Σ-DEFINABILITY IN HEREDITARILY FINITE SUPERSTRUCTURES AND PAIRS OF MODELS

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We consider the problem of being Σ-definable for an uncountable model of a c-simple theory in hereditarily finite superstructures over models of another c-simple theory. A necessary condition is specified in terms of decidable models and the concept of relative indiscernibility introduced in the paper. A criterion is stated for the uncountable model of a c-simple theory to be Σ-definable in superstructures over dense linear orders, and over infinite models of the empty signature. We prove the existence of a c-simple theory (of an infinite signature) every uncountable model of which is not Σ-definable in superstructures over dense linear orders. Also, a criterion is given for a pair of models to be recursively saturated.

INTRODUCTION

In the present paper we study into Σ-definability of algebraic systems in hereditarily finite superstructures, which, in particular, allows us to introduce analogs of the concept of constructibility for uncountable models. Ershov in [1, 2] dealt with the problem of characterizing a class of theories having uncountable models, Σ-definable in hereditarily finite superstructures over dense linear orders. In [1], a criterion of this property was propounded in terms of constructibility of the $^*\omega$-spectrum of a theory. In [2], it was conjectured that all c-simple theories share this property.

In this paper we introduce a concept of relative indiscernibility, and then use it to couch uniform criteria of being Σ-definable for the uncountable model of a c-simple theory in hereditarily finite superstructures over dense linear orders, and over infinite models of the empty signature (with equality). As a consequence it is stated that there exists a c-simple theory (of an infinite signature) every uncountable model of which is not Σ-definable in hereditarily finite superstructures over dense linear orders.

[1]*Supported by RFBR grant No. 02-01-00540, by FP “Universities of Russia” grant UR.04.01.019, and by the Council for Grants (under RF President) and State Aid of Fundamental Science Schools, grant NSh-2069.2003.1.

In dealing with relative indiscernibility, we treat pairs of models whose universes have a non-empty intersection. Dealing in the theory of admissible sets, we also consider pairs of models treated as algebraic systems of a signature obtained by joining (disjoint) signatures of initial models and then adding unary predicate symbols distinguishing universes of those models. As the universe of such a system we take a join of the universes of the initial models, with no limitations on their relative positions. For the thus defined pairs of models, we establish a criterion of being recursively saturated. As a consequence it is shown that a pair formed by models of c-simple theories is recursively saturated. We furnish an example of a theory T (with a prime model $\mathfrak{M}_0$) all of whose models are recursively saturated, but for every model $\mathfrak{M}$ of T, the pair $(\mathfrak{M}_0, \mathfrak{M})$ is recursively saturated iff $\mathfrak{M} \cong \mathfrak{M}_0$. The results obtained allow us to point out an example of models $\mathfrak{M}$ and $\mathfrak{N}$ such that $O(\mathfrak{M}) = O(\mathfrak{N}) = \omega$, but $O(\mathfrak{M}, \mathfrak{N}) > \omega$, where $O(\mathfrak{A})$ is the least ordinal not lying in an admissible set $\mathrm{HYP}(\mathfrak{A})$ (cf. [3, 4]).

The notation and terminology used in the paper are standard and are borrowed from [3-6]. We consider algebraic systems of an at most countable signature, assuming, without loss, that the signature contains predicate symbols only. Writing $\sigma = \langle P_0^{n_0}, \ldots, P_k^{n_k}, \ldots \rangle$ means that P_k is an n_k-ary predicate symbol of the signature σ, and if $f(k) = n_k$ is a computable function, then σ is also said to be computable. If $\mathfrak{A}$ is a model of the signature σ then by $P_k^{\mathfrak{A}}$ we denote the interpretation of a predicate symbol P_k in a model $\mathfrak{A}$, and by $|\mathfrak{A}|$ the universe of $\mathfrak{A}$. Further, let $M^{<\omega}$ be the set of all finite tuples of elements of an arbitrary set M.

1. Σ-DEFINABILITY OVER CLASSES OF MODELS OF c-SIMPLE THEORIES

We recollect the notion of being Σ-definable for an algebraic system in an admissible set (cf. [3]), which generalizes the concept of being constructible. Let $\mathfrak{M}$ be an algebraic system of a computable predicate signature $\langle P_0^{n_0}, \ldots, P_k^{n_k}, \ldots \rangle$ and $\mathbb{A}$ be an admissible set of a signature σ_0.

Definition 1. A system $\mathfrak{M}$ is said to be Σ-*definable in an admissible set* $\mathbb{A}$ if there exists a computable sequence of Σ-formulas in the signature σ_0 such as

$$\Phi(x_0, y),\ \Psi(x_0, x_1, y),\ \Psi^*(x_0, x_1, y),\ \Phi_0(x_0, \ldots, x_{n_0-1}, y),$$

$$\Phi_0^*(x_0, \ldots, x_{n_0-1}, y),\ \ldots,\ \Phi_k(x_0, \ldots, x_{n_k-1}, y),\ \Phi_k^*(x_0, \ldots, x_{n_k-1}, y),\ \ldots,$$

for which the set $M_0 \rightleftharpoons \Phi^{\mathbb{A}}(x_0, a)$ is non-empty for some parameter $a \in A$, and $\eta \rightleftharpoons \Psi^{\mathbb{A}}(x_0, x_1, a) \cap M_0^2$ is a congruence relation on the algebraic system

$$\mathfrak{M}_0 \rightleftharpoons \langle M_0, P_0^{\mathfrak{M}_0}, \ldots, P_k^{\mathfrak{M}_0}, \ldots \rangle,$$

where $P_k^{\mathfrak{M}_0} \rightleftharpoons \Phi_k^{\mathbb{A}}(x_0, \ldots, x_{n_k-1}) \cap M_0^{n_k}$, $k \in \omega$,

$$\Psi^{*\mathbb{A}}(x_0, x_1, a) \cap M_0^2 = M_0^2 \setminus \Psi^{\mathbb{A}}(x_0, x_1, a),$$

$$\Phi_k^{*\mathbb{A}}(x_0, \ldots, x_{n_k-1}, a) \cap M_0^{n_k} = M_0^{n_k} \setminus \Phi_k^{\mathbb{A}}(x_0, \ldots, x_{n_k-1})$$

for all $k \in \omega$, and the system $\mathfrak{M}$ is isomorphic to a quotient system $\mathfrak{M}_0 / \eta$. In this case we say that the given sequence of formulas (with $a \in A$) Σ-*defines* $\mathfrak{M}$ in $\mathbb{A}$.

Below, we will look into the case where an admissible set $\mathbb{A}$ is a hereditarily finite superstructure. Ordinals of any hereditarily finite superstructure are natural numbers only, and so the notion of Σ-definability, in this case, is more close to that of constructibility. For $\mathbb{A} = \mathrm{HF}(\varnothing)$, being Σ-definable in $\mathbb{A}$ coincides with being constructible. The case where $\mathbb{A} = \mathrm{HF}(\mathfrak{M})$ and $\mathfrak{M}$ is an infinite countable model may be reduced to treating relative constructibility in which the concept of being computable with an oracle is involved. Finally, the case where $\mathbb{A}$ is a hereditarily finite superstructure over an uncountable model is of interest because it allows us to introduce a certain analog of (relative) constructibility for uncountable systems also.

Obviously, every algebraic system $\mathfrak{M}$ can be Σ-definable in an appropriate hereditarily finite superstructure — for instance, trivially in $\mathrm{HF}(\mathfrak{M})$. Therefore the question whether $\mathfrak{M}$ is Σ-definable in hereditarily finite superstructures over models in some class K is more challenging. We say that a system $\mathfrak{M}$ is Σ-*definable over a class* K if $\mathfrak{M}$ is Σ-definable in $\mathrm{HF}(\mathfrak{A})$ for some model $\mathfrak{A}$ in K. Below, we will be interested in classes of the form $\mathrm{Mod}(T)$, that is, classes of models for some theories.

