On Mass Problems of Presentability*

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Abstract. We consider the notion of mass problem of presentability for countable structures, and study the relationship between Medvedev and Muchnik reducibilities on such problems and possible ways of syntactically characterizing these reducibilities. Also, we consider the notions of strong and weak presentability dimension and characterize classes of structures with presentability dimensions 1.

1 Basic notions and facts

The main problem we consider in this paper is the relationship between presentations of countable structures on natural numbers and on admissible sets. Most of notations and terminology we use here are standard and corresponds to [4, 1, 13]. We denote the domains of a structures $\mathfrak{M}, \mathfrak{N}, \ldots$ by $M, N.\ldots$. For any arbitrary structure $\mathfrak{M}$ the hereditary finite superstructure $\mathbb{HF}(\mathfrak{M})$, which is the least admissible set containing the domain of $\mathfrak{M}$ as a subset, enables us to study effective (computable) properties of $\mathfrak{M}$ by means of computability theory for admissible sets. The exact definition is as follows: the hereditary finite superstructure $\mathbb{HF}(\mathfrak{M})$ over a structure $\mathfrak{M}$ of signature σ is a structure of signature $\sigma' = \sigma \cup \{U^1, \in^2\}$, whose universe is $HF(M) = \bigcup_{n\in\omega} H_n(M)$, where $H_0(M) = M$, $H_{n+1}(M) = H_n(M) \cup \{a | a \subseteq H_n(M), \mathrm{card}(a) < \omega\}$, the predicate U distinguish the set of the elements of the structure $\mathfrak{M}$ (regarded as urelements), while the relation $\in$ has the usual set theoretic meaning.

In the class of all formulas of signature σ' we define the subclass of Δ_0-formulas as the closure of the class of atomic formulas under $\wedge, \vee, \neg, \rightarrow, \exists x \in y, \forall x \in y$; the class of Σ-formulas is the closure of the class of Δ_0-formulas under $\wedge, \vee, \neg, \rightarrow, \exists x \in y, \forall x \in y$, and the quantifier $\exists x$; the class of Π-formulas is defined in the same way, allowing the quantifier $\forall x$ instead of $\exists x$. A relation on $\mathbb{HF}(\mathfrak{M})$ is called Σ-definable (Π-definable) if it is defined by a corresponding formula, possibly with parameters; it is called Δ-definable if it is Σ- and Π-definable at the same time.

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In all that follows, we consider only countable structures of finite signatures. For a countable structure $\mathfrak{M}$, a presentation of $\mathfrak{M}$ on the natural numbers, or simply *a presentation of* $\mathfrak{M}$, is any structure $\mathcal{C}$ such that $\mathcal{C} \cong \mathfrak{M}$ and the domain of $\mathcal{C}$ is a subset of ω (the relation $=$ is assumed to be a congruence relation on $\mathcal{C}$ and may differ form the normal equality relation on C). We can also treat the atomic diagram of a presentation as a subset of ω, using some Gödel numbering of the atomic formulas of the signature of $\mathfrak{M}$. So any presentation, identified with its atomic diagram, can be considered as a subset of ω.

A *mass problem*, as introduced by Yu.T. Medvedev [7], is any set of total functions from ω to ω. Intuitively, a mass problem can be considered as a set of "solutions" (in form of functions from ω to ω) of some "informal problem". Below we list some examples of mass problems which correspond to well-known informal problems from computability theory:

1) the *problem of solvability* of a set $A \subseteq \omega$ is the mass problem $\mathcal{S}_A = \{\chi_A\}$, where χ_A is the characteristic function of A;
2) the *problem of enumerability* of a set $A \subseteq \omega$ is the mass problem $\mathcal{E}_A = \{f : \omega \to \omega \mid \mathrm{rng}(f) = A\}$;
3) the *problem of separability* of a pair of sets $A, B \subseteq \omega$ is the mass problem $\mathcal{P}_{A,B} = \{f : \omega \to 2 \mid f^{-1}(0) \supseteq A,\ f^{-1}(1) \supseteq B\}$.

In this paper we consider another class of mass problems – problems of presentability, which corresponds to the main informal problem in computable model theory, the problem of presentability of structures on natural numbers. For a structure $\mathfrak{M}$, we consider the set of all possible presentations of $\mathfrak{M}$. The set of characteristic functions of the atomic diagrams of such presentations forms the mass problem

$$\underline{\mathfrak{M}} = \{\ \chi_{D(\mathcal{C})} \mid \mathcal{C} \text{ is a presentation of } \mathfrak{M}\ \}.$$

We call this mass problem *the problem of presentability of* $\mathfrak{M}$.

