A.S. BURNISTOV, A.I. STUKACHEV

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Abstract: We consider a computable model of functionals of finite types used in Montague semantics to represent grammar categories in natural language sentences. The model is based on the notion of Σ-predicates of finite types in admissible sets introduced by Yu.L.Ershov.

Keywords: Montague semantics, functionals of finite types, generalized computability, Σ-predicates, Σ-operators.

1 Introduction

Type theory and functionals of finite types are essentially and fruitfully used in Montague intensional logic for formalizing the basic grammar categories of natural languages. In Montague original works, as well as in works of other researchers in this area of mathematical linguistics, to the authors' knowledge, the complexity issues and algorithmic aspects of objects and constructions of this theory were not considered so far. The functionals of finite types (i.e., functions, functions on functions, etc.) are complex objects and hard to be presented constructively in a way which allows possible applications in (for example) ontological models from computational linguistics.

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Our research, being purely theoretical and based on notions and methods from generalized computability and Ershov-Scott theory of domains and approximation spaces, is aimed to describe some algorithmic properties of these objects. This work continues publications [10, 11, 2] and forthcoming article [13].

In this paper, we present two models of Montague intensional logic. Since they are constructed within the framework of Σ-definability in admissible sets proposed by Yu.L.Ershov, these models can be regarded as effective or computable in a generalized sense. The differences between the models is in the definition of denotation spaces for the basic types of entities and truth values. At first, the simplest possible variant of these spaces is presented. The second model with the ontological space for entities with partial information is briefly described at the end of the paper and studied with more details in [2].

2 Basic Notions

2.1. Montague Intensional Logic. Let e, t and s be some fixed symbols used, correspondingly, as names for basic types of entities and truth values, and for marking an intensional shift, i.e. relativization to a state or situation.

Definition 1. *The set $Types_{IL}$ is defined as follows:*

- $t \in Types_{IL}$*,* $e \in Types_{IL}$*;*
- *if* $a \in Types_{IL}$ *and* $b \in Types_{IL}$ *then* $(a \to b) \in Types_{IL}$*;*
- *if* $a \in Types_{IL}$ *then* $(s \to a) \in Types_{IL}$*.*

The language of intensional logic IL (see [3, 7, 8]) contains countably many constants of any type $a \in Types_{IL}$ and countably many variables of each type $a \in Types_{IL}$.

A model of intensional logic IL is a quadruple $\langle A, W, T, \leq, F\rangle$ such that A, W, T are nonempty sets, $\leq$ is a linear order on T, F is a function defined on the set of constants of IL as described below. Sets W and T correspond to the set of possible worlds and time moments correspondingly.

Definition 2. *The set D_τ of possible denotations of type $\tau \in Types_{IL}$ is defined by induction on complexity of τ:*

- $D_e = A$*,* $D_t = \{0, 1\}$*;*
- $D_{(a\to b)} = D_b^{D_a}$ *(the set of functions from D_a to D_b);*
- $D_{(s\to a)} = D_a^{W\times T}$ *(the set of functions from $W \times T$ to D_a).*

We denote by S_a the set $D_{(s\to a)}$. Function F defines for each constant of type a some element from S_a which is called its *intension*. Elements from D_a are called *extensions* of type a.

Finite types from definition 1 are used to represent grammar categories (parts of speech) of natural languages. Some correspondences between categories and types are listed in Table 1.

For example, proper names correspond to the type $((s \to (e \to t)) \to t)$ – "property of being a property" (or the set of properties true for the individual with this name). Here we do not consider one of the most complex cases, intensional transitive verbs with the type $((s \to ((s \to (e \to t)) \to t)) \to (e \to t)))$.

Extension (the set of denotations) of type a is the set of possible values of the grammar category interpreted by type a in a model of intensional logic. Correspondingly, intension of type a is a function from $W \times T$ to the extension of type a.

TABLE 1. Categories and types of some expressions

Category	Grammar equivalent	Corresponding type	Basic expressions
e	no	e	no
t	sentences	t	no
IV	intransitive verbs	(e → t)	walk, talk
CN	common nouns	(e → t)	man, woman
TV	extensional transitive verbs	(e → (e → t))	love, find
CN/CN	extensional adjectives	((e → t) → (e → t))	tall, young
CN/CN	extensional adverbs	((e → t) → (e → t))	rapidly, slowly
T	noun phrases and proper names	((s → (e → t)) → t)	John, ninety, he
t/t	sentence determinants	((s → t) → t)	necessarily, possibly
IV/t	connective verbs	((s → t) → (e → t))	believe, assert

2.2. Hereditarily Finite Superstructures. Hereditary finite superstructures are the "simplest" examples of models of theory KPU proposed by S.Kripke, R.Platek, J.Barwise and Yu.L.Ershov for studying generalized computability via Σ-definability in admissible sets (see [1, 6, 12]).

By ω we denote the set of natural numbers. For arbitrary set M, we construct the set $HF(M)$ of hereditarily finite sets over M as follows:

$HF_0(M) = \varnothing$;
$HF_{n+1}(M) = \mathcal{P}_\omega(M \cup HF_n(M))$, $n < \omega$
(here $\mathcal{P}_\omega(X)$ is the set of all finite subsets of X);
$HF(M) = \bigcup\limits_{n<\omega} HF_n(M)$.

If $\mathfrak{M}$ is a structure of some relational signature σ then one can define on $M \cup HF(M)$ a structure $\mathbb{HF}(\mathfrak{M})$ of signature $\sigma' = \sigma \cup \{U, \varnothing, \in\}$ ($U, \varnothing, \in$ are some symbols not in σ) with the following interpretation of signature symbols:

$U^{\mathbb{HF}(\mathfrak{M})} = M$;
$P^{\mathbb{HF}(\mathfrak{M})} = P^{\mathfrak{M}}$, $P \in \sigma$;
$\varnothing^{\mathbb{HF}(\mathfrak{M})} = \varnothing \in HF_0(M)$;
$\in^{\mathbb{HF}(\mathfrak{M})} = \in \cap((M \cup HF(M)) \times HF(M))$.

