Effective Model Theory: an Approach via Σ-Definability *

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Abstract. We present an approach to abstract computability based on the notion of Σ-definability, and survey some results on effective model theory which are obtained within this framework.

1 Introduction

We look at the computability over abstract structures via the formalism based on the notion of Σ-definability in admissible sets and, in particular, in hereditarily finite (HF) superstructures. HF-computability, as well as some other different (yet equivalent) approaches to computability over abstract structures provide a natural framework for considering "computability" ("effectiveness") relative to this structure. The survey of results on HF-computability in general can be found in [ErshovPuzarenkoStukachev2011]. Here we present some of the results focused (mostly) on effective model theory.

Considering the effectiveness of model-theoretical constructions, we first look at the notion of interpretation of one structure in another, which seems to be much older than model theory and probably as old as mathematics itself (for examples, one should think of numeral systems, geometries, fields, etc.). Classical computable model theory studies interpretations and presentations of countable structures in the standard model of arithmetics or, equivalently, in the least admissible set $\mathbb{HF}(\varnothing)$. So the notion of Σ-definability of a structure in HF-superstructure over another structure is, on the one hand, an effectivization of the notion of interpretability from model theory, and, on the other hand, a generalization of the notion of constructivizability from computable (or constructive) model theory. Moreover, the notion of Σ-degree of a structure turns out to be well related with the notion of degree (in the sense of Turing or enumeration reducibility) from classical computability theory, as well as with the notion of degree spectra of a (countable) structure. So, the notion of Σ-degree, having an

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advantage of being defined for structures of arbitrary cardinality, is a natural tool for measuring the (relative) complexity of a structure.

It turns out that many of important model-theoretical constructions are natural tools for studying effective model theory. We will mention, besides interpretations, some other typical examples, including Skolem expansions, Marker's extensions, Fraïssé limits, indiscernibles, etc.

Being concerned to our particular approach based on Σ-definability in HF-superstructures, we just mention some closely related approaches, such as search computability [Moschovakis1969b], Montague computability [Montague1967], and BSS-computability [BlumShubSmale1989,AshaevBelyaevMyasnikov1993]. It should be noted that the notion of search computability, as well as the notion of abstract computability in the sense of Montague, are equivalent (in accordance with [Gordon1970]) to the notion of HF-computability.

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2 Basic Definitions and Facts

By ω we denote the set of natural numbers. For arbitrary set M, we construct the collection $HF(M)$ of hereditarily finite sets over M as follows:

$HF_0(M) = \varnothing$;
$HF_{n+1}(M) = \mathcal{P}_\omega(M \cup HF_n(M))$, $n < \omega$
(here $\mathcal{P}_\omega(X)$ is the collection of all finite subsets of X);
$HF(M) = \bigcup\limits_{n<\omega} HF_n(M)$.

For simplicity, we consider structures of relational signatures only, identifying functions with their graphs. We consider at most countable signatures, usually finite but always computable. We assume that each signature is equipped with some fixed Gödel numbering of its (first-order) formulas.

If $\mathfrak{M}$ is a structure of some relation signature σ then one can define on $M \cup HF(M)$ a structure $\mathbb{HF}(\mathfrak{M})$ of signature $\sigma' = \sigma \cup \{U, \varnothing, \in\}$ ($U, \varnothing, \in$ are some symbols not in σ) so that

$U^{\mathbb{HF}(\mathfrak{M})} = M$;
$P^{\mathbb{HF}(\mathfrak{M})} = P^{\mathfrak{M}}$, $P \in \sigma$;
$\varnothing^{\mathbb{HF}(\mathfrak{M})} = \varnothing \in HF_0(M)$;
$\in^{\mathbb{HF}(\mathfrak{M})} = \in \cap ((M \cup HF(M)) \times HF(M))$.

We will also assume that the signature of $\mathbb{HF}(\mathfrak{M})$ contains a binary relational symbol Sat^2 interpreted as the satisfiability relation for the atomic formulas of $\mathfrak{M}$, with respect to the fixed Gödel numbering. In case of finite signatures this additional assumption is not essential.

A class of Δ_0-*formulas* of signature σ' is the least one containing atomic formulas which is closed under $\vee$, $\wedge$, $\to$, $\neg$ and restricted quantifiers $\forall x \in y$

and $\exists x \in y$ ($\forall x \in y\varphi$ and $\exists x \in y\varphi$ are abbreviations for $\forall x(x \in y \to \varphi)$ and $\exists x(x \in y \wedge \varphi)$ respectively).

A class of *Σ-formulas* of signature σ' is the least one containing Δ_0-formulas and closed under $\vee$, $\wedge$, restricted quantifiers $\forall x \in y$, $\exists x \in y$, and $\exists x$.

A *Σ_1-formula* is a formula of kind $\exists u\varphi_0$ where φ_0 is Δ_0-formula. It is known that any Σ-formula is equivalent in the theory KPU (see [Barwise1975]) to some Σ_1-formula.

A *Σ-predicate* is a relation definable by some Σ-formula (possibly with parameters). A *Δ-predicate* is a Σ-predicate whose complement is also Σ. A partial operation is called a *(partial) Σ-function* if its graph is Σ.

There is a useful result which presents the relationship between Σ-definability and definability by infinitary computable formulas.

Theorem 1 ([Vaicenavichyus1989]). *A predicate $P \subseteq M^n$ is Σ-definable in $\mathbb{HF}(\mathfrak{M})$ iff there is a computable family $\varphi_s(\bar{x}, \bar{y})$, $s \in \omega$ ($\bar{x} = (x_0, \ldots, x_k)$, $\bar{y} = (y_0, \ldots, y_{n-1})$), of $\exists$-formulas and a k-tuple $\bar{a} \in M^k$ such that, for any $\bar{b} \in M^n$,*

$$\bar{b} \in P \iff \mathfrak{M} \models \bigvee_{s\in\omega} \varphi_s(\bar{a}, \bar{b})$$

It is usually convenient to use in constructions not only the elements from $\mathbb{HF}(\varnothing) \subseteq \mathbb{HF}(\mathfrak{M})$, but also the elements from $\mathbb{HF}(\mathcal{N})$, where $\mathcal{N}$ is isomorphic to the standard model of arithmetic. To avoid confusion with ordinals from $\mathbb{HF}(\varnothing)$, we denote the domain of $\mathcal{N}$ and its elements as $\underline{\omega}$ and $\underline{n}$, $n \in \omega$, respectively. Since $\mathbb{HF}(\mathcal{N})$ is constructivizable, it can be effectively defined in any hereditarily finite superstructure.

In what follows, we use definitions and constructions from [Ershov1996]. For all $n \in \omega$, $\varkappa \in \mathrm{HF}(\underline{n})$ $\big(\underline{n} = \{\underline{0}, \underline{1}, \ldots, \underline{n-1}\}\big)$, and $\bar{x} \in M^n$, we define an element $\varkappa(\bar{x}) \in \mathrm{HF}(\mathfrak{M})$ as follows. Define a mapping $\lambda_{\bar{x}} : n \to M$ as $\lambda_{\bar{x}}(i) = x_i$, where $\bar{x} = \langle x_0, \ldots, x_{n-1}\rangle$. The mapping $\lambda_{\bar{x}}$ can be uniquely extended to $\lambda^{\omega}_{\bar{x}} : \mathrm{HF}(\underline{n}) \to \mathrm{HF}(\mathfrak{M})$ so that $\lambda^{\omega}_{\bar{x}}(a_0, \ldots, a_k) \rightleftharpoons \big\{\lambda^{\omega}_{\bar{x}}(a_0), \ldots, \lambda^{\omega}_{\bar{x}}(a_k)\big\}$ for each set $\{a_0, \ldots, a_k\} \in \mathrm{HF}(\underline{n})$. Then we put $\varkappa(\bar{x}) \rightleftharpoons \lambda^{\omega}_{\bar{x}}(\varkappa)$.

For every $\varkappa \in \mathrm{HF}(\underline{n})$, we can effectively define a term $t_{\varkappa}(x_0, \ldots, x_{n-1})$ of signature $\langle \{\}, \cup, \varnothing\rangle$ so that, for all elements $x^0_0, \ldots, x^0_{n-1} \in M$, the equality $t_{\varkappa}(x^0_0, \ldots, x^0_{n-1}) = \varkappa(\bar{x}^0)$ is valid.

Hereditarily finite superstructures are the "simplest" admissible sets, from the set-theoretical point of view. Besides of this, Σ-definability in hereditarily finite superstructures is one of the natural approaches for generalizing classical computability theory on natural numbers to the case of computability over arbitrary structures. For the results on computability on admissible sets in general, we refer the reader to [Barwise1975] and [Ershov1996].

3 Semilattices of Σ-Degrees of Structures

Theory of constructive (computable) models is one of the important research areas of classical computability theory, as well as of model theory. Because of the evident cardinality limitations, in classical computable model theory only countable structures are considered. The approach regarding generalized computability as Σ-definability in admissible sets allows to consider structures with arbitrary cardinality.

Hence, for a structure $\mathfrak{M}$ the following problem naturally arise: to describe

– the structures Σ-definable in $\mathbb{HF}(\mathfrak{M})$;
– the structures such that $\mathfrak{M}$ is Σ-definable in their HF-superstructures.

Let us formalize the problems stated above. Let $\mathfrak{M}$ be a structure of a finite predicate signature $\langle P_1^{n_1}, \ldots, P_k^{n_k}\rangle$ and let $\mathbb{A}$ be an admissible set. The following notion is an effectivization of the model-theoretical notion of interpretability of one structure in another, and also a natural generalization of the notion of constructivizability of a (countable) structure on natural numbers.

Definition 1 ([Ershov1985]). $\mathfrak{M}$ *is* Σ-definable in $\mathbb{A}$ *if there exist Σ-formulas*

$$\Phi(x_0, y), \Psi(x_0, x_1, y), \Psi^*(x_0, x_1, y), \Phi_1(x_0, \ldots, x_{n_1-1}, y),$$

$$\Phi_1^*(x_0, \ldots, x_{n_1-1}, y), \ldots, \Phi_k(x_0, \ldots, x_{n_k-1}, y), \Phi_k^*(x_0, \ldots, x_{n_k-1}, y),$$

such that for some parameter $a \in A$, and letting

$$M_0 \leftrightharpoons \Phi^{\mathbb{A}}(x_0, a), \quad \eta \leftrightharpoons \Psi^{\mathbb{A}}(x_0, x_1, a) \cap M_0^2$$

one has that $M_0 \neq \varnothing$ and η is a congruence relation on the structure

$$\mathfrak{M}_0 \leftrightharpoons \langle M_0, P_1^{\mathfrak{M}_0}, \ldots, P_k^{\mathfrak{M}_0}\rangle,$$

where $P_i^{\mathfrak{M}_0} \leftrightharpoons \Phi_i^{\mathbb{A}}(x_0, \ldots, x_{n_i-1}) \cap M_0^{n_i}$ for all $1 \leqslant i < k$,

$$\Psi^{*\mathbb{A}}(x_0, x_1, a) \cap M_0^2 = M_0^2 \setminus \Psi^{\mathbb{A}}(x_0, x_1, a),$$

$$\Phi_i^{*\mathbb{A}}(x_0, \ldots, x_{n_i-1}, a) \cap M_0^{n_i} = M_0^{n_i} \setminus \Phi_i^{\mathbb{A}}(x_0, \ldots, x_{n_i-1})$$

for all $1 \leqslant i < k$, and the structure $\mathfrak{M}$ is isomorphic to the quotient structure $\mathfrak{M}_0/\eta$.

Remark 1. This definition can be naturally generalized to the case of structures with infinite computable signatures. Namely, a structure $\mathfrak{M}$ with a computable predicate signature $\langle P_0^{n_0}, P_1^{n_1}, \ldots\rangle$ is called *Σ-definable in* $\mathbb{A}$ if there exists a computable sequence $\Phi(x_0, y)$, $\Psi(x_0, x_1, y)$, $\Psi^*(x_0, x_1, y)$, $\Phi_0(x_0, \ldots, x_{n_0-1}, y)$, $\Phi_0^*(x_0, \ldots, x_{n_0-1}, y)$, $\ldots$, $\Phi_k(x_0, \ldots, x_{n_k-1}, y)$, $\Phi_k^*(x_0, \ldots, x_{n_k-1}, y)$, $\ldots$ of Σ-formulas and a parameter $a \in A$, which forms a Σ-definition of $\mathfrak{M}$ in $\mathbb{A}$, in the sense of Definition 1.

Also, for a structure with an infinite computable signature, we assume that some Gödel numbering of formulas of this signature is fixed. Σ-reducibility $\leqslant_\Sigma$ is defined as follows: for structures $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \leqslant_\Sigma \mathfrak{B}$ if $\mathfrak{A}$ is Σ-definable in $\mathbb{HF}(\mathfrak{B})$. We assume that the signature of $\mathbb{HF}(\mathfrak{B})$ contains a predicate symbol Sat^2 interpreted by the satisfiability relation for atomic formulas in $\mathfrak{B}$, with respect to a fixed Gödel numbering. In the case of structures with a finite signature this assumption is not essential.

Since, for the sets of natural numbers, Σ-definability in $\mathbb{HF}(\varnothing)$ is equivalent to classical computability, we get the following fact.

Proposition 1. *Let $\mathfrak{M}$ be a countable structure. The following are equivalent:*

1) $\mathfrak{M}$ is constructivizable;
2) $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\varnothing)$.

For structures $\mathfrak{M}$ and $\mathfrak{N}$, we denote by $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ the fact that $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$. From the definition it follows that the relation $\leqslant_\Sigma$ is reflexive and transitive. We now look at the general properties of this relation, regarding it as a kind of effective reducibility on structures.

For any infinite cardinal α, we denote by $\mathcal{K}_\alpha$ the class of structures with cardinality less or equal α.

As usual, preordering $\leqslant_\Sigma$ generates on $\mathcal{K}_\alpha$ a relation of Σ-equivalence: $\mathfrak{A} \equiv_\Sigma \mathfrak{B}$ if $\mathfrak{A} \leqslant_\Sigma \mathfrak{B}$ and $\mathfrak{B} \leqslant_\Sigma \mathfrak{A}$. Classes of Σ-equivalence are called *degrees of Σ-definability*, or *Σ-degrees.* The poset

$$\mathcal{S}_\Sigma(\alpha) = \langle \mathcal{K}_\alpha / \equiv_\Sigma, \leqslant_\Sigma \rangle$$

is an upper semilattice with the least element, which is the degree consisting of computable structures. We denote the Σ-degree of a structure $\mathfrak{A}$ by $[\mathfrak{A}]_\Sigma$. The notion of Σ-degree of a structure is invariant from the choice of a semilattice $\mathcal{S}_\Sigma(\alpha)$, because all infinite structures of the same Σ-degree have the same cardinality. For any structures $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}_\alpha$, $[\mathfrak{A}]_\Sigma \vee [\mathfrak{B}]_\Sigma = [(\mathfrak{A}, \mathfrak{B})]_\Sigma$, where $(\mathfrak{A}, \mathfrak{B})$ is the pair of $\mathfrak{A}$ and $\mathfrak{B}$ in model-theoretical sense.

