# On Inner Constructivizability of Admissible Sets*

Alexey Stukachev

Sobolev Institute of Mathematics, Novosibirsk, Russia
aistu@math.nsc.ru

**Abstract.** We consider a problem of inner constructivizability of admissible sets by means of elements of a bounded rank. For hereditary finite superstructures we find a precise estimates for the rank of inner constructivizability: it is equal to $\omega$ for superstructures over finite structures and less or equal to 2 otherwise. We introduce examples of hereditary finite superstructures with ranks 0, 1, 2. It is shown that hereditary finite superstructure over field of real numbers has rank 1.

Notations and terminology used below are standard and corresponds to [1, 2]. We denote the domains of a structure $\mathfrak{M}$ and KPU-model $\mathcal{A}$ by $M$ and $A$ respectively. Further on, without loss of generality we will consider only structures and KPU-models with predicate signatures.

Let $\mathfrak{M}$ be a structure of computable predicate signature $\langle P_0^{n_0}, \ldots, P_k^{n_k}, \ldots \rangle$, and let $\mathcal{A}$ be a KPU-model, i.e. a structure of signature containing symbols $U^1, \in^2$, which is a model of the system of axioms KPU. Following [1], $\mathfrak{M}$ is called *$\Sigma$-definable (constructivizable) in* $\mathcal{A}$ if there exists a computable sequence of $\Sigma$-formulas

$$\Phi(x_0, y), \Psi(x_0, x_1, y), \Psi^*(x_0, x_1, y), \Phi_0(x_0, \ldots, x_{n_0-1}, y),$$

$$\Phi_0^*(x_0, \ldots, x_{n_0-1}, y), \ldots, \Phi_k(x_0, \ldots, x_{n_k-1}, y), \Phi_k^*(x_0, \ldots, x_{n_k-1}, y), \ldots$$

such that for some parameter $a \in A$, and letting

$$M_0 \leftrightharpoons \Phi^{\mathbb{A}}(x_0, a), \;\; \eta \leftrightharpoons \Psi^{\mathbb{A}}(x_0, x_1, a) \cap M_0^2$$

one has that $M_0 \neq \varnothing$ and $\eta$ is a congruence relation on the structure

$$\mathfrak{M}_0 \leftrightharpoons \langle M_0, P_0^{\mathfrak{M}_0}, \ldots, P_k^{\mathfrak{M}_0}, \ldots \rangle,$$

where $P_k^{\mathfrak{M}_0} \leftrightharpoons \Phi_k^{\mathbb{A}}(x_0, \ldots, x_{n_k-1}) \cap M_0^{n_k}, \; k \in \omega$,

$$\Psi^{*\mathbb{A}}(x_0, x_1, a) \cap M_0^2 = M_0^2 \setminus \Psi^{\mathbb{A}}(x_0, x_1, a),$$

* This work was supported by the INTAS YSF (grant 04-83-3310), the Program "Universities of Russia" (grant UR.04.01.488), the Russian Foundation for Basic Research (grant 05-01-00481a) and the Grant of the President of RF for Young Scientists (grant MK.1239.2005.1).

$$\Phi_k^{*\mathbb{A}}(x_0, \ldots, x_{n_k-1}, a) \cap M_0^{n_k} = M_0^{n_k} \setminus \Phi_k^{\mathbb{A}}(x_0, \ldots, x_{n_k-1})$$

for all $k \in \omega$ and the structure $\mathfrak{M}$ is isomorphic to the quotient structure $\mathfrak{M}_0 / \eta$. In this case we say that the above sequence of formulas together with the parameter $a$ are *$\Sigma$-defining $\mathfrak{M}$ in $\mathcal{A}$*.

In the present setting, however, it would be more convenient to use an equivalent approach based on the notion of $\mathcal{A}$-constructivizability. A mapping (numbering) $\nu : B \to M$ is called an *$\mathcal{A}$-constructivization* of a structure $\mathfrak{M}$ if $B \subseteq A$ is a $\Sigma$-subset and the numbering equivalence relation

$$\eta_\nu = \{\langle b_0, b_1\rangle | b_0, b_1 \in B,\ \mathfrak{M} \models (\nu(b_0) = \nu(b_1))\}$$

as well as the sets

$$\{\langle k, \langle b_0, \ldots, b_{n_k-1}\rangle\ ra | k \in \omega, b_0, \ldots, b_{n_k-1} \in B, \mathfrak{M} \models P_k(\nu(b_0), \ldots, \nu(b_{n_k-1}))\}$$

are $\Delta$-subsets of $\mathcal{A}$. We will say that a structure is *$\mathcal{A}$-constructivizable* if it has an $\mathcal{A}$-constructivization. It is known (see [1]) that a structure $\mathfrak{M}$ is $\Sigma$-definable in a KPU-model $\mathcal{A}$ if and only if $\mathfrak{M}$ is $\mathcal{A}$-constructivizable.