Note that for any presentation $\mathcal{C} \in \mathfrak{M}$ its domain C is effectively recognizable from (more precisely, Turing reducible to) the function $\chi_{D(\mathcal{C})}$, since $c \in C$ iff $(c = c) \in D(\mathcal{C})$. It is also clear that if $\mathcal{C}, \mathcal{D}$ are presentations (maybe of different structures) then $\chi_{D(\mathcal{C})} \leqslant_T \chi_{D(\mathcal{D})}$ if and only if $\chi_{D(\mathcal{C})} \leqslant_e \chi_{D(\mathcal{D})}$.

For a structure $\mathfrak{M}$, one could also study the set

$$\underline{\underline{\mathfrak{M}}} = \{\ \chi^*_{D(\mathcal{C})} \mid \mathcal{C} \text{ is a presentation of } \mathfrak{M}\ \}$$

of partial characteristic functions of the atomic diagrams of all presentations of $\mathfrak{M}$ (recall that, for a set $A \subseteq \omega$, $\chi_A(n) = 0$ if $n \in A$, and $\chi_A(n)$ is undefined otherwise). Any such set is a partial mass problem in the sense of E.Z. Dyment [3], and we will call them *partial problems of presentability.* Such problems, in a different terminology, were considered with respect to classes of finite structures in [2]. In this case enumeration reducibility, the main object of study in [2], is no longer equivalent to Turing reducibility.

In [7] it was introduced a notion of reducibility between mass problems. If $\mathcal{A}$ and $\mathcal{B}$ are mass problems, then $\mathcal{A}$ is said to be *reducible* to $\mathcal{B}$ (denoted by $\mathcal{A} \leqslant \mathcal{B}$), if there exists a recursive operator Ψ such that $\Psi(\mathcal{B}) \subseteq \mathcal{A}$. Informally, $\mathcal{A}$ is reducible to $\mathcal{B}$ if there exists an uniform effective procedure, which, given any "solution" from $\mathcal{B}$, transforms it to some "solution" from $\mathcal{A}$.

The equivalence relation $\equiv$ on mass problem is defined from $\leqslant$ in the usual way: $\mathcal{A} \equiv \mathcal{B}$ if $\mathcal{A} \leqslant \mathcal{B}$ and $\mathcal{B} \leqslant \mathcal{A}$. Equivalence classes of mass problems under $\equiv$ (which are called degrees of difficulty), together with the relation of reducibility $\leqslant$, form a distributive lattice known as the *Medvedev lattice* [7].

There is another important notion of reducibility between mass problems, which was introduced by A.A. Muchnik [9]. Namely, if $\mathcal{A}$ and $\mathcal{B}$ are mass problems, then $\mathcal{A}$ is said to be *weakly reducible* to $\mathcal{B}$ (denoted by $\mathcal{A} \leqslant_w \mathcal{B}$), if, for any $f \in \mathcal{B}$, there is some recursive operator Ψ such that $\Psi(f) \in \mathcal{A}$. So the weak (we will also call it Muchnik) reducibility is obtained from the strong (Medvedev) reducibility by dropping the uniformity requirement. The equivalence relation $\equiv_w$ on mass problem is defined from $\leqslant_w$ in the usual way; equivalence classes of mass problems under $\equiv_w$ with the relation of reducibility $\leqslant_w$ also form a distributive lattice known as the *Muchnik lattice* [9].

There is also another important notion – that of the Dyment lattice [3] – which we recall now. If $\mathcal{A}$ and $\mathcal{B}$ are partial mass problems, $\mathcal{A}$ is said to be *enumeration reducible* (or Dyment reducible) to $\mathcal{B}$ (denoted by $\mathcal{A} \leqslant_e \mathcal{B}$) if for some partial recursive operator Ψ we have $\mathcal{B} \subseteq \mathrm{dom}(\Psi)$ and $\Psi(\mathcal{B}) \subseteq \mathcal{A}$. The Dyment lattice consists of the equivalence classes of partial mass problems under the enumeration reducibility. In the same way as for the Medvedev lattice, we introduce the nonuniform version $\leqslant_{ew}$ of the Dyment reducibility.

In this paper we will consider the reducibilities $\leqslant$ and $\leqslant_w$ for the class of problems of presentability and $\leqslant_e$ and $\leqslant_{ew}$ for the class of partial problems of presentability. There is a syntactical characterization of these reducibilities in the case of problems of enumerability, which follows from a result obtained by A. Selman [11] and rediscovered by M. Rozinas [10]: for any $A, B \subseteq \omega$, $A \leqslant_e B$ if and only if, for any $X \subseteq \omega$, the fact that B is X-c.e. implies that A is X-c.e.. From this theorem we directly obtain that, for any $A, B \subseteq \omega$,

$$\mathcal{E}_A \leqslant_w \mathcal{E}_B \iff \mathcal{E}_A \leqslant \mathcal{E}_B \iff A \leqslant_e B.$$

Besides the syntactical characterization, it implies the fact (observed also in [8]) that Medvedev and Muchnik reducibilities coincide on the class of problems of enumerability.