For a structure $\mathfrak{A} \in \mathcal{K}_\alpha$ and infinite cardinals $\beta \leqslant \alpha$, $\gamma \geqslant \alpha$, the sets

$$I_\beta(\mathfrak{A}) = \{[\mathfrak{B}]_\Sigma \mid \mathfrak{B} \in \mathcal{K}_\beta,\ \mathfrak{B} \leqslant_\Sigma \mathfrak{A}\}, \quad F_\gamma(\mathfrak{A}) = \{[\mathfrak{B}]_\Sigma \mid \mathfrak{B} \in \mathcal{K}_\gamma,\ \mathfrak{A} \leqslant_\Sigma \mathfrak{B}\}$$

are, correspondingly, an ideal in $\mathcal{S}_\Sigma(\beta)$ (principal for $\beta = \alpha$) and a filter in $\mathcal{S}_\Sigma(\gamma)$ (principal for any $\gamma \geqslant \alpha$). The sets $F_\gamma(\mathfrak{A})$ in semilattices $\mathcal{S}_\Sigma(\gamma)$ are natural analogues of the *spectrum* of a structure $\mathfrak{A}$. The sets $I_\beta(\mathfrak{A})$ in semilattices $\mathcal{S}_\Sigma(\beta)$ consist of Σ-degrees of structures Σ-presentable over $\mathfrak{A}$.

A *presentation* of a structure $\mathfrak{M}$ in an admissible set $\mathbb{A}$ is any structure $\mathcal{C}$ which is isomorphic to $\mathfrak{M}$ and whose domain C is a subset of A (relation $=$ is treated as a congruence relation on $\mathcal{C}$, and it may differ from the standard equality relation on $\mathcal{C}$). In what follows, we will identify the presentation $\mathcal{C}$ (more precisely, its atomic diagram) with some subset of A, fixing a Gödel numbering of atomic formulas of the signature $\sigma_\mathfrak{M}$.

Definition 2. A *problem of presentability* of a structure $\mathfrak{M}$ in $\mathbb{A}$ is the set $\mathrm{Pr}_\mathbb{A}(\mathfrak{M})$ consisting of all possible presentations of $\mathfrak{M}$ in $\mathbb{A}$.

Denote by $\underline{\mathfrak{M}}$ the set $\mathrm{Pr}(\mathfrak{M}, \mathbb{HF}(\varnothing))$ of presentations of $\mathfrak{M}$ in the least admissible set.

There exist natural embeddings of the semilattice $\mathcal{D}$ of Turing degrees and the semilattice $\mathcal{D}_e$ of degrees of enumerability of sets of natural numbers into semilattice $\mathcal{S}_\Sigma(\omega)$ (and hence into any semilattice $\mathcal{S}_\Sigma(\alpha)$) via the mappings $i : \mathcal{D} \to \mathcal{S}_\Sigma(\omega)$ and $j : \mathcal{D}_e \to \mathcal{S}_\Sigma(\omega)$ defined below. These definitions show that the notion of Σ-degree of a structure, which is total, i.e. defined for any structure, no matter countable or not, is a natural generalization of the (partial) notion of a degree of a countable structure, introduced in [Richter1981]. Hence, we get that semilattices $\mathcal{S}_\Sigma(\alpha)$ extend in a natural way semilattices $\mathcal{D}$ and $\mathcal{D}_e$.

Definition 3 ([Richter1981,Stukachev2007]). *Let $\mathfrak{M}$ be a countable structure. We say that $\mathfrak{M}$ has a degree (e-degree) if there exists the least degree in the set of T-degrees (e-degrees) of all possible presentations of $\mathfrak{M}$ on natural numbers.*

Using the equivalence of "∀-recursiveness" and "∃-definability", in the sense of [Lacombe1964], [Moschovakis1969a], see also [AshKnightManasseSlaman1989], [Chisholm1990], we get following result.

Theorem 2 ([Stukachev2007]). *For a countable structure $\mathfrak{M}$, the following are equivalent:*

1) $\mathfrak{M}$ has a degree (e-degree);
2) there exists a presentation $\mathcal{C} \in \underline{\mathfrak{M}}$ which is a Δ-subset (Σ-subset) of $\mathbb{HF}(\mathfrak{M})$.

We define mappings $i : \mathcal{D} \to \mathcal{S}_\Sigma(\omega)$ and $j : \mathcal{D}_e \to \mathcal{S}_\Sigma(\omega)$ in the following way: for every degree $\mathbf{a} \in \mathcal{D}$, put

$$i(\mathbf{a}) = [\mathfrak{M}_\mathbf{a}]_\Sigma, \text{ where } \mathfrak{M}_\mathbf{a} \text{ is any structure having degree } \mathbf{a}.$$

Similarly, for every e-degree $\mathbf{b} \in \mathcal{D}_e$, put

$$j(\mathbf{b}) = [\mathfrak{M}_\mathbf{b}]_\Sigma, \text{ where } \mathfrak{M}_\mathbf{b} \text{ is any structure having } e\text{-degree } \mathbf{b}.$$

Lemma 1. *The mappings i and j are well-defined: for any (e-)degree $\mathbf{a}$ there are structures having (e-)degree $\mathbf{a}$. Moreover, for any countable structures $\mathfrak{M}$ and $\mathfrak{N}$,*
1) if $\mathfrak{M}$ has (e-)degree $\mathbf{a}$ and $\mathfrak{M} \equiv_\Sigma \mathfrak{N}$, then $\mathfrak{N}$ also has (e-)degree $\mathbf{a}$;
2) if $\mathfrak{M}$ and $\mathfrak{N}$ have the same (e-)degree then $\mathfrak{M} \equiv_\Sigma \mathfrak{N}$.

(Notice, however, that the property of having a (e-)degree is not closed downwards w.r.t. $\leqslant_\Sigma$.) Usually, we just write $\mathbf{a}$ instead of $i(\mathbf{a})$.

For a structure $\mathfrak{A}$, a *Σ-jump* of $\mathfrak{A}$ is the structure

$$\mathfrak{A}' = (\mathbb{HF}(\mathfrak{A}), \Sigma\text{-Sat}^{\mathbb{HF}(\mathfrak{A})}),$$

where Σ-Sat$^{\mathbb{HF}(\mathfrak{A})}$ denotes the satisfiability relation for the set of Σ-formulas in $\mathbb{HF}(\mathfrak{A})$. The definition of Σ-jump is persistent w.r.t. the Σ-equivalence: for any structures $\mathfrak{A}$ and $\mathfrak{B}$, from $\mathfrak{A} \equiv_\Sigma \mathfrak{B}$ it follows that $\mathfrak{A}' \equiv_\Sigma \mathfrak{B}'$. Hence, we may assume that semilattices $\mathcal{S}_\Sigma(\alpha)$ are equipped with the jump operation.

Remark 2. In a similar way the jump operation was introduced in [Baleva2006] for the semilattice of s-degrees of countable structures. Also, in the same way a notion of the jump of an admissible set with respect to various effective reducibilities was introduced in [Morozov2004,Puzarenko2009].

The operation of Σ-jump agrees with the jump operations for Turing and enumeration degrees w.r.t. the natural embeddings i and j: if a structure $\mathfrak{A}$ has a (e-)degree $\mathbf{a}$, then the structure $\mathfrak{A}'$ has (e-)degree $\mathbf{a}'$.

Proposition 2. *The mappings $i : \mathcal{D} \to \mathcal{S}_\Sigma$ and $j : \mathcal{D}_e \to \mathcal{S}_\Sigma$ are embeddings preserving* 0, $\vee$ *and the jump operation.*

The existence of an embedding of $\mathcal{D}$ in $\mathcal{S}_\Sigma$ was first noted in [Khisamiev2004a].

The jump inversion theorem from the classical computability theory can also be generalized to the case of the semilattices of Σ-degrees of structures.

Theorem 3 ([Stukachev2009,Stukachev2010]). *Let $\mathfrak{A}$ be a structure such that* $\mathbf{0}' \leqslant_\Sigma \mathfrak{A}$. *Then there exists a structure $\mathfrak{B}$ such that*

$$\mathfrak{B}' \equiv_\Sigma \mathfrak{A}.$$

The proof uses Marker's extensions in the similar form as proposed in [GoncharovKhoussainov2004] and used in [Soskova2007,SoskovSoskova2009]. It should be noted that relation of Σ-reducibility, being defined on structures of arbitrary cardinality, in the case of countable structures can be viewed as the strongest reducibility in the hierarchy of effective reducibilities on structures [Stukachev2007], [Stukachev2008] (see Section 5). One of the weak reducibilities in this hierarchy is the Muchnik reducibility. In [Soskova2007,SoskovSoskova2009], the jump inversion theorem is proved for the semilattices of degrees of presentability of countable structures with respect to the Muchnik reducibility. As a corollary of Theorem 3, we get the jump inversion theorem for all known effective reducibilities on countable structures.

It follows from the definition of the Σ-jump that $\mathfrak{A} \vee \mathbf{0}' \leqslant_{\boldsymbol{\Sigma}} \mathfrak{A}'$ for any structure $\mathfrak{A}$. We say that a structure $\mathfrak{A}$ is *generalized low* with respect to the Σ-jump (in short, *generalized Σ-low*) if

$$\mathfrak{A}' \equiv_\Sigma \mathfrak{A} \vee \mathbf{0}'.$$

It turns out that the class of structures with a *c-simple* theory (i.e., model complete, ω-categorical, decidable, and with a decidable set of atoms) constitutes a series of examples (in arbitrary cardinality) of generalized Σ-low structures.

Theorem 4 ([StukachevTA]). *For a structure $\mathfrak{A}$, if* $\mathrm{Th}(\mathfrak{A})$ *is c-simple, then* $\mathfrak{A}' \equiv_\Sigma \mathfrak{A} \vee \mathbf{0}'$.

So, a jump of a structure $\mathfrak{A}$ in the sense of [Montalban2009] is any expansion $(\mathfrak{A}, P_0, P_1, \ldots)$, where $P_0, P_1, \ldots$ is a complete set of Π^c_1-relations on $\mathfrak{A}$ As an example, one can always choose an expansion

$$\mathfrak{A}^* = (\mathfrak{A}, P_1, P_2, \ldots),$$

where $P_1, P_2, \ldots$ is the list of all relations on $\mathfrak{A}$ which are Π-definable in $\mathbb{HF}(\mathfrak{A})$, corresponding to some computable numbering $\Phi_1, \Phi_2, \ldots$ of all Π-formulas of signature $\sigma'_{\mathfrak{A}}$. As follows from the results on the equivalence of computable infinitary formulas and Π-formulas [Vaicenavichyus1989], $\mathfrak{A}^*$ is a jump of this structure in the sense of [Montalban2009]. However, for a given structure there can be many different jumps in this sense: for example, any dense linear order is a jump of itself, see [Montalban2009]. We propose to call by a *weak jump* of a structure any jump of that structure in the sense of [Montalban2009]. The terminology we use is justified by the relationship between a weak jump of a structure $\mathfrak{A}$ and the Σ-jump of $\mathfrak{A}$. Exactly, there is

Proposition 3. *For any structure $\mathfrak{A}$ and any of its weak jumps, $\mathfrak{A}^*$,*

$$\mathfrak{A}' \equiv_\Sigma \mathfrak{A}^* \vee \mathbf{0}'.$$

Proof. Reducibility $\mathfrak{A}^* \vee \mathbf{0}' \leqslant_{\boldsymbol{\Sigma}} \mathfrak{A}'$ is evident from the the equivalence of computable infinitary formulas and Π-formulas [Vaicenavichyus1989]. To prove the reverse reducibility, we use notations from [Ershov1996]. Any Π-subset $P \subseteq \mathbb{HF}(\mathfrak{A})$ can be represented in the form $P = \cup_{\varkappa \in \mathrm{HF}(\omega)} P_\varkappa$, where, for any $\varkappa \in \mathrm{HF}(\omega)$, $P_\varkappa = \{\varkappa(\overline{a}) \mid \mathfrak{A} \models \Phi_\varkappa(\overline{a})\}$, $\Phi_\varkappa$ is a computable conjunction of $\forall$-formulas of signature σ (with parameters $\bar{c} \in A^{<\omega}$), and $\{\Phi_\varkappa \mid \varkappa \in \mathrm{HF}(\omega)\}$ is a computable family.

Consider an element $\varkappa(\overline{a}) \in \mathbb{HF}(\mathfrak{A})$, where $\varkappa \in \mathrm{HF}(\omega)$ and $\overline{a} \in A^{<\omega}$ are urelements. We have $\varkappa(\bar{a}) \in P$ if and only if $\mathfrak{A} \models \wedge_{n\in\omega} \forall \bar{y}_n \varphi_{\varkappa,n}(\bar{c}, \bar{y}_n, \bar{a})$. Since in $\mathbb{HF}(\mathfrak{A}^*)$ one can effectively (with $\mathbf{0}'$) find a Σ-formula equivalent to the Π^c_1-formula described above, we get that the relation $\Sigma\text{-Sat}^{\mathbb{HF}(\mathfrak{A})}$ is Δ-definable in $\mathbb{HF}(\mathfrak{A}^*)$ with $\mathbf{0}'$.

As it was already mentioned, the original definition from [Montalban2009] is not quite precise, but it is precise in the sense that any two jumps of a given structure are Σ-equivalent above $\mathbf{0}'$. Namely, exactly from the definition follows

Corollary 1. *If $\mathfrak{B}_1, \mathfrak{B}_2$ are the weak jumps of a structure $\mathfrak{A}$ then*

$$\mathfrak{B}_1 \vee \mathbf{0}' \equiv_\Sigma \mathfrak{B}_2 \vee \mathbf{0}'.$$

We mention the results on "Σ-universality" (i.e., universality w.r.t. the Σ-equivalence) of the two classes of posets — graphs and lattices.

Proposition 4 ([StukachevTA]). *For any structure $\mathfrak{A}$, there is an irreflexive graph $G_\mathfrak{A}$ such that $\mathfrak{A} \equiv_\Sigma G_\mathfrak{A}$.*

Proposition 5 ([StukachevTA]). *For any structure $\mathfrak{A}$, there is a lattice $L_\mathfrak{A}$ such that $\mathfrak{A} \equiv_\Sigma L_\mathfrak{A}$.*

It should be noted that the class of linear orders is not Σ-universal (see the results on local constructivizability below).