A class of Δ_0-*formulas* of signature σ' is the least one containing atomic formulas which is closed under $\vee$, $\wedge$, $\rightarrow$, $\neg$ and bounded quantifiers $\forall x \in y$ and $\exists x \in y$ ($\forall x \in y\, \varphi$ and $\exists x \in y\, \varphi$ are abbreviations for $\forall x(x \in y \rightarrow \varphi)$ and $\exists x(x \in y \wedge \varphi)$ respectively).

A class of Σ-*formulas* of signature σ' is the least one containing Δ_0-formulas and closed under $\vee$, $\wedge$, bounded quantifiers $\forall x \in y$, $\exists x \in y$, and $\exists x$. As usual, a set is called Σ-definable if it is definable by some Σ-formula with parameters, and Δ-definable if it and its complement are Σ-definable.

2.3. Ershov-Scott Functional Spaces. To construct an effective model of Montague intensional logic we apply the domain theory proposed by D.S. Scott [9] and the theory of functional spaces of finite types proposed by Yu.L. Ershov [4, 5, 6]. The definitions below are from [6].

Let $\mathbb{A}$ be a model of KPU (see [6]). If $a \in A$ then $p_l^* a = \{b \mid \exists c(\langle b, c\rangle \in a)\}$, $p_r^* a = \{b \mid \exists c(\langle c, b\rangle \in a)\}$. If $B \subseteq A$ then $B^* = \{b \mid b \subseteq B \text{ and } b \in A\}$.

The notion of effectively presented functional space is based on the general

Definition 3. *Quadruple* $\mathfrak{B} = \langle B, \leq, Cons, \sqcup\rangle$ *is called an* f-base on $\mathbb{A}$ *(see* [4, 5]*) if the following holds:*

1) B *is a* Δ*-definable subset of* $\mathbb{A}$*;*
2) $\leq$ *is a* Δ*-definable preorder on* B*;*
   let $[B]$ *be the quotient of set* B *by the equivalence relation* $\equiv$ *induced by the preorder* $\leq$ *(*$b_0 \equiv b_1 \Leftrightarrow b_0 \leq b_1$ *and* $b_1 \leq b_0$*); as usual,* $[b]$ *denotes the element of* $[B]$ *which is the equivalence class of* $b \in B$*; if* $C \subseteq B$ *then* $[C] = \{[b] \mid b \in C\}$*; we also use* $\leq$ *to denote the preorder induced on* $[B]$ *by the original preorder* $\leq$*;*
3) $Cons$ *is a* Δ*-definable subset of* $B^* \setminus \{\emptyset\}$*, and for any* $b_* \in B^*$ *holds*

$$b_* \in Cons \Leftrightarrow (\exists b \in B)(\forall b^{'} \in b_*)(b^{'} \leq b);$$

4) $\sqcup : Cons \to B$ *is a* Σ*-definable function such that* $[\sqcup b_*]$ *for any* $b_* \in Cons$ *is the least upper bound of* $[b_*] \subseteq [B]$ *in* $\langle [B], \leq\rangle$*.*

Definition 4. *Let* $\mathfrak{B}_0 = \langle B_0, \leq_0, Cons_0, \sqcup_0\rangle$ *and* $\mathfrak{B}_1 = \langle B_1, \leq_1, Cons_1, \sqcup_1\rangle$ *be some* f*-bases on* $\mathbb{A}$*. A direct product* $\mathfrak{B}_1 \times \mathfrak{B}_2$ *of* $\mathfrak{B}_0$ *and* $\mathfrak{B}_1$ *is the* f*-base* $\langle B_0 \times B_1, \leq, Cons, \sqcup\rangle$*, where* $\leq$*,* $Cons$ *and* $\sqcup$ *are defined as follows:*

1) $\langle b_0, b_1\rangle \leq \langle b_0^{'}, b_1^{'}\rangle$ *iff* $b_0 \leq_0 b_0^{'}$ *and* $b_1 \leq_1 b_1^{'}$ *for every* $b_0, b_0^{'} \in B_0$ *and every* $b_1, b_1^{'} \in B_1$*;*
2) $b_* \in Cons$ *iff* $p_l^*(b_*) \in Cons_0$ *and* $p_r^*(b_*) \in Cons_1$ *for every* $b_* \in (B_0 \times B_1)^*$*;*
3) $\sqcup b_* \leftrightharpoons \langle \sqcup_0 p_l^*(b_*), \sqcup_1 p_r^*(b_*)\rangle$ *for every* $b_* \in Cons$*.*

In case the set $Cons$ of mutually consistent fragments (approximations) should be as large as possible, we need

Definition 5. *Quadruple* $\mathfrak{B} = \langle B, b_0, \leq, \sqcup\rangle$ *is called an* f^*-base on $\mathbb{A}$ *if* $\langle B, \leq, B^* \setminus \{\emptyset\}, \sqcup\rangle$ *is an* f*-base on* $\mathbb{A}$*,* $[b_0]$ *is the least element in* $\langle [B], \leq\rangle$ *and* $\sqcup\emptyset = b_0$*.*

In general, the range (the set of possible values) of a functional can be arbitrary, so the notion of f^*-base is used in the following

Definition 6. *Let* $\mathfrak{B}_0 = \langle B_0, \leq_0, Cons_0, \sqcup_0\rangle$ *be an* f*-base,* $\mathfrak{B}_1 = \langle B_1, b_1, \leq_1, \sqcup_1\rangle$ *be an* f^**-base. A functional product* $F(\mathfrak{B}_0, \mathfrak{B}_1)$ *of* f*-base* $\mathfrak{B}_0$ *and* f^**-base* $\mathfrak{B}_1$ *is the* f^**-base* $\langle (B_0 \times B_1)^*, \emptyset, \leq, \sqcup\rangle$*, where* $\leq$ *and* $\sqcup$ *are defined as follows:*