A theory, T, is said to be *c-simple* if it is countably categorical, model complete, decidable, and has a decidable set of complete formulas (cf. [3]). Throughout the paper, we assume that the signature of a c-simple theory is computable (i.e., not necessarily finite). Due to being ω-categorical, every c-simple theory has a unique (up to isomorphism) countable model; moreover, such has a decidable model, which is unique up to computable isomorphism. A model is said to be *decidable* if its universe is a computable set of natural numbers and all of its definable relations are likewise (uniformly) computable. A model is *computable* if its universe is also a computable set of natural numbers, but only relations definable by atomic formulas are uniformly computable. Along with the above concepts, we also use notions of a constructible model and of a strongly constructible model (cf. [7]).

In terms of decidable models, we formulate a necessary condition for a c-simple theory to have an uncountable model, Σ-definable over a class of models of another c-simple theory. To do this, below we introduce a general concept, which is meaningful for an arbitrary pair of models with intersecting universes.

Definition 2. If $\mathfrak{M}$ and $\mathfrak{N}$ are some algebraic systems then $I \subseteq |\mathfrak{M}| \cap |\mathfrak{N}|$ is called the set of $\mathfrak{M}$-*indiscernibles in* $\mathfrak{N}$ provided that

$$\langle \mathfrak{M}, i_0, \ldots, i_n \rangle \equiv \langle \mathfrak{M}, i'_0, \ldots, i'_n \rangle \Rightarrow \langle \mathfrak{N}, i_0, \ldots, i_n \rangle \equiv \langle \mathfrak{N}, i'_0, \ldots, i'_n \rangle,$$

for any $i_0, \ldots, i_n, i'_0, \ldots, i'_n \in I$.

THEOREM 1. Let T_1 and T_2 be c-simple theories. If T_2 has an uncountable model Σ-definable over a class $\mathrm{Mod}(T_1)$ then there exist decidable models $\mathfrak{M}$ and $\mathfrak{N}$ of T_1 and T_2, respectively, such that $\mathfrak{N}$ contains an infinite computable set of $\mathfrak{M}^*$-indiscernibles, where $\mathfrak{M}^*$ is an expansion of $\mathfrak{M}$ by finitely many constants.

Proof. Let some uncountable model of T_2 be Σ-definable in a hereditarily finite superstructure over some (uncountable) model $\mathfrak{M}'$ of T_1 by a sequence of Σ-formulas such as

$$\Gamma = \langle \Phi, \Psi, \Psi^*, \Phi_0, \Phi^*_0, \ldots, \Phi_k, \Phi^*_k, \ldots \rangle,$$

where Ψ and Ψ^* define an equality relation; moreover, there is no loss of generality in assuming that the parameter is $\bar{m}' \in |\mathfrak{M}'|^{<\omega}$, a tuple of urelements. Let $\mathfrak{M}$ be a decidable model of T_1. Since T_1 is c-simple, there is a tuple $\bar{m}_0$ of elements of $\mathfrak{M}$ such that $\langle \mathfrak{M}, \bar{m}_0 \rangle \equiv \langle \mathfrak{M}', \bar{m}' \rangle$. Every model of a c-simple theory is sufficiently saturated, and so $\langle \mathrm{HF}(\mathfrak{M}), \bar{m}_0 \rangle \equiv \langle \mathrm{HF}(\mathfrak{M}'), \bar{m}' \rangle$. (Recall that $\mathfrak{M}_0$ is said to be *sufficiently saturated* if $\mathfrak{M}_0 \preccurlyeq \mathfrak{M}_1$ and $\mathrm{HF}(\mathfrak{M}_0) \preccurlyeq \mathrm{HF}(\mathfrak{M}_1)$ for some ω-saturated model $\mathfrak{M}_1$; see [3].) Therefore the sequence Γ of formulas with a parameter $\bar{m}_0$ properly defines in $\mathrm{HF}(\mathfrak{M})$ a model $\mathfrak{N}'$, which will be a model of T_2. The sequence Γ of formulas with the given tuple of parameters cannot define a model with a finite universe (otherwise, a model defined by that tuple in $\mathrm{HF}(\mathfrak{M}')$ would also be finite), and so $\mathfrak{N}'$ will be a countable model of T_2. Moreover, given any strong constructivization of $\mathfrak{M}$, Γ allows us to construct a constructivization of $\mathfrak{N}'$. In view of the method of construction opted for $\mathfrak{N}'$, we write $\Gamma(\mathrm{HF}(\mathfrak{M}), \bar{m}_0)$.

It is known that every element of a hereditarily finite superstructure $\mathrm{HF}(\mathfrak{M})$ is representable as the value of a term $t_{\varkappa}(\bar{m})$, where $\bar{m} \in |\mathfrak{M}|^{<\omega}$ is a tuple of urelements, and $\varkappa \in \mathrm{HF}(\omega)$ (cf. [3]). We claim that there exist an element $\varkappa \in \mathrm{HF}(\omega)$, a tuple $\bar{m}_1 \in |\mathfrak{M}|^{<\omega}$, and an infinite set $X \subseteq M$ such that $\mathrm{HF}(\mathfrak{M}) \models \Psi^*(t_\varkappa(m, \bar{m}_1), t_\varkappa(m', \bar{m}_1), \bar{m}_0)$, for any distinct m and m' in X. Indeed, otherwise (since $\mathfrak{M}$ is a prime model of T_1), Γ would define at most countable models over any models of T_1.

Since $\mathfrak{M}$ is decidable, and Ψ^* is a Σ-formula, we can find an infinite computable set $I \subseteq X$. To do this, it suffices to take an arbitrary $x_0 \in X$, find (effectively) $x_1 = \mu x(\mathrm{HF}(\mathfrak{M}) \models \Psi^*(x_0, x_1, \bar{m}_0))$, and proceed further, to arrive eventually at $I \leftrightharpoons \{x_0, x_1, \ldots\}$.

Of importance is the following property. Let $\mathfrak{M}_0$ be a sufficiently saturated model. If $a_0, a_1 \in \mathrm{HF}(\mathfrak{M}_0)$ then types of the elements a_1 and a_2 coincide in $\mathrm{HF}(\mathfrak{M}_0)$ iff there exist $n \in \omega$, $\varkappa \in \mathrm{HF}(n)$, and $\bar{m}_0, \bar{m}_1 \in M_0^n$ such that $a_0 = t_\varkappa(\bar{m}_0)$, $a_1 = t_\varkappa(\bar{m}_1)$, and the types of $\bar{m}_0$ and $\bar{m}_1$ coincide in $\mathfrak{M}_0$ (for a proof, see [3]). Since $\mathfrak{M}$ is sufficiently saturated, for any $i_0, \ldots, i_n, i'_0, \ldots, i'_n \in I$, $\langle \mathfrak{M}, \bar{m}_2, i_0, \ldots, i_n \rangle \equiv \langle \mathfrak{M}, \bar{m}_2, i'_0, \ldots, i'_n \rangle$ implies

$$\langle \mathrm{HF}(\mathfrak{M}), t_\varkappa(i_0, \bar{m}_1), \ldots, t_\varkappa(i_n, \bar{m}_1) \rangle \equiv \langle \mathrm{HF}(\mathfrak{M}), t_\varkappa(i'_0, \bar{m}_1), \ldots, t_\varkappa(i'_n, \bar{m}_1) \rangle,$$

where the tuple $\bar{m}_2$ is the concatenation of $\bar{m}_0$ and $\bar{m}_1$. The model $\mathfrak{N}'$ is defined in $\mathrm{HF}(\mathfrak{M})$ by a sequence of Σ-formulas; so, using an arbitrary constructivization μ of $\mathfrak{M}$, we can construct a constructivization ν of a hereditarily finite superstructure $\mathrm{HF}(\mathfrak{M})$, for which $\mu^{-1}(i) = \nu^{-1}(t_\varkappa(i, \bar{m}_0))$ with all $i \in I$. Based on this constructivization, it is easy to construct a decidable model $\mathfrak{N} \cong \mathfrak{N}'$ such that I is an infinite computable set of $\langle \mathfrak{M}, \bar{m}_2 \rangle$-indiscernibles in $\mathfrak{N}$.