4 Effective Self-Presentations of Admissible Sets

In Theorem 2, we have already seen examples of non-trivial effective self-presentations of admissible sets of kind $\mathbb{HF}(\mathfrak{M})$, there $\mathfrak{M}$ is a countable structure. Namely, $\mathbb{HF}(\mathfrak{M})$ has a 'pure' copy (i.e., a copy which is a subset of $\mathbb{HF}(\varnothing)$) as a Δ-subset (resp., Σ-subset) of $\mathbb{HF}(\mathfrak{M})$ if and only if $\mathfrak{M}$ have a degree (resp., an e-degree). Now, we consider effective self-presentations of admissible sets which are, in some sense, as close to the ground structure as possible.

In some cases, for a structures $\mathfrak{A}$ and $\mathfrak{B}$ one can say more than just state the fact that $\mathfrak{A} \leqslant_\Sigma \mathfrak{B}$. For example, it is obvious that $\mathbb{HF}(\mathfrak{A}) \leqslant_\Sigma \mathfrak{A}$ for any $\mathfrak{A}$, but, in case of the standard model of arithmetics $\mathbb{N}$, much stronger result is true: $\mathbb{HF}(\mathbb{N})$ is Σ-definable within $\mathbb{N}$, not using the elements of the superstructure.

In particular, a natural additional restriction on Σ-definability of structures in admissible sets is the restriction on the rank of elements used in this process. To describe the situation formally, we now give some definitions.

Fix some signature σ, and let P be an unary predicate symbol not in σ. For any formula Φ of the signature $\sigma \cup \{\in\}$, with the bounded quantifiers of the form $\forall x \in t$ and $\exists x \in t$, we define by induction the *relativization* Φ^P of Φ by P:
— if Φ is an atomic formula, put $\Phi^P = \Phi$;
— if $\Phi = (\Phi_1 * \Phi_2)$, $* \in \{\wedge, \vee, \rightarrow\}$, put $\Phi^P = (\Phi_1^P * \Phi_2^P)$;
— if $\Phi = \neg\Psi$, put $\Phi^P = \neg\Phi^P$;
— if $\Phi = (Qx \in y)\Psi$, $Q \in \{\forall, \exists\}$, put $\Psi^P = (Qx \in y)\Psi^P$;
— if $\Phi = \exists x\Psi$, put $\Phi^P = \exists x(P(x) \wedge \Psi^P)$;
— if $\Phi = \forall x\Psi$, put $\Phi^P = \forall x(P(x) \rightarrow \Psi^P)$.

Let now $\mathbb{A}$ be an admissible set, $B \subseteq A$ be some transitive subset of $\mathbb{A}$, and $\Phi(x_0, \ldots, x_{n-1})$ be a formula of the signature $\sigma_{\mathbb{A}}$. Define the set

$$(\Phi(x_0, \ldots, x_{n-1}))^B = \{\langle a_0, \ldots, a_{n-1}\rangle \in A^n \mid \langle \mathbb{A}, B\rangle \models \Phi^P(a_0, \ldots, a_{n-1})\}.$$

Definition 4 ([Stukachev2005]). *Let $\mathbb{A}$ be an admissible set, $B \subseteq A$ be some transitive subset of $\mathbb{A}$. A structure of a computable predicate signature $\langle P_0^{n_0}, P_1^{n_1}, \ldots\rangle$ is called* Σ-definable in $\mathbb{A}$ inside B *if there exist a computable sequence*

$$\Phi(x_0, y), \Psi(x_0, x_1, y), \Psi^*(x_0, x_1, y), \Phi_0(x_0, \ldots, x_{n_0-1}, y),$$

$$\Phi_0^*(x_0, \ldots, x_{n_0-1}, y), \ldots, \Phi_k(x_0, \ldots, x_{n_k-1}, y), \Phi_k^*(x_0, \ldots, x_{n_k-1}, y), \ldots$$

of Σ-formulas of $\sigma_{\mathbb{A}}$, and a parameter $b \in B$, such that, for the sets

$$M_0 \leftrightharpoons \Phi^B(x_0, b),\ M_0 \subseteq B,\ \ \eta \leftrightharpoons \Psi^B(x_0, x_1, b) \cap M_0^2,$$

the following holds: $M_0 \neq \varnothing$, η is a congruence relation on the structure

$$\mathfrak{M}_0 \leftrightharpoons \langle M_0, P_0^{\mathfrak{M}_0}, \ldots, P_k^{\mathfrak{M}_0}, \ldots\rangle,$$

where $P_k^{\mathfrak{M}_0} \leftrightharpoons (\Phi_k(x_0, \ldots, x_{n_k-1}))^B \cap M_0^{n_k},\ k \in \omega$,

$$(\Psi^*(x_0, x_1, a))^B \cap M_0^2 = M_0^2 \setminus (\Psi(x_0, x_1, a))^B,$$

$$(\Phi_k^*(x_0,\dots,x_{n_k-1},a))^B \cap M_0^{n_k} = M_0^{n_k} \setminus (\Phi_k(x_0,\dots,x_{n_k-1}))^B$$

for any $k \in \omega$, and the quotient structure $\mathfrak{M}$ is isomorphic to $\mathfrak{M}_0 / \eta$.

For an admissible set $\mathbb{A}$ and a subset $B \subseteq A$, define the ordinal $\mathrm{rnk}(B)$ as follows:

$$\mathrm{rnk}(B) = \sup\{\mathrm{rnk}(b) | b \in B\},$$

where $\mathrm{rnk}(b)$ is the rank of the set b in $\mathbb{A}$ [Barwise1975].

Definition 5 ([Stukachev2005]). *The rank of inner constructivizability of an admissible set $\mathbb{A}$ is the ordinal*

$$\mathrm{cr}(\mathbb{A}) = \inf\{\mathrm{rnk}(B) \mid \mathbb{A} \textit{ is } \Sigma\textit{-definable in } \mathbb{A} \textit{ inside } B\}.$$

The next theorem gives the precise estimate for the rank of inner constructivizability of hereditarily finite superstructures. It can be viewed as an effective analogue of some results from [Montague1967] on definability in higher order languages.

Theorem 5 ([Stukachev2005]). *Let $\mathfrak{M}$ be a structure of a computable signature.*
1) If $\mathfrak{M}$ is finite then $\mathrm{cr}(\mathbb{HF}(\mathfrak{M})) = \omega$.
2) If $\mathfrak{M}$ is infinite then $\mathrm{cr}(\mathbb{HF}(\mathfrak{M})) \leqslant 2$.

As a corollary of Theorem 5 we get the following. For a structures $\mathfrak{M}$, $\mathfrak{N}$, and a natural number $n \in \omega$, we denote by $\mathfrak{M} \leqslant_\Sigma^n \mathfrak{N}$ the fact that $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$ inside the subset consisting of all elements with the rank less or equal n. If $\mathfrak{N}$ is an infinite structure then

$$\mathfrak{M} \leqslant_\Sigma^n \mathfrak{N} \text{ if and only if } \mathfrak{M} \leqslant_\Sigma \mathfrak{N}$$

for any $\mathfrak{M}$ and any $n \geqslant 2$.

Typical examples of structures $\mathfrak{M}$ with $\mathrm{cr}(\mathbb{HF}(\mathfrak{M})) = 2$ are infinite structures with the empty signature, dense linear orders, and, a more interesting one, the structure $\langle \omega, s\rangle$ of natural numbers with the successor function. This fact follows from the next proposition, taking into account the decidability of $\mathrm{Th}_{\mathrm{WM}}(\langle \omega, s\rangle)$, where $\mathrm{Th}_{\mathrm{WM}}(\mathfrak{M})$ is the weak monadic second-order theory of $\mathfrak{M}$.

Proposition 6 ([Stukachev2005]). *If $\mathrm{Th}_{\mathrm{WM}}(\mathfrak{M})$ is decidable then*

$$\mathrm{cr}(\mathbb{HF}(\mathfrak{M})) = 2.$$

An example of a structure $\mathfrak{M}$ with $\mathrm{cr}(\mathbb{HF}(\mathfrak{M})) = 0$ is, obviously, the standard model of the arithmetic. An example of a structure which hereditary finite superstructure has rank of inner constructivizability 1 is the field $\mathbb{R}$ of real numbers.

Proposition 7 ([Stukachev2005]). $\mathrm{cr}(\mathbb{HF}(\mathbb{R})) = 1$.

Another natural special type of a Σ-presentation of a structure $\mathfrak{M}$ in an admissible set $\mathbb{A}$, s.t. $M \subseteq U(\mathbb{A})$, is a Σ-presentation preserving the domain of a structure. For a signature σ and an ordinal $n \leqslant \omega$, we denote by $Form_n(\sigma)$ the set of (finite first-order) formulas of the signature σ, which have a prenex normal form with no more than n alterating groups of quantifiers.

We assume that, for any signature considered, some Gödel numbering $\lceil \cdot \rceil$ of its terms and formulas is fixed.

Definition 6. Let $\mathfrak{M}$ be a structure of a finite signature σ, $\mathbb{A}$ an admissible set, and let $M \subseteq U(A)$. The structure $\mathfrak{M}$ is *n-decidable in* $\mathbb{A}$ ($n \leqslant \omega$) if

$$\{\langle \lceil \varphi \rceil, \overline{m} \rangle \mid \varphi \in Form_n(\sigma), \overline{m} \in M^{<\omega}, \mathfrak{M} \models \varphi(\overline{m})\}$$

is Δ-definable in $\mathbb{A}$.

A structure $\mathfrak{M}$ is *computable in* $\mathbb{A}$ if $\mathfrak{M}$ is 0-decidable in $\mathbb{A}$, and *decidable in* $\mathbb{A}$ if $\mathfrak{M}$ is ω-decidable in $\mathbb{A}$. It is easy to prove that, if $\mathrm{Th}(\mathfrak{M})$ is regular, then $\mathfrak{M}$ is decidable in $\mathbb{HF}(\mathfrak{M})$.

The decidability is rather a strong condition. For example, there is

Proposition 8. *A liner order $\mathfrak{L}$ is 1-decidable in $\mathbb{HF}(\mathfrak{L})$ if and only if $\mathfrak{L}$ is a sum of a finite number of dense linear orders and points.*

A structure $\mathfrak{M}$ of signature σ is *n-complete* [ErshovGoncharov2000] ($n \leqslant \omega$) if for any formula $\varphi(\overline{x}) \in Form_n(\sigma)$ and for any $\overline{m} \in M^{<\omega}$ such that $\mathfrak{M} \models \varphi(\overline{m})$ there exists a $\exists$-formula $\psi(\overline{x})$ such that $\mathfrak{M} \models \psi(\overline{m})$ and $\mathfrak{M} \models \forall \overline{x}(\psi(\overline{x}) \to \varphi(\overline{x}))$. The following proposition follows immediately from the definitions.

Proposition 9. *1) Suppose $\mathfrak{M}$ is n-decidable in $\mathbb{HF}(\mathfrak{M})$ ($n \leqslant \omega$). Then $\mathfrak{M}$ is n-complete in some expansion of $\mathfrak{M}$ by a finite number of constants.*
2) Suppose $\mathfrak{M}$ is n-complete and $\mathrm{Th}(\mathfrak{M})$ is decidable. Then $\mathfrak{M}$ is n-decidable in $\mathbb{HF}(\mathfrak{M})$.

Suppose $\mathfrak{M}$ is 1-decidable in $\mathbb{HF}(\mathfrak{M})$. Then $\mathbb{HF}(\mathfrak{M})$ has a universal Σ-function and the reduction property, but not necessarily the uniformization property (see [Ershov1996]).

A structure $\mathfrak{A}$ is *$s\Sigma$-definable* [Stukachev2009,Stukachev2010] in $\mathbb{HF}(\mathfrak{B})$ (denoted by $\mathfrak{A} \leqslant_{s\Sigma} \mathfrak{B}$) if $A \subseteq \mathrm{HF}(B)$ is a Σ-subset in $\mathbb{HF}(\mathfrak{B})$, and all the signature relations and functions of $\mathfrak{A}$ are Δ-definable in $\mathbb{HF}(\mathfrak{B})$. Relation $\leqslant_{s\Sigma}$ is reflexive and transitive, under some additional assumptions on the structures being considered [StukachevTA]. For countable structures, in a slightly different form, $s\Sigma$-reducibility was introduced in [Baleva2006].

We write $\mathfrak{A} <_{s\Sigma} \mathfrak{B}$ to denote the fact that $\mathfrak{A} \leqslant_{s\Sigma} \mathfrak{B}$ and $\mathfrak{B} \nleqslant_{s\Sigma} \mathfrak{A}$.

In [Stukachev2009,Stukachev2010] it was noted that

$$\mathfrak{A} <_{s\Sigma} \mathfrak{A}'$$

for any structure $\mathfrak{A}$, no matter countable or not. It means that the operation of Σ-jump has no fixed points with respect to $s\Sigma$-reducibility, one of the strongest effective reducibilities on countable structures (see [Stukachev2007,Stukachev2008]).

However, the relation of $s\Sigma$-reducibility is not persistent relative to the isomorphism, and the corresponding relation of $s\Sigma$-equivalence is stronger than the isomorphism relation. Also, the embeddings of the semilattices of Turing and enumeration degrees into the semilattice of $s\Sigma$-degrees are not so natural as the corresponding embeddings in the case of the semilattices of Σ-degrees [Stukachev2007,Stukachev2008].

However, $s\Sigma$-reducibility is useful in studying generalized computability. Actually, it was implicitly used in [Stukachev1997] to formulate a criterion of the uniformization property.

Recall that an admissible set $\mathbb{A}$ is said to satisfy

- *reduction* if, for any Σ-subsets B_0 and B_1, there are disjoint Σ-subsets $C_0 \subseteq B_0$ and $C_1 \subseteq B_1$ such that $C_0 \cup C_1 = B_0 \cup B_1$.
- *uniformization* if, for any binary Σ-predicate R on $\mathbb{A}$, there is a partial Σ-function $\varphi(x)$ with $\delta\varphi = \mathrm{Pr}_1(R)$ and $\Gamma_\varphi \subseteq R$.

Suppose $\mathfrak{M}$ has a *regular* (i.e., model complete and decidable [Ershov1996]) first-order theory. Then $\mathfrak{M}$ is 1-decidable in $\mathbb{HF}(\mathfrak{M})$, and hence has a universal (partial, single-valued) Σ-function and satisfies reduction, but not necessarily uniformization. It turns out that the relation of Σ-reducibility (in fact, of $s\Sigma$-equivalence) can be used to provide a criterion.

Recall that a theory T of signature σ is said to be a *theory with definable Skolem functions* [Dries1984], provided that, for each formula $\varphi(x_0, \ldots, x_n)$ of signature σ, there exists a formula $\psi(x_0, \ldots, x_n)$ of the same signature such that

$$T \vdash \forall x_1 \ldots \forall x_n \Big[\exists x_0\, \varphi(x_0, \ldots, x_n) \to \exists\,!\, x_0\, \big(\varphi(x_0, \ldots, x_n) \wedge \psi(x_0, \ldots, x_n)\big)\Big].$$

Actually, the requirement of definability of Skolem functions is too stringent. Let $\mathfrak{M}$ be a structure of signature σ, and let σ' denotes the signature of $\mathbb{HF}(\mathfrak{M})$. In formulas of signature σ', we conventially distinguish between variables with values in the set of urelements and general variables, i.e. variables whose values may be arbitrary elements of an admissible set. In what follows, given a formula of signature σ, we assume all its variables, free or bounded, to be variables for urelements.

