Effective Reducibilities on Structures and Degrees of Presentability*

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Abstract. In this survey paper we consider presentations of structures in admissible sets and some effective reducibilities between structures and their degrees of presentability. Degrees of Σ-definability of structures, as well as degrees of presentability with respect to different effective reducibilities, are natural measures of complexity which are total, i.e. defined for any structure. We also consider properties of structures invariant under various effective reducibilities, and study how a degree of presentability of a structure depends from a domain for presentations (i.e. from the choice of an admissible set).

In this survey paper we consider presentations of structures in admissible sets and some effective reducibilities between structures and their degrees of presentability. The main object of study are semilattices of degrees of Σ-definability, which can be considered as a theoretical model of object-oriented programming, based on a generalization of oracle computability regarding oracles, as well as the results of computation, as abstract structures. On the other hand, the notion of Σ-definability of a structure in an admissible set is an effectivization of one of central notions of model theory, the notion of interpretability of one structure in another, and, at the same time, a generalization of the notion of constructivizability of a structure on natural numbers. We show that the semilattices of Turing and enumeration degrees of subsets of natural numbers are embeddable in a natural way into the semilattices of degrees of Σ-definability. The notion of a structure having a degree, known in computable model theory, gives only a partial measure of complexity, since there are a lot of structures which do not have a degree. Degrees of Σ-definability, as well as degrees of presentability with respect to different effective reducibilities, are natural measures of complexity which are total, i.e. defined for any structure. We can also consider properties of structures invariant under various effective reducibilities, and study how a degree of presentability of a structure depends from a domain for presentations (i.e. from the choice of an admissible set).

Most of notations and terminology we use here are standard and corresponds to $[4, 3, 11]$. We denote the domains of a structure $\mathfrak{M}$ by M, and its signature by

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$\sigma_{\mathfrak{M}}$. In all that follows we consider only structures with computable signatures. For arbitrary structure $\mathfrak{M}$, the hereditary finite superstructure $\mathbb{HF}(\mathfrak{M})$, which is the least admissible set containing M as a subset, is defined as a structure of signature $\sigma'_{\mathfrak{M}} = \sigma_{\mathfrak{M}} \cup \{U^1, \in^2, \mathrm{Sat}^2\}$, whose universe is $HF(M) = \bigcup_{n\in\omega} H_n(M)$, where $H_0(M) = M$, $H_{n+1}(M) = H_n(M) \cup \{a | a \subseteq H_n(M), \mathrm{card}(a) < \omega\}$, the predicate U distinguish the set of the elements of the set M (regarded as urelements), the relation $\in$ has the usual set theoretic meaning, and the interpretation of Sat is the set $\{\langle k, \bar{m}\rangle | \mathfrak{M} \models P_k(\bar{m})\}$. We add Sat to the signature since $\sigma_{\mathfrak{M}}$ is allowed to be infinite. In case when $\sigma_{\mathfrak{M}}$ is finite this is not necessary.

In the class of all formulas of signature $\sigma'_{\mathfrak{M}}$ the subclass of Δ_0-formulas is defined as the closure of the class of atomic formulas under $\wedge, \vee, \neg, \rightarrow$, and bounded quantifiers $\exists x \in y, \forall x \in y$; the class of Σ-formulas is the closure of the class of Δ_0-formulas under $\wedge, \vee, \neg, \rightarrow, \exists x \in y, \forall x \in y$, and the quantifier $\exists x$; the class of Π-formulas is defined in the same way, allowing the quantifier $\forall x$ instead of $\exists x$. A relation on $\mathbb{HF}(\mathfrak{M})$ is called Σ-definable (Π-definable) if it is defined by a corresponding formula, possibly with parameters; it is called Δ-definable if it is Σ- and Π-definable.

Let $\mathfrak{M}$ be a structure of relational signature $\langle P_0^{n_0}, \dots, P_{k-1}^{n_{k-1}}\rangle$ (the restriction to relational signature is not essential and stands only for simplicity) and let $\mathbb{A}$ be an admissible set (see [4, 3] for definition).