Let $\mathcal{A}$ be a KPU-model of signature $\sigma_{\mathcal{A}}$ and let $\Theta$ be a $\Sigma$-formula of the same signature. For arbitrary $\Sigma$-formula $\Phi$ of the signature $\sigma_{\mathcal{A}}$ the *relativization* $\Phi^\Theta$ of formula $\Phi$ by formula $\Theta$ is defined inductively:

if $\Phi$ is an atomic formula then $\Phi^\Theta$ is equal to $\Phi$;
if $\Phi$ is of the form $\neg\Psi$ then $\Phi^\Theta$ is equal to $\neg(\Psi^\Theta)$;
if $\Phi$ is of the form $(\Psi_1 * \Psi_2)$ , $* \in \{\wedge, \vee, \to\}$, then $\Phi^\Theta$ is equal to $(\Psi_1^\Theta * \Psi_2^\Theta)$;
if $\Phi$ is of the form $(Qx \in y)\Psi$, $Q \in \{\forall, \exists\}$, then $\Phi^\Theta$ is equal to $(Qx \in y)\Psi^\Theta$;
if $\Phi$ is of the form $\exists x\Psi$ then $\Phi^\Theta$ is equal to $\exists x(\Theta(x) \wedge \Psi^\Theta)$.

It is clear that $\Phi^\Theta$ is a $\Sigma$-formula of signature $\sigma_{\mathcal{A}}$.

**Definition 1.** *Let $\mathcal{A}$ be a* KPU*-model of computable predicate signature $\sigma_{\mathcal{A}} = \langle U^1, \in^2, P_0^{n_0}, \ldots\rangle$, and let $B \subseteq A$ be a transitive $\Sigma$-subset defined in $\mathcal{A}$ by some $\Sigma$-formula $\Theta$ of the signature $\sigma_{\mathcal{A}}$ which contains parameters only from $B$. $\mathcal{A}$ is said to be constructivizable inside $B$ if there is a computable sequence $\Phi(\bar{x}_0, \bar{y})$, $\Phi_=(\bar{x}_0, \bar{x}_1, \bar{y})$, $\Psi_=(\bar{x}_0, \bar{x}_1, \bar{y})$, $\Phi_\in(\bar{x}_0, \bar{x}_1, \bar{y})$, $\Psi_\in(\bar{x}_0, \bar{x}_1, \bar{y})$, $\Phi_U(\bar{x}_0, \bar{y})$, $\Psi_U(\bar{x}_0, \bar{y})$, $\Phi_{P_0}(\bar{x}_0, \ldots, \bar{x}_{n_0-1}, \bar{y})$, $\Psi_{P_0}(\bar{x}_0, \ldots, \bar{x}_{n_0-1}, \bar{y}), \ldots$ of $\Sigma$-formulas (tuples $\bar{x}_0$, $\bar{x}_1$ ... are supposed to be of the same length $k$ – a dimension of the constructivization, tuple $\bar{y}$ is of the length $l$), and a tuple of parameters $\bar{b} \in B^l$ such that $\{\bar{a} \in A | \mathcal{A} \models \Phi^\Theta(\bar{a}, \bar{b})\} \subseteq B^k$ and the sequence of the relativized formulas $\langle \Phi^\Theta, (\Phi_=)^\Theta, (\Psi_=)^\Theta, (\Phi_\in)^\Theta, (\Psi_\in)^\Theta, (\Phi_U)^\Theta, (\Psi_U)^\Theta, \Phi_{P_0}^\Theta, \Psi_{P_0}^\Theta, \ldots\rangle$ with parameters $\bar{b}$ are $\Sigma$-defining the KPU-model $\mathcal{A}$ in $\mathcal{A}$.*

The above notion can also be reformulated in terms of constructivizations, so we will usually speak about *$\mathcal{A}$-constructivizations of $\mathfrak{M}$ inside* $B$. Note also that because of the requirement on parameters to be elements from $B$ we could not, in general, replace in the above definition a tuple $\bar{b}$ by a single $b \in B$.