A structure $\mathfrak{M}$ is said to have a *Σ-definable Skolem functions* [Stukachev1997], provided that, given any formula $\varphi(x_0, \ldots, x_n)$ of signature σ, we can effectively find a Σ-formula $\psi(x_0, \ldots, x_n)$ of signature σ' such that

$$\mathbb{HF}(\mathfrak{M}) \models \forall x_1 \ldots \forall x_n \Big[\exists x_0 \varphi(x_0, \ldots, x_n) \to \exists!\, x_0 \big(\varphi(x_0, \ldots, x_n) \wedge \psi(x_0, \ldots, x_n)\big)\Big]$$

(recall that $x_0, \ldots, x_n$ together with all bounded variables in φ are variables for urelements).

The above definition can be easily expressed in terms of $s\Sigma$-equivalence. Let $\mathfrak{M}$ be a structure of signature σ and let signature σ_{Skolem} consist of all symbols of σ and new functional symbols $f_\varphi(x_1, \ldots, x_n)$ for all formulas $\varphi(x_0, x_1, \ldots, x_n)$ of signature σ. The structure $\mathfrak{M}^S$ of signature σ_{Skolem} is called a (non-iterated)

Skolem expansion of $\mathfrak{M}$ if $M^S = M$, $\mathfrak{M} \upharpoonright_\sigma = \mathfrak{M}^S \upharpoonright_\sigma$, and for any formula $\varphi(x_0, x_1, \ldots, x_n)$ of signature σ

$$\mathfrak{M}^S \models \forall x_1 \ldots \forall x_n (\exists x \varphi(x, x_1, \ldots, x_n) \to \varphi(f_\varphi(x_1, \ldots, x_n), x_1, \ldots, x_n)).$$

It is easy to see that, if $\mathfrak{M}$ is a structure with a regular theory, then $\mathfrak{M}$ is a structure with Σ-definable Skolem functions if and only if, for some Skolem expansion $\mathfrak{M}^S$ of $\mathfrak{M}$ holds

$$\mathfrak{M}^S \equiv_{s\Sigma} \mathfrak{M}.$$

Skolem expansion $\mathfrak{M}^S$ of a structure $\mathfrak{M}$ is *well-defined* if for every formula $\varphi(x_0, x_1, \ldots, x_n)$ of signature σ, every $\overline{m} \in M^n$, and every permutation ρ of the set $\{1, \ldots, n\}$,

$$\mathfrak{M} \models (\varphi(x_0, \overline{m}) \leftrightarrow \varphi(x_0, \rho(\overline{m}))) \text{ implies } \mathfrak{M}^S \models (f_\varphi(\overline{m}) = f_\varphi(\rho(\overline{m}))),$$

where $\rho(\overline{m}) = \langle m_{\rho(1)}, \ldots, m_{\rho(n)} \rangle$.

Recall that $\mathbb{HF}(\mathfrak{M})$ has the *uniformization property* if for every Σ-predicate $E \subseteq \mathrm{HF}(M) \times \mathrm{HF}(M)$ there exists a Σ-function F such that the following assertions are valid:

1) $\mathrm{dom}(F) = \mathrm{Pr}_1(E)$,

2) $\mathrm{graph}(F) \subseteq E$,

where $\mathrm{dom}(F) = \{x \mid F(x)\downarrow\}$, $\mathrm{graph}(F) = \{\langle x, y\rangle \mid F(x) = y\}$, and $\mathrm{Pr}_1(E) = \{x \mid \exists\, y(\langle x, y\rangle \in E)\}$.

The next theorem is a reformulation (and correction) of the main result from [Stukachev1997]. (Unfortunately, the property of well-definedness for Skolem expansions was not explicitly stated there, yet it was implicitly used in the text.) This theorem gives a natural (and useful) example of using $s\Sigma$-reducibility on structures and Proposition 10 can be viewed as a natural (and non-trivial) example of $s\Sigma$-equivalence.

Theorem 6. *Let* $\mathfrak{M}$ *be a structure with a regular theory.* $\mathbb{HF}(\mathfrak{M})$ *satisfies the uniformization property if and only if, for some well-defined Skolem expansion* $\mathfrak{M}^S$ *of* $\mathfrak{M}$*, holds*

$$\mathfrak{M}^S \equiv_{s\Sigma} \mathfrak{M}.$$

Proof. Fix a Gödel numbering of formulas of signature σ' which distinguishes variables for urelements. The Gödel number of a formula φ is denoted by $[\varphi]$. Note that if T is a regular theory then, by model completeness, each formula of its signature is T-equivalent to some $\exists$-formula and, moreover, by decidability, this formula can be found effectively (henceforth, by effectiveness we mean existence of an appropriate computable function on the set of Gödel numbers).

Let us first prove the necessity. Suppose that $\mathbb{HF}(\mathfrak{M})$ satisfies the uniformization property. Let $\varphi(x_0, x_1, \ldots, x_n)$ be an arbitrary formula of signature σ, and let $\psi(x_0, x_1, \ldots, x_n)$ be some $\exists$-formula of signature σ which is $\mathrm{Th}(\mathfrak{M})$-equivalent to φ (such a formula exists and can be found effectively since $\mathrm{Th}(\mathfrak{M})$ is regular). First, define a binary predicate G_0 as follows:

$$G_0 = \{\langle a, b\rangle | a = \langle [\varphi], m_1, \ldots, m_{n-1}\rangle, b = \{\langle [\varphi], m_{\rho(1)}, \ldots, m_{\rho(n-1)}\rangle |$$

$$|\rho \in S_{n-1} \text{ is a permutation such that}$$
$$\mathfrak{M} \models (\varphi(x_0, m_1, \ldots, m_{n-1}) \leftrightarrow \varphi(x_0, m_{\rho(1)}, \ldots, m_{\rho(n-1)}))\}\}.$$

From the regularity it follows that G_0 is a Σ-predicate on $\mathbb{HF}(\mathfrak{M})$. Let $F_0(x)$ be a Σ-function which uniformizes G_0. Second, define a binary Σ-predicate G_1 on $\mathbb{HF}(\mathfrak{M})$ as follows:

$$\langle a, m\rangle \in G_1 \iff \Big(a = \big\langle [\varphi], m_1, \ldots, m_{n-1}\big\rangle\Big) \wedge$$

$$\wedge(\psi \text{ is an } \exists\text{-formula equivalent to } \varphi) \wedge \Sigma\text{-Sat}\big([\psi], \langle m, m_1, \ldots, m_{n-1}\rangle\big).$$

Again, there exists a Σ-function F_1 that uniformizes G_1. The function

$$f_\varphi(x_1, \ldots, x_{n-1}) = \lambda x_1.\ \ldots\ \lambda x_{n-1}.\, F_1\Big(F_0(\langle [\varphi], x_1, \ldots, x_{n-1}\rangle)\Big)$$

is a Skolem function for the formula φ, which is well-defined by the construction.

Let us prove the sufficiency. Hereinafter, let T be a regular theory of signature σ and let $\mathfrak{M} = \big\langle M, \sigma^{\mathfrak{M}}\big\rangle$ be a model of T with a well-definable Skolem expansion $\mathfrak{M}^S$ such that $\mathfrak{M}^S \equiv_{s\Sigma} \mathfrak{M}$.

Lemma 2. *Suppose that P is a definable n-ary predicate over $\mathfrak{M}$. Then each formula defining P can be effectively transformed into a Σ-formula, with parameters used in the definition of P and in the definition of $\mathfrak{M}^S$, that defines a predicate Q on $\mathbb{HF}(\mathfrak{M})$ such that*

1) if $P = \varnothing$ then $Q = \varnothing$,

2) if $P \neq \varnothing$ then $Q = \{\bar{x}\}$, $\bar{x} \in P$.

Proof. The case $n = 1$ is evidently follows from the existence of a Skolem expansion $\mathfrak{M}^S$ with the property $\mathfrak{M}^S \equiv_{s\Sigma} \mathfrak{M}$; so, assume that $n > 1$ and that the statement is true for all $k < n$. Suppose that the predicate P is defined by a formula $\varphi(x_0, \ldots, x_{n-1}, \bar{y})$ of signature σ with parameters $\overline{m}$. Given the predicate

$$X = \big\{x_0 \bigm| \mathfrak{M} \models \exists x_1 \ldots \exists x_{n-1}\, \varphi(x_0, \ldots, x_{n-1}, \overline{m})\big\},$$

we can effectively find by induction a Σ-formula $\Phi(x, \bar{y})$ that defines a single element in X (we can also assume that all the parameters used are urelements). Consider the predicate

$$Y = \Big\{\langle x_1, \ldots, x_{n-1}\rangle \in M^{n-1} \Bigm| \mathbb{HF}(\mathfrak{M}) \models \exists x_0\big(\varphi(\bar{x}, \overline{m}) \wedge \Phi(x_0, \overline{m})\big)\Big\}.$$

By Lemma 3, $\mathbb{HF}(\mathfrak{M}) \models \Phi(x_0, \overline{m}) \iff \mathfrak{M} \models \bigvee_{i\in\omega} \varphi_i(x_0, \overline{m})$, where φ_i are formulas of signature σ and the set $\big\{[\varphi_i] \bigm| i \in \omega\big\}$ is computably enumerable. Whence

$$Y = \Big\{\langle x_1, \ldots, x_{n-1}\rangle \in M^{n-1} \Bigm| \mathfrak{M} \models \bigvee_{i\in\omega} \exists x_0\big(\varphi(\bar{x}, \overline{m}) \wedge \varphi_i(x_0, \overline{m})\big)\Big\}.$$

Assume that $i_0 = \mu i\Big(\mathfrak{M} \models \exists \bar{x}\big(\varphi(\bar{x}, \overline{m}) \wedge \varphi_i(x_0, \overline{m})\big)\Big)$; by regularity, i_0 is defined by a $\varSigma$-formula in $\mathbb{HF}(\mathfrak{M})$. Since the formula $\varPhi(x_0, \overline{m})$ is true for at most one element, we have

$$Y = \Big\{\langle x_1, \ldots, x_{n-1}\rangle \in M^{n-1} \;\Big|\; \mathfrak{M} \models \exists x_0\big(\varphi(\bar{x}, \overline{m}) \wedge \varphi_{i_0}(x_0, \overline{m})\big)\Big\}.$$

We find by induction a $\varSigma$-formula $\varPsi(x_1, \ldots, x_{n-1}, \bar{y})$ that defines a single element in Y, with the same restrictions on the parameters. The required predicate Q is defined by the $\varSigma$-formula $\varPhi(x_0, \bar{y}) \wedge \varPsi(x_1, \ldots, x_{n-1}, \bar{y})$. The lemma is proven.

Define a function $h : \omega \to \mathrm{HF}(\underline{\omega})$. For each $n \in \omega$, we put

$$h(n) = \begin{cases} \underline{n_1}, & \text{if } n = c(0, n_1), \\ \{h(n_1)\}, & \text{if } n = c(1, n_1), \\ h(n_1) \cup h(n_2), & \text{if } n = c(2, c(n_1, n_2)), l(n_1) \cdot l(n_2) > 0, \text{ and } n_1 < n_2 \\ \varnothing, & \text{otherwise,} \end{cases} \tag{1}$$

where $c(n, m) = \frac{(n+m)^2 + 3n + m}{2}$ is Cantor's bijection, and $l(n)$ and $r(n)$ are the left and right projections. It is easy to see by definition that h is a numbering of $\mathrm{HF}(\underline{\omega})$, and since ω is a $\varDelta$-subset in $\mathrm{HF}(\mathfrak{M})$, we conclude that, in terms of [Ershov1996], h is an $\mathbb{HF}(\mathfrak{M})$-constructivization of $\mathrm{HF}(\underline{\omega})$. Thus, $\mathrm{HF}(\underline{\omega})$ can be effectively defined in each superstructure.

Lemma 3. *Suppose that $\varphi(x)$ is a $\varDelta_0$-formula of signature σ' and let $\varkappa \in \mathrm{HF}(n)$. Then we can effectively find a formula $\varphi^*(x_0, \ldots, x_{n-1})$ of signature σ so that, for each valuation $\gamma : \{x_0, \ldots, x_{n-1}\} \to M$,*

$$\mathbb{HF}(\mathfrak{M}) \models \varphi(x)^x_{t_\varkappa(\bar{x})}[\gamma] \iff \mathfrak{M} \models \varphi^*(x_0, \ldots, x_{n-1})[\gamma].$$

Proof. Given a formula $\varphi(x)$ and element $\varkappa \in \mathrm{HF}(n)$, we construct a formula $\varphi^x_\varkappa(x_0, \ldots, x_{n-1})$ of signature $\sigma' \cup \big\{\varnothing, \{\}, \cup\big\}$ as follows:

1) if $\varphi = \varphi_1 \, q \, \varphi_2$, $q \in \{\vee, \wedge, \to\}$, then $\varphi^x_\varkappa \rightleftharpoons (\varphi_1)^x_\varkappa \, q \, (\varphi_2)^x_\varkappa$

2) if $\varphi = \neg\varphi_1$ then $\varphi^x_\varkappa \rightleftharpoons \neg(\varphi_1)^x_\varkappa$

3) if $\varphi = (t_1 \, p \, t_2)$, $p \in \{\in, =\}$, then $\varphi^x_\varkappa \rightleftharpoons (t_1 \, p \, t_2)^x_{t_\varkappa(\bar{x})}$

4) if $\varphi = \exists y \in x(\varphi_1)$ then $\varphi^x_\varkappa \rightleftharpoons \bigvee\limits_{\varkappa' \in \varkappa} ((\varphi_1)^y_{\varkappa'})^x_\varkappa$

5) if $\varphi = \forall y \in x(\varphi_1)$ then $\varphi^x_\varkappa \rightleftharpoons \bigwedge\limits_{\varkappa' \in \varkappa} ((\varphi_1)^y_{\varkappa'})^x_\varkappa$

6) if $\varphi = U(x)$ then $\varphi^x_\varkappa \rightleftharpoons \begin{cases} \tau, & \text{if } \varkappa \in n \\ \neg\tau, & \text{otherwise} \end{cases}$

7) if $\varphi = P(t_0, \ldots, t_k)$, $P \in \sigma$, then $\varphi^x_\varkappa \rightleftharpoons \begin{cases} P(t_0, \ldots t_k)^x_{t_\varkappa(\bar{x})}, & \text{if } \varkappa \in n \\ \neg\tau, & \text{otherwise} \end{cases}$

where τ denotes the statement $\exists x(x = x)$ (without loss of generality we may assume that σ does not contain functional symbols).