It is clear that (strong) Medvedev reducibility always implies (weak) Muchnik reducibility: for any mass problems $\mathcal{A}, \mathcal{B}$, $\mathcal{A} \leqslant \mathcal{B} \Rightarrow \mathcal{A} \leqslant_w \mathcal{B}$. In [9] was established a sufficient condition under which these reducibilities are equivalent. In this paper we will consider the problem of describing the relationship between uniform and nonuniform reducibilities in the case of mass problems of presentability.

We recall a sufficient condition from [9]. By a finite function we will mean a function $\tilde{f} : n \to \omega$, where $n < \omega$. An *open interval* is a mass problem of the form

$\mathcal{I}_{\tilde f} = \{f : \omega \to \omega \mid \tilde f \subseteq f\}$ for some finite function $\tilde f$. The Baire topology on ω^ω is generated by open intervals as a basis for the class of the open sets. A mass problem is called *closed* if it is a closed subset of ω^ω in the Baire topology. A mass problem $\mathcal{A}$ is called *uniform* if, for any open interval $\mathcal{I}_{\tilde f}$ such that $\mathcal{A} \cap \mathcal{I}_{\tilde f} \neq \varnothing$, we have $\mathcal{A} \cap \mathcal{I}_{\tilde f} \leqslant \mathcal{A}$.

Let $\mathcal{A}$ be a mass problem. We define a game of two players on $\mathcal{A}$ as follows. At the first step, the first player chooses an open interval $\mathcal{I}_{\tilde f_1}$ such that $\mathcal{A} \cap \mathcal{I}_{\tilde f_1} \neq \varnothing$. At the second step, the second player chooses an open interval $\mathcal{I}_{\tilde f_2}$ such that $\mathcal{A} \cap \mathcal{I}_{\tilde f_1} \cap \mathcal{I}_{\tilde f_2} \neq \varnothing$. At the third step the first player chooses an open interval $\mathcal{I}_{\tilde f_3}$ such that $\mathcal{A} \cap \mathcal{I}_{\tilde f_1} \cap \mathcal{I}_{\tilde f_2} \cap \mathcal{I}_{\tilde f_3} \neq \varnothing$, and so on. The second player wins if the intersection of $\mathcal{I}_{\tilde f_1}, \mathcal{I}_{\tilde f_2}, \mathcal{I}_{\tilde f_3}, \ldots$ is a single function from $\mathcal{A}$. A mass problem $\mathcal{A}$ is called *winning* if the second player always has a winning strategy. Now the sufficient condition from [9] can be stated in the following

Theorem 1 (A.A. Muchnik [9]). *Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems. If $\mathcal{A}$ is closed and $\mathcal{B}$ is uniform and winning, then*

$$\mathcal{A} \leqslant \mathcal{B} \iff \mathcal{A} \leqslant_w \mathcal{B}.$$

Of course these requirements are rather strong, because of the generality of the situation. In fact, most restricting is the requirement of closeness, which makes it difficult to use this criterion in some special cases. For example, in the case of problems of enumerability it was shown in [9] that, for any $A \subseteq \omega$, $\mathcal{E}_A$ is uniform and winning, but closed if and only if $\mathrm{card}(A) \leqslant 1$. So the above sufficient condition can not be applied to problems of enumerability. In spite of this, we have seen that in this case these reducibilities coincide. The condition from [9] is of no use also in the case of problems of presentability. One of the requirements hold for free – we have

Lemma 1. *Any mass problem of presentability is uniform.*

Proof. For a structure $\mathfrak{M}$, let $\tilde f$ be a finite function such that $\mathcal{I}_{\tilde f} \cap \underline{\mathfrak{M}} \neq \varnothing$. It means that $\tilde f$ represents some finite part of the atomic diagram of $\mathfrak{M}$. We describe an effective procedure which transforms any $\mathcal{C} \in \underline{\mathfrak{M}}$ to the presentation in $\underline{\mathfrak{M}} \cap \mathcal{I}_{\tilde f}$. We effectively enumerate all finite pieces of the atomic diagram of $\mathcal{C}$ until we find the piece isomorphic to one represented by $\tilde f$, and then apply to the domain of $\mathcal{C}$ a finite permutation witnessing this isomorphism.

Consider now the property of closeness. It is easy to prove the following

Lemma 2. *Let $\mathfrak{M}$ be a structure of relational signature. If $\underline{\mathfrak{M}}$ is closed then, for any countable structure $\mathfrak{N}$ of the same signature as $\mathfrak{M}$, such that $\mathfrak{N} \not\cong \mathfrak{M}$, there exists an $\exists$-sentence φ with the following properties:*

1) $\mathfrak{N} \models \varphi$;

2) for any structure $\mathfrak{N}'$ (of the same signature as $\mathfrak{M}$), $\mathfrak{N}' \models \varphi$ implies that $\mathfrak{N}' \not\cong \mathfrak{M}$.