1) $f_0 \leq f_1$ *iff* $\forall b_0 \in p_l^* f_0(\sqcup_1\{b_1 \mid \exists b_0^{'} \in p_l^* f_0(b_0^{'} \leq_0 b_0$ *and* $\langle b_0^{'}, b_1\rangle \in f_0)\} \leq_1$ $\leq_1 \sqcup_1\{b_1 \mid \exists b_0^{'} \in p_l^{'} f_1(b_0^{'} \leq_0 b_0$ *and* $\langle b_0^{'}, b_1\rangle \in f_1)\})$ *for* $f_0, f_1 \in (B_0 \times B_1)^*$*;*
2) $\sqcup f_* \leftrightharpoons \cup f_*$ *for every* $f_* \in ((B_0 \times B_1)^*)^*$*.*

Definition 7. *For an* f*-base* $\mathfrak{B} = \langle B, \leq, Cons, \sqcup\rangle$*, the family* $I_\Sigma(\mathfrak{B})$ *of* Σ*-ideals in* $\mathfrak{B}$ *consists of nonempty* Σ*-definable subsets* $C \subseteq B$ *such that*

1) *from* $c \in C, b \in B, b \leq c$ *it follows that* $b \in C$*;*
2) *from* $c \in C^*$ *it follows that* $c \in Cons$ *and* $\sqcup c \in C$*.*

We define a topology on the set $I_\Sigma(\mathfrak{B})$ by fixing the basis

$$V_b \leftrightharpoons \{C \mid C \in I_\Sigma(\mathfrak{B}), b \in C\}, b \in B.$$

The set $I_\Sigma(\mathfrak{B})$ together with the topology specified above is called the *space of* Σ*-ideals* of f-base $\mathfrak{B}$. The space $I_\Sigma(\mathfrak{B})$ is a topological T_0-space.

Let $\mathfrak{B}_0$ be an f-base and let $\mathfrak{B}_1$ be an f^*-base. For any ideal I of f-base $F(\mathfrak{B}_0, \mathfrak{B}_1)$ we can define the continuous function $f_I : I_\Sigma(\mathfrak{B}_0) \to I_\Sigma(\mathfrak{B}_1)$ as follows. Let $I_0 \in I_\Sigma(\mathfrak{B}_0)$. We define

$$f_I(I_0) \leftrightharpoons \{b_1 \,|\, b_1 \in B_1, (\exists c^* \in I)(\exists b_0 \in I_0)\exists b_1^{'}(b_1 \leq_1 b_1^{'} \text{ and } \langle b_0, b_1^{'}\rangle \in c^*)\}.$$

If $\{\langle b_0, b_1\rangle\} \in I, b_0 \in I_0$, then $b_1 \in f_I(I_0)$.

The mapping $I \to f_I$ from $I_\Sigma(F(\mathfrak{B}_0, \mathfrak{B}_1))$ to $C(I_\Sigma(\mathfrak{B}_0), I_\Sigma(\mathfrak{B}_1))$ (the set of all continuous functions from the space $I_\Sigma(\mathfrak{B}_0)$ to the space $I_\Sigma(\mathfrak{B}_1)$) is injective.

To introduce the simplest example of spaces for entities and truth values, let $\mathfrak{A} = \langle A, =, P_1(A), \cup \restriction P_1(A)\rangle$, where $P_1(A) \leftrightharpoons \{\{a\} \,|\, a \in A\}$. This quadruple is an f-base with $I_\Sigma(\mathfrak{A}) = P_1(A)$. Also, let α be an arbitrary ordinal in $\mathbb{A}$ and let $\mathfrak{B}_\alpha = \langle \alpha, \emptyset, \subseteq, \cup \restriction \alpha\rangle$. This quadruple is an f^*-base with $I_\Sigma(\mathfrak{B}_\alpha) = (\alpha + 1) \setminus \emptyset$. Further on we consider the case $\alpha = 2$.

Definition 8. *The set of functional types $Types_f$ together with its proper subset $PTypes_f$ are defined as follows:*

1) $\boldsymbol{o} \in Types_f \setminus PTypes_t, B \in PTypes_f \subseteq Types_f$;
2) *if* $\tau_0, \tau_1 \in Types_f(PTypes_f)$ *then* $(\tau_0 \times \tau_1) \in Types_f(PTypes_f)$;
3) *if* $\tau_0 \in Types_f, \tau_1 \in PTypes_f$ *then* $(\tau_0 \to \tau_1) \in PTypes_f$.

Definition 9. *For every type $\tau \in Types_f$, the f-base $\mathcal{F}_\tau$ is defined by induction on the complexity of τ:*

1) $\mathcal{F}_{\boldsymbol{o}} \leftrightharpoons \mathfrak{A}$, $\mathcal{F}_B \leftrightharpoons \mathfrak{B}_2$;
2) $\mathcal{F}_{(\tau_0 \times \tau_1)} \leftrightharpoons \mathcal{F}_{\tau_0} \times \mathcal{F}_{\tau_1}$;
3) $\mathcal{F}_{(\tau_0 \to \tau_1)} \leftrightharpoons F(\mathcal{F}_{\tau_0}, \mathcal{F}_{\tau_1})$.

If $\tau \in PTypes_f$ then $\mathcal{F}_\tau$ is an f^*-base.

Definition 10. *By a Σ-predicate of type $\tau \in Types_f$ on A we mean an arbitrary element of $I_\Sigma(\mathcal{F}_\tau)$.*

The propositions below easily follow from the definitions. Here $\Sigma(A)$ denotes the set of all Σ-definable subsets of $\mathbb{A}$.

