Presentations of Structures in Admissible Sets

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Abstract. We consider copies and constructivizations of structures in admissible sets. It is well known that in classical computable model theory (on natural numbers) these approaches are equivalent: a structure has computable (decidable) copy if and only if it is constructivizable (strongly constructivizable). However, in admissible sets the "if" part of this statement is not true in general. In the first section we survey results about copies in hereditary finite superstructures and definability (so called syntactical conditions of intrinsically computable properties). The second section is devoted to constructivizations of uncountable structures in "simplest" uncountable admissible sets. The third section contains some results on constructivizations of admissible sets within themselves.

1 Copies of Structures in Admissible Sets

We denote by $F(\sigma)$ the set of finite first order formulas of a signature σ. We also fix some Gödel numbering $\ulcorner\cdot\urcorner : F(\sigma) \to \omega$ ($\ulcorner\varphi\urcorner$ – Gödel number of φ). In all that follows we consider only computable signatures and suppose that Gödel numberings are effective. We also denote by $F_n(\sigma)$ ($n \leqslant \omega$) a set of (finite first order) formulas of signature σ with no more than n alterating groups of quantifiers in prenex normal form. $F_0(\sigma)$ is a set of quantifier-free formulas of signature σ.

Let $\mathfrak{M}$ be a structure of signature σ, $\mathbb{A}$ an admissible set, and let $M \subseteq A$. Then the atomic diagram

$$D(\mathfrak{M}) = \{\langle\ulcorner\varphi\urcorner, \bar{m}\rangle \mid \varphi \in F_0(\sigma) - \text{atomic formula}, \bar{m} \in M^{<\omega},\ \mathfrak{M} \models \varphi(\bar{m})\}$$

is a subset of A.

Definition 1. *Let $\mathfrak{M}$ be a structure of computable structure σ, $\mathbb{A}$ an admissible set, and let $M \subseteq A$. Structure $\mathfrak{M}$ is n-decidable in $\mathbb{A}$ ($n \leqslant \omega$) if*

$$\{\langle\ulcorner\varphi\urcorner, \bar{m}\rangle \mid \varphi \in F_n(\sigma), \bar{m} \in M^{<\omega}, \mathfrak{M} \models \varphi(\bar{m})\}$$

is Δ-definable in $\mathbb{A}$.

Structure $\mathfrak{M}$ is *computable in* $\mathbb{A}$ if $\mathfrak{M}$ is 0-decidable in $\mathbb{A}$, and *decidable in* $\mathbb{A}$ if $\mathfrak{M}$ is ω-decidable in $\mathbb{A}$. It is obvious that if $\mathfrak{M}$ is n-decidable in $\mathbb{A}$ for some n then M is Δ-definable in $\mathbb{A}$.

Structure $\mathfrak{M}$ is computable (decidable) in classical sense if and only if it is computable (decidable) in the least admissible set $\mathbb{HF}(\varnothing)$.

Definition 2. *$F : P(A)^n \to P(A)$ is a Σ-operator if there exists a Σ-formula $\Phi(x_0, \dots, x_{n-1}, y)$ such that for any $S_0, \dots, S_{n-1} \in P(A)$*

$$F(S_0, \dots, S_{n-1}) = \{a \mid \exists a_0 \dots \exists a_{n-1}(\bigwedge_{i<n} a_i \subseteq S_i \wedge \mathbb{A} \models \Phi(a_0, \dots, a_{n-1}, a))\}.$$

Let $F : P(A)^n \to P(A)$ be a Σ-operator, $\delta_c(F)$ – a set of elements of $P(A)^n$ in which F is strongly continous [1]. It is easy to show that in $\mathbb{HF}(\mathfrak{M})$ any subset belongs to $\delta_c(F)$ for any Σ-operator F.