From this lemma we get

Theorem 2. *Let $\mathfrak{M}$ be a structure of relational signature. $\underline{\mathfrak{M}}$ is a closed mass problem if and only if* $\mathrm{card}(M) = 1$.

Proof. It is enough to prove that $\underline{\mathfrak{M}}$ is not closed in the case then $\mathrm{card}(M) \geqslant 2$. So let $\mathfrak{M}' \subsetneq \mathfrak{M}$ be a proper finite substructure (it exists because there are no functional symbols in the signature). Of course we have $\mathfrak{M}' \ncong \mathfrak{M}$, but any $\exists$-sentence which is true on $\mathfrak{M}'$ is also true on $\mathfrak{M}$. From Lemma 2 it follows that $\underline{\mathfrak{M}}$ is not closed.

2 Medvedev and Muchnik reducibilities in the case of problems of presentability

We now look at the relationship between problems of presentability and some other mass problems. Considering the problems of enumerability, in [15] we obtain, by applying results and techniques due to J.F. Knight [5], the following result, which is in some way analogous to Selman-Rozinas Theorem.

Theorem 3. *Let $\mathfrak{M}$ be a structure, and $A \subseteq \omega$, $A \neq \varnothing$. Then the following are equivalent:*

1) $\mathcal{E}_A \leqslant_w \underline{\mathfrak{M}}$;
2) $\mathcal{E}_A \leqslant \underline{(\mathfrak{M}, \bar{m})}$ for some $\bar{m} \in M^{<\omega}$;
3) A is Σ-definable in $\mathbb{HF}(\mathfrak{M})$.

As an immediate consequence of Theorem 3 we get

Theorem 4. *Let $\mathfrak{M}$ be a structure, and $A \subseteq \omega$. Then the following are equivalent:*

1) $\mathcal{S}_A \leqslant_w \underline{\mathfrak{M}}$;
2) $\mathcal{S}_A \leqslant \underline{(\mathfrak{M}, \bar{m})}$ for some $\bar{m} \in M^{<\omega}$;
3) A is Δ-definable in $\mathbb{HF}(\mathfrak{M})$.

Proof. Follows from Theorem 3, because, for any mass problem $\mathcal{B}$, $\mathcal{S}_A \leqslant_w \mathcal{B}$ if and only if $\mathcal{E}_A \leqslant_w \mathcal{B}$ and $\mathcal{E}_{\overline{A}} \leqslant_w \mathcal{B}$ (the same hold also for $\leqslant$).

Consider now the relations of Medvedev and Muchnik reducibility for the class of mass problems of presentability. Let $\mathfrak{M}$ be a structure of relational signature $\langle P_0^{n_0}, \ldots, P_k^{n_{k-1}} \rangle$ (the restriction to predicative signature is not essential and stands only for simplicity) and let $\mathbb{A}$ be an admissible set (see [4, 1] for definition).

Definition 1 (Yu.L. Ershov [4]). *$\mathfrak{M}$ is said to be Σ-definable in $\mathbb{A}$ if there exist Σ-formulas*

$$\Phi(x_0, y), \Psi(x_0, x_1, y), \Psi^*(x_0, x_1, y), \Phi_0(x_0, \ldots, x_{n_0-1}, y),$$

$$\Phi_0^*(x_0, \ldots, x_{n_0-1}, y), \ldots, \Phi_{k-1}(x_0, \ldots, x_{n_{k-1}-1}, y), \Phi_{k-1}^*(x_0, \ldots, x_{n_{k-1}-1}, y)$$

such that for some parameter $a \in A$*, and letting* $M_0 \rightleftharpoons \Phi^{\mathbb{A}}(x_0, a)$*,* $\eta \rightleftharpoons \Psi^{\mathbb{A}}(x_0, x_1, a) \cap M_0^2$*, one has that* $M_0 \neq \varnothing$ *and* η *is a congruence relation on the structure*

$$\mathfrak{M}_0 \rightleftharpoons \langle M_0; P_0^{\mathfrak{M}_0}, \dots, P_{k-1}^{\mathfrak{M}_0} \rangle,$$

where $P_i^{\mathfrak{M}_0} \rightleftharpoons \Phi_i^{\mathbb{A}}(x_0, \dots, x_{n_i-1}) \cap M_0^{n_i}$ *for all* $i < k$*,* $\Psi^{*\mathbb{A}}(x_0, x_1, a) \cap M_0^2 = M_0^2 \setminus \Psi^{\mathbb{A}}(x_0, x_1, a)$*,* $\Phi_i^{*\mathbb{A}}(x_0, \dots, x_{n_i-1}, a) \cap M_0^{n_i} = M_0^{n_i} \setminus \Phi_i^{\mathbb{A}}(x_0, \dots, x_{n_i-1})$ *for all* $i < k$*, and the structure* $\mathfrak{M}$ *is isomorphic to the quotient structure* $\mathfrak{M}_0 / \eta$*.*