In the same way, under the same conditions on $B$, we call a subset $C \subseteq A$ to be $\Sigma$-definable in $\mathcal{A}$ *inside* $B$, if $C$ is defined in $\mathcal{A}$ by means of $\Phi^\Theta$ for some $\Sigma$-formula $\Phi$ with parameters from $B$.

Suppose now that $\mathcal{A}$ is an admissible set, i.e. a KPU-model in which the set of ordinals is well-founded (see [1]). If, for any $a \in \mathcal{A}$, $\mathrm{rnk}(a)$ denotes the rank of $a$, we can define a notion of rank for arbitrary subset $B \subseteq A$ in the following way: $\mathrm{rnk}(B) = \sup\{\mathrm{rnk}(b) | b \in B\}$.

**Definition 2.** *The rank of inner constructivizability of an admissible set $\mathcal{A}$ is the ordinal*

$$cr(\mathcal{A}) = \inf\{\mathrm{rnk}(B) | \mathcal{A} \textit{ is constructivizable inside } B\}.$$

The theorem below gives the precise estimates for the rank of inner constructivizability for admissible sets of form $\mathbb{H}F(\mathfrak{M})$ — hereditary finite structures.

**Theorem 1.** *Let $\mathfrak{M}$ be a structure of computable signature. Then*
*1) if $\mathfrak{M}$ is finite then $cr(\mathbb{H}F(\mathfrak{M})) = \omega$,*
*2) if $\mathfrak{M}$ is infinite then $cr(\mathbb{H}F(\mathfrak{M})) \leqslant 2$.*

We now begin the proof. Assume as usual that for any $n \in \omega$ $HF_n(M)$ is the set of all elements from $\mathbb{H}F(\mathfrak{M})$ with rank less or equal to $n$. It is easy to see that in case when $\mathfrak{M}$ is finite, $HF_n(M)$ is finite for all $n \in \omega$, and hence $\mathbb{H}F(\mathfrak{M})$ is not constructivizable inside $HF_n(\mathfrak{M})$ for any $n \in \omega$, thus the first statement is true. The second statement comes from the following

**Theorem 2.** *If $\mathfrak{M}$ is infinite then the hereditary finite structure $\mathbb{H}F(\mathfrak{M})$ is constructivizable inside $HF_2(M)$.*

*Proof.* First, we construct an $\mathbb{H}F(\mathfrak{M})$-constructivization $\nu$ inside $HF_2(M)$ of the standard model of arithmetic $\mathcal{N} = \langle \omega, \leq, +, \cdot, s, 0\rangle$. For this we will use the cardinal presentation of natural numbers on the set $M$: with any $n \in \omega$ we connect the collection of all subsets of $M$ containing exactly $n$ elements, i.e.

$$\nu^{-1}(n) \leftrightharpoons \{a \subseteq M \mid \mathrm{card}(a) = n\}.$$

Thus defined numeration $\nu$ is called a cardinal numeration. Relative to this numeration, two subsets of $M$ represents the same natural number if there exists a bijection from one subset onto another. We will represent functions whose domains are finite subsets of $M$ by means of elements of $\mathbb{H}F(\mathfrak{M})$ with rank 2. Namely, any function $f = \{\langle u_0, v_0\rangle, \ldots, \langle u_n, v_n\rangle\}$ is uniquely determined by any element (of rank 2) of the form

$$\{w_0, \ldots, w_n, \{u_0, w_0\}, \ldots, \{u_n, w_n\}, \{u_0, v_0, w_0\}, \ldots, \{u_n, v_n, w_n\}\},$$

where $w_0, \ldots, w_n \in M \setminus \{u_0, \ldots, u_n, v_0, \ldots, v_n\}$ are pairwise different (such elements do exist since $M$ is infinite). Let $C_f$ be the set of all such presentations of $f$, and let

$$C = \cup\{C_f \mid f \text{ is a finite function with } \mathrm{dom}(f) \subseteq M \text{ and } \mathrm{rng}(f) \subseteq M\}.$$