Definition 3. *Suppose B, C are subsets of an admissible set $\mathbb{A}$. B is Σ-reducible to C ($B \leqslant_\Sigma C$) if there exists binary Σ-operator F_0 such that $\langle C, A \setminus C\rangle \in \delta_c(F_0)$ and $B = F_0(C, A \setminus C)$. If except that there exists binary Σ-operator F_1 such that $\langle C, A \setminus C\rangle \in \delta_c(F_1)$ and $A \setminus B = F_1(C, A \setminus C)$ then B is said to be $T\Sigma$-reducible to C ($B \leqslant_{T\Sigma} C$).*

Let $\mathbb{A}$ be an admissible set, $\mathfrak{M}$ a structure such that $M \subseteq A$, and let $P \subseteq M^n$. P is *relatively computable in* $\mathbb{A}$ if P is $T\Sigma$-reducible to $D(\mathfrak{M})$ in $\mathbb{A}$, and *relatively c.e. in* $\mathbb{A}$ if P is Σ-reducible to $D(\mathfrak{M})$ in $\mathbb{A}$.

Definition 4. *Let $\mathfrak{M}$ be a structure of computable signature σ, $\mathbb{A}$ an admissible set, and let $M \subseteq A$. Structure $\mathfrak{M}$ is relatively n-decidable in $\mathbb{A}$ ($n \leqslant \omega$) if*

$$\{\langle\ulcorner\varphi\urcorner, \bar{m}\rangle \mid \varphi \in F_n(\sigma), \bar{m} \in M^{<\omega}, \mathfrak{M} \models \varphi(\bar{m})\}$$

is $T\Sigma$-reducible to $D(\mathfrak{M})$ in $\mathbb{A}$.

Definition 5. *A copy of a structure $\mathfrak{M}$ in an admissible set $\mathbb{A}$ is a structure $\mathfrak{N}$ such that $\mathfrak{N} \simeq \mathfrak{M}$ and $N \subseteq A$.*

Theorem 1 (Ash, Knight, Manasse, Slaman [2], Chisholm [3]). *Let $\mathfrak{M}$ be a countable structure and let $P \subseteq M^n$. Then the following are equivalent:*
- *P is Σ-definable in $\mathbb{HF}(\mathfrak{M})$;*
- *for any copy $\mathfrak{N}$ of $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{M})$ $P^{\mathfrak{N}}$ is relatively c.e.;*
- *for any copy $\mathfrak{N}$ of $\mathfrak{M}$ in $\mathbb{HF}(\varnothing)$ $P^{\mathfrak{N}}$ is relatively c.e..*

Theorem 2 (Goncharov [4], Manasse [5]). *There exists a countable structure $\mathfrak{M}$ with computable copy in $\mathbb{HF}(\varnothing)$ and $P \subseteq M$ such that*
- *for any computable copy $\mathfrak{N}$ of $\mathfrak{M}$ in $\mathbb{HF}(\varnothing)$ $P^{\mathfrak{N}}$ is c.e.;*
- *P is not Σ-definable in $\mathbb{HF}(\mathfrak{M})$.*

Theorem 3. *Let $\mathfrak{M}$ be a countable structure, $n \leqslant \omega$. Then the following are equivalent:*
- *$\mathfrak{M}$ is n-decidable in $\mathbb{HF}(\mathfrak{M})$;*
- *any copy of $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{M})$ is relatively n-decidable;*
- *any copy of $\mathfrak{M}$ in $\mathbb{HF}(\varnothing)$ is relatively n-decidable.*

Theorem 4 (Nurtazin [6]). *Let $\mathfrak{M}$ be a countable structure with computable copy in $\mathbb{HF}(\varnothing)$, $n \leqslant \omega$. Then the following are equivalent:*
- *$\mathfrak{M}$ is n-decidable in $\mathbb{HF}(\mathfrak{M})$;*
- *any copy of $\mathfrak{M}$ in $\mathbb{HF}(\varnothing)$ is relatively n-decidable;*
- *any computable copy of $\mathfrak{M}$ in $\mathbb{HF}(\varnothing)$ is n-decidable.*