If, in addition, there exists a Σ*-formula* $\Phi^*(x_0, y)$ *such that* $\mathbb{A} \models \forall x_0(\Phi^*(x_0, a) \leftrightarrow \neg\Phi(x_0, a))$*, then* $\mathfrak{M}$ *is said to be* Δ*-definable in* $\mathbb{A}$*. We say that* $\mathfrak{M}$ *is* Δ*-definable in* $\mathbb{A}$ *with no parameters if the above hold for* $a = \varnothing$*.*

It is easy to show that, if we allow parameters, $\mathfrak{M}$ is Σ-definable in $\mathbb{A}$ if and only if $\mathfrak{M}$ is Δ-definable in $\mathbb{A}$. However, this is not so if we restrict ourselves to definitions with no parameters.

Given arbitrary structures $\mathfrak{M}$ and $\mathfrak{N}$, consider the following properties:

1) $\underline{\mathfrak{M}} \leqslant_w \underline{\mathfrak{N}}$;
2) $\underline{\mathfrak{M}} \leqslant \underline{(\mathfrak{N}, \bar{n})}$ for some $\bar{n} \in N^{<\omega}$;
3) $\mathfrak{M}$ is Δ-definable in $\mathbb{HF}(\mathfrak{N})$.

It is easy to see that, for any $\mathfrak{M}$ and $\mathfrak{N}$, 3 implies 2 and 2 implies 3. To prove that 3 ⇒ 2, suppose that $\mathfrak{M}$ is Δ-definable in $\mathbb{HF}(\mathfrak{N})$ by means of some sequence Γ of Σ-formulas with parameters $\bar{n} \in N^{<\omega}$ (without loss of generality we may assume that all parameters are elements of N). Then a recursive operator witnessing that $\underline{\mathfrak{M}} \leqslant \underline{(\mathfrak{N}, \bar{n})}$ can be defined from Γ, using the fact that for witnessing the truth of a Σ-formula in $\mathbb{HF}(\mathfrak{N}, \bar{n})$ it is enough to provide a finite subset of the atomic diagram of $(\mathfrak{N}, \bar{n})$ together with some natural number. To prove that 2 ⇒ 1, note that, for any presentation of $\mathfrak{N}$, distinguishing in it any tuple of representatives of $\bar{n}$ and applying the s-m-n Theorem to an operator witnessing that $\underline{\mathfrak{M}} \leqslant \underline{(\mathfrak{N}, \bar{n})}$, we get an operator which maps this presentation into some presentation of $\mathfrak{M}$.

We now distinguish the class of structures $\mathfrak{N}$ for which the conditions 1, 2 and 3 are equivalent for any structure $\mathfrak{M}$. The next important notion was introduced by L. Richter in [16]. A structure $\mathfrak{M}$ is said to *have degree* **d** if

$$\mathbf{d} = \min\{\deg_T(\mathcal{C}) \mid \mathcal{C} \text{ is a presentation of } \mathfrak{M}\}.$$

The original definition from [16] was formulated with respect to presentations with domains ω only, but it is easy to see that, for any $\mathfrak{M}$ and any its presentation $\mathcal{C}$, there is a presentation $\mathcal{C}'$ of $\mathfrak{M}$, with ω as the domain, such that $\mathcal{C}' \leqslant_T \mathcal{C}$. So our definition coincides with that of Richter. There are examples of structures which have or fail to have a degree (see [16]). Below we show that the class of the structures having a degree is naturally described in terms of effective definability in admissible superstructures.

Theorem 5. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be a structures, and let $\mathfrak{N}$ has a degree. Then the following are equivalent:*

1) $\underline{\mathfrak{M}} \leqslant_w \underline{\mathfrak{N}}$*;*
2) $\underline{\mathfrak{M}} \leqslant \underline{(\mathfrak{N}, \bar{n})}$ *for some* $\bar{n} \in N^{<\omega}$*;*
3) $\mathfrak{M}$ *is* Δ*-definable in* $\mathbb{HF}(\mathfrak{N})$*.*

The only thing we need to prove is the implication $1 \Rightarrow 3$. For this we will use the following result, which characterizes the class of structures having a degree by means of definability in hereditary finite superstructures. For arbitrary countable structure $\mathfrak{M}$ of a signature σ, we consider its expansion $\mathfrak{M}'$ as a structure of the signature $\sigma \cup \{s^1; 0\}$, where s is the symbol of an unary function and 0 is a constant symbol, such that

$$\langle M, s^{\mathfrak{M}'}, 0^{\mathfrak{M}'} \rangle \cong \langle \omega, s, 0 \rangle.$$

Any such structure $\mathfrak{M}'$ is called an *s-expansion* of $\mathfrak{M}$.