Now, we distinguish a subclass of the class of c-simple theories such that the necessary condition for uncountable models to be Σ-definable is also sufficient. To do this, we consider a relativized version of the Ryll-Nardzewski function. For any ω-categorical model $\mathfrak{A}$ and any subset $X \subseteq A$, define a function $R_X^{\mathfrak{A}} : \omega \to \omega$ as follows: for every $n \in \omega$, let $R_X^{\mathfrak{A}}(n)$ be the number of n-types realized in $\mathfrak{A}$ by the elements of X. For simplicity, replace $R_{|\mathfrak{A}|}^{\mathfrak{A}}$ by $R^{\mathfrak{A}}$. We say that an ω-categorical theory has *wide models* if $R_X^{\mathfrak{A}} = R^{\mathfrak{A}}$ for (any) model $\mathfrak{A}$ of that theory and for any infinite set $X \subseteq |\mathfrak{A}|$. In other words, every infinite subset of a wide model realizes all types of this model's elementary theory. A theory, T_E, of infinite models of the empty signature (with equality) and a theory, T_{DLO}, of dense linear orders without endpoints are examples of c-simple theories all models of which are wide.

THEOREM 2. Let T_1 and T_2 be c-simple theories; T_1 has wide models. In this case an uncountable model of T_2 is Σ-definable over the class $\mathrm{Mod}(T_1)$ if and only if there exist decidable models $\mathfrak{M}$ and $\mathfrak{N}$ of T_1 and T_2, respectively, such that $\mathfrak{N}$ contains an infinite computable set of $\mathfrak{M}^*$-indiscernibles, where $\mathfrak{M}^*$ is an expansion of $\mathfrak{M}$ by finitely many constants.

Proof. The necessity was shown in Theorem 1, and so we need only argue for sufficiency. Let $\mathfrak{M}$ and $\mathfrak{N}$ be decidable models of, respectively, T_1 and T_2 of signatures σ_1 and σ_2 and let $\mathfrak{N}$ possess an infinite computable set $I \subseteq |\mathfrak{N}|$ of $\mathfrak{M}^*$-indiscernibles, where $\mathfrak{M}^* = \langle \mathfrak{M}, \bar{m}_0 \rangle$, $\bar{m}_0 \in |\mathfrak{M}|^{<\omega}$. We build up a constructivization of $\mathfrak{N}$, taking as the universe the Skolem hull of a set $|\mathfrak{M}|$ relative to T_2. (For this, it is required that first we "project"the set $|\mathfrak{M}|$ onto a set $I \subseteq |\mathfrak{M}| \cap |\mathfrak{N}|$). In the course of the construction we arrive at a sequence Γ of Σ-formulas, which for an appropriate model $\mathfrak{M}'$ of T_1 of arbitrarily large cardinality, defines in $\mathrm{HF}(\mathfrak{M}')$ a model of T_2 of the same cardinality. In defining on $|\mathfrak{M}|$ a submodel structure of some model of T_2 by projecting onto I, it is required that the model $\mathfrak{M}$ be wide. A Skolem term corresponding to a formula $\exists y \varphi(\bar{x}, y)$ of the signature σ_2 is denoted by $t_\varphi(\bar{x})$; Skolem terms have an effective representation in any hereditarily finite superstructure due to σ_2 being computable. In constructing, new Skolem terms are added only if a given formula fails to be satisfied by any other element in the Skolem hull at a given step.

Construction

For every step t, we effectively define the following: the set, S_t, forming part of the Skolem closure of $|\mathfrak{M}|$ relative to T_2; the function, $p_t : S_t^{<\omega} \to (S_t \upharpoonright I)^{<\omega}$, where $S_t \upharpoonright I$ is a subset of S_t forming an appropriate part of the Skolem closure of a set I; the set, F_t, which is the full diagram of a set S_t in the signature σ_2. With an arbitrary model $\mathfrak{A}$, we associate the model $\mathfrak{A}^{<\omega}$, whose universe is the set $|\mathfrak{A}|^{<\omega}$ and whose signature consists of a binary relation $\sim$ and a binary function $\hat{}$ defined as follows: $\bar{a}_1 \sim \bar{a}_2$ iff tuples $\bar{a}_1$ and $\bar{a}_2$ are equal in length, $\langle \mathfrak{A}, \bar{a}_1 \rangle \equiv \langle \mathfrak{A}, \bar{a}_2 \rangle$, and given a pair of tuples $\bar{a}_1$ and $\bar{a}_2$, the function $\hat{}$ yields a tuple $\bar{a}_1\hat{}\bar{a}_2$, which is their concatenation. If $\mathfrak{A}$ is a countable model of a c-simple theory then $\mathfrak{A}^{<\omega}$ is constructible.

We fix some constructivization μ of $\mathfrak{M}^{<\omega}$, a constructivization ν of $\mathfrak{N}^{<\omega}$, and some computable Gödel numbering $\{\varphi_n(\bar{x}) \mid n \in \omega\}$ of formulas in the signature σ_2.

Step 0. Letting $S_0 \leftrightharpoons |\mathfrak{M}|$, for any tuple $\bar{m} \in |\mathfrak{M}|^{<\omega}$, we put $p_0(\bar{m}) \leftrightharpoons \bar{n}$, where $\bar{n}$ is a tuple of elements of I such that its number with respect to μ is least possible and its type in $\mathfrak{M}^*$ is the same as $\bar{m}$'s (i.e., the condition that $\bar{m}_0\hat{}\bar{m} \sim \bar{m}_0\hat{}\bar{n}$, which is verified effectively, is satisfied). Now, we define (effectively) the set

$$F_0 \leftrightharpoons \{\varphi(\bar{m}) \mid \bar{m} \in |\mathfrak{M}|^{<\omega},\ \varphi \text{ is a formula in the signature } \sigma_2,\ \mathfrak{N} \models \varphi(p_0(\bar{m}))\}.$$

Step $t+1$. Assume that sets S_t and F_t and the function p_t are already constructed. For tuples of elements in S_t, the concept of being equivalent relative to $\mathfrak{M}^*$ is defined as follows: the tuples $\bar{s}_1$ and $\bar{s}_2$ in $S_t^{<\omega}$ are equivalent relative to $\mathfrak{M}^*$ if they are equal in length and $p_t(\bar{s}_1) = p_t(\bar{s}_2)$. For the case where $\bar{m}_1, \bar{m}_2 \in S_0$, $\bar{m}_1$ and $\bar{m}_2$ are equivalent relative to $\mathfrak{M}^*$ iff they are of equal length and $\langle \mathfrak{M}^*, \bar{m}_1 \rangle \equiv \langle \mathfrak{M}^*, \bar{m}_2 \rangle$.

Since T_1 is ω-categorical, for every formula $\varphi_n(\bar{x}, y)$, $n < t$, in the signature σ_2, there exist not more than finitely many tuples $\bar{s}$ of elements of S_t which are pairwise non-equivalent relative to $\mathfrak{M}^*$ and are such that $\exists y \varphi_k(\bar{s}, y) \in F_t$. Let $\{\langle \varphi_{n_k}, \bar{s}_k \rangle \mid 1 \leqslant k \leqslant k_0\}$ be a list of all such formulas with suitable tuples. We introduce intermediate sets S_t^k, F_t^k and a function p_t^k, for all $k \leqslant k_0$, setting $S_t^0 \leftrightharpoons S_t$, $F_t^0 \leftrightharpoons F_t$, $p_t^0 \leftrightharpoons p_t$, and for every $k \leqslant k_0$, execute the following:

Stage k. Let $S_t^k \leftrightharpoons S_t^{k-1}$, $F_t^k \leftrightharpoons F_t^{k-1}$, and $p_t^k \leftrightharpoons p_t^{k-1}$. We determine (effectively) whether there exists an element $c \in S_t^{k-1}$ such that $\varphi_{n_k}(\bar{s}_k \hat{} c) \in F_t^{k-1}$. (The effectiveness is established by induction on t. For $t = 0$, this is clear from the following observation: I is the set of $\mathfrak{M}^*$-indiscernibles in $\mathfrak{N}$ and $\mathfrak{M}^*$ is ω-categorical; so, to verify that a given formula is not realized by the elements of I, we can take finitely many steps, covering all possible $\mathfrak{M}^*$-types of the potential witnesses for the given formula to be realized in $\mathfrak{N}$.) If such an element exists, we are done away; otherwise, we proceed as follows.