Next, for any pair of terms t_0, t_1 of signature $\langle\varnothing, \{\}, \cup\rangle$ over variables for urelements $x_0, \dots, x_{n-1}$, we can effectively define formulas Φ_{t_0,t_1} and Ψ_{t_0,t_1} of empty signature so that $\mathrm{FV}(\Phi_{t_0,t_1}) = \mathrm{FV}(\Psi_{t_0,t_1}) = \mathrm{FV}(t_0) \cup \mathrm{FV}(t_1)$ and, for each valuation $\gamma : \mathrm{FV}(t_0 = t_1) \to M$, the following statements be true:

$$t_0^{\langle\mathbb{HF}(\mathfrak{M}),\{\},\cup\rangle}[\gamma] \in t_1^{\langle\mathbb{HF}(\mathfrak{M}),\{\},\cup\rangle}[\gamma] \iff \mathfrak{M} \models \Phi_{t_0,t_1}[\gamma]$$
$$t_0^{\langle\mathbb{HF}(\mathfrak{M}),\{\},\cup\rangle}[\gamma] \subseteq t_1^{\langle\mathbb{HF}(\mathfrak{M}),\{\},\cup\rangle}[\gamma] \iff \mathfrak{M} \models \Psi_{t_0,t_1}[\gamma]$$

(see [Ershov1996] for a proof). The formula $\varphi^*(\bar{x})$ is obtained from $\varphi^x_\varkappa(\bar{x})$ by replacing the subformulas of kind $(t_0 \in t_1)$ by Φ_{t_0,t_1} and the subformulas of kind $(t_0 = t_1)$ by $(\Psi_{t_0,t_1} \wedge \Psi_{t_1,t_0})$. The lemma is proven.

Lemma 3 can be easily extended to formulas with several variables. This lemma also implies that we can restrict our consideration to formulas with parameters in M only.

Assume that $\Phi(x, \overline{m})$ is a Δ_0-formula of signature σ' with parameters $\overline{m}$ in M. For each $n \in \omega$, we define the set

$$\mathrm{H}_n \rightleftharpoons \{\varkappa \in \mathrm{HF}(n) \mid \mathrm{HF}(\mathfrak{M}) \models \exists x_0 \dots \exists x_{n-1} (\Phi(x, \overline{m}))^x_{t_\varkappa(\bar{x})}\}$$

and put $\mathrm{H} \rightleftharpoons \bigcup_{n\in\omega} \mathrm{H}_n$. The following lemma is valid:

Lemma 4. *The set* H *is a* Δ*-subset in* $\mathbb{HF}(\mathfrak{M})$.

Proof. Let $\overline{\mathrm{H}}_n \rightleftharpoons \mathrm{HF}(n) \setminus \mathrm{H}_n$, $\overline{\mathrm{H}} \rightleftharpoons \mathrm{HF}(\omega) \setminus \mathrm{H}$; then $\overline{\mathrm{H}} = \bigcup_{n\in\omega} \overline{\mathrm{H}}_n$. So, it suffices to prove that H_n is a Δ-subset in $\mathbb{HF}(\mathfrak{M})$.

Making use of Lemma 3, given a formula Φ and an element $\varkappa$, we effectively find a formula $\Psi_\varkappa(\bar{x}, \overline{m})$ of signature σ such that

$$\varkappa \in \mathrm{H}_n \iff \mathfrak{M} \models \exists x_0 \dots \exists x_{n-1} \Psi_\varkappa(\bar{x}, \overline{m}).$$

By regularity, given the formula $\exists \bar{x}\, \Psi_\varkappa(\bar{x}, \bar{y})$, we can effectively find an $\exists$-formula $\Theta_\varkappa(\bar{y})$ equivalent to it. Thus,

$$\varkappa \in \mathrm{H}_n \iff \mathbb{HF}(\mathfrak{M}) \models \Sigma\text{-Sat}\big([\Theta_\varkappa], \overline{m}\big).$$

The case $\varkappa \in \overline{\mathrm{H}}_n$ is handled similarly. The lemma is proven.

So, let $E \subseteq \mathrm{HF}(M) \times \mathrm{HF}(M)$ be an arbitrary Σ-predicate. Without loss of generality, we may assume that the predicate $E(x, y)$ is defined by a formula $\exists z\, \Phi(x, y, z, \overline{m})$, where $\Phi(x, y, z, \overline{m})$ is a Δ_0-formula with parameters $\overline{m}$ in M.

It is evident that $\mathrm{Pr}_1(E)$ is a Σ-predicate. Indeed, consider the Δ_0-formula

$$\Psi(x, t, \overline{m}) \rightleftharpoons \exists u \in t\, \exists v \in t\, \exists y \in u\, \exists z \in v\, \big(t = \langle y, z\rangle \wedge \Phi(x, y, z, \overline{m})\big).$$

It is clear that $x \in \mathrm{Pr}_1(E) \iff \mathbb{HF}(\mathfrak{M}) \models \exists t\, \Psi(x, t, \overline{m})$.

For each $a \in \mathrm{HF}(M)$, there exist $n \in \omega$, $\varkappa \in \mathrm{HF}(\underline{n})$, and $a_0, \dots, a_{n-1} \in M$ such that $a = \varkappa(\bar{a})$. (Here is the point where we assume the tuple $\bar{a}$ being ordered in some way, but the ordering does not matter, because the Skolem expansion is well-defined.) Let $x^* \in \mathrm{HF}(\mathfrak{M})$, $x^* = \varkappa_0(\bar{x})$, where $\varkappa_0 \in \mathrm{HF}(\underline{l})$, $\bar{x} = \langle x_0, \ldots, x_{l-1}\rangle \in M^l$. In the same way as in Lemma 3, we define the sets

$$H_n \rightleftharpoons \{\ \varkappa \in \mathrm{HF}(\underline{n}) \mid \mathbb{HF}(\mathfrak{M}) \models \exists t_0 \ldots \exists t_{n-1} (\Psi(x^*, t, \overline{m}))^t_{t_\varkappa(\bar{t})}\}$$

for all $n \in \omega$ and put $\mathrm{H} \rightleftharpoons \bigcup_{n\in\omega} \mathrm{H}_n$.

If $x^* \in \mathrm{Pr}_1(E)$ then the set $\{t \mid \mathbb{HF}(\mathfrak{M}) \models \Psi(x^*, t, \overline{m})\}$ is nonempty; hence, the set H is nonempty too. In this case, the element $\varkappa_1 \in \mathrm{H}$ minimal in the sense of the enumeration h above is uniquely defined. In other words, $\varkappa_1$ is taken so as to satisfy the following conditions:

$$\exists k\Big((k \in \omega) \wedge \big(\varkappa_1 = h(k)\big) \wedge (\varkappa_1 \in \mathrm{H}) \wedge \forall k' < k\big(h(k') \notin \mathrm{H}\big)\Big).$$

By virtue of Lemma 4, this condition is expressed in $\mathbb{HF}(\mathfrak{M})$ by some Σ-formula $\Psi_1(\varkappa_1, x^*, \overline{m})$.

Suppose that $\varkappa_1 \in \mathrm{HF}(\underline{n})$. Consider the set

$$T = \Big\{\langle t_0, \ldots, t_{n-1}\rangle \in M^n \ \Big|\ \mathbb{HF}(\mathfrak{M}) \models \Psi(x^*, t, \overline{m})^t_{\varkappa_1(\bar{t})}\Big\}.$$

By Lemma 3 we can effectively construct a formula $\Theta(\bar{x}, \bar{t}, \bar{y})$ of signature σ so that, for each valuation $\gamma\ \{\bar{x}, \bar{t}\} \to M$, the following be true:

$$\mathbb{HF}(\mathfrak{M}) \models \Psi(x, t, \overline{m})^{x,\ \ \ \ t}_{\varkappa_0(\bar{x}),\ \varkappa_1(\bar{t})}[\gamma] \iff \mathfrak{M} \models \Theta(\bar{x}, \bar{t}, \overline{m})[\gamma].$$

By Lemma 2, given the formula Θ, we can effectively find an $\exists$-formula $\Theta^*(\bar{x}, \bar{t}, \overline{m})$ that defines a unique element $\overline{t^*}$ in the set T.

The element $t^* \rightleftharpoons \varkappa_1(\overline{t^*})$ satisfies the formula $\Psi(x^*, t^*, \overline{m})$; hence, it has the form $t^* = \langle y^*, z^*\rangle$ and, moreover, $\langle x^*, y^*\rangle \in E$. We put $F(x^*) \rightleftharpoons y^*$ by definition.

The required Σ-function F is defined as follows: let $F(x^*) = y^*$ if

$$\mathbb{HF}(\mathfrak{M}) \models \exists t^* \exists z^* \exists \varkappa_0 \exists \varkappa_1 \big((t^* = \langle y^*, z^*\rangle) \wedge \Psi_1(\varkappa_1, x^*, \overline{m}) \wedge$$

$$\wedge \Sigma\text{-}Sat([\exists x_0 \ldots \exists x_{l-1} \exists t_0 \ldots \exists t_{n-1} (x^* = \varkappa_0(\bar{x}) \wedge$$

$$\wedge t^* = \varkappa_1(\bar{t}) \wedge \Theta^*(\bar{x}, \bar{t}, \bar{y}))], \overline{m})\big).$$

One of the important corollaries of this criterion follows from the next result.

Proposition 10 ([Stukachev1997]). *There exist well-defined Skolem expansions $\mathbb{R}^S$ and $(\mathbb{Q}_p)^S$, of the fields $\mathbb{R}$ and $\mathbb{Q}_p$, such that $\mathbb{R}^S \equiv_{s\Sigma} \mathbb{R}$ and $(\mathbb{Q}_p)^S \equiv_{s\Sigma} \mathbb{Q}_p$.*

Corollary 2 ([Stukachev1997]). *Structures $\mathbb{HF}(\mathbb{R})$ and $\mathbb{HF}(\mathbb{Q}_p)$ satisfy uniformization and have a universal Σ-function.*

For $\mathbb{HF}(\mathbb{R})$, the uniformization property and existence of a universal Σ-function was independently proved by the author in [Stukachev1996] and by M.V. Korovina in [Korovina1996]. In [Ershov1996], a general sufficien condition for an admissible set to have a universal Σ-function was found, which implies the existence of a universal Σ-function for $\mathbb{HF}(\mathfrak{M})$, there $\mathrm{Th}(\mathfrak{M})$ is regular.

The role of parameters in the Σ-definition of a structure is rather important. For example, as it is easy to see, any countable structure is Σ-definable in $\mathbb{HF}(\mathbb{R})$, where $\mathbb{R}$ is the field of real numbers. The case of Σ-definability without parameters turned out to be quite interesting, as it was shown recently in [MorozovKorovina2008].

Theorem 7 ([MorozovKorovina2008]). *Suppose a countable structure $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathbb{R})$ without parameters. Then $\mathfrak{M}$ has a hyperarithmetic presentation.*

This estimate is precise, as follows from the next theorem.

Theorem 8 ([MorozovKorovina2008]). *For any $\delta < \omega_1^{CK}$ there is a countable structure $\mathfrak{M}$ such that*

1) $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathbb{R})$ without parameters;
2) for any $H \subseteq \omega$ such that $\mathfrak{M}$ has an H-computable presentation, $0^{(\delta)} \leqslant_T H$.

In case we fix some restrictions on the cardinality of the congruence classes, the estimate of complexity becomes much lower.

Theorem 9 ([MorozovKorovina2008]). *Let $\mathfrak{M}$ be a countable structure with a finite signature. The following are equivalent:*

1) $\mathfrak{M}$ is Σ-definable without parameters in $\mathbb{HF}(\mathbb{R})$, and all congruence classes are at least countable;
2) $\mathfrak{M}$ is computable.

5 Degrees of Presentability of Structures in Admissible Sets

The relation $\leqslant_\Sigma$ of Σ-reducibility, which is defined on structures of arbitrary cardinality, in the case of countable structures can be viewed as the strongest reducibility in the hierarchy of effective reducibilities on structures, as it was shown in [Stukachev2007,Stukachev2008]. We overview some of the results in this field.

Let $\mathbb{A}$ be an admissible set. A mapping $F : P(A)^n \to P(A)$ ($n \in \omega$) is called a *Σ-operator* [Ershov1996] if if there exists a Σ-formula $\Phi(x_0, \dots, x_{n-1}, y)$ of signature $\sigma_{\mathbb{A}}$ such that, for any $S_0, \dots, S_{n-1} \in P(A)$,

$$F(S_0, \dots, S_{n-1}) = \{a | \exists a_0, \dots, a_{n-1} \in A(\bigwedge_{i<n} a_i \subseteq S_i \wedge \mathbb{A} \models \Phi(a_0, \dots, a_{n-1}, a))\}.$$

The next condition is necessary for the transitiveness of the reducibilities defined below. An operator $F : P(A) \to P(A)$ is called *strongly continuous in* $S \in P(A)$ if, for any $a \subseteq F(S)$, $a \in A$, there exists an $a' \subseteq S$, $a' \in A$, such that $a \subseteq F(a')$ (this definition can be easily generalized for the case of operators with the number of arguments more than 1).

For an operator $F : P(A)^n \to P(A)$, we denote by $\delta_c(F)$ the set of elements of $P(A)^n$ in which F is strongly continuous. A set $S \in P(A)^n$ is called a Σ_**-set* if $S \in \delta_c(F)$ for any Σ-operator $F : P(A)^n \to P(A)$. It is easy to verify that in admissible sets of the kind $\mathbb{HF}(\mathfrak{M})$ any subset is a Σ_*-set. However, in general this is not so: for example, in [Stukachev2002] were studied Σ_*-sets in $\mathbb{HYP}(\mathbb{L})$, where $\mathbb{L}$ is a dense linear order. Even in this simplest case the class of Σ_*-sets is non-trivial.

Let $B, C \subseteq A$. Below are reducibilities that are direct generalizations of e- and T-reducibilities on natural numbers:

1) $B \leqslant_{e\Sigma} C$, if there is a unary Σ-operator F for which $C \in \delta_c(F)$ and $B = F(C)$;

2) $B \leqslant_{T\Sigma} C$, if there are binary Σ-operators F_0 and F_1 such that $\langle C, A \setminus C\rangle \in \delta_c(F_0) \cap \delta_c(F_1)$ and $B = F_0(C, A \setminus C)$, $A \setminus B = F_1(C, A \setminus C)$.