Definition 1 (Yu.L. Ershov [4]). *$\mathfrak{M}$ is said to be Σ-definable in $\mathbb{A}$ if there exist Σ-formulas*

$$\Phi(x_0, y), \Psi(x_0, x_1, y), \Psi^*(x_0, x_1, y), \Phi_0(x_0, \dots, x_{n_0-1}, y),$$

$$\Phi_0^*(x_0, \dots, x_{n_0-1}, y), \dots, \Phi_{k-1}(x_0, \dots, x_{n_{k-1}-1}, y), \Phi_{k-1}^*(x_0, \dots, x_{n_{k-1}-1}, y)$$

of signature $\sigma_{\mathbb{A}}$, such that for some parameter $a \in A$, and letting $M_0 \leftrightharpoons \Phi^{\mathbb{A}}(x_0, a)$, $\eta \leftrightharpoons \Psi^{\mathbb{A}}(x_0, x_1, a) \cap M_0^2$, one has that $M_0 \neq \varnothing$ and η is a congruence relation on the structure

$$\mathfrak{M}_0 \leftrightharpoons \langle M_0; P_0^{\mathfrak{M}_0}, \dots, P_{k-1}^{\mathfrak{M}_0}\rangle,$$

where $P_i^{\mathfrak{M}_0} \leftrightharpoons \Phi_i^{\mathbb{A}}(x_0, \dots, x_{n_i-1}) \cap M_0^{n_i}$ for all $i < k$, $\Psi^{*\mathbb{A}}(x_0, x_1, a) \cap M_0^2 = M_0^2 \setminus \Psi^{\mathbb{A}}(x_0, x_1, a)$, $\Phi_i^{*\mathbb{A}}(x_0, \dots, x_{n_i-1}, a) \cap M_0^{n_i} = M_0^{n_i} \setminus \Phi_i^{\mathbb{A}}(x_0, \dots, x_{n_i-1})$ for all $i < k$, and the structure $\mathfrak{M}$ is isomorphic to the quotient structure $\mathfrak{M}_0 / \eta$.

If, in addition, there exists a Σ-formula $\Phi^*(x_0, y)$ such that $\mathbb{A} \models \forall x_0(\Phi^*(x_0, a) \leftrightarrow \neg\Phi(x_0, a))$, then $\mathfrak{M}$ is said to be Δ-definable in $\mathbb{A}$. We say that $\mathfrak{M}$ is Δ-definable in $\mathbb{A}$ with no parameters if the above hold for $a = \varnothing$.

It is easy to show that, if we allow parameters, $\mathfrak{M}$ is Σ-definable in $\mathbb{A}$ if and only if $\mathfrak{M}$ is Δ-definable in $\mathbb{A}$. However, this is not so if we restrict ourselves to definitions with no parameters.

For structures $\mathfrak{M}$ and $\mathfrak{N}$, by $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ we denote the fact that $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$. It is easy to see that the relation $\leqslant_\Sigma$ is reflexive and transitive. For arbitrary infinite cardinal α, let $\mathcal{K}_\alpha$ be the class of all structures of cardinality $\leqslant \alpha$. We define on $\mathcal{K}_\alpha$ an equivalence relation $\equiv_\Sigma$ as follows: for $\mathfrak{M}, \mathfrak{N} \in \mathcal{K}_\alpha$, $\mathfrak{M} \equiv_\Sigma \mathfrak{N}$ if $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ and $\mathfrak{N} \leqslant_\Sigma \mathfrak{M}$. A structure

$$\mathcal{S}_\Sigma(\alpha) = \langle \mathcal{K}_\alpha / \equiv_\Sigma, \leqslant_\Sigma \rangle$$

is an upper semilattice with the least element, which is the degree consisting of constructivizable structures, and, for any $\mathfrak{M}, \mathfrak{N} \in \mathcal{K}_\alpha$, $[\mathfrak{M}]_\Sigma \vee [\mathfrak{N}]_\Sigma = [(\mathfrak{M}, \mathfrak{N})]_\Sigma$, where $(\mathfrak{M}, \mathfrak{N})$ denotes the model-theoretic pair of $\mathfrak{M}$ and $\mathfrak{N}$. For simplicity, $\mathcal{S}_\Sigma(\omega)$ will be denoted by $\mathcal{S}_\Sigma$.