Theorem 6. *For a structure $\mathfrak{M}$ the following are equivalent:*

1) $\mathfrak{M}$ *has a degree;*
2) some presentation of $\mathfrak{M}$ *is* Δ*-definable in* $\mathbb{HF}(\mathfrak{M})$ *(as a subset of* ω*);*
3) some s-expansion of $\mathfrak{M}$ *is* Δ*-definable in* $\mathbb{HF}(\mathfrak{M})$*.*

Proof. $2 \Rightarrow 3$. Let $\mathcal{C} \in \underline{\mathfrak{M}}$ be such that $\mathcal{C}$ is Δ-definable in $\mathbb{HF}(\mathfrak{M})$. It is easy to define by $\mathcal{C}$ the corresponding s-expansion of $\mathfrak{M}$, which therefore would be Δ-definable in $\mathbb{HF}(\mathfrak{M})$.

$3 \Rightarrow 2$. Suppose $\mathfrak{M}'$ is Δ-definable in $\mathbb{HF}(\mathfrak{M})$. We will show that in this case some $\mathcal{C} \in \underline{\mathfrak{M}}$ is Δ-definable in $\mathbb{HF}(\mathfrak{M})$, with domain of $\mathcal{C}$ equal to ω. We estimate an isomorphism f from $\mathfrak{M}'$ (more precisely, from its presentation in $\mathbb{HF}(\mathfrak{M})$) to $\mathcal{C}$, which will be Δ-defanable in $\mathbb{HF}(\mathfrak{M})$, in the following way: for any $a \in HF(M)$ and any $n \in \omega$, let $f(a) = n$ if and only if there are $a_0, \ldots, a_n \in HF(M)$ such that, accordingly to the given presentation of $\mathfrak{M}'$, $a_0 = 0^{\mathfrak{M}'}$, $a_1 = s^{\mathfrak{M}'}(a_0), \ldots$, $a = a_n = s^{\mathfrak{M}'}(a_{n-1})$.

$2 \Rightarrow 1$. Suppose that, for some $\mathcal{C} \in \underline{\mathfrak{M}}$, the atomic diagram of $\mathcal{C}$ is Δ-definable in $\mathbb{HF}(\mathfrak{N})$ with parameters $\bar{n} \in N^{<\omega}$ (again, we may assume that all of the parameters are from N). But from this we immediately obtain that $\mathcal{C} \leqslant_T \mathcal{C}'$ for any $\mathcal{C}' \in \underline{\mathfrak{M}}$. Indeed, the recursive operators witnessing this are derived from the Σ-formulas defining $\mathcal{C}$.

$1 \Rightarrow 2$. Suppose that there is some $\mathcal{C} \in \underline{\mathfrak{M}}$ such that $\mathcal{C} \leqslant_T \mathcal{C}'$ for any $\mathcal{C}' \in \underline{\mathfrak{M}}$. This is equivalent to saying that, in terms of the mass problems, $\mathcal{S}_{\mathcal{C}} \leqslant_w \underline{\mathfrak{M}}$. So, by the Theorem 4, $\mathcal{C}$ is Δ-definable in $\mathbb{HF}(\mathfrak{M})$ (as a subset of ω).

Finally, let us prove the implication $1 \Rightarrow 3$ of the Theorem 5. So suppose $\mathfrak{M}$ is such that $\underline{\mathfrak{M}} \leqslant_w \underline{\mathfrak{N}}$. Let also fix some $\mathcal{C}_0 \in \underline{\mathfrak{N}}$ such that $\mathcal{C}_0$ is Δ-definable in $\mathbb{HF}(\mathfrak{N})$. Then from $\underline{\mathfrak{M}} \leqslant_w \underline{\mathfrak{N}}$ it follows that there is a presentation $\mathcal{C} \in \mathfrak{M}$ such that $\mathcal{C} \leqslant_T \mathcal{C}_0$. Since $\mathcal{C}_0$ is Δ-definable in $\mathbb{HF}(\mathfrak{N})$, the same is true for $\mathcal{C}$, hence it follows that $\mathfrak{M}$ is Δ-definable in $\mathbb{HF}(\mathfrak{M})$ via the presentation $\mathcal{C}$.

In [15] we show that the requirement that a structure $\mathfrak{N}$ have a degree in the Theorem 5 is essential and can not be dropped. For this we use the fact (obtained independently by S. Wehner [17] and T. Slaman [12]) that there exist structures which mass problems of presentability belongs to the least non-zero degree of difficulty in the Medvedev lattice.

Now we introduce some class of structures for which, considering their problems of presentability, Medvedev and Muchnik reducibilities are equivalent. In fact, we adjust the notion of uniformity to our model theoretical setting.