It is clear that $C \subseteq HF_2(M)$ and, moreover, $C$ is a $\Delta_0$-subset in $\mathbb{H}F(\mathfrak{M})$. It is easy to write down $\Delta_0$-formulas defining, for any element $c \in C$ which represents some finite function $f_c$, the sets $\mathrm{dom}(f_c)$ and $\mathrm{rng}(f_c)$ — the domain and the range of $f_c$ respectively, and a $\Delta_0$-formula which is true if and only if $f_c$ is a bijection. So it follows that the numeration equivalence relation for the cardinal numeration $\nu$ is $\Sigma$-definable inside $HF_2(M)$: for any finite $a, b \subseteq M$

$$\begin{aligned}\nu(a) = \nu(b) \iff & \exists c \in C\ ((f_c \text{ is a bijection}) \\ & \wedge(\mathrm{dom}(f_c) = a) \wedge (\mathrm{rng}(f_c) = b)).\end{aligned}$$

In the same way, for the natural order relation $\leq$ we have

$$\nu(a) \leq \nu(b) \iff \exists a' \in HF_1(M)\ ((\nu(a') = \nu(a)) \wedge (a' \subseteq b)),$$

$$\begin{aligned}\nu(a) < \nu(b) \iff & \exists a', b' \in HF_1(M)\ ((\nu(a') = \nu(a)) \wedge (b = a' \cup b') \\ & \wedge(a' \cap b' = \varnothing) \wedge (b' \neq \varnothing)),\end{aligned}$$

hence, since $\nu(a) \neq \nu(b)$ iff $((\nu(a) < \nu(b)) \vee (\nu(b) < \nu(a)))$, we get that both the numbering equivalence relation and the order relation are $\Delta$-definable inside $HF_2(M)$.

For the operations of addition and multiplication we have that

$$\begin{aligned}\nu(a) + \nu(b) = \nu(c) \iff & \exists a', b' \in HF_1(M)\ ((\nu(a') = \nu(a)) \wedge (\nu(b) = \nu(b')) \\ & \wedge(c = a' \cup b') \wedge (a' \cap b' = \varnothing)\end{aligned}$$

$$\begin{aligned}\nu(a) \cdot \nu(b) = \nu(c) \iff & \exists c' \in HF_2(M)\ ((\cup c' = c) \wedge (\text{"}c' = \{a'_1, \ldots, a'_{\nu(b)}\}\text{"}) \\ & \wedge(\text{"}a'_i \cap a'_j = \varnothing \text{ then } i \neq j\text{"}) \\ & \wedge(\text{"}\nu(a'_i) = \nu(a) \text{ for all } i\text{"})),\end{aligned}$$

where "$c' = \{a'_1, \ldots, a'_{\nu(b)}\}$" denotes the formula

$$\exists c'' \in HF_1(M)\ ((\nu(c'') = \nu(b)) \wedge \forall a' \in c' \exists! x \in a' (x \in c'')).$$

Thus, relative to the cardinal numbering $\nu$, the operations of additions and multiplication of natural numbers are $\Delta$-definable inside $HF_2(M)$.

Recall that for arbitrary structure $\mathfrak{M}$ a *coding scheme* [4] $\mathcal{C}$ consists of a set $N^{\mathcal{C}} \subseteq M$ and a linear order $<^{\mathcal{C}}$ on $N^{\mathcal{C}}$ such that

$$\langle N^{\mathcal{C}}, <^{\mathcal{C}} \rangle \simeq \langle \omega, < \rangle,$$

and an injective mapping $\pi^{\mathcal{C}}$ from the set of all finite sequences of elements of $M$ into $M$. For a given coding scheme $\mathcal{C}$ we will denote by $\dot{0}, \dot{1}, \dot{2}, \ldots$ the corresponding elements of $N^{\mathcal{C}}$, relative to $<^{\mathcal{C}}$. With $\mathcal{C}$ we will also consider the predicate $Seq^{\mathcal{C}}(x)$ which is true in case then $x = \pi^{\mathcal{C}}(\emptyset)$ or $x = \pi(\langle m_0, \ldots, m_n \rangle)$ for some $m_0, \ldots, m_n \in M$, and functions $lh^{\mathcal{C}}(x)$, $pr^{\mathcal{C}}(x, \dot{m})$, which gives correspondingly the length and the $m$-th element of the tuple with code $x$, and gives $\dot{0}$ in case of mismatch of the arguments. A structure $\mathfrak{M}$ is called *acceptable* [4] if it has a coding scheme $\mathcal{C}$ such that functions and relations $N^{\mathcal{C}}$, $<^{\mathcal{C}}$, $Seq^{\mathcal{C}}$, $lh^{\mathcal{C}}$, $pr^{\mathcal{C}}$ are definable in $\mathfrak{M}$.