The previous theorem shows that in case of decidability it is impossible to construct an analog of Goncharov-Manasse example from theorem 2. About the existence of relatively decidable copies, there is

Theorem 5 (Harizanov, Knight, Morozov [7]). *Let $\mathfrak{M}$ be a countable structure. Then in $\mathbb{HF}(\varnothing)$ there exists a relatively decidable copy of $\mathfrak{M}$.*

We prove the following

Theorem 6. *Let $\mathfrak{M}$ be a structure of computable signature. Then in $\mathbb{HF}(\mathfrak{M})$ there exists a relatively decidable copy of $\mathfrak{M}$.*

Suppose $\mathfrak{M}$ is arbitrary (possibly uncountable) structure of computable signature, $\mathcal{S}$ – structure of empty signature of the same cardinality as $\mathfrak{M}$.

Conjecture 1. There exists a relatively decidable copy of $\mathfrak{M}$ in $\mathbb{HF}(\mathcal{S})$.

Conjecture 2. For any $n \leqslant \omega$ the following are equivalent:
– $\mathfrak{M}$ is n-decidable in $\mathbb{HF}(\mathfrak{M})$;
– any copy of $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{M})$ is relatively n-decidable;
– any copy of $\mathfrak{M}$ in $\mathbb{HF}(\mathcal{S})$ is relatively n-decidable.

Conjecture 3. Suppose $\mathfrak{M}$ is a structure with computable copy in $\mathbb{HF}(\mathcal{S})$, $n \leqslant \omega$. Then the following are equivalent:
– $\mathfrak{M}$ is n-decidable in $\mathbb{HF}(\mathfrak{M})$;
– any copy of $\mathfrak{M}$ in $\mathbb{HF}(\mathcal{S})$ is relatively n-decidable;
– any computable copy of $\mathfrak{M}$ in $\mathbb{HF}(\mathcal{S})$ is n-decidable.

A theory T is *regular* [1] if it is model complete and decidable.

Proposition 1. *If $Th(\mathfrak{M})$ is regular then $\mathfrak{M}$ is decidable in $\mathbb{HF}(\mathfrak{M})$.*

Example 1. $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{C}$ are structures with regular elementary theories.

We describe decidable linear orders in the following way:

Theorem 7. *A linear order $\mathfrak{L}$ is 1-decidable in $\mathbb{HF}(\mathfrak{L})$ iff $\mathfrak{L}$ is a sum of a finite number of dense linear orders and points.*

A structure $\mathfrak{M}$ is *n-complete* [4] ($n \leqslant \omega$) if for any formula $\varphi(\bar{x}) \in F_n(\sigma)$ and for any $\bar{m} \in M^{<\omega}$ s.t. $\mathfrak{M} \models \varphi(\bar{m})$ there exists a $\exists$-formula $\psi(\bar{x})$ such that $\mathfrak{M} \models \psi(\bar{m})$ and $\mathfrak{M} \models \forall\bar{x}(\psi(\bar{x}) \to \varphi(\bar{x}))$.

Proposition 2. *Suppose $\mathfrak{M}$ is n-decidable in $\mathbb{HF}(\mathfrak{M})$ ($n \leqslant \omega$). Then $\mathfrak{M}$ is n-complete in some constant expansion.*

Proposition 3. *Suppose $\mathfrak{M}$ is n-complete and $Th(\mathfrak{M})$ is decidable. Then $\mathfrak{M}$ is n-decidable in $\mathbb{HF}(\mathfrak{M})$.*

Suppose $\mathfrak{M}$ is 1-decidable in $\mathbb{HF}(\mathfrak{M})$. Then $\mathbb{HF}(\mathfrak{M})$ has universal Σ-function and reduction property, but not necessarily uniformization property.