Definition 2. *A structure $\mathfrak{M}$ is called $*$-uniform if $\underline{(\mathfrak{M}, \bar{m})} \leqslant \underline{\mathfrak{M}}$ for any $\bar{m} \in M^{<\omega}$.*

From Theorem 5 we immediately obtain

Corollary 1. *If $\mathfrak{N}$ is $*$-uniform and has a degree then, for any structure $\mathfrak{M}$,*

$$\underline{\mathfrak{M}} \leqslant \underline{\mathfrak{N}} \iff \underline{\mathfrak{M}} \leqslant_w \underline{\mathfrak{N}}.$$

We remind the following definition from the model theory: a structure $\mathfrak{M}$ is called *ultrahomogeneous* if any isomorphism between finitely generated substructures of $\mathfrak{M}$ can be extended to an automorphism of $\mathfrak{M}$. It is clear that, if $\mathfrak{M}$ is homogeneous structure of relational signature, then $\mathfrak{M}$ is $*$-uniform. Also clear that, if $\mathfrak{M}$ is constructivizable (i.e. have a computable presentation), then $\mathfrak{M}$ is $*$-uniform. We establish now an example of nonhomogeneous and nonconstructivizable structure which is $*$-uniform.

Lemma 3. *If $\alpha_1, \ldots, \alpha_n$ are constructive ordinals, then $\langle \omega_1^{CK}; \leqslant, \alpha_1, \ldots, \alpha_n \rangle$ is Δ-definable in $\mathbb{HF}(\langle \omega_1^{CK}, \leqslant \rangle)$ with no parameters.*

Proof. We use the fact that $\alpha + \omega_1^{CK} = \omega_1^{CK}$ for any constructive ordinal α. Indeed, if α is constructive, then so is $\alpha \cdot \omega$, hence $\alpha \cdot \omega < \omega_1^{CK}$. But $\alpha + \alpha \cdot \omega = \alpha \cdot \omega$, so $\alpha + \omega_1^{CK} = \omega_1^{CK}$.

Let us suppose that $\alpha_1 < \ldots < \alpha_n$, for simplicity. Since all of these ordinals are constructive, the structure $\langle \alpha_n; \leqslant, \alpha_1, \ldots, \alpha_{n-1} \rangle$ is Δ-definable in $\mathbb{HF}(\varnothing)$ (with no parameters, of course). So the sum $\langle \alpha_n + \omega_1^{CK}; \leqslant, \alpha_1, \ldots, \alpha_n \rangle$ is Δ-definable in $\mathbb{HF}(\langle \omega_1^{CK}; \leqslant \rangle)$ with no parameters. By the fact mentioned above, the lemma is proved.

Corollary 2. *Suppose $\alpha_1, \ldots, \alpha_n \in \omega_1^{CK}$ are constructive ordinals. Then $\underline{\langle \omega_1^{CK}; \leqslant, \bar{\alpha} \rangle} \leqslant \underline{\langle \omega_1^{CK}, \leqslant \rangle}$. As a consequence, $\langle \omega_1^{CK}; \leqslant \rangle$ is $*$-uniform.*

3 Partial mass problems of presentability and *e*-reducibility

We will say that a structure $\mathfrak{M}$ has *e-degree* $\mathbf{d}$ if

$$\mathbf{d} = \min\{\deg_e(\mathcal{C}) \mid \mathcal{C} \text{ is a presentation of } \mathfrak{M}\}.$$

The following theorem gives the syntactical characterization for structures with an *e*-degree.

Theorem 7. *For a structure $\mathfrak{M}$ the following are equivalent:*

1) $\mathfrak{M}$ has an e-degree;
2) some presentation of $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{M})$ (as a subset of ω).

Proof. Analogous to the proof of Theorem 6.

As an immediate consequence of this theorem and Theorem 6 we get

Proposition 1. *If $\mathfrak{M}$ has a degree then $\mathfrak{M}$ has an e-degree.*

There are examples (implicitly presented in [16]) of structures which have an e-degree but does not have a degree. The analog of Theorem 5 for the partial mass problems of presentability is the following

Theorem 8. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be a structures, and let $\mathfrak{N}$ has an e-degree. The following are equivalent:*

1) $\underset{--}{\mathfrak{M}} \leqslant_{ew} \underset{--}{\mathfrak{N}}$;
2) $\underset{--}{\mathfrak{M}} \leqslant_{e} \underset{---}{(\mathfrak{N}, \bar{n})}$ for some $\bar{n} \in N^{<\omega}$;
3) $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$.

Proof. Analogous to the proof of Theorem 5.