Let $\mathfrak{M}$ be a structure of a computable signature and let $\mathbb{A}$ be an admissible set. A *presentation* of $\mathfrak{M}$ in $\mathbb{A}$ is any structure $\mathcal{C}$ such that $\mathcal{C} \cong \mathfrak{M}$ and the domain of $\mathcal{C}$ is a subset of A (the relation $=$ is assumed to be a congruence relation on $\mathcal{C}$ and may differ form the normal equality relation on C). We can treat (the atomic diagram of) a presentation $\mathcal{C}$ as a subset of A, using some Gödel numbering of the atomic formulas of the signature of $\mathfrak{M}$.

Definition 2. *The problem of presentability of* $\mathfrak{M}$ *in* $\mathbb{A}$ *is the set* $\mathrm{Pr}(\mathfrak{M}, \mathbb{A})$ *consisting of all possible presentations of* $\mathfrak{M}$ *in* $\mathbb{A}$*:*

$$\mathrm{Pr}(\mathfrak{M}, \mathbb{A}) = \{ \mathcal{C} \mid \mathcal{C} \textit{ is a presentation of } \mathfrak{M} \textit{ in } \mathbb{A} \}.$$

We denote by $\underline{\mathfrak{M}}$ the set $\mathrm{Pr}(\mathfrak{M}, \mathbb{HF}(\varnothing))$ of all presentations of $\mathfrak{M}$ in the least admissible set. It is well-known (see [3, 4]) that computability (i.e. effective definability) in $\mathbb{HF}(\varnothing)$ is equivalent to the classical computability on natural numbers.

There are natural embeddings of the semilattices $\mathcal{D}$ and $\mathcal{D}_e$ of Turing and enumeration degrees into $\mathcal{S}_\Sigma$. To show this, we use the notion of a structure having a degree, introduced by L. Richter [9].

Definition 3. *Let* $\mathfrak{M}$ *be a countable structure.* $\mathfrak{M}$ *is said to have a degree (e-degree) if there exists a least degree in the class of* T*-degrees (e-degrees) of all possible presentations of* $\mathfrak{M}$ *on natural numbers.*

Using the classical result connecting $\forall$-computability and $\exists$-definability, first proved by Y.N. Moschovakis [7] and lately rediscovered and generalized by J. Knight [1], in [15] was proved the following

Theorem 1. *For a countable* $\mathfrak{M}$*,* $\mathfrak{M}$ *has a degree (e-degree) iff, for some* $\mathcal{C} \in \underline{\mathfrak{M}}$*,* $\mathcal{C}$ *is* Δ*-definable (*Σ*-definable) in* $\mathbb{HF}(\mathfrak{M})$*.*

Define mappings $i : \mathcal{D} \to \mathcal{S}_\Sigma$ and $j : \mathcal{D}_e \to \mathcal{S}_\Sigma$ as follows: for any $\mathbf{a} \in \mathcal{D}$, let

$$i(\mathbf{a}) = [\mathfrak{M}_\mathbf{a}]_\Sigma, \text{ where } \mathfrak{M}_\mathbf{a} \text{ is some structure with degree } \mathbf{a}.$$

In the same way, for any $\mathbf{b} \in \mathcal{D}_e$, let

$$j(\mathbf{b}) = [\mathfrak{M}_\mathbf{b}]_\Sigma, \text{ where } \mathfrak{M}_\mathbf{b} \text{ is some structure with e-degree } \mathbf{b}.$$