Let $\mathfrak{M}$ be a structure of signature σ and let signature σ_* consists of all symbols of σ and new functional symbols $f_\varphi(x_1, \ldots, x_n)$ for all existential formulas $\varphi(x_0, x_1, \ldots, x_n)$ of signature σ. Structure $\mathfrak{M}_*$ of signature σ_* is called *existential Skolem expansion of* $\mathfrak{M}$ if $|\mathfrak{M}_*| = |\mathfrak{M}|$, $\mathfrak{M} \restriction_\sigma = \mathfrak{M}_* \restriction_\sigma$ and for any existential formula $\varphi(x_0, x_1, \ldots, x_n)$ of signature σ

$$\mathfrak{M}_* \models \forall x_1 \ldots \forall x_n (\exists x \varphi(x, x_1, \ldots, x_n) \to \varphi(f_\varphi(x_1, \ldots, x_n), x_1, \ldots, x_n)).$$

The next theorem is a generalization of the main result from [12].

Theorem 8. *Suppose $\mathfrak{M}$ is 1-decidable in $\mathbb{HF}(\mathfrak{M})$. Then $\mathbb{HF}(\mathfrak{M})$ has uniformization property iff some existential Skolem expansion of $\mathfrak{M}$ is computable in $\mathbb{HF}(\mathfrak{M})$.*

Theorem 9. *For any $n \in \omega$ there exists ω-categorical structure $\mathfrak{M}$ such that $\mathfrak{M}$ is n-decidable in $\mathbb{HF}(\mathfrak{M})$ but not $(n+1)$-decidable $\mathbb{HF}(\mathfrak{M})$. There also exists ω-categorical structure $\mathfrak{M}$ such that for any $n \in \omega$ $\mathfrak{M}$ is n-decidable in $\mathbb{HF}(\mathfrak{M})$ but $\mathfrak{M}$ is not decidable in $\mathbb{HF}(\mathfrak{M})$.*

Admissible set $\mathbb{A}$ is *quasiresolvable* [1] if there exists a sequence $B_0 \subseteq B_1 \subseteq \ldots \subseteq B_\alpha \subseteq \ldots$, $\alpha \in Ord\,\mathbb{A}$ of transitive subsets of A such that $\cup_{\alpha \in Ord\,\mathbb{A}} B_\alpha = A$, and subsets $\{\langle \alpha, a \rangle | a \in B_\alpha\}$ and

$$\{\langle \alpha, \ulcorner \Phi \urcorner, \bar{a} \rangle \mid \alpha \in Ord\,\mathbb{A}, \Phi(\bar{x}) \in F(\sigma_{\mathbb{A}}), \bar{a} \in B_\alpha^{<\omega},\ \mathbb{A} \restriction B_\alpha \models \Phi(\bar{a})\}$$

are Δ-definable in $\mathbb{A}$.

Admissible set $\mathbb{A}$ is *1-quasiresolvable* if there exists a sequence $B_0 \subseteq B_1 \subseteq \ldots \subseteq B_\alpha \subseteq \ldots$, $\alpha \in Ord\,\mathbb{A}$ of transitive subsets of A such that $\cup_{\alpha \in Ord\,\mathbb{A}} B_\alpha = A$, and subsets $\{\langle \alpha, a \rangle | a \in B_\alpha\}$ and

$$\{\langle \alpha, \ulcorner \Phi \urcorner, \bar{a} \rangle \mid \alpha \in Ord\,\mathbb{A}, \Phi(\bar{x})\ -\ \Pi\text{-formula of } \sigma_{\mathbb{A}}, \bar{a} \in B_\alpha^{<\omega},\ \mathbb{A} \restriction B_\alpha \models \Phi(\bar{a})\}$$

are Δ-definable in $\mathbb{A}$.

If admissible set $\mathbb{A}$ is 1-quasiresolvable then $\mathbb{A}$ has universal Σ-function and reduction property [1]. If $\mathfrak{M}$ is (1-)decidable in $\mathbb{HF}(\mathfrak{M})$ then $\mathbb{HF}(\mathfrak{M})$ is (1-)quasiresolvable. The converse is not true in general.