Add to S_t^k all Skolem terms equivalent to a Skolem term $t_{\varphi_{n_k}}(\bar{s}_k)$ relative to $\mathfrak{M}^*$, that is, all terms of the form $t_{\varphi_{n_k}}(\bar{s})$ such that $p_t^{k-1}(\bar{s}) = p_t^{k-1}(\bar{s}_k)$. Redefine the function p_t^k on S_t^k by setting $p_t^k(t_{\varphi_{n_k}}(\bar{s})) \leftrightharpoons t_{\varphi_{n_k}}(p_t^{k-1}(\bar{s}))$ for all the new terms. The set F_t^k is redefined thus: for any newly added Skolem term $t_{\varphi_{n_k}}(\bar{s})$ and for every formula θ complete relative to T_2 in the signature σ_2 with $\mathrm{lh}(\bar{s}) + 1$ variables (here, $\mathrm{lh}(\bar{s})$ is the length of $\bar{s}$), we add to F_t^k a formula $\theta(\bar{s}, t_{\varphi_{n_k}}(\bar{s}))$, provided that θ has the least Gödel number among all complete formulas ρ in $\mathrm{lh}(\bar{s}) + 1$ variables for which

$$\exists y (\rho(\bar{s}, y) \wedge \varphi_{n_k}(\bar{s}, y)) \in F_t^{k-1}.$$

Further, for an arbitrary tuple $\bar{s} \in S_t^k$, we add a formula $\theta(\bar{s})$ to F_t^k, if θ is complete relative to T_2 in the signature σ_2, has a least Gödel number, and is such that $\theta(\bar{s})$ is consistent relative to T_2 with all formulas in $F_t^k \restriction \bar{s}$, where $F_t^k \restriction \bar{s} \leftrightharpoons \{\varphi(\bar{s}') \mid \varphi(\bar{s}') \in F_t,\ \bar{s}' \in (\mathrm{sp}(\bar{s}))^{<\omega}\}$ and the function sp is defined inductively thus: put $\mathrm{sp}(m) \leftrightharpoons \{m\}$ for $m \in M$, $\mathrm{sp}(t_\varphi(\bar{s})) \leftrightharpoons \{t_\varphi(\bar{s})\} \cup \mathrm{sp}(\bar{s})$ for all Skolem terms in S_t, and finally, put $\mathrm{sp}(\langle s_1, \ldots, s_n \rangle) \leftrightharpoons \mathrm{sp}(s_1) \cup \ldots \cup \mathrm{sp}(s_n)$ for tuples.

Then we also require that the set F_t^k obtained as above be closed under logical deducibility relative to T_2. Our description shows that F_t^k is defined inductively; so, it is effective by the Gandy theorem (cf. [3, 4]). Stage k is completed.

In order to end off step $t+1$, we need only put $S_{t+1} \leftrightharpoons S_t^{k_0}$, $F_{t+1} \leftrightharpoons F_t^{k_0}$, and $p_{t+1} \leftrightharpoons p_t^{k_0}$.

The construction is completed. Its properties immediately imply that the resulting model with the universe $\bigcup\limits_{t\in\omega} S_t$ and (full) diagram $\bigcup\limits_{t\in\omega} F_t$ is a model of T_2. Consequently, given a constructivization of $\mathfrak{M}^{<\omega}$, we can build up a strong constructivization of $\mathfrak{N}$. Obviously, such a construction can be realized via a computable sequence of Σ-formulas in the signature $\sigma_1 \cup \{\in, U\}$, which defines an uncountable model of T_2 in a hereditarily finite superstructure over a suitable model $\mathfrak{M}'$ of T_1. In fact, let $\theta(x, \bar{y})$ be a complete formula in T_1 such that the set $I_\theta \leftrightharpoons \{i \in I \mid \mathfrak{M} \models \theta(i, \bar{m}_0)\}$ is infinite. (Such a formula exists in virtue of T_1 being ω-categorical). Take a model $\mathfrak{M}'$ and a tuple $\bar{m}'$ of its elements such that $\theta(x, \bar{m}')$ defines an uncountable subset in $\mathfrak{M}'$. If Γ is a sequence of Σ-formulas defined via the process above, then the properties of the construction imply that $\Gamma(\mathrm{HF}(\mathfrak{M}'), \bar{m}')$ is an uncountable model of T_2. Thus the uncountable model of T_2 is Σ-definable over $\mathrm{Mod}(T_1)$.

Actually we can somewhat extend the range of application of the previous theorem. For instance, no dense linear order with endpoints can be a wide model, though it is easy to verify that the statement that an uncountable model of a c-simple theory is Σ-definable would equally do for dense linear orders with or without endpoints. Theorem 2 remains true if we weaken the requirement on a countable model $\mathfrak{M}$ of T_1 specified in its formulation. Namely, if the set of all n-types realizable in $\mathfrak{M}$ by elements of a set $X \subseteq |\mathfrak{M}|$ is denoted by $S_X^{\mathfrak{M}}(n)$ then it suffices to impose on $\mathfrak{M}$ the following restrictions: for any infinite set $X \subseteq |\mathfrak{M}|$, there exists a (infinite) definable set $D \subseteq |\mathfrak{M}|$ such that $S_X^{\mathfrak{M}} = S_D^{\mathfrak{M}}$. The conditions $S_X^{\mathfrak{M}} = S^{\mathfrak{M}}$ and $R_X^{\mathfrak{M}} = R^{\mathfrak{M}}$ are equivalent, and so every wide model possesses this property. A proof in this — more general — case differs from the previous one only in that at the initial step, we need take, not the whole set $|\mathfrak{M}|$, but its definable subset D. (Recall that for every model of a c-simple theory, the Σ-definability of its subsets in a hereditarily finite superstructure is equivalent to ordinary definability.)

From the Ramsey theorem, using the argument of the Ehrenfeucht–Mostowski theorem, we infer the following property of ω-categorical models: if $\mathfrak{M}$ is an ω-categorical model then, for any infinite set $I \subseteq |\mathfrak{M}|$ and any tuple $\bar{m}_0 \in |\mathfrak{M}|^{<\omega}$, there exists an infinite set $J \subseteq I$ such that

$$\langle \mathfrak{M}, \bar{j}_1 \rangle \equiv \langle \mathfrak{M}, \bar{j}_2 \rangle \Rightarrow \langle \mathfrak{M}^*, \bar{j}_1 \rangle \equiv \langle \mathfrak{M}^*, \bar{j}_2 \rangle,$$

where $\mathfrak{M}^* = \langle \mathfrak{M}, \bar{m}_0 \rangle$ for all tuples $\bar{j}_1$ and $\bar{j}_2$ of elements in J.

An effective version of this property is as follows. We say that a c-simple theory T admits an *effective elimination of constants* if every decidable model $\mathfrak{M}$ of T satisfies the following: for every infinite computable set $I \subseteq |\mathfrak{M}|$ and every tuple $\bar{m}_0 \in |\mathfrak{M}|^{<\omega}$, there exists an infinite computable set $J \subseteq I$ such that for all tuples $\bar{j}_1$ and $\bar{j}_2$ of elements in J,

$$\langle \mathfrak{M}, \bar{j}_1 \rangle \equiv \langle \mathfrak{M}, \bar{j}_2 \rangle \Rightarrow \langle \mathfrak{M}^*, \bar{j}_1 \rangle \equiv \langle \mathfrak{M}^*, \bar{j}_2 \rangle,$$

where $\mathfrak{M}^* = \langle \mathfrak{M}, \bar{m}_0 \rangle$ and J is determined from I and $\bar{m}_0$ effectively.

Proposition 1. Theories T_{DLO} and T_E admit an effective elimination of constants.