Lemma 1. *The definitions above are correct: for any countable structures* $\mathfrak{M}$ *and* $\mathfrak{N}$*, if* $\mathfrak{M}$ *has (e-)degree* $\mathbf{a}$ *and* $\mathfrak{M} \equiv_\Sigma \mathfrak{N}$*, then* $\mathfrak{N}$ *also has the same (e-)degree* $\mathbf{a}$*.*

Proof. Suppose, for example, that $\mathfrak{M}$ has an e-degree $\mathbf{a}$. Since $\mathbb{HF}(\mathfrak{M}) \leqslant_\Sigma \mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ and some $\mathcal{C} \in \underline{\mathfrak{M}}$ is Σ-definable in $\mathbb{HF}(\mathfrak{M})$, we get that $\mathcal{C}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$. Now, since $\mathfrak{N} \leqslant_\Sigma \mathfrak{M}$, by applying this Σ-definition to $\mathcal{C}$ we get $\mathcal{C}' \in \underline{\mathfrak{N}}$ s.t. $\mathcal{C}' \leqslant_e \mathcal{C}$, and hence $\mathcal{C}'$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$. So $\mathfrak{N}$ has an e-degree which is less or equal to the e-degree of $\mathfrak{M}$. The same argument in the opposite direction shows that these degrees coincide.

However, in general the property of having a degree is not closed downwards with respect to $\leqslant_\Sigma$. In the same way as Lemma 1 was proved, we can prove

Proposition 1. *Mappings $i : \mathcal{D} \to \mathcal{S}_\Sigma$ and $j : \mathcal{D}_e \to \mathcal{S}_\Sigma$ are semilattice embeddings preserving* 0 *and* $\vee$.

(The embedability of $\mathcal{D}$ into $\mathcal{S}_\Sigma$ was previously noted by A.N. Khisamiev [6].)

Let $\mathbb{A}$ be an arbitrary admissible set. A mapping $F : P(A)^n \to P(A)$ ($n \in \omega$) is called a *Σ-operator*[4] if there is a Σ-formula $\Phi(x_0, \ldots, x_{n-1}, y)$ of signature $\sigma_\mathbb{A}$ such that for all $S_0, \ldots, S_{n-1} \in P(A)$

$$F(S_0, \ldots, S_{n-1}) = \{\, a \mid \exists a_0, \ldots, a_{n-1} \in A(\bigwedge_{i<n} a_i \subseteq S_i \wedge \mathbb{A} \models \Phi(a_0, \ldots, a_{n-1}, a))\}.$$

To guarantee transitiveness of reducibilities defined below, we need the following notion. An operator $F : P(A) \to P(A)$ is *strongly continuous* in $S \in P(A)$, if for any $a \subseteq F(S)$, $a \in A$,there exists $a' \subseteq S$, $a' \in A$, s.t. $a \subseteq F(a')$ (this definition can be easily modified for operators of arbitrary arity).

For operator $F : P(A)^n \to P(A)$, $\delta_\mathrm{c}(F)$ is the set of elements of $P(A)^n$ in which F is strongly continuous. A set $S \in P(A)^n$ is called a *Σ_*-set* if $S \in \delta_c(F)$ for any Σ-operator $F : P(A)^n \to P(A)$. It is easy to show that in $\mathbb{HF}(\mathfrak{M})$ any subset is a Σ_*-set. However, in general this is not so: for example, in [12] were studied Σ_*-sets of urelements in admissible set of kind $\mathbb{HYP}(\mathbb{L})$, where $\mathbb{L}$ is a dense linear order. Even in this simplest case the class of Σ_*-sets is nontrivial.