Theorem 10. *Suppose $\mathfrak{M}$ is ω-categorical. Then*

1) $\mathfrak{M}$ is decidable in $\mathbb{HF}(\mathfrak{M})$ iff $\mathbb{HF}(\mathfrak{M})$ is quasiresolvable;
2) $\mathfrak{M}$ is 1-decidable in $\mathbb{HF}(\mathfrak{M})$ iff $\mathbb{HF}(\mathfrak{M})$ is 1-quasiresolvable.

2 Constructivizations of Structures in Admissible Sets

Let $\mathfrak{N}$ be a structure of relational computable signature $\langle P_0^{n_0}, \ldots, P_k^{n_k}, \ldots \rangle$ and let $\mathbb{A}$ be an admissible set.

Definition 6 (Ershov [1]). *$\mathfrak{N}$ is Σ-definable (constructivizable) in $\mathbb{A}$ if there exists a computable sequence of Σ-formulas*

$$\Phi(x_0,y),\Psi(x_0,x_1,y),\Psi^*(x_0,x_1,y),\Phi_0(x_0,\dots,x_{n_0-1},y),$$

$$\Phi_0^*(x_0,\dots,x_{n_0-1},y),\dots,\Phi_k(x_0,\dots,x_{n_k-1},y),\Phi_k^*(x_0,\dots,x_{n_k-1},y),\dots$$

such that for some parameter $a \in A$

$$N_0 \leftrightharpoons \Phi^{\mathbb{A}}(x_0,a) \neq \varnothing,\ \ \eta \leftrightharpoons \Psi^{\mathbb{A}}(x_0,x_1,a) \cap N_0^2$$

is a congruence relation on the structure

$$\mathfrak{N}_0 \leftrightharpoons \langle N_0, P_0^{\mathfrak{N}_0},\dots,P_k^{\mathfrak{N}_0},\dots\rangle,$$

where $P_k^{\mathfrak{N}_0} \leftrightharpoons \Phi_k^{\mathbb{A}}(x_0,\dots,x_{n_k-1}) \cap N_0^{n_k},\ k \in \omega$,

$$\Psi^{*\mathbb{A}}(x_0,x_1,a) \cap N_0^2 = N_0^2 \setminus \Psi^{\mathbb{A}}(x_0,x_1,a),$$

$$\Phi_k^{*\mathbb{A}}(x_0,\dots,x_{n_k-1},a) \cap N_0^{n_k} = N_0^{n_k} \setminus \Phi_k^{\mathbb{A}}(x_0,\dots,x_{n_k-1})$$

for all $k \in \omega$ and the structure $\mathfrak{N}$ is isomorphic to the quotient structure $\mathfrak{N}_0/\eta$.

Definition 7. *A theory T is c-simple [1] if it is ω-categorical, model complete, decidable and has decidable set of complete formulas.*

Conjecture 4 (Ershov [8]). If T is c-simple theory then some uncountable model of T is Σ-definable in $\mathbb{HF}(\mathfrak{L})$ for some (uncountable) dense linear order $\mathfrak{L}$.

Theorem 11 (Schmerl [9]). *If $\mathfrak{A}$ is countably infinite, ω-categorical structure, then there is a linear order $<$ of A with order type of rationals such that $\langle\mathfrak{A},<\rangle$ is ω-categorical.*

Definition 8. *For arbitrary structures $\mathfrak{A}$ and $\mathfrak{B}$ a set $I \subseteq A \cap B$ is called a set of $\mathfrak{A}$-indiscernibles in $\mathfrak{B}$ if for any tuples $\bar{i},\bar{i}' \in I^{<\omega}$ of the same length*

$$\langle\mathfrak{A},\bar{i}\rangle \equiv \langle\mathfrak{A},\bar{i}'\rangle \text{ implies } \langle\mathfrak{B},\bar{i}\rangle \equiv \langle\mathfrak{B},\bar{i}'\rangle.$$

Let T and T' be c-simple theories. If some uncountable model of T' is Σ-definable in HF-superstructure over some model of T, then there are decidable models $\mathfrak{A}$ and $\mathfrak{B}$ of T and T' respectively such that there is an infinite computable set of $\mathfrak{A}^*$-indiscernibles in $\mathfrak{B}$, where $\mathfrak{B}^*$ is expansion of $\mathfrak{B}$ by finite number of constants.