Let $\mathbb{A}$ be an admissible set. We define uniform reducibilities on subsets of A, which are direct generalizations of Medvedev, Muchnik, and Dyment reducibilities on mass problems [Sorbi1996]. Let $\mathcal{X}, \mathcal{Y} \subseteq P(A)$. Then:

1) $\mathcal{X}$ is *Medvedev reducible* to $\mathcal{Y}$ ($\mathcal{X} \leqslant_s \mathcal{Y}$) if there are binary Σ-operators F_0 and F_1 such that for all $Y \in \mathcal{Y}$, $\langle Y, A \setminus Y\rangle \in \delta_c(F_0) \cap \delta_c(F_1)$, and for some $X \in \mathcal{X}$, $X = F_0(Y, A \setminus Y)$ and $A \setminus X = F_1(Y, A \setminus Y)$;

2) $\mathcal{X}$ is *Dyment reducible* to $\mathcal{Y}$ ($\mathcal{X} \leqslant_e \mathcal{Y}$) if there is a unary Σ-operator F such that, for all $Y \in \mathcal{Y}$, $Y \in \delta_c(F)$ and $F(\mathcal{Y}) \subseteq \mathcal{X}$;

3) $\mathcal{X}$ is *Muchnik reducible* to $\mathcal{Y}$ ($\mathcal{X} \leqslant_w \mathcal{Y}$) if for any $Y \in \mathcal{Y}$ there are binary Σ-operators F_0 and F_1 such that $\langle Y, A \setminus Y\rangle \in \delta_c(F_0) \cap \delta_c(F_1)$ and, for some $X \in \mathcal{X}$, $X = F_0(Y, A \setminus Y)$ and $A \setminus X = F_1(Y, A \setminus Y)$;

4) $\mathcal{X}$ is *weakly Dyment reducible* to $\mathcal{Y}$ ($\mathcal{X} \leqslant_{ew} \mathcal{Y}$) if for any $Y \in \mathcal{Y}$ there is a unary Σ-operator F such that $Y \in \delta_c(F)$ and $F(\mathcal{Y}) \subseteq \mathcal{X}$.

For an admissible set $\mathbb{A}$ and a symbol $r \in \{e, s, w, ew\}$, we denote by $\mathcal{M}_r(\mathbb{A})$ the degree structure $\langle P(P(A))/\equiv_r, \leqslant_r\rangle$. For simplicity, we use the notation $\mathcal{M}_r$ instead of $\mathcal{M}_r(\mathbb{HF}(\varnothing))$. All structures of the kind $\mathcal{M}_r(\mathbb{A})$ are lattices with 0 and 1, moreover, $\mathcal{M}, \mathcal{M}_e, \mathcal{M}_w$ are isomorphic to the Medvedev, Dyment, and Muchnik lattices, respectively.

Proposition 11. *For any admissible set* $\mathbb{A}$ *and any reducibility symbol* $r \in \{e, s, w, ew\}$*, the structure*

$$\mathcal{M}_r(\mathbb{A}) = \langle P(P(A))/\equiv_r, \leqslant_r\rangle$$

is a distributive lattice with 0 *and* 1*.*

Proof. Fix some admissible set $\mathbb{A}$, together with some reducibility symbol $r \in \{e, s, w, ew\}$. For arbitrary $a \in A$ and $X \subseteq A$, we denote by $a * X$ the set

$\{\langle a,x\rangle | x\in X\}$. For any $\mathcal{X},\mathcal{Y}\subseteq P(A)$, define, as in the classical case,

$$\mathcal{X}\vee\mathcal{Y}=\{X\oplus Y|X\in\mathcal{X},Y\in\mathcal{Y}\},\ \mathcal{X}\wedge\mathcal{Y}=0*\mathcal{X}\cup 1*\mathcal{Y}.$$

It is easy to check that $\mathcal{X}\vee\mathcal{Y}$ and $\mathcal{X}\wedge\mathcal{Y}$ are the l.u.b. and the g.l.b. for $\mathcal{X}$ and $\mathcal{Y}$, correspondingly. We define $[\mathcal{X}]_r\vee[\mathcal{Y}]_r=[\mathcal{X}\vee\mathcal{Y}]_r$ and $[\mathcal{X}]_r\wedge[\mathcal{Y}]_r=[\mathcal{X}\wedge\mathcal{Y}]_r$. To prove the distributiveness, it is enough to check it straightforwardly as in the classical case. As usual, $\mathbf{1}=[\varnothing]_r$ and $\mathbf{0}=[\{\varnothing\}]_r$.

Recall [Stukachev2007] that the problem of presentability of a structure $\mathfrak{M}$ in an admissible set $\mathbb{A}$ is the mass problem

$$\mathrm{Pr}_{\mathbb{A}}(\mathfrak{M})=\{\mathcal{C}\,|\,\mathcal{C}\subseteq A,\,\mathcal{C}\simeq\mathfrak{M}\}.$$

Here, we identify presentations of structures with the atomic diagrams of these presentations and assume that some Gödel numbering for the signature symbols is fixed. We denote by $\mathfrak{M}\leqslant_r^{\mathbb{A}}\mathfrak{N}$ the fact that $\mathrm{Pr}_{\mathbb{A}}(\mathfrak{M})\leqslant_r\mathrm{Pr}_{\mathbb{A}}(\mathfrak{N})$. For an admissible set $\mathbb{A}$ and a reducibility symbol $r\in\{e,s,\,w,ew\}$, preorder $\leqslant_r^{\mathbb{A}}$ generates the semilattice $\mathcal{S}_r(\mathbb{A})$ of degrees of presentability of structures (with cardinality $\leqslant\mathrm{card}(\mathbb{A})$) in $\mathbb{A}$ w.r.t. reducibility r.

Recall that we denote by $\underline{\mathfrak{M}}$ the set $\mathrm{Pr}(\mathfrak{M},\mathbb{HF}(\varnothing))$ of presentations of $\mathfrak{M}$ in the least admissible set $\mathbb{HF}(\varnothing)$.

For a countable structure $\mathfrak{M}$, we consider the following classes consisting of structures that are effectively reducible to $\mathfrak{M}$:
$\mathcal{K}_\Sigma(\mathfrak{M})=\{\mathfrak{N}\mid\mathfrak{N}\leqslant_\Sigma\mathfrak{M}\}$,
$\mathcal{K}_e(\mathfrak{M})=\{\mathfrak{N}\mid\underline{\mathfrak{N}}\leqslant_e\underline{(\mathfrak{M},\bar{m})}$ for some $\bar{m}\in M^{<\omega}\}$,
$\mathcal{K}_s(\mathfrak{M})=\{\mathfrak{N}\mid\underline{\mathfrak{N}}\leqslant_s\underline{(\mathfrak{M},\bar{m})}$ for some $\bar{m}\in M^{<\omega}\}$,
$\mathcal{K}_{ew}(\mathfrak{M})=\{\mathfrak{N}\mid\underline{\mathfrak{N}}\leqslant_{ew}\underline{\mathfrak{M}}\}$,
$\mathcal{K}_w(\mathfrak{M})=\{\mathfrak{N}\mid\underline{\mathfrak{N}}\leqslant_w\underline{\mathfrak{M}}\}$.

It is known [Stukachev2008] that for any structure $\mathfrak{M}$, the following inclusions hold:

$$\mathcal{K}_\Sigma(\mathfrak{M})\subseteq\mathcal{K}_e(\mathfrak{M})\subseteq\mathcal{K}_s(\mathfrak{M})\subseteq\mathcal{K}_w(\mathfrak{M}),$$

and

$$\mathcal{K}_e(\mathfrak{M})\subseteq\mathcal{K}_{ew}(\mathfrak{M})\subseteq\mathcal{K}_w(\mathfrak{M}).$$

In general, all these inclusions are proper [Kalimullin2006,Kalimullin2009].

For any $r\in\{e,s,w,ew\}$, we define a relation $\leqslant_r$ on $\mathcal{K}_\omega$ by setting $\mathfrak{M}\leqslant_r\mathfrak{N}$ iff $\mathcal{K}_r(\mathfrak{M})\subseteq\mathcal{K}_r(\mathfrak{N})$ and letting $\mathcal{S}_r=\langle\mathcal{K}_\omega/\equiv_r,\leqslant_r\rangle$ be the structure of degrees of presentability corresponding to this relation.

Theorem 10 ([Stukachev2007]). *For any $r\in\{e,s,w,ew\}$, the structure $\mathcal{S}_r$ is an upper semilattice with 0, and the following embeddings ($\hookrightarrow$) and homomorphisms ($\to$) hold:*

$$\mathcal{D}\hookrightarrow\mathcal{D}_e\hookrightarrow\mathcal{S}_\Sigma\to\mathcal{S}_e\to\mathcal{S}\hookrightarrow\mathcal{M}.$$

The next theorem clarifies the role of Σ-reducibility as the strongest possible effective reducibility on structures. Moreover, it gives an "existential" characterization (item 1 below) of a "universal" sentence (item 2).

Theorem 11 ([StukachevTA]). *For any structures $\mathfrak{M}$, $\mathfrak{N}$, and any reducibility symbol $r \in \{e, ew\}$, the following are equivalent:*

1) $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$;
2) for any admissible set $\mathbb{A}$, $\mathfrak{M} \leqslant_r^{\mathbb{A}} \mathfrak{N}$.

Remark. The cardinalities of $\mathfrak{M}$, $\mathfrak{N}$, and $\mathbb{A}$ in this theorem are arbitrary: $\mathrm{Pr}_{\mathbb{A}}(\mathfrak{M}) = \varnothing$ (i.e., the "hardest" problem) if $\mathrm{card}(\mathbb{A}) < \mathrm{card}(\mathfrak{M})$.

As a corollary of the Jump Inversion Theorem for the semilattices of Σ-degrees and Theorem 11, we get

Proposition 12 ([StukachevTA]). *Suppose $\mathbb{A}$ is an admissible set, and r is a reducibility symbol, $r \in \{e, ew\}$. Let $\mathfrak{A}$ be a structure such that $\mathbf{0}' \leqslant_\Sigma \mathfrak{A}$. There exists a structure $\mathfrak{B}$ for which*

$$\mathfrak{B}' \equiv_r^{\mathbb{A}} \mathfrak{A}.$$

Proposition 13 ([StukachevTA]). *Let $\mathbb{A}$ be an admissible set with $\mathbf{0}' \in \mathbf{\Delta}(\mathbb{A})$. Then, for any generalized Σ-low structure $\mathfrak{A}$ and for any effective reducibility $r \in \{e, ew\}$,*

$$\mathfrak{A}' \equiv_r^{\mathbb{A}} \mathfrak{A}.$$

Proof. It is sufficient to consider the case of $\mathfrak{A}$ being a generalized Σ-low structure with $\mathrm{card}(\mathfrak{A}) \leqslant \mathrm{card}(\mathbb{A})$. Since there is a natural homomorphism from $\mathcal{S}_\Sigma(\mathrm{card}(\mathfrak{A}))$ into $\mathcal{M}_r^{\mathbb{A}}$, we get the desired statement.

Remark 3. Existence of the fixed points for the operation of Σ-jump w.r.t. Muchnik reducibility (i.e., existence of structures $\mathfrak{A}$ with the property $\mathfrak{A} \equiv_w \mathfrak{A}'$) was announced by the author in his talk at the CiE2009 Conference. However, the proof turned to be more complicated than he assumed. The corrected proof, valid for all abstract effective reducibilities, will appear in [StukachevTA].

Remark 4. If an admissible set $\mathbb{A}$ is recursively listed [Barwise1975] then Theorems 11, 12, and Proposition 13 are true also for reducibilities $r \in \{s, w\}$, since in this case an analogue of Theorem 2 from [Stukachev2007] is valid.

For arbitrary structures $\mathfrak{M}$ and $\mathfrak{M}'$ with the same signature and any $n \in \omega$, we denote by $\mathfrak{M} \equiv_n^{\mathrm{HF}} \mathfrak{M}'$ the fact that $\mathbb{HF}(\mathfrak{M}) \equiv_n \mathbb{HF}(\mathfrak{M}')$, and by $\mathfrak{M} \preccurlyeq_n^{\mathrm{HF}} \mathfrak{M}'$ the fact that $\mathbb{HF}(\mathfrak{M}) \preccurlyeq_n \mathbb{HF}(\mathfrak{M}')$. It is easy to check that, for $n < 2$, $\mathfrak{M} \equiv_n^{\mathrm{HF}} \mathfrak{M}'$ if and only if $\mathfrak{M} \equiv_n \mathfrak{M}'$. In case $n = 2$, $\mathfrak{M} \equiv_2^{\mathrm{HF}} \mathfrak{M}'$ if and only if, for any computable sequence $\{\varphi_{mn}(\overline{x}_m, \overline{y}_n) | m, n \in \omega\}$ of quantifier-free formulas of signature $\sigma_{\mathfrak{M}}$,

$$\mathfrak{M}' \models \bigvee_{m \in \omega} \exists \overline{x}_m \bigwedge_{n \in \omega} \forall \overline{y}_n \varphi_{mn}(\overline{x}_m, \overline{y}_n)$$

if and only if the same sentence is true in $\mathfrak{M}$.

For arbitrary structures $\mathfrak{M}$ and $\mathfrak{N}$, we denote by $\mathfrak{M} \leqslant_\exists \mathfrak{N}$ the fact that, for any tuple $\overline{m} \in M^{<\omega}$, there exists a tuple $\overline{n} \in N^{<\omega}$ such that $\mathrm{Th}_\exists(\mathfrak{M}, \overline{m}) \leqslant_e \mathrm{Th}_\exists(\mathfrak{N}, \overline{n})$. In particular, if $\mathfrak{M}$ is locally constructivizable then $\mathfrak{M} \leqslant_\exists \mathfrak{N}$ for any structure $\mathfrak{N}$. As it was noted in [Ershov1996], if $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ and $\mathfrak{N}$ is locally constructivizable then $\mathfrak{M}$ is also locally constructivizable. A straightforward generalization of this fact is as follows: $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ implies $\mathfrak{M} \leqslant_\exists \mathfrak{N}$.

Definition 7 ([Stukachev2007]). A structure $\mathfrak{M}$ is *uniformly locally constructivizable of level* n $(1 < n \leqslant \omega)$ if there exists a constructivizable structure $\mathfrak{N}$ for which $\mathfrak{M} \preccurlyeq_n^{\mathrm{HF}} \mathfrak{N}$.

For instance, the structure $\langle \omega_1^{CK}, \leqslant \rangle$ is uniformly locally constructivizable of level ω since $\langle \omega_1^{CK}, \leqslant \rangle \preccurlyeq^{\mathrm{HF}} \langle \omega_1^{CK}(1+\eta), \leqslant \rangle$, where the last ordering (known as the *Harrison ordering*) is constructivizable.

Proposition 14 ([Stukachev2008]). *If $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ and a structure $\mathfrak{N}$ is (uniformly) locally constructivizable of level n $(1 < n \leqslant \omega)$, then $\mathfrak{M}$ is also (uniformly) locally constructivizable of level n.*

The next proposition holds that a class of locally constructivizable (of level 1) countable structures is closed downward w.r.t. $\leqslant_w$, which is weakest among the reducibilities under consideration.