Suppose $B, C \subseteq A$. The following reducibilities are natural generalizations of e- and T-reducibilities on natural numbers:

1) *B is $e\Sigma$-reducible to C* ($B \leqslant_{e\Sigma} C$) if there exists a unary Σ-operator F such that $C \in \delta_c(F)$ and $B = F(C)$;

2) *B is $T\Sigma$-reducible to C* ($B \leqslant_{T\Sigma} C$) if there exist binary Σ-operators F_0 and F_1 such that $\langle C, A \setminus C\rangle \in \delta_c(F_0) \cap \delta_c(F_1)$ for which $B = F_0(C, A \setminus C)$ and $A \setminus B = F_1(C, A \setminus C)$.

Also, we consider some uniform reducibilities on families of subsets of A, naturally generalizing Medvedev, Muchnik and Dyment reducibilities on mass problems [11]. Suppose $\mathcal{X}, \mathcal{Y} \subseteq P(A)$, then

1) *$\mathcal{X}$ is Medvedev reducible to $\mathcal{Y}$* ($\mathcal{X} \leqslant \mathcal{Y}$) if there exist binary Σ-operators F_0 and F_1 such that, for all $Y \in \mathcal{Y}$, $\langle Y, A \setminus Y\rangle \in \delta_c(F_0) \cap \delta_c(F_1)$ and, for some $X \in \mathcal{X}$, $X = F_0(Y, A \setminus Y)$ and $A \setminus X = F_1(Y, A \setminus Y)$;

2) $\mathcal{X}$ *is Dyment reducible to* $\mathcal{Y}$ ($\mathcal{X} \leqslant_e \mathcal{Y}$) if there exists a unary Σ-operator F such that, for all $Y \in \mathcal{Y}$, $Y \in \delta_c(F)$ and $F(\mathcal{Y}) \subseteq \mathcal{X}$;

3) $\mathcal{X}$ *is Muchnik reducible to* $\mathcal{Y}$ ($\mathcal{X} \leqslant_w \mathcal{Y}$) if, for any $Y \in \mathcal{Y}$, there exist binary Σ-operators F_0 and F_1 such that $\langle Y, A \setminus Y\rangle \in \delta_c(F_0) \cap \delta_c(F_1)$ and, for some $X \in \mathcal{X}$, $X = F_0(Y, A \setminus Y)$ and $A \setminus X = F_1(Y, A \setminus Y)$;

4) $\mathcal{X}$ *is nonuniformly Dyment reducible to* $\mathcal{Y}$ ($\mathcal{X} \leqslant_e \mathcal{Y}$) if for all $Y \in \mathcal{Y}$ there exists a unary Σ-operator F such that $Y \in \delta_c(F)$ and $F(\mathcal{Y}) \subseteq \mathcal{X}$.

For any admissible set $\mathbb{A}$ and any $* \in \{e, \ , w, ew\}$, let $\mathcal{M}_*(\mathbb{A})$ denotes the structure $\langle P(P(A))/\equiv_*, \leqslant_*\rangle$. For simplicity, we write $\mathcal{M}_*$ instead of $\mathcal{M}_*(\mathbb{HF}(\varnothing))$. Each of $\mathcal{M}_*(\mathbb{A})$ is a lattice with 0 and 1, and lattices $\mathcal{M}, \mathcal{M}_e, \mathcal{M}_w$ are isomorphic to the Medvedev, Dyment and Muchnik lattices, respectively.

For a countable structure $\mathfrak{M}$ consider the cones of structures reducible to it:
$\mathcal{K}_\Sigma(\mathfrak{M}) = \{\mathfrak{N} \mid \mathfrak{N} \leqslant_\Sigma \mathfrak{M}\}$,
$\mathcal{K}_e(\mathfrak{M}) = \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant_e \underline{(\mathfrak{M}, \bar{m})}$ for some $\bar{m} \in M^{<\omega}\}$,
$\mathcal{K}(\mathfrak{M}) = \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant \underline{(\mathfrak{M}, \bar{m})}$ for some $\bar{m} \in M^{<\omega}\}$,
$\mathcal{K}_{ew}(\mathfrak{M}) = \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant_{ew} \underline{\mathfrak{M}}\}$, $\mathcal{K}_w(\mathfrak{M}) = \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant_w \underline{\mathfrak{M}}\}$.