Proof. Let $\langle L, < \rangle$ be a decidable dense linear order and $\bar{l} = \langle l_0, \ldots, l_n \rangle \in L^{<\omega}$. If k is the number of pairwise distinct elements of $\bar{l}$ then these elements partition L into finitely many intervals $U_0, \ldots, U_k$; so, for any infinite set $I \subseteq L$, there is an interval U_i, $i \leqslant k$, such that $J \rightleftharpoons I \cap U_i$. Obviously, for any $\bar{j}_1, \bar{j}_2 \in J^{<\omega}$, $\langle L, <, \bar{j}_1 \rangle \equiv \langle L, <, \bar{j}_2 \rangle$ implies $\langle L, <, \bar{l}, \bar{j}_1 \rangle \equiv \langle L, <, \bar{l}, \bar{j}_2 \rangle$. Since J is the intersection of I and a definable subset of L, J is computable if so is I.

Now, let $\langle S \rangle$ be a decidable infinite model of the empty signature, $\bar{s} = \langle s_0, \ldots, s_n \rangle \in S^{<\omega}$, and $I \subseteq S$ be an infinite computable set. It suffices to put $J \rightleftharpoons I \setminus \{s_0, \ldots, s_n\}$.

The two preceding statements immediately imply criteria determining whether an uncountable model of a c-simple theory is Σ-definable over a class of dense linear orders and over a class of infinite models of the empty signature. In correspondence with the general notion of $\mathfrak{M}$-indiscernibility in both of these two cases are the concepts of ordered indiscernibility and of total indiscernibility, which are well known in model theory (cf. [6]). Let T_{DLO} be the theory of dense linear orders and T_E be the theory of infinite models of the empty signature.

We say that a subset of a computable model is *computable* if it is a computable subset of natural numbers; an ordered subset is *computable* if the order relation on it is computable.

THEOREM 3. Let T be a c-simple theory. Then:

(1) T has an uncountable model Σ-definable over the class $\mathrm{Mod}(T_{\mathrm{DLO}})$ iff some decidable model of T contains an infinite computable set of orderly indiscernible elements.

(2) T has an uncountable model Σ-definable over the class $\mathrm{Mod}(T_E)$ iff some decidable model of T contains an infinite computable set of totally indiscernible elements.

The **proof** follows immediately from Theorem 2 and Prop. 1.

Ershov in [2] came up with the hypothesis that every c-simple theory T has an uncountable model, Σ definable over the class $\mathrm{Mod}(T_{\mathrm{DLO}})$. Appealing to Theorem 3, however, we can point out an example of a c-simple theory (of an infinite signature) for which this is not the case. To do this, we make use of a construction in [8, 9].

Let $\mathcal{T} \subseteq 2^{<\omega}$ be a binary tree; here, $2 = \{0, 1\}$. Denote by $P(\mathcal{T})$ the set of infinite paths in this tree. For a model $\mathfrak{M}$ with the universe ω, $\mathcal{I}(\mathfrak{M})$ denotes the set of all finite sequences of orderly indiscernible elements in $\mathfrak{M}$, that is, $\mathcal{I}(\mathfrak{M}) \subseteq \omega^{<\omega}$ and the order is defined by the successor relation in a sequence. We say that the problem of searching an infinite path in $\mathcal{T}$ is *effectively equivalent* to searching an infinite sequence of orderly indiscernible elements in $\mathfrak{M}$, and write $P(\mathcal{T}) \approx \mathcal{I}(\mathfrak{M})$, if there exist $e, f \in \omega$ such that:

(i) $\varphi_e^I \in P(\mathcal{T})$ if $I \in \mathcal{I}(\mathfrak{M})$;

(ii) $\varphi_f^\pi \in \mathcal{I}(\mathfrak{M})$ if $\pi \in P(\mathcal{T})$;

(iii) $\varphi_e^I = \pi$ for all $\pi \in P(\mathcal{T})$ if $\varphi_f^\pi = I$,

where $\{\varphi_n \mid n \in \omega\}$ is some computable numbering of all unary partial computable functions with an oracle (cf. [5]).

THEOREM 4 [9]. For every infinite computable binary tree $\mathcal{T}$, there exists a decidable model complete ω-categorical theory T with a decidable set of complete formulas such that $P(\mathcal{T}) \approx \mathcal{I}(\mathfrak{M})$, for any decidable model $\mathfrak{M}$ of T.

Let $\mathcal{T}_0$ be an infinite recursive binary tree without infinite recursive branches and T_0 be the theory constructed from $\mathcal{T}_0$. If $\mathfrak{M}_0$ is a countable model of T_0 then every infinite set of orderly indiscernible elements of $\mathfrak{M}_0$ is non-computable. By Theorem 3, therefore, no uncountable model of T_0 can be Σ-definable over $\mathrm{Mod}(T_{\mathrm{DLO}})$. At the same time, T_0 is a c-simple theory. Thus we arrive at

THEOREM 5. There exists a c-simple theory T_0 every uncountable model of which is not Σ-definable over the class $\mathrm{Mod}(T_{\mathrm{DLO}})$.

The theory obtained by using the construction in [7] has an infinite signature. Whether it is possible to construct a c-simple theory of a finite signature satisfying the assumption of Theorem 5 is not known.

Therefore it seems interesting to consider the following question: Is it true that for any c-simple theory T (of a finite signature), there exists a c-simple theory T' such that every uncountable model of T' is not Σ-definable over

$\mathrm{Mod}(T)$. In this connection, it is worth mentioning the following consequence of Theorem 1.

If T_1 and T_2 are c-simple theories, and an uncountable model of T_2 is Σ-definable over $\mathrm{Mod}(T_2)$, then the necessary condition of being Σ-definable for uncountable models implies that there exist decidable models $\mathfrak{M} \models T_1$ and $\mathfrak{N} \models T_2$ such that for some infinite computable set $I \subseteq |\mathfrak{M}| \cap |\mathfrak{N}|$, $R_I^{\mathfrak{N}}(n) \leqslant R_I^{\mathfrak{M}}(n)$ for all $n \in \omega$.

To conclude this section, we couch yet another notion. Let K_1 be some class of models of an arbitrary finite signature σ_1 and K_2 be some class of models of a computable predicate signature $\sigma_2 = \langle P_0^{n_0}, \ldots, P_k^{n_k}, \ldots \rangle$. The class K_2 is said to be *spectrally Σ-definable over* K_1 if there exists a computable sequence

$$\Gamma = \langle \Phi, \Psi, \Psi^*, \Phi_0, \Phi_0^*, \ldots, \Phi_k, \Phi_k^*, \ldots \rangle$$

of Σ-formulas in the signature $\sigma_1 \cup \{\in, U\}$ such that for any model $\mathfrak{M}$ in K_1 and for an element $a \in \mathrm{HF}(\mathfrak{M})$, the sequence Γ of formulas with a parameter a properly defines in $\mathrm{HF}(\mathfrak{M})$ a model of the signature σ_2 belonging to K_2, and the following condition holds:

$$\mathrm{Sp}(\Gamma(K_1)) = \mathrm{Sp}(K_2),$$

where $\mathrm{Sp}(K)$ stands for the class of cardinalities of models in K and $\Gamma(K)$ denotes the class of all models that are Σ-definable in hereditarily finite superstructures over models of K by a sequence Γ of formulas with an arbitrary parameter.

Proposition 2. If T_1 and T_2 are c-simple then the class $\mathrm{Mod}(T_2)$ is spectrally Σ-definable over $\mathrm{Mod}(T_1)$ if and only if some uncountable model of T_2 is Σ-definable over $\mathrm{Mod}(T_1)$.