For some c-simple theories this necessary condition of Σ-definability is also sufficient. We denote by T_{DLO} theory of dense linear order and by T_E theory of infinite models of equality.

Theorem 12 ([13]). *Let T be c-simple theory and $\mathfrak{A}$ be any decidable model of T. Then*

1) T has uncountable model which is Σ-definable in $\mathbb{HF}(\mathfrak{L})$ for some $\mathfrak{L} \models T_{DLO}$ iff there exists an infinite computable set of order indiscernibles in $\mathfrak{A}$;
2) T has uncountable model which is Σ-definable in $\mathbb{HF}(\mathcal{S})$ for some $\mathcal{S} \models T_E$ iff there exists an infinite computable set of total indiscernibles in $\mathfrak{A}$.

Theorem 13 (Kierstead, Remmel [10]). *There exists c-simple theory T s.t. any infinite set of (order) indiscernibles in decidable model of T is not computable.*

By using this result we obtain a counterexample for Ershov conjecture.

Corollary 1 ([13]). *There exists c-simple theory (of infinite signature) such that none of it's uncountable models is Σ-definable in $\mathbb{HF}(\mathfrak{L})$, where $\mathfrak{L}$ is a dense linear order.*

Conjecture 5. For any c-simple theory T there exists a c-simple theory T' such that for any uncountable $\mathfrak{M} \models T$ and $\mathfrak{M}' \models T'$ $\mathfrak{M}'$ is not Σ-definable in $\mathbb{HF}(\mathfrak{M})$.

3 Inner Constructivizability of Admissible Sets

Consider a signature σ and let P be unary predicate symbol not in σ. For QR-formula (i.e. formula which possibly contain restricted quantifiers of kind $\forall x \in y$ and $\exists x \in y$) Φ of signature $\sigma \cup \{\in\}$ we define inductively *relativization* Φ^P of formula Φ by predicate P :

- if Φ is atomic then $\Phi^P = \Phi$;
- if $\Phi = (\Phi_1 * \Phi_2)$, $* \in \{\wedge, \vee, \rightarrow\}$ then $\Phi^P = (\Phi_1^P * \Phi_2^P)$;
- if $\Phi = \neg\Psi$ then $\Phi^P = \neg\Phi^P$;
- if $\Phi = (Qx \in y)\Psi$, $Q \in \{\forall, \exists\}$ then $\Psi^P = (Qx \in y)\Psi^P$;
- if $\Phi = \exists x\Psi$ then $\Phi^P = \exists x(P(x) \wedge \Psi^P)$;
- if $\Phi = \forall x\Psi$ then $\Phi^P = \forall x(P(x) \rightarrow \Psi^P)$.

In case then $\mathbb{A}$ is admissible set, $B \subseteq A$ and $\Phi(x_0, \dots, x_{n-1})$ is a QR-formula of signature $\sigma_{\mathbb{A}}$, we define

$$(\Phi(x_0, \dots, x_{n-1}))^B = \{\langle a_0, \dots, a_{n-1}\rangle \in A^n \mid \langle \mathbb{A}, B\rangle \models \Phi^P(a_0, \dots, a_{n-1})\}.$$