Proposition 15 ([Stukachev2008]). *Let $\mathfrak{M}$ and $\mathfrak{N}$ be structures. Then $\mathfrak{N} \leqslant_\exists \mathfrak{M}$ if $\mathfrak{N} \in \mathcal{K}_w(\mathfrak{M})$. In particular, if $\mathfrak{M}$ is locally constructivizable, then every structure $\mathfrak{N} \in \mathcal{K}_w(\mathfrak{M})$ is also locally constructivizable.*

A pair $(\mathfrak{M}, \mathfrak{N})$ is locally constructivizable iff so are $\mathfrak{M}$ and $\mathfrak{N}$; therefore, a set of degrees generated by locally constructivizable structures is an ideal in semilattices $\mathcal{S}_r$, $r \in \{\Sigma, e, s, w, ew\}$. Classes of locally constructivizable structures of level n, $n > 1$, however, are downward closed w.r.t. $\leqslant_\Sigma$ only (so they form initial segments in $\mathcal{S}_\Sigma$). For weaker reducibilities, this is not the case.

Theorem 12 ([Stukachev2008]). *There exists a countable structure $\mathfrak{M}_0$ which is locally constructivizable of level 1 exactly and is such that $\mathfrak{M}_0 \leqslant \mathfrak{M}$ for every nonconstructivizable countable structure $\mathfrak{M}$. If $\mathfrak{M}$ is locally constructivizable of level $n > 1$ but is not constructivizable, then $\mathcal{K}_\Sigma(\mathfrak{M}) \subsetneq \mathcal{K}(\mathfrak{M})$.*

The proof makes use of the result (obtained by T. Slaman [Slaman1998], and, independently, S. Wehner [Wehner1998]) which states that there exists a structure whose problem of presentability belongs to the least nonzero degree of the Medvedev lattice (which, in particular, means that a semilattice $\mathcal{S}$ of degrees of presentability has a least nonzero element). Every such structure is locally constructivizable. Namely, in [Stukachev2007] was proved the following fact:

Theorem 13. *There exist a countable structure $\mathfrak{M}$ and a unary relation $P \subseteq M$ for which $\underline{(\mathfrak{M}, P)} \equiv_s \underline{\mathfrak{M}}$ but $(\mathfrak{M}, P) \nleqslant_\Sigma \mathfrak{M}$.*

This theorem is of interest in connection with the following result from [AshKnightManasseSlaman1989,Chisholm1990]: for any countable structure $\mathfrak{M}$, a relation $P \subseteq M^n$, $n \in \omega$, is Σ-definable in $\mathbb{HF}(\mathfrak{M})$ iff $P^{\mathcal{C}}$ is $\mathcal{C} \upharpoonright \sigma_{\mathfrak{M}}$-c.e. for every $\mathcal{C} \in \underline{(\mathfrak{M}, P)}$.

The next result from [Stukachev2007] gives some sufficient conditions for the equality of the principal ideals generated by a structure $\mathfrak{M}$ with respect to different effective reducibilities.

Theorem 14. *If $\mathfrak{M}$ has a degree then $\mathcal{K}_{\Sigma}(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M}) = \mathcal{K}_s(\mathfrak{M}) = \mathcal{K}_w(\mathfrak{M})$. If $\mathfrak{M}$ has an e-degree then $\mathcal{K}_{\Sigma}(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M}) = \mathcal{K}_{ew}(\mathfrak{M})$.*

For an admissible set $\mathbb{A}$ and a structures $\mathfrak{M}$, consider the class

$$\mathcal{K}_s^{\mathbb{A}}(\mathfrak{M}) = \{\mathfrak{M}' \mid \mathrm{Pr}_{\mathbb{A}}(\mathfrak{M}') \leqslant_s \mathrm{Pr}_{\mathbb{A}}(\mathfrak{M}, \bar{m}) \text{ for some } \bar{m} \in M^{<\omega}\}.$$

Classes $\mathcal{K}_e^{\mathbb{A}}(\mathfrak{M})$, $\mathcal{K}_w^{\mathbb{A}}(\mathfrak{M})$, and $\mathcal{K}_{ew}^{\mathbb{A}}(\mathfrak{M})$ are defined similarly.

Proposition 16 ([Stukachev2007]). *Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable structures and let $\mathfrak{N}$ be a structure of the empty signature, or dense linear order. Then $\mathcal{K}_{\Sigma}(\mathfrak{M}) = \mathcal{K}_e^{\mathbb{HF}(\mathfrak{N})}(\mathfrak{M}) = \mathcal{K}_s^{\mathbb{HF}(\mathfrak{N})}(\mathfrak{M})$.*

As a consequence, there exist natural isomorphisms between a semilattice $\mathcal{S}_{\Sigma}$ of degrees of Σ-definability and semilattices $\mathcal{S}_r^{\mathbb{HF}(\mathfrak{N})}$ of degrees of presentability, where $\mathfrak{N}$ is a countable structure of empty signature, or dense linear order.

We mention one more result using the equivalence of "$\forall$-recursiveness" and "$\exists$-definability", based on results from [Lacombe1964], [Moschovakis1969a], and [AshKnightManasseSlaman1989,Chisholm1990].

Theorem 15 ([Stukachev2007]). *For any countable structures $\mathfrak{M}$ and $\mathfrak{N}$ and any relation $R \subseteq \mathbb{HF}(\mathfrak{N})$, the following conditions are equivalent:*

1) $R \leqslant_{e\Sigma} \mathcal{C}$ for every presentation $\mathcal{C}$ of $\mathfrak{M}$ in the admissible set $\mathbb{HF}(\mathfrak{N})$;
2) R is Σ-definable in $\mathbb{HF}(\mathfrak{M}, \mathfrak{N})$, where $(\mathfrak{M}, \mathfrak{N})$ is the pair of $\mathfrak{M}$ and $\mathfrak{N}$).

Definition 8. Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable structures. Structure $\mathfrak{M}$ *has a degree* (an *e-degree*) *over structure* $\mathfrak{N}$ if there exists a least degree among all $T\Sigma$-degrees ($e\Sigma$-degrees) of all possible presentations of $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{N})$.

An immediate consequence of Theorem 15 is the following generalization of item 2:

Theorem 16. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable structures. Then the conditions below are equivalent:*

1) $\mathfrak{M}$ has a degree (an e-degree) over $\mathfrak{N}$;
2) some presentation $\mathcal{C} \subseteq HF(N)$ of $\mathfrak{M}$ is Δ-subset (Σ-subset) in $\mathbb{HF}(\mathfrak{M}, \mathfrak{N})$.

As a corollary, for $\mathfrak{M} \leqslant_\exists \mathfrak{N}$, the structure $\mathfrak{M}$ has a degree (an e-degree) over $\mathfrak{N}$ iff $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$. It is also true that if $\mathfrak{M}$ has a degree (an e-degree) over $\mathfrak{N}$, and $\mathfrak{N} \leqslant_\Sigma \mathfrak{N}'$, then $\mathfrak{M}$ has a degree (an e-degree) over $\mathfrak{N}'$. Furthermore, we have for any countable structure $\mathfrak{A}$, there exists a structure $\mathfrak{M}$ which has a degree but is not Σ-definable in $\mathbb{HF}(\mathfrak{A})$.

As in the nonrelativized case, we have the following result:

Theorem 17. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable structures. If $\mathfrak{M}$ has a degree over $\mathfrak{N}$, then $\mathcal{K}_\Sigma(\mathfrak{M}) = \mathcal{K}_e^{\mathbb{HF}(\mathfrak{N})}(\mathfrak{M}) = \mathcal{K}_s^{\mathbb{HF}(\mathfrak{N})}(\mathfrak{M})$. If $\mathfrak{M}$ has an e-degree over $\mathfrak{N}$, then $\mathcal{K}_\Sigma(\mathfrak{M}) = \mathcal{K}_e^{\mathbb{HF}(\mathfrak{N})}(\mathfrak{M})$.*

6 Effective Presentations of Special Structures

As it was already mentioned, cardinality boundaries are unavoidable in the classical theory of computability (CTC). Numberings allow to use CTC for countable objects. Admissible sets of the form $\mathbb{HF}(\mathfrak{M})$ can have an arbitrary cardinality. Hence, the following question naturally arise: does there exists a "reasonably good" theory T such that the class of admissible sets of the form $\mathbb{HF}(\mathfrak{M})$, with $\mathfrak{M} \models T$, allows to extend, in some natural way, the classical theory CTC to the case of objects with an arbitrary cardinality.

Recall that a theory T of a finite signature is called *regular* [Ershov1996] if it is decidable and model complete.

Remark 5. Let T be a regular theory. Then, for any formula $\Phi(\overline{x})$ of the signature of T, there exists an $\exists$-formula $\Psi(\overline{x})$ which is equivalent (w.r.t. the theory T) to $\Phi(\overline{x})$. Moreover, $\Psi(\overline{x})$ can be found effectively from $\Phi(\overline{x})$.

Recall that a theory T is called *c-simple* (constructively simple) [Ershov1996] if it is regular, ω-categorical, and has a decidable set of the complete formulas.

Remark 6. In [Ershov1996] such theories were called simple, but this terminology was simultaneously used in model theory for a different notion.

In the definition of a c-simple theory, ω-categoricity gives the uniqueness, up to an isomorphism, of a countable model of such theory. Model completeness, decidability of a theory, and decidability of the set of its complete formulas, guarantee the autostability of every constructivization of this countable theory, i.e., the uniqueness of the "computability" on its countable models.

Furthermore, if T is a c-simple theory, $\mathfrak{M}_0$ and $\mathfrak{M}_1$ are any models of T ($\mathfrak{M}_i \models T$, $i = 0, 1$), then $\mathbb{HF}(\mathfrak{M}_0) \equiv \mathbb{HF}(\mathfrak{M}_1)$, since the models of ω-categorical theories are saturated enough ([Ershov1996]).

Henceforth, for a c-simple theory T, the class of admissible sets of the form $\mathbb{HF}(\mathfrak{M})$, $\mathfrak{M} \models T$, extends "uniformly" the classical theory of computability for arbitrary infinite cardinalities.

An example of a c-simple theory is the theory T_E of infinite structures with the empty signature. But this theory is too "weak", if we regard a theory T

being "strong" in case there are enough of uncountable structures Σ-definable in $\mathbb{HF}(\mathfrak{M})$, $\mathfrak{M} \models T$. The reason of the "weakness" of T_E is the following property: for an arbitrary set X and arbitrary permutation f on X, f can be extended (in a unique way) to an automorphism f^* of $\mathbb{HF}(X)$.

Following [Ershov1985,Ershov1995,Ershov1996], we present a characterization of the theories having uncountable models which are Σ-definable in $\mathbb{HF}(\mathfrak{L})$ for $\mathfrak{L} \models T_{DLO}$.

The category ${}^*\omega$ is defined as follows. Its objects are the sets of the form $[\mathbf{n}] \rightleftharpoons \{0, 1, \ldots, n-1\}$, $n \in \omega$ ($[\mathbf{0}] \rightleftharpoons \varnothing$), and its morphisms are order-preserving embeddings. It should be noted that there is a unique morphism from $[\mathbf{0}]$ into $[\mathbf{n}]$ for any $n \in \omega$.

Definition 9. *By a ${}^*\omega$-spectrum we mean any functor S from the category ${}^*\omega$ into the category Mod^*_σ of structures (of some fixed signature σ), whose morphisms are all possible embeddings.*

To define a ${}^*\omega$-spectrum S, it is necessary to give an infinite sequence $\mathfrak{M}_0$, $\mathfrak{M}_1$, ..., $\mathfrak{M}_n$, ..., $n \in \omega$, of structures of signature σ, and associate with each order-preserving embeddings $\mu : [\mathbf{n}] \to [\mathbf{m}]$ an embedding $\mu_* : \mathfrak{M}_n \to \mathfrak{M}_m$ so that, if $\mu_0 : [\mathbf{n}] \to [\mathbf{m}]$ and $\mu_1 : [\mathbf{m}] \to [\mathbf{k}]$, $n \leqslant m \leqslant k \in \omega$, are morphisms of the category ${}^*\omega$, then $(\mu_0\mu_1)_* = \mu_{1*}\mu_{0*}$, and if $\mu : [\mathbf{n}] \to [\mathbf{n}]$ is the unique morphism from $[\mathbf{n}]$ into $[\mathbf{n}]$ ($= \mathrm{id}_{[\mathbf{n}]}$), then $\mu_* = \mathrm{id}_{\mathfrak{M}_n} : \mathfrak{M}_n \to \mathfrak{M}_n$, $n \in \omega$.

If the ${}^*\omega$-spectrum $S = \{\mathfrak{M}_n, \mu_* | n \in \omega, \mu \in \mathrm{Mor}^*\omega\}$ has been defined, then for any linearly ordered set L, it is possible to define the structure $\mathfrak{M}_L(\mathfrak{M}^S_L)$ as a direct limit $\lim_{\boldsymbol{L_0}} \mathfrak{M}'_{L_0}$ of the spectrum

$$\{\mathfrak{M}'_{L_0}, \varphi_{L_0,L_1} \mid L_0 \subseteq L_1 \subseteq L,\ L_1 \text{ is finite}\},$$

where $\mathfrak{M}'_{L_0} \rightleftharpoons \mathfrak{M}_n$, if $L_0 \subseteq L$ is finite and $|L_0| = n$, and the embedding $\varphi_{L_0,L_1} : \mathfrak{M}'_{L_0} \to \mathfrak{M}'_{L_1}$ is defined for finite $L_0 \subseteq L_1 (\subseteq L)$ as follows: if $L_1 = \{l_0 < l_1 < \ldots < l_{m-1}\}$ and $L_0 = \{l_{i_0} < l_{i_1} < \ldots < l_{i_{n-1}}\}$ (in which case $0 \leqslant i_0 < i_1 < \ldots < i_{n-1} \leqslant m$) and $\mu : [\mathbf{n}] \to [\mathbf{m}]$ is defined as $\mu(j) \rightleftharpoons i_j$, $j < n$, then

$$\varphi_{L_0,L_1} \rightleftharpoons \mu_* : \mathfrak{M}'_{L_0} = \mathfrak{M}_n \to \mathfrak{M}_m = \mathfrak{M}'_{L_1}.$$

If $L \subseteq L'$ are linearly ordered sets, then the structure $\mathfrak{M}_L$ can be identified with a substructure of $\mathfrak{M}_{L'}$ in a natural way.

Any isomorphism between linearly ordered sets L and L' induces an isomorphism between $\mathfrak{M}_L$ and $\mathfrak{M}_{L'}$. Also if $L \subseteq L'$ are dense linear orders without endpoints, then $\mathfrak{M}_L \preccurlyeq \mathfrak{M}_{L'}$. As a corollary, if L and L' are dense linear orders without endpoints, then $\mathfrak{M}_L \equiv \mathfrak{M}_{L'}$.