Lemma 1. *For any $n > 0$ there is a natural bijective correspondence between Σ-predicates of type $\boldsymbol{o}^n \to B$ and n-ary Σ-predicates on A.*

Proposition 1. *A mapping $F : \Sigma(A) \to \Sigma(A)$ is a restriction of a Σ-operator if and only if F is continuous with respect to the strong topology and there is a Σ-function $f : A \to A$ such that $F(Q_{u,a}) = Q_{u,f(a)}$ for all $a \in A$.*

Proposition 2. *For a family $S \subseteq \Sigma(A)$ the following are equivalent:*

(1) *S is represented by a Σ-predicate of type $((\boldsymbol{o} \to B) \to B)$;*
(2) *there is a Σ-formula $\Phi(P^+)$ of signature $\sigma \cup \langle P^1\rangle$ such that*

$$S = \{Q \,|\, Q \in \Sigma(A), \langle \mathbb{A}, Q\rangle \models \Phi(P)\}.$$

Proposition 3. *There is a natural bijective correspondence between Σ-predicates of type $((\boldsymbol{o} \to B) \to (\boldsymbol{o} \to B))$ and unary Σ-operators.*

3 Σ-predicates of Finite Types in $\mathbb{HF}(\mathbb{R})$ and Intensional Logic

For arbitrary model $\mathbb{A}$ of KPU, the simplest example of spaces for entities and truth values are the f-base

$$\mathfrak{A} = \langle A, =, P_1(A), \cup \restriction P_1(A) \rangle$$

and the f^*-base

$$\mathfrak{B}_2 = \langle 2, \emptyset, \subseteq, \cup \restriction 2 \rangle.$$

Further on, we restrict ourselves to the case $\mathbb{A} = \mathbb{HF}(\mathbb{R})$, where $\mathbb{R}$ is the ordered field of real numbers. This choice is motivated by the fact that the reals are very natural to use for representing the scale of time and for coding. Let $W = P_1(\mathbb{R})$. We also fix the f-base

$$\mathfrak{W} = \langle W, =, P_1(W), \cup \restriction P_1(W) \rangle.$$

The set of ideals is defined to be the set of singletons from W, i.e. $P_1(P_1(\mathbb{R}))$.

Singletons from $\mathbb{HF}(\mathbb{R})$ (Σ-ideals of f-base $\mathfrak{A}$) correspond to basic entities, objects of type e. Real numbers (urelements) represent possible worlds. In particular, each possible world can be considered as a substructure of the whole structure with some partial information about the universe.

We give analogues of definitions 8 and 9.

Definition 11. *The set $Types$ together with its proper subset $PTypes$ are defined as follows:*

1) $e \in Types \setminus PTypes$, $t \in PTypes$;
2) *if* $a \in Types$ *and* $b \in PTypes$, *then* $(a \to b) \in PTypes$;
3) *if* $a \in PTypes$ *then* $(s \to a) \in PTypes$.

Definition 12. *For every $a \in Types$, the f-base $\mathcal{F}_a$ is defined as follows:*

1) $\mathcal{F}_e \leftrightharpoons \mathfrak{A}$, $\mathcal{F}_t \leftrightharpoons \mathfrak{B}_2$;
2) $\mathcal{F}_{(a\to b)} \leftrightharpoons F(\mathcal{F}_a, \mathcal{F}_b)$;
3) $\mathcal{F}_{(s\to a)} \leftrightharpoons F(\mathfrak{W}, \mathcal{F}_a)$.

For every type $a \in Types$, Σ-predicates of this type are introduced by Definition 10.

Definition 13. *The set of possible denotations D_τ of type $\tau \in Types$ is the set of all Σ-predicates of type τ.*

Note that $\mathfrak{A}$ is not precisely an f^*-base (there is no minimal element in $\mathfrak{A}$, but A^* can be considered as $Cons$). Nevertheless, we define a functional product

$$\mathcal{F}_{(s\to e)} \leftrightharpoons F(\mathfrak{W}, \mathcal{F}_e).$$

Since in $\mathbb{HF}(\mathbb{R})$ holds the uniformization property [12], for every Σ-ideal I of type $(s \to e)$ there is a corresponding (partial) Σ-function $f_I : P_1(\mathbb{R}) \to A$. Hence the following is true:

Lemma 2. *There is a natural bijective correspondence between Σ-predicates of type $(s \to e)$ and Σ-functions from $P_1(\mathbb{R})$ to A.*

As a corollary of Lemma 1 and Lemma 2 we get the following

Lemma 3. *There is a natural bijective correspondence between Σ-predicates of type $(s \to t)$ and unary Σ-predicates on $P_1(\mathbb{R})$.*

Proposition 4. *For a family $S \subseteq \Sigma(P_1(\mathbb{R}))$ the following are equivalent:*

(1) *S is represented by a Σ-predicate of type $((s \to t) \to t)$;*
(2) *there exists a Σ-formula $\Phi(P^+)$ of signature $\sigma \cup \langle P^1 \rangle$ such that*

$$S = \{Q \mid Q \in \Sigma(P_1(\mathbb{R})), \langle \mathbb{A}, Q\rangle \models \Phi(P^+)\}.$$

We obtain the correspondence for type $((s \to t) \to (e \to t))$ analogous to Proposition 3.

Lemma 4. *There is a natural bijective correspondence between Σ-predicates of type $((s \to t) \to (e \to t))$ and unary Σ-operators on A.*

Proof. If $I \in I_\Sigma(F(\mathfrak{W}, \mathfrak{B}_2))$, by Lemma 3 we get that I can be uniquely constructed from some $Q \in \Sigma(P_1(\mathbb{R}))$. We denote $I = I_Q$ and assume that elements from $I_\Sigma(F(\mathfrak{W}, \mathfrak{B}_2))$ are of the form I_Q for a suitable $Q \in \Sigma(P_1(\mathbb{R}))$. In the same way (by Lemma 1), elements from $I_\Sigma(F(\mathfrak{A}, \mathfrak{B}_2))$ are of the form I_Q for a suitable $Q \in \Sigma(A)$.

With any Σ-predicate I of type $((s \to t) \to (e \to t))$, a continuous mapping $f_I : I_\Sigma(F(\mathfrak{W}, \mathfrak{B}_2)) \to I_\Sigma(F(\mathfrak{A}, \mathfrak{B}_2))$ is naturally associated. Hence for f_I there is a unique mapping $F_I : \Sigma(P_1(\mathbb{R})) \to \Sigma(A)$ such that for any $Q \in \Sigma(P_1(\mathbb{R}))$ holds $f_I(I^{'}_Q) = I^{'}_{F_I(Q)}$ for $I^{'}_Q \in I_\Sigma(F(\mathfrak{W}, \mathfrak{B}_2))$ and for $I^{'}_{F_I(Q)} \in I_\Sigma(F(\mathfrak{A}, \mathfrak{B}_2))$.