Proof. The necessity being obvious, we need only argue for sufficiency. Suppose that for some model $\mathfrak{M}'$ of T_1, the uncountable model of T_2 is definable in $\mathrm{HF}(\mathfrak{M})$ by a sequence Γ of Σ-formulas, in which case we may assume that the parameter of the formulas in Γ is a tuple $\bar{m}' \in |\mathfrak{M}'|^{<\omega}$ of urelements. Since T_2 is c-simple it contains a computable model $\mathfrak{N}_0$ which is, obviously, Σ-definable in any hereditarily finite superstructure. In view of this, we may define a sequence Γ^* of Σ-formulas such that for any model $\mathfrak{M}$ of T_2 and any element $a \in \mathrm{HF}(\mathfrak{M})$,

$$\Gamma^*(\mathrm{HF}(\mathfrak{M}), a) \leftrightharpoons \begin{cases} \Gamma(\mathrm{HF}(\mathfrak{M}), a) & \text{if } a = \bar{m} \text{ and } \langle \mathfrak{M}, \bar{m} \rangle \equiv \langle \mathfrak{M}', \bar{m}' \rangle; \\ \mathfrak{N}_0 & \text{otherwise.} \end{cases}$$

This condition can be verified effectively since T_1 is also a c-simple theory. From $\langle \mathfrak{M}, \bar{m}\rangle \equiv \langle \mathfrak{M}', \bar{m}'\rangle$ it follows that $\langle \mathrm{HF}(\mathfrak{M}), \bar{m}\rangle \equiv \langle \mathrm{HF}(\mathfrak{M}'), \bar{m}'\rangle$, and so the sequence Γ^* of formulas in a hereditarily finite superstructure over any model of T_1, for any parameter, defines a model of T_2 properly. The fact that a model of any infinite cardinality can be defined in a similar fashion is established as in the proof of Theorem 2.

2. PAIRS OF RECURSIVELY SATURATED MODELS

Let $\sigma_1 = \langle P_0^{n_0}, \ldots, P_k^{n_k}, \ldots\rangle$ and $\sigma_2 = \langle Q_0^{m_0}, \ldots, Q_l^{m_l}, \ldots\rangle$ be predicate signatures (we may assume that $\sigma_1 \cap \sigma_2 = \varnothing$) and $\mathfrak{M}$ and $\mathfrak{N}$ be models of signatures σ_1 and σ_2, respectively. By a *pair* $(\mathfrak{M}, \mathfrak{N})$ we mean a model of the signature $\sigma \leftrightharpoons \langle M^1, N^1, P_0^{n_0}, \ldots, P_k^{n_k}, \ldots, Q_0^{m_0}, \ldots, Q_l^{m_l}, \ldots\rangle$, whose universe is the union $|\mathfrak{M}| \cup |\mathfrak{N}|$ and predicate symbols are interpreted thus: $M^{(\mathfrak{M},\mathfrak{N})} = |\mathfrak{M}|$, $N^{(\mathfrak{M},\mathfrak{N})} = |\mathfrak{N}|$, $P_i^{(\mathfrak{M},\mathfrak{N})} = P_i^{\mathfrak{M}}$, $i = 1, \ldots, k, \ldots$, $Q_j^{(\mathfrak{M},\mathfrak{N})} = Q_j^{\mathfrak{N}}$, $j = 1, \ldots, l, \ldots$.

Fixing some Gödel numberings of formulas in the signatures σ_1, σ_2, and σ, we identify arbitrary sets of formulas in these signatures with corresponding sets of their Gödel numbers. In particular, a set of formulas is said to be recursive if the set of Gödel numbers of those formulas is recursive (provided that σ_1 and σ_2 are computable). Throughout this section, for uniformity of the terminology adhered to here, we use the term "recursive" instead of "computable" in describing properties of the objects in question.

An algebraic system $\mathfrak{A}$ of a computable signature σ' is said to be *recursively saturated* if, for any finite tuple $\bar{a}$ of elements of $|\mathfrak{A}|$, every recursive set of formulas (with a same set of free variables) of the signature $\sigma' \cup \langle \bar{a}\rangle$ locally realizable in $(\mathfrak{A}, \bar{a})$ is realizable in $(\mathfrak{A}, \bar{a})$. Relativizing this definition, we arrive at a concept of an X-recursively saturated model for an arbitrary set $X \subseteq \omega$ (in which case sets of formulas recursive with an oracle X are treated).

THEOREM 6. Let $\mathfrak{M}$ and $\mathfrak{N}$ be models of computable signatures. Then the model $(\mathfrak{M}, \mathfrak{N})$ is recursively saturated if and only if:

(1) $\mathfrak{M}$ is $\mathrm{Th}(\mathfrak{N}, \bar{n})$-recursively saturated, for all $\bar{n} \in |\mathfrak{N}|^{<\omega}$;

(2) $\mathfrak{N}$ is $\mathrm{Th}(\mathfrak{M}, \bar{m})$-recursively saturated, for all $\bar{m} \in |\mathfrak{M}|^{<\omega}$.

Proof. Let σ_1 and σ_2 be signatures of $\mathfrak{M}$ and $\mathfrak{N}$, respectively. (Without loss, these can be conceived of as predicate signatures.) Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ satisfy (1) and (2), respectively. We claim that $(\mathfrak{M}, \mathfrak{N})$ is recursively saturated. Assume that $\{\theta^k(\bar{z}) \mid k \in \omega\}$ is a recursive set of formulas in the signature σ, which is locally realized in $(\mathfrak{M}, \mathfrak{N})$. There is no loss of generality in assuming that $\theta^{k+1}(\bar{z}) \to \theta^k(\bar{z})$ holds for any $k \in \omega$ (in which case it is required that we pass to a set of formulas $\theta^k_*(\bar{z}) \leftrightharpoons \theta^0(\bar{z}) \wedge \ldots \wedge \theta^k(\bar{z})$, $k \in \omega$).

For convenience of the presentation, below, instead of unary predicates M and N, distinguishing the universes $|\mathfrak{M}|$ and $|\mathfrak{N}|$, we treat a language with variables of two kinds: $\bar{x}$ and $\bar{m}$, for variables and constants corresponding to the elements of $\mathfrak{M}$, and $\bar{y}$ and $\bar{n}$ for the elements of $\mathfrak{N}$. In what follows, it is assumed that the set of formulas in question has the form $\{\theta^k(\bar{x}, \bar{y}) \mid k \in \omega\}$, and that all bound variables in these formulas are also of one of the two kinds possible. In fact, every formula $\theta(\ldots, z, \ldots)$ in $(\mathfrak{M}, \mathfrak{N})$ is equivalent to the disjunction $(M(z) \wedge \theta) \vee (N(z) \wedge \theta)$, or, in our notation, to $\theta(\ldots, x, \ldots) \vee \theta(\ldots, y, \ldots)$.

Given any formula $\theta^k(\bar{x}, \bar{y})$, we can effectively find its prenex normal form. In view of $P_i(\ldots, z, \ldots) \wedge N(z) \equiv Q_j(\ldots, z, \ldots) \wedge M(z) \equiv \neg(z = z)$, every disjunctive term $\theta^k_i(\bar{x}_i, \bar{y}_i)$ in the prenex normal form matrix is equivalent to the conjunction $\varphi^k_i(\bar{x}_i) \wedge \psi^k_i(\bar{y}_i)$, where $\varphi^k_i(\bar{x}_i)$ and $\psi^k_i(\bar{y}_i)$ are elementary conjunctions containing only predicates and variables defined on, respectively, $\mathfrak{M}$ and $\mathfrak{N}$. We describe the procedure allowing of quantifiers in the quantifier prefix to be carried over inside the matrix, in the course of which the prenex normal form of $\theta^k(\bar{x}, \bar{y})$, via a chain of equivalent transformations, turns into a formula of the form $(\varphi^k_1(\bar{x}) \wedge \psi^k_1(\bar{y})) \vee \ldots \vee (\varphi^k_{n_k}(\bar{x}) \wedge \psi^k_{n_k}(\bar{y}))$, where $\varphi^k_i(\bar{x})$ and $\psi^k_i(\bar{y})$ are arbitrary formulas all of whose predicates as well as free and bound variables are defined on $\mathfrak{M}$ and $\mathfrak{N}$, respectively. The quantifiers $\exists x$ and $\exists y$ are carried over inside the disjunction in the obvious manner because

$$\exists y(\theta^k_1(\bar{x}_1, \bar{y}_1) \vee \ldots \vee \theta^k_{n_k}(\bar{x}_{n_k}, \bar{y}_{n_k})) \equiv$$
$$(\varphi^k_1(\bar{x}_1) \wedge \exists y \psi^k_1(\bar{y}_1)) \vee \ldots \vee (\varphi^k_{n_k}(\bar{x}_{n_k}) \wedge \exists y \psi^k_{n_k}(\bar{y}_{n_k}))$$