We introduce the (multivalued) coding scheme $\mathcal{C}_*$ for coding finite sequences of elements from $M$ by elements from $HF_2(M)$, such that $N^{\mathcal{C}_*} = \nu^{-1}(\omega)$, and

$Seq^{\mathcal{C}_*}$, $lh^{\mathcal{C}_*}$ and $pr^{\mathcal{C}_*}$ are $\Delta$-definable in $\mathbb{H}F(\mathfrak{M})$ inside $HF_2(M)$. The set of codes of the tuple $\langle m_0, \ldots, m_k\rangle\ inM^{k+1}$ in the coding scheme $\mathcal{C}_*$ is equal, by definition, to the set of all elements of the form

$$\{\{m_0, u_0\}, \ldots, \{m_k, u_0, \ldots, u_k\}, u_0, \ldots, u_k\},$$

there $u_0, \ldots, u_k$ are pairwise different elements from $M$ such that $\{u_0, \ldots, u_k\} \cap \{m_0, \ldots, m_k\} = \varnothing$. It is easy to see that the relation $Seq^{\mathcal{C}_*}$ and the functions $lh^{\mathcal{C}_*}$ and $pr^{\mathcal{C}_*}$ are $\Delta$-definable inside $HF_2(M)$.

Having the cardinal $\mathbb{H}F(\mathfrak{M})$-constructivization of the standard model of arithmetic $\mathcal{N}$, the coding scheme $\mathcal{C}_*$ and arbitrary constructivization $\gamma$ (in sense of the classical theory of constructive models) of the admissible set $\mathbb{H}F(\omega)$, we construct the $\mathbb{H}F(\mathfrak{M})$-constructivization $\nu_*$ of $\mathbb{H}F(\mathfrak{M})$ inside $HF_2(M)$ in the following way. Suppose $a \in \mathbb{H}F(\mathfrak{M})$; we let $(\nu_*)^{-1}(a)$ to be equal to the set of all elements of the form

$$\{a_\varkappa, \{m_0, u_0\}, \ldots, \{m_k, u_0, \ldots, u_k\}, u_0, \ldots, u_k\},$$

where $\varkappa \in HF(\omega)$ and $m_0, \ldots, m_k \in M$ are such that $a = \varkappa(m_0, \ldots, m_k)$ in the notations of [1], the set $a_\varkappa \subseteq M$ satisfies the condition $\nu(a_\varkappa) = \gamma^{-1}(\varkappa)$, and elements $u_0, \ldots, u_k$ from $M$ are pairwise different and $\{u_0, \ldots, u_k\} \cap \{m_0, \ldots, m_k\} = \{u_0, \ldots, u_k\} \cap a_\varkappa = \{m_0, \ldots, m_k\} \cap a_\varkappa = \varnothing$.

The numeration $\nu_*$ defined in such way is, in fact, a constructivization of $\mathbb{H}F(\mathfrak{M})$ inside $HF_2(M)$. Indeed, the equality relation and the membership relation are defined by mutual recursion in the following way:

$$\varkappa_1(\bar{m}_1) \in \varkappa_2(\bar{m}_2) \iff \exists \varkappa' \in \varkappa_2(\varkappa_1(\bar{m}_1) = \varkappa'(\bar{m}_2)),$$

$$\varkappa_1(\bar{m}_1) \subseteq \varkappa_2(\bar{m}_2) \iff \forall \varkappa' \in \varkappa_1 \exists \varkappa'' \in \varkappa_2(\varkappa'(\bar{m}_1) = \varkappa''(\bar{m}_2)),$$

$$\varkappa_1(\bar{m}_1) = \varkappa_2(\bar{m}_2) \iff (\varkappa_1(\bar{m}_1) \subseteq \varkappa_2(\bar{m}_2)) \wedge (\varkappa_2(\bar{m}_2) \subseteq \varkappa_1(\bar{m}_1)).$$

Since the recursive part of this definition corresponds to the preimage of the set of natural numbers $\nu^{-1}(\omega)$, there exist $\Sigma$-formulas which define the numeration equivalence relation and the preimage of the membership relation for $\nu_*$ inside $HF_2(M)$.