Definition 9. *A structure $\mathfrak{M}$ of computable predicate signature $\langle P_0^{n_0}, P_1^{n_1}, \dots\rangle$ is constructivizable in an admissible set $\mathbb{A}$ inside $B \subseteq A$ if there exists computable sequence of formulas*

$$\Phi(x_0, y), \Psi(x_0, x_1, y), \Psi^*(x_0, x_1, y), \Phi_0(x_0, \dots, x_{n_0-1}, y),$$

$$\Phi_0^*(x_0, \dots, x_{n_0-1}, y), \dots, \Phi_k(x_0, \dots, x_{n_k-1}, y), \Phi_k^*(x_0, \dots, x_{n_k-1}, y), \dots$$

and $b \in B$ such that

$$M_0 \risingdotseq \Phi^B(x_0, b) \neq \varnothing,\ M_0 \subseteq B,\ \ \eta \risingdotseq \Psi^B(x_0, x_1, b) \cap M_0^2$$

is a congruence relatin on the structure

$$\mathfrak{M}_0 \risingdotseq \langle M_0, P_0^{\mathfrak{M}_0}, \dots, P_k^{\mathfrak{M}_0}, \dots\rangle,$$

where $P_k^{\mathfrak{M}_0} \rightleftharpoons (\Phi_k(x_0, \ldots , x_{n_k-1}))^B \cap M_0^{n_k},\ k \in \omega$,

$$(\Psi^*(x_0, x_1, a))^B \cap M_0^2 = M_0^2 \setminus (\Psi(x_0, x_1, a))^B,$$

$$(\Phi_k^*(x_0, \ldots , x_{n_k-1}, a))^B \cap M_0^{n_k} = M_0^{n_k} \setminus (\Phi_k(x_0, \ldots , x_{n_k-1}))^B$$

for all $k \in \omega$*, and* $\mathfrak{M}$ *is isomorphic to* $\mathfrak{M}_0 \diagup \eta$.

If $\mathbb{A}$ is an admissible set then for arbitrary $B \subseteq A$ we define $\mathrm{rnk}(B)$ in the usual way:

$$\mathrm{rnk}(B) = \sup\{\mathrm{rnk}(b) | b \in B\}.$$

Definition 10. *Rank of inner constructivizability of an admissible set* $\mathbb{A}$ *is an ordinal*

$$cr(\mathbb{A}) = \inf\{\mathrm{rnk}(B) \mid \mathbb{A} \textit{ is constructivizable in } \mathbb{A} \textit{ inside } B\}.$$

The next theorem gives the precise estimates of the rank of inner constructivizability for hereditary finite superstructures.

Theorem 14 ([14]). *Suppose* $\mathfrak{M}$ *is a structure of computable signature. Then*
1) if $\mathfrak{M}$ *is finite then* $cr(\mathbb{HF}(\mathfrak{M})) = \omega$,
2) if $\mathfrak{M}$ *is infinite then* $cr(\mathbb{HF}(\mathfrak{M})) \leqslant 2$.

From this theorem we obtain effective analogs of some results from [11] about definability in multisorted languages.

Examples of structures $\mathfrak{M}$ for which $cr(\mathbb{HF}(\mathfrak{M})) = 2$ are infinite models of empty signature, dense linear orders, and, more interesting, the structure $\langle \omega, s\rangle$ of natural numbers with successor function. Indeed, if we denote by $Th_{WM}(\mathfrak{M})$ a theory of $\mathfrak{M}$ in the language of weak monadic second order logic, then the following lemma is true:

Lemma 1. *If* $Th_{WM}(\mathfrak{M})$ *is decidable then* $cr(\mathbb{HF}(\mathfrak{M})) = 2$.

From Büchi result about decidability of $Th_{WM}(\langle \omega, s\rangle)$ and the previous lemma we get that

$$cr(\mathbb{HF}(\langle \omega, s\rangle)) = 2.$$

An example of structure $\mathfrak{M}$ for which $cr(\mathbb{HF}(\mathfrak{M})) = 0$ is, obviously, the standard model of arithmetic $\mathbb{N}$. An example of structure for which rank of inner constructivizability is equal to 1 is the field $\mathbb{R}$ of real numbers.

Theorem 15 ([14]).

$$cr(\mathbb{HF}(\mathbb{R})) = 1.$$

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