(similarly for $\exists x$). For the quantifiers $\forall x$ and $\forall y$, we have

$$\forall y(\theta^k_1(\bar{x}_1, \bar{y}_1) \vee \ldots \vee \theta^k_{n_k}(\bar{x}_{n_k}, \bar{y}_{n_k})) \equiv \bigvee_{S \subseteq \{1, \ldots, n_k\}} \left(\bigwedge_{s \in S} \varphi^k_s(\bar{x}_s) \wedge \left(\forall y \left(\bigvee_{s \in S} \psi^k_s(\bar{y}_s) \right) \right) \right).$$

Exercising this process with all quantifiers in the quantifier prefix of the prenex normal form of $\theta^k(\bar{x}, \bar{y})$, we ultimately arrive at a formula of the form

$$(\varphi^k_1(\bar{x}, \bar{m}) \wedge \psi^k_1(\bar{y}, \bar{n})) \vee \ldots \vee (\varphi^k_{n_k}(\bar{x}, \bar{m}) \wedge \psi^k_{n_k}(\bar{y}, \bar{n})),$$

where $\bar{m}$ and $\bar{n}$ are tuples of parameters in $\mathfrak{M}$ and $\mathfrak{N}$, respectively, occurring in the formulas $\theta^k(\bar{x}, \bar{y})$. For every $k \in \omega$, put

$$\Psi_k(\bar{y}, \bar{n}) \leftrightharpoons \bigvee_{S \in \mathcal{S}_k} \bigwedge_{s \in S} \psi^k_s(\bar{y}, \bar{n}),$$

where $\mathcal{S}_k = \left\{ S \subseteq \{1, \ldots, n_k\} \;\middle|\; \mathfrak{M} \models \exists \bar{x} \left(\bigvee_{s \in S} \varphi_s^k(\bar{x}, \bar{m}) \right) \right\}$ by definition. The initial type $\{\theta^k(\bar{x}, \bar{y}) \mid k \in \omega\}$ is locally realized in $(\mathfrak{M}, \mathfrak{N})$; so, $\mathcal{S}_n \neq \varnothing$ for all $n \in \omega$. The set $\{\Psi_k(\bar{y}, \bar{n}) \mid k \in \omega\}$ is $\mathrm{Th}(\mathfrak{M}, \bar{m})$-recursive, and by assumption, it is locally realized in $\mathfrak{N}$. Since $\mathfrak{N}$ is $\mathrm{Th}(\mathfrak{M}, \bar{m})$-recursively saturated, this type is realized in $\mathfrak{N}$ by some tuple $\bar{c}$ of elements. Consider formulas such as

$$\Phi_k(\bar{x}, \bar{m}) \rightleftharpoons \bigvee_{s \in S_k(\bar{c})} \varphi_s^k(\bar{x}, \bar{m}),$$

where $S_k(\bar{c}) = \{l \in \{1, \ldots, n_k\} \mid \mathfrak{N} \models \psi_l^k(\bar{c}, \bar{n})\}$. The set $\{\Phi_k(\bar{x}, \bar{m}) \mid k \in \omega\}$ is $\mathrm{Th}(\mathfrak{N}, \bar{n}, \bar{c})$-recursive, and it is locally realized in $\mathfrak{M}$ by the choice of $\bar{c}$. Since $\mathfrak{M}$ is $\mathrm{Th}(\mathfrak{M}, \bar{n}, \bar{c})$-recursively saturated, there exists a tuple $\bar{a}$ of elements in $\mathfrak{M}$ such that $\mathfrak{M} \models \Phi_k(\bar{a}, \bar{m})$ for all $k \in \omega$. Thus $\langle \bar{a}, \bar{c} \rangle$ realizes type $\{\theta^k(\bar{x}, \bar{y}, \bar{m}, \bar{n}) \mid k \in \omega\}$ in $(\mathfrak{M}, \mathfrak{N})$, as required.

To argue for the way back, assume that the model $(\mathfrak{M}, \mathfrak{N})$ is recursively saturated. Let $\bar{n}$ be an arbitrary tuple of elements in $|\mathfrak{N}|$ and let $Q = \gamma(\mathrm{Th}(\mathfrak{N}, \bar{n}))$, where γ is some Gödel numbering of formulas in the signature σ_2. We claim that $\mathfrak{M}$ is Q-recursively saturated. Let $\{\varphi_k(\bar{x}, \bar{m}) \mid k \in \omega\}$ be a Q-recursive set of formulas (with parameters $\bar{m}$ in $|\mathfrak{M}|$), which we represent as

$$\{\theta_k(\bar{x}, \bar{m}) \mid \exists D_u \subseteq Q \; \langle k, u \rangle \in W_z\},$$

for some z, where $\theta_k = \gamma^{-1}(k)$ (since Q is a complete type, the quantifier $\exists D_v \subseteq N \setminus Q$ can be dropped). Then the fact that this set is (locally) realized in $\mathfrak{M}$ is equivalent to the recursive set

$$\{(\theta_k(\bar{x}, \bar{m}) \wedge \psi_u(\bar{n})) \mid \langle k, u \rangle \in W_z\}$$

of formulas of the signature σ, where $\psi_u = \gamma(i_1) \wedge \ldots \wedge \gamma(i_n)$ for $D_u = \{i_1, \ldots, i_n\}$, being (locally) realized in $(\mathfrak{M}, \mathfrak{N})$. Since $(\mathfrak{M}, \mathfrak{N})$ is recursively saturated, the condition that this set is locally realizable implies that it is also realized in $(\mathfrak{M}, \mathfrak{N})$, which is equivalent to the initial type being realizable in $\mathfrak{M}$. Similarly we can state that $\mathfrak{N}$ is P-recursively saturated for all P of the form $\mathrm{Th}(\mathfrak{M}, \bar{m})$.

We say that a model $\mathfrak{M}$ of a signature σ is *locally decidable* if $\mathrm{Th}(\mathfrak{M}, \bar{m})$ is decidable for any tuple $\bar{m}$ of elements in $|\mathfrak{M}|$. In particular, every model of a c-simple theory is locally decidable. A consequence of Theorem 6 is the following:

COROLLARY 1. Let models $\mathfrak{M}$ and $\mathfrak{N}$ be locally decidable. The pair $(\mathfrak{M}, \mathfrak{N})$ is recursively saturated if and only if $\mathfrak{M}$ and $\mathfrak{N}$ are recursively saturated.

Proposition 3. Let $\mathfrak{M}$ and $\mathfrak{N}$ be recursively saturated so that $\mathrm{Th}(\mathfrak{M}) = \mathrm{Th}(\mathfrak{N})$ and $\mathfrak{M}$ is locally decidable. The pair $(\mathfrak{M}, \mathfrak{N})$ is recursively saturated if and only if $\mathfrak{N}$ is locally decidable.

Proof. Let $(\mathfrak{M}, \mathfrak{N})$ be recursively saturated. Assume that $\mathfrak{N}$ is not locally decidable, that is, there exists a tuple $\bar{n}$ of elements in $|\mathfrak{N}|$ such that $\mathrm{Th}(\mathfrak{N}, \bar{n})$ is not decidable. Since $\mathrm{Th}(\mathfrak{M}) = \mathrm{Th}(\mathfrak{N})$, a type realizable in $\mathfrak{N}$ by $\bar{n}$ is also a type relative to $\mathrm{Th}(\mathfrak{M})$, and, since $\mathfrak{M}$ is $\mathrm{Th}(\mathfrak{N}, \bar{n})$-saturated by Theorem 1, this type should be realized in $\mathfrak{M}$ by some tuple $\bar{m}$. Hence $\mathrm{Th}(\mathfrak{M}, \bar{m}) = \mathrm{Th}(\mathfrak{N}, \bar{n})$, which contradicts the assumption that $\mathfrak{M}$ is locally decidable.

Conversely, let $\mathfrak{N}$ be locally decidable. In this instance $(\mathfrak{M}, \mathfrak{N})$ is recursively saturated in view of Corollary 1.