It was proved in [15] that, for any structure $\mathfrak{M}$,

$$\mathcal{K}_\Sigma(\mathfrak{M}) \subseteq \mathcal{K}_e(\mathfrak{M}) \subseteq \mathcal{K}(\mathfrak{M}) \subseteq \mathcal{K}_w(\mathfrak{M}),$$

as well as $\mathcal{K}_e(\mathfrak{M}) \subseteq \mathcal{K}_{ew}(\mathfrak{M}) \subseteq \mathcal{K}_w(\mathfrak{M})$. In general, all these inclusions are proper.

For any $* \in \{e, \ , w, ew\}$, define the relation $\leqslant_*$ on $\mathcal{K}_\omega$ in the following way: $\mathfrak{M} \leqslant_* \mathfrak{N}$ if and only if $\mathcal{K}_*(\mathfrak{M}) \subseteq \mathcal{K}_*(\mathfrak{N})$, and let $\mathcal{S}_* = \langle \mathcal{K}_\omega/\equiv_*, \leqslant_*\rangle$ be a structure of degrees of presentability corresponding to this reducibility relation.

Theorem 2 ([15]). *Each of* $\mathcal{S}_*$*,* $* \in \{e, \ , w, ew\}$*, is an upper semilattice with* 0*, and there are following embeddings* ($\hookrightarrow$) *and homomorphisms* ($\to$)

$$\mathcal{D} \hookrightarrow \mathcal{D}_e \hookrightarrow \mathcal{S}_\Sigma \to \mathcal{S}_e \to \mathcal{S} \hookrightarrow \mathcal{M}.$$

For arbitrary structures $\mathfrak{M}$ and $\mathfrak{M}'$ of the same signature and any $n \in \omega$, we denote by $\mathfrak{M} \equiv_n^{\mathrm{HF}} \mathfrak{M}'$ the fact that $\mathbb{HF}(\mathfrak{M}) \equiv_n \mathbb{HF}(\mathfrak{M}')$. It is easy to verify that, for $n < 2$, $\mathfrak{M} \equiv_n^{\mathrm{HF}} \mathfrak{M}'$ if and only if $\mathfrak{M} \equiv_n \mathfrak{M}'$. For $n = 2$, $\mathfrak{M} \equiv_2^{\mathrm{HF}} \mathfrak{M}'$ if and only if, for any computable sequence $\{\varphi_{mn}(\bar{x}_m, \bar{y}_n) | m, n \in \omega\}$ of quantifier-free formulas of signature $\sigma_{\mathfrak{M}}$,

$$\mathfrak{M}' \models \bigvee_{m\in\omega} \exists \bar{x}_m \bigwedge_{n\in\omega} \forall \bar{y}_n \varphi_{mn}(\bar{x}_m, \bar{y}_n)$$

if and only if the same infinitary sentence is true in $\mathfrak{M}$.