Since F_I is continuous with respect to weak topology because of the continuity of f_I, it is enough to show that the set

$$\Gamma^*_{F_I} = \{\langle a, b\rangle \mid a \in A^*, b \in F_I(a)\}$$

is a Σ-subset.

For arbitrary $a \in A^*$, we have

$$b \in F_I(a) \Leftrightarrow \exists c(c \in I^{'}_{F_I(a)} \,\&\, \langle b, 1\rangle \in c),$$

where

$$c \in I^{'}_{F_I(a)} \Leftrightarrow \exists c^* \exists b_0 \exists b_1 (\forall x \in b_0 (\exists a_0 (x = \langle a_0, 0\rangle) \ \vee\ \exists a_1 (a_1 \in a \text{ and}$$
$$\text{and } x = \langle a_1, 1\rangle))) \text{ and } c \leq b_1 \text{ and } \langle b_0, b_1\rangle \in c^*).$$

Hence $\Gamma^*_{F_I}$ is a Σ-subset and F_I is a restriction of some Σ-operator.

On the other hand, if $F : P(A) \to P(A)$ is a Σ-operator then from the bijective correspondence between the ideals of f^*-base $F(F(\mathfrak{W}, \mathfrak{B}_2), F(\mathfrak{A}, \mathfrak{B}_2))$ and continuous mappings from $I_\Sigma(F(\mathfrak{W}, \mathfrak{B}_2))$ to $I_\Sigma(F(\mathfrak{A}, \mathfrak{B}_2))$, an operator F (more exactly, its restriction on $P_1(\mathbb{R})$) uniquely determines the ideal I_F such that for all $Q \in \Sigma(P_1(\mathbb{R}))$ holds $f_{I_F}(I^{'}_Q) = I^{'}_{F(Q)}$. □

Now we describe Σ-predicates of type $(s \to (e \to t))$. Let $\Phi(x, a)$ be a Σ-formula with parameters, and let $Q \subseteq A$. We associate with Φ and with set Q the family of subsets $S^Q_\Phi = \{\Phi^{\mathbb{A}}(x, a) \mid a \in Q\}$. The set Q is called the set of indices of S and denoted by $Ind(S)$. If possible, we will omit indices Q and Φ.

Lemma 5. *There is a natural bijective correspondence between Σ-predicates of type $(s \to (e \to t))$ and families $S^{Ind(S)}_\Phi$ of Σ-subsets defined by Σ-formulas with parameters, such that $Ind(S) \subseteq P_1(\mathbb{R})$ is a Σ-subset.*

Proof. Let I be a Σ-predicate of type $(s \to (e \to t))$, $I \subseteq (P_1(\mathbb{R}) \times (A \times 2)^*)^*$. Let

$$\Phi(x, a) \leftrightharpoons (\exists f \in I)(\exists f^{'} \in f)(p_l f^{'} = a \text{ and } \langle x, 1\rangle \in p_r f^{'}),$$
$$\Psi(x) \leftrightharpoons (\exists f \in I)(\exists f^{'} \in f)(p_l f^{'} = x \text{ and } \exists y(\langle y, 1\rangle \in p_r f^{'})).$$

We define $Q = \Psi^{\mathbb{A}}(x)$, $S_I = \{\Phi^{\mathbb{A}}(x, w) \mid w \in Q\}$. Then $S_I = S_\Phi^Q$ is a family of Σ-subsets which is defined by a Σ-formula $\Phi(x, a)$ with a parameter, which set of indices $Ind(S)(= Q) \subseteq P_1(R)$ is a Σ-subset.

Let $S_\Phi^{Ind(S)}$ be a family of Σ-subsets which is defined by a Σ-formula $\Phi(x, a)$ with a parameter, which set of indices $Ind(S) \subseteq P_1(\mathbb{R})$ is a Σ-subset. Let

$$I_S = (\{\langle w, \{\langle a, 0\rangle \mid a \in A\}^*\rangle \mid w \in P_1(\mathbb{R})\} \bigcup$$

$$\bigcup\{\langle w, (\{\langle a, 0\rangle \mid a \in A\} \cup \{\langle a, 1\rangle \mid a \in \Phi^{\mathbb{A}}(x, w)\})^*\rangle \mid w \in Ind(S)\})^*.$$

We prove that this is a Σ-ideal. It is clear that $I_S \subseteq (P_1(\mathbb{R}) \times (A \times 2)^*)^*$ and that condition 2 for ideals is satisfied. Check condition 1: let $f \in I_S$, $g \in (P_1(\mathbb{R}) \times (A \times 2)^*)^*$ and $g \leq f$. We need to show that in that case $g \in I_S$. For this, it is enough to show that

$$g \subseteq \{\langle w, \{\langle a, 0\rangle \mid a \in A\}^*\rangle \mid w \in P_1(\mathbb{R})\} \bigcup$$

$$\bigcup\{\langle w, (\{\langle a, 0\rangle \mid a \in A\} \cup \{\langle a, 1\rangle \mid a \in \Phi^{\mathbb{A}}(x, w)\})^*\rangle \mid w \in Ind(S)\}.$$

Let $w \notin Ind(S)$ and there is an a_0 such that $\langle w, a_0\rangle \in g$. Since $g \leq f$ (and by the definition of I_S), from $\langle a_1, \epsilon\rangle \in a_0$ it follows that $\epsilon = 0$. Hence

$$\langle w, a_0\rangle \in \{\langle w, \{\langle a, 0\rangle \mid a \in A\}^*\rangle \mid w \in P_1(\mathbb{R})\}.$$

Let $w \in Ind(S)$ and there is an a_0 such that $\langle w, a_0\rangle \in g$. Let $\langle a_1, \epsilon\rangle \in a_0$. If $\epsilon = 0$ then $\langle a_1, \epsilon\rangle \in \{\langle a, 0\rangle \mid a \in A\}$. On the other hand, if $\epsilon = 1$, then since $g \leq f$ there exists b such that $\langle w, b\rangle \in f$ and $\langle a_1, 1\rangle \in b$. By the definition of I_S we get that $\langle a_1, \epsilon\rangle \in \{\langle a, 1\rangle \mid a \in \Phi^{\mathbb{A}}(x, w)\}$. Hence

$$g \subseteq \{\langle w, \{\langle a, 0\rangle \mid a \in A\}^*\rangle \mid w \in P_1(\mathbb{R})\} \bigcup$$

$$\bigcup\{\langle w, (\{\langle a, 0\rangle \mid a \in A\} \cup \{\langle a, 1\rangle \mid a \in \Phi^{\mathbb{A}}(x, w)\})^*\rangle \mid w \in Ind(S)\}$$

and $g \in I_S$.