An example of structure $\mathfrak{M}$ with $cr(\mathbb{H}F(\mathfrak{M})) = 2$ is any infinite model of the theory of equality or, more interesting, the structure $\langle \omega, s\rangle$ of natural numbers with successor function. Indeed, if we denote by $Th_{WM}(\mathfrak{M})$ the weak monadic second order logic theory of a structure $\mathfrak{M}$, the following lemma is true.

**Lemma 1.** *If $\mathfrak{M}$ is infinite and $Th_{WM}(\mathfrak{M})$ is decidable, then $cr(\mathbb{H}F(\mathfrak{M})) = 2$.*

*Proof.* Suppose, for a contradiction, that $cr(\mathbb{H}F(\mathfrak{M})) < 2$. Then, in particular, the standard model of arithmetic $\mathcal{N}$ is $\mathbb{H}F(\mathfrak{M})$-constructivizable inside $HF_1(M)$, hence $\mathcal{N}$ is interpretable in $\mathfrak{M}$ by means of weak monadic second order logic. So $Th(\mathcal{N}) \leq_m Th_{WM}(\mathfrak{M})$, and from the decidability of $Th_{WM}(\mathfrak{M})$ follows decidability of the elementary theory of the standard model of arithmetic, a contradiction.

From the well-known result of Büchi [3] about decidability of $Th_{WM}(\langle\omega, s\rangle)$ and the previous lemma we get that $cr(\mathbb{H}F(\langle\omega, s\rangle)) = 2$.

An example of structure $\mathfrak{M}$ with $cr(\mathbb{H}F(\mathfrak{M})) = 0$ is, obviously, the standard model of arithmetic $\mathcal{N}$. An example of a structure which hereditary finite superstructure has rank of inner constructivizability 1, is the field $\mathcal{R}$ of real numbers. First, we establish one general result.

**Lemma 2.** *If $\mathbb{P}$ is a field of characteristic 0 then the standard model of arithmetic is constructivizable in $\mathbb{H}F(\mathbb{P})$ inside $HF_1(\mathbb{P})$.*

*Proof.* We build an $\mathbb{H}F(\mathbb{P})$-constructivization $\mu$ of the standard model of arithmetic $\langle\ omega, \leq, +, \cdot, s, 0\rangle$ inside $HF_1(\mathbb{P})$. Since $\mathbb{P}$ is a field of characteristic 0, the set of natural numbers $\mathcal{N} = \{0, 1, 1+1, \ldots\}$ is a subset of $\mathbb{P}$. As the requested constructivization we take a mapping $\mu : \mathcal{N} \to \omega$ defined as follows: $\mu^{-1}(n) = \underbrace{1 + \ldots + 1}_{n}$ for all $n \in \omega$.

The set of natural numbers $\mathcal{N} \subseteq \mathbb{P}$ is $\Sigma$-definable in $\mathbb{H}F(\mathbb{P})$ inside $HF_1(\mathbb{P})$: for $t \in \mathbb{P}$ we have

$$\begin{aligned} t \in \mathcal{N} \iff \mathbb{H}F(\mathbb{P}) \models \exists a\ ((a \subseteq \mathbb{P}) \wedge (0 \in a) \wedge \forall x \in a \\ (x \neq 0 \to \exists y \in a(x = y+1)) \wedge (t = \max(a)), \end{aligned}$$

where $t = \max(a)$ denotes the formula $\neg(t+1 \in a)$. The numeration equivalence relation for $\mu$ coinsides with the equality relation on $\mathcal{N}$, the order relation is $\Delta$-definable in $\mathbb{H}F(\mathbb{P})$ inside $HF_1(\mathbb{P})$: for $n, m \in \mathcal{N}$

$$\begin{aligned} \mu(n) \leq \mu(m) \iff \mathbb{H}F(\mathbb{P}) \models \exists a \exists b\ ((a = \{0, 1, \ldots, n\}) \\ \wedge (b = \{0, 1, \ldots, m\}) \wedge (a \subseteq b)), \end{aligned}$$

$$\mu(n) \not\leq \mu(m) \iff \mu(m) < \mu(n) \iff (\mu(m) \leq \mu(n)) \wedge (n \neq m)).$$

The operations of addition and multiplication on $\mathcal{N}$ are induced by the corresponding operations of the field $\mathbb{P}$, and so they are $\Delta$-definable in $\mathbb{H}F(\mathbb{P})$ inside $HF_1(\mathbb{P})$.