Let μ_0 and μ_1 be morphisms from $[\mathbf{1}]$ into $[\mathbf{2}]$ such that $\mu_0(0) = 0$ and $\mu_1(0) = 1$. The condition

$$\mu_{0*} \neq \mu_{1*}. \quad (*)$$

is sufficient for $|\mathfrak{M}^S_L| \geqslant |L|$ to hold for any linearly ordered set L.

Definition 10. *A system of numberings $\nu_n : \omega \to M_n$, $n \in \omega$, is called a computable sequence of constructivization*

$$(\mathfrak{M}_0, \nu_0), (\mathfrak{M}_1, \nu_1), \ldots, (\mathfrak{M}_n, \nu_n), \ldots, \quad n \in \omega,$$

if the following conditions hold (we assume that the signature σ of the structures $\mathfrak{M}_0, \mathfrak{M}_1, \ldots$ is finite and without function symbols):

1) *$E \rightleftharpoons \{\langle n, m_0, m_1\rangle | n, m_0, m_1 \in \omega, \nu_n(m_0) = \nu_n(m_1)\}$ is a Δ-predicate on ω;*
2) *$N_P \rightleftharpoons \{\bar{n} = \langle n_0, n_1, \ldots, n_k\rangle | \bar{n} \in \omega^{k+1}, \langle \nu_{n_0}(n_1), \ldots, \nu_{n_0}(n_k)\rangle \in P^{\mathfrak{M}_{n_0}}\}$ is a Δ-predicate on ω for any (k-ary) predicate symbol $P \in \sigma$;*
3) *for any constant symbol $c \in \sigma$ there exists a Σ-function $f_c : \omega \to \omega$ such that $c^{\mathfrak{M}_n} = \nu_n f_c(n)$.*

Every morphism $\mu : [\mathbf{n}] \to [\mathbf{m}]$ of the category ${}^*\omega$ is uniquely defined by the number m and the subset $\mu([\mathbf{n}]) \subseteq [\mathbf{m}]$. This remark allows one to define a one-to-one correspondence $\mu^* : \Delta \to \mathrm{Mor}^*\omega$ between the subset $\Delta \rightleftharpoons \{n | n \in \omega, r(n) < 2^{l(n)}\} \subseteq \omega$ and the set $\mathrm{Mor}^*\omega$, provided that $n \in \Delta$ is assumed to code the morphism $\mu : [\mathbf{k}] \to [\mathbf{l}]$ such that $l = l(n)$ and $r(n)$ is the number of the subset $\mu([\mathbf{k}]) \subseteq [\mathbf{l}] = [\mathbf{l}(\mathbf{n})]$ in some standard listing of the finite subsets of ω (here, $l(n)$ and $r(n)$ are the left and right projections). It is evident that Δ is a Δ-subset of ω.

Definition 11. *Let $S = \{\mathfrak{M}_n, \mu_* | n \in \omega, \mu \in \mathrm{Mor}^*\omega\}$ be a ${}^*\omega$-spectrum. By a constructivization of S we mean any computable sequence of constructivizations*

$$(\mathfrak{M}_0, \nu_0), (\mathfrak{M}_1, \nu_1), \ldots, (\mathfrak{M}_n, \nu_n), \ldots, \quad n \in \omega,$$

together with a Σ-function $f : \Delta \times \omega \to \omega$ such that, for any $n, m, k \in \omega$ and $\mu : [\mathbf{n}] \to [\mathbf{m}] \in \mathrm{Mor}^*\omega$, if $n^* \in \Delta$ is such that $\mu^*(n^*) = \mu$, then $\mu_*\nu_n(k) = \nu_m f(n^*, k)$.

A ${}^*\omega$-spectrum S is called constructivizable if there exists a constructivization for it.

Theorem 18 ([Ershov1996]). *Let L be a dense linear order without endpoints. A theory T has an uncountable model Σ-definable in $\mathbb{HF}(L)$ if and only if there exists a constructivizable ${}^*\omega$-spectrum S, satisfying condition ($*$), and such that $\mathfrak{M}_L^S \models T$.*

One of the important corollaries of this theorem is the first part of the following result, showing that the field $\mathbb{C}$ of complex numbers is rather "simple". The second part shows that $\mathbb{C}$ is not "too simple".

Theorem 19 ([Ershov1996]).

1) *$\mathbb{C}$ is Σ-definable in $\mathbb{HF}(\mathfrak{L})$ for any dense linear order $\mathfrak{L}$ of size continuum;*
2) *$\mathbb{C}$ is not Σ-definable in $\mathbb{HF}(\mathcal{S})$ for any structure $\mathcal{S}$ with empty signature.*

Another example of a c-simple theory is the theory T_{DLO} of dense linear orders (without endpoints). This theory seems to be quite reasonable candidate for a "correct extension of CTC for arbitrary cardinalities". Below we present two different characterizations of the theories having uncountable models which are Σ-definable in $\mathbb{HF}(\mathfrak{L})$ $\mathfrak{L} \models T_{DLO}$.

We now formalize a desired property of T_{DLO} to be the "strongest" in the class of c-simple theories.

Conjecture 1 ([Ershov1998]). Suppose a theory T has an uncountable model which is Σ-definable in $\mathbb{HF}(\mathfrak{M})$, for some structure $\mathfrak{M}$ with a c-simple theory. Then T has an uncountable model which is Σ-definable in $\mathbb{HF}(\mathfrak{L})$ for some $\mathfrak{L} \models T_{DLO}$.

It is an open question whether this conjecture is equivalent to the following one (which is its formal consequence).

Conjecture 2. Any c-simple theory has an uncountable model which is Σ-definable in $\mathbb{HF}(\mathfrak{L})$ for some $\mathfrak{L} \models T_{DLO}$.

It turned out that there are counterexample to Conjectures 1 and 2. The next definition is a generalization of the model-theoretical notions of order and total indiscernibility.

Definition 12. *For structures $\mathfrak{A}$, $\mathfrak{B}$ and some $k > 0$, a set $I \subseteq A^k \cap B$ is called a* set of $\mathfrak{A}$-indiscernibles in $\mathfrak{B}$ *(with dimension k) if for any pair of tuples $\bar{i}, \bar{i}' \in I^{<\omega}$ with the same length,*

$$\langle \mathfrak{A}, \bar{i} \rangle \equiv \langle \mathfrak{A}, \bar{i}' \rangle \text{ implies } \langle \mathfrak{B}, \bar{i} \rangle \equiv \langle \mathfrak{B}, \bar{i}' \rangle.$$

Proposition 17 ([Stukachev2010]). *Suppose $\mathfrak{A}$ is an uncountable structure, a structure $\mathfrak{B}$ is saturated enough and locally constructivizable of level ω, and let $\mathfrak{A} \leqslant_\Sigma \mathfrak{B}$. There exist computable structures $\mathfrak{A}_0$ and $\mathfrak{B}_0$ such that $\mathfrak{A}_0 \equiv \mathfrak{A}$, $\mathfrak{B}_0 \equiv \mathfrak{B}$, and there is an infinite computable set of $(\mathfrak{B}_0, \bar{b}_0)$-indiscernibles in $\mathfrak{A}_0$ with dimension k, for some $k > 0$ and $\bar{b}_0 \in (B_0)^{<\omega}$.*

For certain c-simple theories this necessary condition of Σ-definability of uncountable models can be simplified (by assuming the dimension to be equal 1). Namely, for theory T_{DLO} of dense linear orders without endpoints, and the theory T_E of infinite structures with empty signature, there is the following theorem which is a correct part of the corresponding incorrect statement from [Stukachev2004] (the proof can be found in [Stukachev2010]).

Theorem 20. *Let T be a c-simple theory, and let $\mathfrak{A}$ be any computable model of T. Then*

1) *if there exists an uncountable $\mathfrak{M} \models T$ such that $\mathfrak{M} \leqslant_\Sigma \mathfrak{L}$, $\mathfrak{L} \models T_{DLO}$, then there exists an infinite computable set of order indiscernibles in $\mathfrak{A}$ (with dimension 1);*

2) *if there exists an uncountable $\mathfrak{M} \models T$ such that $\mathfrak{M} \leqslant_\Sigma \mathcal{S}$, $\mathcal{S} \models T_E$, then there exists an infinite computable set of total indiscernibles in $\mathfrak{A}$ (with dimension 1).*

In the case of an infinite signature the counterexample to Conjecture 2 was obtained using the construction from [KiersteadRemmel1985] together with Theorem 20.

Theorem 21 ([Stukachev2004]). *There is a sc-simple theory T of an infinite computable signature, such that, for any uncountable $\mathfrak{A} \models T$ and any $\mathfrak{L} \models T_{DLO}$, we have $\mathfrak{A} \nleqslant_\Sigma \mathfrak{L}$.*

Using Theorem 21 and the Hrushovski's construction which allow to interpret countably categorical structures with an infinite signature in countably categorical structures with a finite signature, we get the following result.

Theorem 22 ([StukachevTA]). *There is a c-simple theory T with a finite signature, such that, for any uncountable $\mathfrak{A} \models T$ and any $\mathfrak{L} \models T_{DLO}$, we have $\mathfrak{A} \nleqslant_\Sigma \mathfrak{L}$.*

It is known that Conjecture 2 is true for rather a "rich" class of c-simple theories (see Theorem 23 below).

Definition 13. *Let $n \in \omega$. A (first-order) theory T is called* n-discrete *if any finite type of T is uniquely determined by its n-subtypes.*

A theory T is called *discrete* if it is n-discrete for some $n \in \omega$. If T is n-discrete and has a finite number of n-types then T is ω-categorical and submodel complete in some expansion by a finite number of definable predicates. Any regular n-discrete theory with a finite number of n-types is c-simple. Also, any submodel complete theory of a finite relational signature is n-discrete with a finite number of n-types, for some $n \in \omega$, and any ω-categorical submodel complete theory of a finite signature is n-discrete with a finite number of n-types, for some $n \in \omega$.

A theory T is called *sc-simple* [Stukachev2010] if it is ω-categorical, submodel complete, decidable, and has a decidable set of the complete formulas. Henceforth, a theory (of a finite signature) is sc-simple if it is c-simple and submodel complete.

As corollary of the Ehrenfeucht-Mostowski Theorem we get the next result.

Proposition 18 ([Stukachev2010]). *If T is a sc-simple theory of a finite signature then, in any computable model of T, there exists an infinite computable set of order indiscernibles.*

Using this fact, we get a partial positive answer to Conjecture 2.

Theorem 23 ([Stukachev2010]). *Let T be sc-simple theory of a finite signature. There exists an uncountable model $\mathfrak{A}$ of T such that $\mathfrak{A} \leqslant_\Sigma \mathfrak{L}$, $\mathfrak{L} \models T_{DLO}$.*

We now present some examples of *sc*-simple theories constructed via Fraïssé limits.

Let $FinGraph$ be the class of all finite symmetric graphs. It is easy to check that this class satisfies the properties HP, JEP, AP, and has a ULF-computable presentation.

Definition 14. *A symmetric graph* $\mathfrak{A}$ *is called* random *if, for any finite* $X, Y \subseteq A$ *such that* $X \cap Y = \emptyset$*, there is a vertex* $v \in A \setminus (X \cup Y)$ *such that* v *is adjacent with all vertices from* X *and not adjacent with vertices from* Y*.*

Proposition 19 ([Hodges1993]). *If* $\mathfrak{A}$ *is the Fraïssé limit of the class* $FinGraph$ *then* $\mathfrak{A}$ *is a random graph. As a corollary,* $\mathrm{Th}(\mathfrak{A})$ *is sc-simple.*

Let σ be a finite predicate signature. The class $Fin(\sigma)$ of all finite structures of signature σ satisfies the properties HP, JEP, AP, and has a ULF-computable presentation.

Definition 15. *Let* σ *be a finite predicate signature. A* random structure $Ran(\sigma)$ *of signature* σ *is the Fraïssé limit of the class* $Fin(\sigma)$*.*

A structure $\mathfrak{A}$ is called *locally constructivizable* [Ershov1996] if $\mathrm{Th}_{\exists}(\mathfrak{A}, \bar{\mathrm{a}})$ is c.e. for any $\bar{a} \in A^{<\omega}$. It is easy to verify that a structure $\mathfrak{A}$ is locally constructivizable if and only if, for any $\bar{a} \in A^{<\omega}$, there exist a constructivizable structure $\mathfrak{B}$ and a tuple $\bar{b} \in B^{<\omega}$ such that $(\mathfrak{A}, \bar{a}) \equiv_1 (\mathfrak{B}, \bar{b})$ (or, which is the same, $\mathbb{HF}(\mathfrak{A}, \bar{a}) \equiv_1 \mathbb{HF}(\mathfrak{B}, \bar{b})$). Symbol $\equiv_\alpha$, here and further on, denotes elementary equivalence w.r.t. the class of formulas with less than α groups of alternating groups of quantifieres in the prenex normal form ($0 \leqslant \alpha \leqslant \omega$). Henceforth, the next definition is a generalization of the notion of local constructivizability.

Definition 16 ([Stukachev2008]). *A structure* $\mathfrak{A}$ *is called* locally constructivizable of level α *($0 < \alpha \leqslant \omega$) if for any* $\bar{a} \in A^{<\omega}$ *there exists a constructivizable structure* $\mathfrak{B}$ *and a tuple* $\bar{b} \in B^{<\omega}$ *such that*

$$\mathbb{HF}(\mathfrak{A}, \bar{a}) \equiv_\alpha \mathbb{HF}(\mathfrak{B}, \bar{b}).$$

Local constructivizability of any level is preserved by Σ-definability.

Proposition 20 ([Stukachev2008]). *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be such that* $\mathfrak{A} \leqslant_\Sigma \mathfrak{B}$ *and* $\mathfrak{B}$ *is locally constructivizable of level* α*,* $0 < \alpha \leqslant \omega$*. Then* $\mathfrak{A}$ *is also locally constructivizable of level* α*.*

A structure $\mathfrak{A}_0$ is called *saturated enough* [Ershov1996] if there exists an ω-saturated structure $\mathfrak{A}_1$ such that $\mathfrak{A}_0 \preccurlyeq \mathfrak{A}_1$ and $\mathbb{HF}(\mathfrak{A}_0) \preccurlyeq \mathbb{HF}(\mathfrak{A}_1)$. Any structure with a *c*-simple theory is saturated enough and locally constructivizable of level ω. Moreover, its countable "computable simulation", in the terminology from [MillerMulcahey2008], is unique up to the computable isomorphism. The situation is different in the case of regular theories: there are structures with a regular theory, which are not locally constructivizable even of level 1. For example, consider the fields $\mathbb{R}$ and $\mathbb{Q}_p$ of real and *p*-adic numbers.

Corollary 3 ([Ershov1996]). *For any linear order $\mathfrak{L}$, fields $\mathbb{R}$ and $\mathbb{Q}_p$ are not Σ-definable in $\mathbb{HF}(\mathfrak{L})$.*

We conclude with the list of references which are relevant to the topics discussed in the paper.

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