Using the correspondences described above we can get for any family $S_\Phi^{Ind(S)}$ of Σ-subsets that $S_\Phi^{Ind(S)} = S_{I_S}$. On reverse, for any Σ-predicate I of type $(s \to (e \to t))$ we get that $I = I_{S_I}$. □

Corollary 1. *There is a natural bijective correspondence between Σ-predicates of type $(e \to (e \to t))$ and families $S_\Phi^{Ind(S)}$ of Σ-subsets defined by Σ-formulas with parameters such that $Ind(S) \subseteq A$ is a Σ-subset.*

Remark: objects of type $(s \to (e \to t))$ and $(e \to (e \to t))$ can also be considered as binary Σ-predicates (in case of type $(s \to (e \to t))$ they are subsets of $P_1(\mathbb{R}) \times A$). This is equivalent to the correspondence described above and will be also convenient to use.

The set of all families described in Proposition 5 is denoted by $\Sigma^{'}(A)$.

Using the proof of Proposition 2 we describe Σ-predicates of type $((s \to (e \to t)) \to t)$: they correspond to definable families of families of Σ-subsets obtained above.

We use the following notation: if $S_\Phi^{Ind(S)} \in \Sigma^{'}(A)$ then

$$C_S = \{\langle w, a\rangle : w \in Ind(S), a \in (\Phi^{\mathbb{A}}(x, w))^*\}.$$

Lemma 6. *For a family $K \subseteq \Sigma^{'}(A)$ the following are equivalent:*

(1) *K corresponds to a Σ-predicate of type $((s \to (e \to t)) \to t)$;*

(2) *there is a Σ-formula $\Phi(P^+)$ of signature $\sigma \cup \langle P^1 \rangle$ such that*

$$K = \{S \mid S \in \Sigma^{'}(A), \langle \mathbb{A}, C_S \rangle \vDash \Phi(P^+)\}.$$

Proof. Let I be a Σ-predicate of type $((s \to (e \to t)) \to t)$, $I \subseteq ((P_1(\mathbb{R}) \times (A \times \times 2)^*)^* \times 2)^*$. We connect with I the family

$$K_I = \{S \mid S \subseteq \Sigma^{'}(A), f_I(I_S) = 2\}$$

(here I_S is the Σ-predicate from Proposition 5).

$(1 \Rightarrow 2)$. Let

$$K = \{S \mid S \subseteq \Sigma^{'}(A), \langle \mathbb{A}, C_S \rangle \vDash \Phi(P^+)\},$$

where

$$\Phi(P^+) \leftrightharpoons \exists x \exists y \exists z [x \in I \text{ and } \langle y, 1 \rangle \in x \text{ and } (\forall w \in p_l^* y)(\forall a \in p_r^* y)(\langle w, a \rangle \in y \to$$
$$\to \exists z^{'} \subseteq z (a = z^{'} \times \{1\} \text{ and } P(\langle w, z^{'} \rangle))))].$$

We prove that $K = K_I$.

$(\subseteq)$. If $S \in K$ then by the definition there are $x \in I$, $y \in (P_1(\mathbb{R}) \times (A \times 2)^*)^*$ and $z \in A$ such that $\langle y, 1 \rangle \in x$ and from $\langle w, a \rangle \in y$ follows that $a = z^{'} \times 1$ for some $z^{'} \subseteq z$. Besides, for all such pairs we have $P(\langle w, z^{'} \rangle)$, hence $y \in I_S$. By the definition of mapping f_I we get $1 \in f_I(I_S)$, and hence $S \in K_I$.

$(\supseteq)$ Let $S \in K_I$. Then $f_I(I_S) = 2$ and there are $c^* \in I$, $b_0 \in I_S$ such that $\langle b_0, 1 \rangle \in I$. Define the sets b^w and b as follows:

$$b^w = \{a \mid \exists a_0 (\langle w, a_0 \rangle \in b_0 \text{ and } \langle a, 1 \rangle \in a_0)\}$$

for $w \in p_l^* b_0$,

$$b = \{\{w\} \times (b^w \times \{1\}) \mid w \in p_l^* b_0 \text{ and } b^w \neq \emptyset\}.$$

Let us show that $\{\langle b, 1 \rangle\} \in I$. Since $\{\langle b_2, 1 \rangle\} \leq \{\langle b_2, 1 \rangle\} \cup c*$, it is enough to show that $\{\langle b_2, 1 \rangle\} \cup c^* \in I$. We show that $\{\langle b_2, 1 \rangle\} \cup c^* \leq c^*$. Since $\langle b_0, 1 \rangle \in c^*$, it is enough to find a pair $\langle b_1, 1 \rangle \in c^*$ such that $b_1 \leq b$ (the orders belong to corresponding f-bases). We can take $b_1 = b_0$: if for some a, a_0 it is true that $\langle w, a \rangle \in b_0$ and $\langle a_0, 1 \rangle \in a$, by the definition of $a_0 \in b^w$ there exists c such that $\langle w, c \rangle \in b$ and $\langle a_0, 1 \rangle \in c$. Hence $b_0 \leq b$ (and also $b \leq b_0$).