Appealing to Theorem 6, we furnish an example of a pair of models $\mathfrak{M}$ and $\mathfrak{N}$ such that $\mathfrak{M}$ and $\mathfrak{N}$ are recursively saturated, but $(\mathfrak{M}, \mathfrak{N})$ is not. The models under construction, among other things, are elementary equivalent. Take a signature $\sigma \leftrightharpoons \{P^1_\varepsilon \mid \varepsilon \in E\}$, $E = \{0,1\}^{<\omega}$, consisting of countably many unary predicates indexed by finite sequences of 0's and 1's. Let $D \subseteq E$ be an infinite recursive binary tree without infinite recursive branches. Such a tree was used in [3] to construct a theory, T_D, with the following set of axioms:

$\forall x P_\Lambda(x)$,

$\forall x(P_{\varepsilon 0}(x) \vee P_{\varepsilon 1}(x) \to P_\varepsilon(x))$, $\varepsilon \in E$,

$\forall x((P_{\varepsilon 0}(x) \to \neg P_{\varepsilon 1}(x)) \wedge (P_{\varepsilon_1}(x) \to \neg P_{\varepsilon_0}(x)))$, $\varepsilon \in E$,

$\exists x(P_\varepsilon(x) \wedge \neg P_{\varepsilon 0}(x) \wedge \neg P_{\varepsilon 1}(x))$, $\varepsilon \in D$,

$\forall x \neg P_\varepsilon(x)$, $\varepsilon \in E \setminus D$,

$\forall x \forall y(P_\varepsilon(x) \wedge \neg P_{\varepsilon 0}(x) \wedge \neg P_{\varepsilon 1}(x) \wedge P_\varepsilon(y) \wedge \neg P_{\varepsilon 0}(y) \wedge \neg P_{\varepsilon 1}(y) \to x = y)$, $\varepsilon \in E$.

The theory T_D is complete and decidable. Since D has no infinite recursive branches, it follows that every model of T_D is recursively saturated. By the same token, the sole locally decidable model of T_D is its prime model — $\mathfrak{M}_0$. Therefore if $\mathfrak{M}$ is a model of T_D non-isomorphic to $\mathfrak{M}_0$, then $(\mathfrak{M}, \mathfrak{N})$ is not recursively saturated by Prop. 6. Thus we have

Proposition 4. If $\mathfrak{M}_0$ is a prime model of T_D then, for any model $\mathfrak{M}$ of T_D, the model $(\mathfrak{M}, \mathfrak{M}_0)$ is recursively saturated if and only if $\mathfrak{M} \cong \mathfrak{M}_0$.

Let $\mathfrak{M}$ be a model of a finite signature; then an admissible set $\mathrm{HYP}(\mathfrak{M})$ is defined. If $O(\mathfrak{M})$ is the least ordinal not in $\mathrm{HYP}(\mathfrak{M})$ then $\mathfrak{M}$ is recursively saturated iff $O(\mathfrak{M}) = \omega$ (cf. [3, 4]). We know that every model has a recursively saturated elementary extension, and so there exist recursively saturated models whose elementary theory is as complex as is wished. If we fix the thus obtained model $\mathfrak{M}$ with a sufficiently complex elementary theory then, choosing a recursively saturated but not $\mathrm{Th}(\mathfrak{M})$-recursively saturated model $\mathfrak{N}$ (such exists by [10]) and using Theorem 6, we can ascertain that the pair $(\mathfrak{M}, \mathfrak{N})$ is not recursively saturated. We thus arrive at the following:

Proposition 5. There exist models $\mathfrak{M}$ and $\mathfrak{N}$ with finite signatures such that $O(\mathfrak{M}) = O(\mathfrak{N}) = \omega$, but $O(\mathfrak{M}, \mathfrak{N}) > \omega$.

Denote by $pp(\mathrm{HYP}(\mathfrak{M}))$ the pure part of an admissible set $\mathrm{HYP}(\mathfrak{M})$, that is, the set of elements whose transitive closure is freed of urelements. Consider the case where $\mathfrak{M}$ is a recursively saturated model. In this instance we can fix a computable numbering $\nu : \omega \to pp(\mathrm{HYP}(\mathfrak{M}))$ (such numberings are all computably equivalent). Pure Σ-subsets of $\mathrm{HYP}(\mathfrak{M})$ for the case where $\mathrm{Th}(\mathfrak{M})$ is a c-simple theory are described in the following:

LEMMA 1. Let $T = \mathrm{Th}(\mathfrak{M})$ be a c-simple theory. An arbitrary subset $P \subseteq pp(\mathrm{HYP}(\mathfrak{M}))$ is a Σ-subset of $\mathrm{HYP}(\mathfrak{M})$ if and only if $\nu^{-1}(P)$ is computably enumerable.

Proof. Let $P \subseteq pp(\mathrm{HYP}(\mathfrak{M}))$ be defined by a Σ-formula $\Phi(x, \bar{c})$ with a tuple $\bar{c}$ of parameters. Given Φ, we can effectively construct an $\exists$-formula $\Phi^*(x)$ in the signature $\langle +, \cdot, 0, 1\rangle$ such that for any $x_0 \in pp(\mathrm{HYP}(\mathfrak{M}))$,

$$\mathrm{HYP}(\mathfrak{M}) \models \Phi(x_0, \bar{c}) \Leftrightarrow \mathbb{N} \models \Phi^*(\nu^{-1}(x_0)).$$

In fact, if $\mathrm{Th}(\mathfrak{M})$ is c-simple, then the set $\mathrm{HYP}(\mathfrak{M})$ being admissible is Σ-definable in $\mathrm{HF}(\mathfrak{M})$ (cf. [11, 12]), and the relevant result for $\mathrm{HF}(\mathfrak{M})$ can be applied to it.

The statement from [11, 12] used in the proof of the previous lemma implies that for a model $\mathfrak{M}$ of a c-simple theory, an arbitrary algebraic system $\mathfrak{A}$ is Σ-definable in $\mathrm{HYP}(\mathfrak{M})$ iff $\mathfrak{A}$ is Σ-definable in $\mathrm{HF}(\mathfrak{M})$.

In [13], the concept of being $\Sigma_{\mathbb{A}}$-saturated was defined for an arbitrary admissible set $\mathbb{A}$. Namely, a model $\mathfrak{N}$ of a signature σ is said to be $\Sigma_{\mathbb{A}}$-*saturated* if, for every set $p(\bar{x}, \bar{y})$ of formulas in the signature σ that is Σ-definable in $\mathbb{A}$, the fact that every $\mathbb{A}$-finite subset $q(\bar{x}, \bar{n})$ is realizable in $\mathfrak{N}$ implies that $p(\bar{x}, \bar{n})$ is realizable in $\mathfrak{N}$, where $\bar{n} \in |\mathfrak{N}|^{<\omega}$ is a tuple of parameters.

For any models $\mathfrak{M}$ and $\mathfrak{N}$, we consider the following conditions:

(1) $\mathfrak{N}$ is recursively saturated;

(2) $\mathfrak{N}$ is $\mathrm{Th}(\mathfrak{M}, \bar{m})$-recursively saturated, for all $\bar{m} \in |\mathfrak{M}|^{<\omega}$.

(3) $\mathfrak{N}$ is $\Sigma_{\mathrm{HYP}(\mathfrak{M})}$-saturated.

For any $\mathfrak{M}$ and $\mathfrak{N}$, (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1). In the general case, however, the reverse implications fail. We distinguish a class of models for which the three conditions are equivalent.

Proposition 6. If $\mathrm{Th}(\mathfrak{M})$ is a c-simple theory then

$$(1) \Leftrightarrow (2) \Leftrightarrow (3)$$

for any model $\mathfrak{N}$.

Proof. Suppose that $\mathfrak{M}$ satisfies the conditions of the proposition. Then $\mathrm{Th}(\mathfrak{M}, \bar{a})$ is decidable for any tuple $\bar{a} \in M^{<\omega}$, and so $(1) \Rightarrow (2)$. It remains to show that the property of $\mathfrak{N}$ being recursively saturated implies that $\mathfrak{N}$ is $\Sigma_{\mathrm{HYP}(\mathfrak{M})}$-saturated. This is so since every pure Σ-subset of $\mathrm{HYP}(\mathfrak{M})$, for the case where $\mathrm{Th}(\mathfrak{M})$ is c-simple, is computably enumerable in the sense of Lemma 1.

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