Theorem 9. *For any structures $\mathfrak{M}, \mathfrak{N}$,*

$$\underset{--}{\mathfrak{M}} \leqslant_e \underset{--}{\mathfrak{N}} \text{ implies } \underline{\mathfrak{M}} \leqslant \underline{\mathfrak{N}}, \text{ and } \underset{--}{\mathfrak{M}} \leqslant_{ew} \underset{--}{\mathfrak{N}} \text{ implies } \underline{\mathfrak{M}} \leqslant_w \underline{\mathfrak{N}},$$

Proof. Analogous to the proof of Lemma 1. Suppose, for example, that $\underset{--}{\mathfrak{M}} \leqslant_e \underset{--}{\mathfrak{N}}$ by means of the partial recursive operator Ψ. It is easy to build from Ψ a partial recursive operator Ψ', defined on $\underline{\mathfrak{N}}$, such that for any $f \in \underline{\mathfrak{N}}$ we have $\Psi'(f) = \Psi(f')$, where $f' \in \underset{--}{\mathfrak{N}}$ corresponds to f. So $\Psi'(\underline{\mathfrak{N}}) \subseteq \underset{--}{\mathfrak{M}}$. We describe an effective procedure which transforms any characteristic function $f \in \underline{\mathfrak{N}}$ to the characteristic function of some presentation $\mathcal{C}(f)$ of $\mathfrak{M}$. We define the domain of $\mathcal{C}(f)$ together with the bijection π which maps it onto the domain of the presentation defined by f. Namely, at the step s we define the subset $C_s \supseteq C_{s-1}$ of the domain $\mathcal{C}(f)$ as follows: consider all numbers from 0 to s which are not in $\pi(C_{s-1})$; add the number s to C_s and the pair $\langle s, c\rangle$ to π if and only if $c \leqslant s, c \notin \pi(C_{s-1})$ is the least for which there exist a finite set $D_k \subseteq f$ with the number $k \leqslant s$ in some fixed enumeration, for which $\langle (c = c), 1\rangle \in \Psi'(D_k)$. The construction defined above gives the domain C_f and bijection π which define the characteristic function of desired presentation.

4 Presentability dimensions

It is reasonable, having a problem of presentability, which consists of all possible presentations of some structure, to try to find a subset of it, with the same properties with respect to Medvedev (Muchnik) reducibility, which is as small as possible.

Definition 3. *A countable structure* $\mathfrak{M}$ *is said to have (strong) presentability dimension* α *(denote* $\text{Pr-dim}(\mathfrak{M}) = \alpha$*), where* α *is a cardinal, if* $\underline{\mathfrak{M}} \equiv \mathcal{B}$ *for some* $\mathcal{B} \subseteq \underline{\mathfrak{M}}$, $\text{card}(\mathcal{B}) = \alpha$, *and* α *is the least cardinal satisfying these conditions.*

In the same way we can introduce the notion of weak presentability dimension $\text{Pr-dim}_w(\mathfrak{M})$, changing in the above definition $\equiv$ to $\equiv_w$. It is clear that for any (countable) structure $\mathfrak{M}$ we have

$$1 \leqslant \text{Pr-dim}_w(\mathfrak{M}) \leqslant \text{Pr-dim}(\mathfrak{M}) \leqslant 2^\omega.$$

It is also easy to see that, for any structure $\mathfrak{M}$, $\text{Pr-dim}_w(\mathfrak{M}) = 1$ if and only if $\mathfrak{M}$ has a degree. Next, there is the following

Theorem 10. *For a structure* $\mathfrak{M}$ *the following are equivalent:*

1) $\text{Pr-dim}_w(\mathfrak{M}) = 1$;
2) $\text{Pr-dim}(\mathfrak{M}, \bar{m}) = 1$ *for some* $\bar{m} \in M^{<\omega}$.

Proof. Immediately follows from Theorem 6.

So, a structure has a degree if and only if some of its constant expansions has a strong degree.

Corollary 3. *If* $\mathfrak{M}$ *is* $*$*-uniform then*

$$\text{Pr-dim}(\mathfrak{M}) = 1 \iff \text{Pr-dim}_w(\mathfrak{M}) = 1.$$

Let $\mathfrak{M}$ be a structure, and suppose that some presentation of $\mathfrak{M}$ is Δ-definable in $\mathbb{HF}(\mathfrak{M})$ with no parameters. Then $\text{Pr-dim}(\mathfrak{M}) = 1$. The author does not know whether this sufficient condition is also necessary or not.

The following question also seems reasonable: are there structures of finite or countable strong presentability dimension, i.e. is there $\mathfrak{M}$ such that

$$1 < \text{Pr-dim}(\mathfrak{M}) \leqslant \omega?$$

For such $\mathfrak{M}$ we necessarily must have $\text{Pr-dim}_w(\mathfrak{M}) = 1$. Indeed, this follows from the inequality $\text{Pr-dim}_w(\mathfrak{M}) \leqslant \text{Pr-dim}(\mathfrak{M})$ and the next result observed independently by J.F. Knight [6] and I.N. Soskov [14]: for any $\mathfrak{M}$, $\text{Pr-dim}_w(\mathfrak{M})$ is either 1 or uncountable. From this we immediately obtain that, for any $\mathfrak{M}$, $\text{Pr-dim}(\mathfrak{M})$ is either 1 or infinite. Recently I. Kalimullin (personal communication) showed that there are structures with strong presentability dimension ω.

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