So, $\{\langle b, 1 \rangle\} \in I$. Now take

$$x = \{\langle b, 1 \rangle\},\ y = b,\ z = \bigcup_{w \in p_l^* b_0} b^w.$$

Then in $\langle \mathbb{A}, C_S \rangle$ it is true that

$$x \in I \text{ and } \langle y, 1 \rangle \in x \text{ and } (\forall w \in p_l^* y)(\forall a \in p_r^* y)(\langle w, a \rangle \in y \to$$
$$\to \exists z^{'} \subseteq z (a = z^{'} \times \{1\} \text{ and } P(\langle w, z^{'} \rangle)))).$$

Hence $S \in K$.

$(2 \Rightarrow 1)$. Let $K = \{S \mid S \subseteq \Sigma^{'}(A), \langle \mathbb{A}, C_S \rangle \vDash \Phi(P^+)\}$ for some Σ-formula $\Phi(P^+)$. Let $\Phi^*(x) = [\Phi(P^+)]_x^P$ (this means that all atomic subformulas of kind $P(y)$ are substituted by $y \in x$). Prove first that

$$K = \{S \mid S \subseteq \Sigma^{'}(A), \exists a (\Phi^*(a) \text{ and } a \subseteq C_S)\}.$$

Let $\Psi(a, P^+) = (\Phi(P^+) \ \& \ a = \emptyset)$. Consider the operator

$$\Gamma_\Psi(Q) = \{a \in A \mid \langle \mathbb{A}, S \rangle \vDash \Psi(a, P^+)\}.$$

TABLE 2. Intensional Logic Types and $\mathbb{HF}(\mathbb{R})$

Category	Grammar equivalent	Type	Object in $\mathbb{HF}(\mathbb{R})$
e	no	e	sets $\{a\}$ for $a \in \mathbb{HF}(\mathbb{R})$
t	sentenses	t	no
IV	intransitive verbs	(e $\to$ t)	unary Σ-predicates
CN	common nouns	(e $\to$ t)	unary Σ-predicates
TV	existential transitive verbs	(e $\to$ (e $\to$ t))	binary Σ-operators
CN/CN	existential adjectives	((e $\to$ t) $\to$ (e $\to$ t))	Σ-operators
CN/CN	existential adverbs	((e $\to$ t) $\to$ (e $\to$ t))	Σ-predicates
T	noun phrases and proper names	((s $\to$ (e $\to$ t)) $\to$ t)	Σ-definable families of binary Σ-predicates
t/t	sentence determiners	((s $\to$ t) $\to$ t)	Σ-definable families of Σ-predicates on $P_1(\mathbb{R})$
IV/t	connective verbs	((s $\to$ t) $\to$ (e $\to$ t))	Σ-operators

For the corresponding Σ-operator F_Ψ such that for every $Q \subseteq \Sigma(A)$ holds $\Gamma_\Psi(Q) = F_\Psi(Q)$, we have

$$F_\Psi(Q) = \{a \in A \,|\, \exists y(y \subseteq Q \text{ and } \Psi(a,y)\}.$$

Hence for F_Ψ we have $S \in K \Leftrightarrow F_\Psi(C_S) = \{\emptyset\}$, and so

$$K = \{S \,|\, S \subseteq \Sigma^{'}(A), \exists a(\Phi^*(a) \text{ and } a \subseteq C_S\}.$$

Now define the ideal I as follows:

$$I = \{c \in ((P_1(R) \times (A \times 2)^*)^* \times 2)^* \,|\, \exists a(\Phi^*(a) \wedge$$

$$\wedge c \leq \{\langle\{\langle w, b \times \{1\}\rangle \,|\, \langle w, b\rangle \in a\}, 1\rangle\})\}$$

(it is easy to check it is really an ideal). We show that $K = K_I$.

($\subseteq$). Let $S \in K$. Then there is an a such that $a \subseteq C_S$ and $\Phi^*(a)$. Let

$$c^* = \{\langle\{\langle w, b \times \{1\}\rangle \,|\, \langle w, b\rangle \in a\}, 1\rangle\}.$$

Then $c^* \in I$, but

$$b_0 = \{\langle w, b \times \{1\}\rangle \,|\, \langle w, b\rangle \in a\} \in I_S,$$

hence there are $c^* \in I$ and $b_0 \in I_S$ such that $\langle b_0, 1\rangle \in c^*$. Henceforth, $f_I(I_S) = 2$ and $S \in K_I$.

($\supseteq$). Let $S \in K_I$. Then $f_I(I_S) = 2$ and there are $c^* \in I$ and $b_0 \in I_S$ such that $\langle b_0, 1\rangle \in c^*$. Since $c^* \in I$, there is an a such that $\Phi^*(a)$ and

$$c^* \leq \{\langle\{\langle w, b \times \{1\}\rangle \,|\, \langle w, b\rangle \in a\}, 1\rangle\}(= B).$$

Since B is a singleton and $\langle b_0, 1\rangle \in c^*$, we have

$$\{\langle w, b \times \{1\}\rangle \,|\, \langle w, b\rangle \in a\} \leq b_0.$$

Hence

$$\{\langle w, b \times \{1\}\rangle \,|\, \langle w, b\rangle \in a\} \in I_S$$

and $a \subseteq C_S$, that means that $S \in K$. □

All these correspondences are summarized in Table 2.

3.1. Analysis of Natural Language Sentences. Consider some simple examples of English sentences by means of Σ-predicates of finite types. Truth value of any sentence is regarded relative to a possible world $w \in P_1(\mathbb{R})$. However, we consider the simplest absolute case first.

According to Table 1, the proper name **John** has the type $((s \to (e \to t)) \to t)$. But the semantics of the sentence

(1) **John walks**.

is the same in case we consider **John** as an object of type e and in case we consider it as an object of type $((s \to (e \to t)) \to t)$. From the grammar point of view the latter case is more appropriate: in sentences of kind "subject + predicate" the subject is regarded as a functor which receives a predicate as an argument. This is not so when **John** is considered as an object of type e.