**Corollary 1.** *If $\mathbb{P}$ is a field of characteristic 0 then the weak monadic second order theory $Th_{WM}(\mathbb{P})$ is undecidable. In particular, weak monadic second order theories $Th_{WM}(\mathbb{R})$, $Th_{WM}(\mathbb{Q}_p)$ and $Th_{WM}(\mathbb{C})$ are undecidable.*

**Theorem 3.** $cr(\mathbb{H}F(\mathbb{R})) = 1$.

*Proof.* By Lemma 2, the standard model of arithmetic is constructivizable in $\mathbb{H}F(\mathbb{R})$ inside $HF_1(\mathbb{R})$. For the existence of a constructivization of $\mathbb{H}F(\mathbb{R})$ inside $HF_1(\mathbb{R})$ necessary and sufficient condition is the existence of a $\Sigma$-definable inside $HF_1(\mathbb{R})$ coding scheme for finite sequences of reals.

We introduce the coding scheme for finite sequences of reals by the pairs of finite sets of reals. A tuple $\langle a_0, \ldots, a_{n-1}\rangle \in \mathbb{R}^n$ is represented by the set of pairs $\langle\{a_0, \ldots, a_{n-1}\}, \{q_0, \ldots, q_{n-1}\}\rangle$, where elements $q_0, \ldots, q_{n-1} \in \mathbb{R}$ are defined in the following way: we find the least distance $d = \min\{|a_i - a_j|\ | i, j < n, a_i \neq a_j\}$

between distinct elements of the tuple and let $q_i = a_i + \frac{d}{2^{i+2}}$ for all $i < n$ (under this assumption $q_0, \ldots, q_{n-1}$ are pairwise different even in case then some of $a_0, \ldots, a_{n-1}$ are equal. The set of pairs coding finite sequences of reals is $\Sigma$-definable inside $HF_1(\mathbb{R})$ since there exists the corresponding constructivization of natural numbers. The projecting function is $\Delta$-definable inside $HF_1(\mathbb{R})$: $a_i = pr(\langle\{a_0, \ldots, a_{n-1}\}, \{q_0, \ldots, q_{n-1}\}\rangle, \mu^{-1}(i))$ if and only if there exists $q_i \in \{q_0, \ldots, q_{n-1}\}$ such that $|a_i - q_i| = \frac{d}{2^{i+2}}$. In the same way it is easy to show that the function $lh$ in the described coding scheme is also $\Delta$-definable inside $HF_1(\mathbb{R})$.

We define the constructivization $\mu_*$ of the admissible set $\mathbb{H}F(\mathbb{R})$ inside $HF_1(\mathbb{R})$ in the following way. Suppose $a \in \mathbb{H}F(\mathbb{R})$; we let $(\mu_*)^{-1}(a)$ to be equal to the set of all triples of the form

$$\langle \mu^{-1}(\gamma(\varkappa)), \{a_0, \ldots, a_n\}, \{q_0, \ldots, q_n\}\rangle,$$

where $\varkappa \in HF(\omega)$ $a_0, \ldots, a_n \in \mathbb{R}$ are such that $a = \varkappa(a_0, \ldots, a_n)$, $\gamma : \omega \to HF(\omega)$ is a constructivization of the admissible set $\mathbb{H}F(\omega)$ and the pair $\langle\{a_0, \ldots, a_n\}, \{q_0, \ldots, q_n\}\rangle$ is coding the tuple $\langle a_0, \ldots, a_n\rangle$ in the coding scheme described above.

The mapping $\mu_*$ thus defined is a constructivization (of dimension 3) of the admissible set $\mathbb{H}F(\mathbb{R})$ inside $HF_1(\mathbb{R})$.

From the Theorem 1, in particular, follows some constructive analogs of some results (namely, of the Theorems 18, 19, 20) from [5] about the definability in multisorted languages, where the type of a variable describes the rank of its possible values.

## References

1. Yu.L. Ershov: Definability and Computability. Plenum. New York. (1996)
2. J. Barwise: Admissible Sets and Strucrures. Springer-Velag. Berlin. (1975)
3. J.R. Buchi: Weak second order arithmetic and finite automata. Z. Math. Logik Grundl. Math.**6** (1960) 66-92
4. Y.N. Moschovakis: Elementary induction on abstract structures. Amsterdam. (1974)
5. R. Montague: Recursion theory as a branch of model theory. Proceedings of the Third International Congress for Logic, Methodology and Philosophy of Science. Amsterdam. (1967) 63-86