Yet consider **John** as an object of type e (a set $\{j\}$ for some $j \in \mathbb{HF}(\mathbb{R})$), and (intransitive) verb **walk** as an object of type $(e \to t)$ (unary Σ-predicate **walk'**). The truth value of this sentence is equivalent to the truth value of Σ-formula

$$\{j\} \in \textbf{walk'}$$

in $\mathbb{HF}(\mathbb{R})$.

Consider now the sentence

(2) **John loves Mary**.

As in the previous case, names **John** and **Mary** are considered as objets of type e ($\{j\}$ and $\{m\}$ respectively). Transitive verb **love** is considered as an object of type $(e \to (e \to t))$, hence it is interpreted by some binary Σ-predicate **love'**. Hence the truth value of this sentence is equivalent to the truth value of Σ-formula

$$\langle \{j\}, \{m\} \rangle \in \textbf{love'}$$

in $\mathbb{HF}(\mathbb{R})$.

In addition, consider the quantified sentence

(3) **Every fish walks**.

The common noun **fish** is an object of type $(e \to t)$, that is, the set of all fishes, and is denoted by **fish'**. The quantifier can be regarded from the usual point of view in first-order logic: the truth value of sentence **Every fish walks** is equivalent to the truth value of formula

$$\forall x (x \in \textbf{fish'} \to x \in \textbf{walk'})$$

in $\mathbb{HF}(\mathbb{R})$. This, however, is not a Σ-formula. But from the point of the complexity, checking the truth of this formula is a finite search through the domain used by the model.

Now we relativize our model via the concept of possible worlds and consider the truth values of sentences (1) and (2) in a given possible world $w \in P_1(\mathbb{R})$. First we look at sentence (1). Previous interpretations for **John** and **walk** remain the same, but now for checking the truth value it becomes necessary to connect with objects j and **walk'** their intensions, i.e., objects of types $(s \to e)$ and$(s \to (e \to t))$. These are, correspondingly, a Σ-function $\hat{}j : P_1(\mathbb{R}) \to HF(\mathbb{R})$ and a binary Σ-predicate $\hat{}$**walk'** $\subseteq P_1(\mathbb{R}) \times (HF(\mathbb{R}) \cup \mathbb{R})$. The truth value of this sentence in world w is equivalent to the truth value of Σ-formula (with a parameter)

$$\langle w, \hat{}j(w) \rangle \in \hat{}\textbf{walk'}$$

in $\mathbb{HF}(\mathbb{R})$.

In the same way, the truth value of sentence (2) in possible world w is equivalent to the truth value of Σ-formula (with a parameter)

$$\langle w, \hat{}j(w), \hat{}m(w)\rangle \in \hat{}\mathbf{love'}$$

in $\mathbb{HF}(\mathbb{R})$.

4 Ontological Models

In models described above, the space D_t of truth values consists of elements 0 and 1 with natural ordering $(0 < 1)$. The intuition is that 0 means "the property is not true now but probably will become true in the future", while "1" means "the property true now and forever". One can consider a different type of models of intensional logic. The main difference is that these models use other spaces of entities and truth values. Let

$$D_t = \{0, 1, \bot, \top\}$$

and the ordering is defined as follows: $\bot < 0$, $\bot < 1$, $0 < \top$, $1 < \top$ while 0 and 1 are uncomparable. Intuitively, 0, 1 and $\bot$ correspond to *no, yes*, and *unknown*. The element $\top$ corresponds to inconsistency of data and is necessary for constructing f^*-spaces. We set

$$D_e = (\mathbb{R} \cup \{\bot\})^{<\omega}.$$

In particular, $(D_t)^{<\omega} \subseteq D_e$. Intuitively, every element from D_e is interpreted as a tuple (sequence) of properties of this element. Properties can be discrete (in our case, binary) and continuous, i.e. described by some measure. We assume that even positions of elements from D_e correspond to binary properties (the values in these positions belong to D_t and compared via the corresponding ordering), while odd positions correspond to continuous properties (the values in these positions belong to $\mathbb{R} \cup \{\bot\}$ and compared as follows: $\bot < a$ for all $a \in \mathbb{R}$, a and b are uncomparable for $a, b \in \mathbb{R}$, $a \neq b$).

The ordering on D_e is defined as follows: for $\alpha, \beta \in D_e$, $\alpha \leqslant \beta$ if and only if the length of the tuple α is less or equal to the length of the tuple β and $\alpha(i) \leqslant \beta(i)$ for all $i \leqslant lh(\alpha)$.

As in models described previously, it is possible to interprete by Σ-definable objects in $\mathbb{HF}(\mathbb{R})$ the corresponding objects (types) of intensional logic. For example, a common noun "man" (of type $(e \to t)$) can be interpreted by Σ-predicate $(\alpha(human) = 1) \wedge (\alpha(gender) = \text{"}M''\text{)}$ for some fixed in this model positions $human, gender \in \omega$. In the same way, an adjective "tall" (type $(e \to t) \to (e \to t)$) can be interpreted by Σ-operator H such that, for example, $\alpha \in H(\text{man}) \iff \alpha(height) \geqslant 180$, $\alpha \in H(\text{woman}) \iff \alpha(height) \geqslant 175$, $\alpha \in H(\text{chair}) \iff \alpha(height) \geqslant 120$ Again, $height \in \omega$ is some fixed position.

In our version of ontological models the scale of time is identified with the ordered set of real numbers: $T = \mathbb{R}$. The set of possible worlds is identified with the set of natural numbers: $W = \mathbb{N}$. Entities of world n are identified with the tuples of length $n+1$. The moment of time in which a given entity is described by the corresponding tuples is identified with the value of the leftmost position of these tuples.

The detailed description of this type of models can be given in the same way as before and was presented in [2].

5 Conclusion and Acknowledgements

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Artem Sergeevich Burnistov
Mines Paris, PSL University,
60 bd Saint-Michel,
75006, Paris, France
Email address: a.burnistov@alumni.nsu.ru

Alexey Ilyich Stukachev
Novosibirsk State University,
Pirogova str., 1,
630090, Novosibirsk, Russia
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
Email address: a.stukachev@g.nsu.ru, aistu@math.nsc.ru