A structure $\mathfrak{M}$ is called *locally constructivizable* [4] if $\mathrm{Th}_\exists(\mathfrak{M}, \bar{m})$ is c.e. for any $\bar{m} \in M^{<\omega}$, or, equivalently, for any $\bar{m} \in M^{<\omega}$ there exists a constructivizable structure $\mathfrak{N}$ and $\bar{n} \in N^{<\omega}$ such that $\mathrm{Th}_\exists(\mathfrak{M}, \bar{m}) = \mathrm{Th}_\exists(\mathfrak{N}, \bar{n})$. For structures $\mathfrak{M}$ and $\mathfrak{N}$, we denote by $\mathfrak{M} \leqslant_\exists \mathfrak{N}$ the fact that for any $\bar{m} \in M^{<\omega}$ there is $\bar{n} \in N^{<\omega}$ such that $\mathrm{Th}_\exists(\mathfrak{M}, \bar{m}) \leqslant_e \mathrm{Th}_\exists(\mathfrak{N}, \bar{n})$. In particular, if $\mathfrak{M}$ is locally constructivizable then $\mathfrak{M} \leqslant_\exists \mathfrak{N}$ for any $\mathfrak{N}$. As it was first observed in [4], if $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$, and $\mathfrak{N}$ is locally constructivizable, then $\mathfrak{M}$ is also locally constructivizable. In fact, this observation can be easily generalized as follows: if $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ then $\mathfrak{M} \leqslant_\exists \mathfrak{N}$. For stating a series of another necessary conditions of Σ-definability (which are useful for proving negative results about reducibility $\leqslant_\Sigma$) we need the following

Definition 4. *A structure $\mathfrak{M}$ is called locally constructivizable of level n ($1 < n \leqslant \omega$), if, for any tuple $\bar{m} \in M^{<\omega}$, there exist a constructivizable structure $\mathfrak{N}$ and a tuple $\bar{n} \in N^{<\omega}$ such that $(\mathfrak{M}, \bar{m}) \equiv_n^{\mathrm{HF}} (\mathfrak{N}, \bar{n})$. A countable structure $\mathfrak{M}$ is called uniformly locally constructivizable of level n ($1 < n \leqslant \omega$) if there exists a constructivizable structure $\mathfrak{N}$ such that $\mathfrak{M} \preccurlyeq_n^{\mathrm{HF}} \mathfrak{N}$.*

For example, well-ordering $\langle \omega_1^{CK}, \leqslant \rangle$ is uniformly locally constructivizable of level ω, since $\langle \omega_1^{CK}, \leqslant \rangle \preccurlyeq^{\mathrm{HF}} \langle \omega_1^{CK}(1 + \eta), \leqslant \rangle$, there the last ordering is well-known computable Harrison ordering.

Proposition 2. *If $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ and $\mathfrak{N}$ is (uniformly) locally constructivizable of level n ($1 < n \leqslant \omega$) then $\mathfrak{M}$ is also (uniformly) locally constructivizable of level n.*

The next result shows that the class of locally constructivizable (of level 1) countable structures is closed downwards with respect to $\leqslant_w$, i.e. weakest of considered effective reducibilities.

Proposition 3 ([15]). *For arbitrary structures $\mathfrak{M}$ and $\mathfrak{N}$, $\mathfrak{N} \in \mathcal{K}_w(\mathfrak{M})$ implies that $\mathfrak{N} \leqslant_\exists \mathfrak{M}$. In particular, if $\mathfrak{M}$ is locally constructivizable, then any $\mathfrak{N} \in \mathcal{K}_w(\mathfrak{M})$ is also locally constructivizable.*

Since $(\mathfrak{M}, \mathfrak{N})$ is locally constructivizable in case then $\mathfrak{M}$ and $\mathfrak{N}$ are locally constructivizable, the sets of degrees generated by locally constructivizable structures form the ideals in semilattices $\mathcal{S}_*$, $* \in \{\Sigma, e, \ , w, ew\}$. The classes of locally constructivizable structures of level n for $n > 1$, however, are closed downwards only with respect to $\leqslant_\Sigma$ (and so form initial segments in $\mathcal{S}_\Sigma$). For weaker reducibilities this is not so. For example, there is

Theorem 3 ([15]). *If a structure $\mathfrak{M}$ is locally constructivizable of level $n > 1$ and not constructivizable, then there is a structure $\mathfrak{M}_0 \in \mathcal{K}(\mathfrak{M})$ which is locally constructivizable of level 1 sharply. In particular, $\mathcal{K}_\Sigma(\mathfrak{M}) \subsetneq \mathcal{K}(\mathfrak{M})$.*

The proof of Theorem 3 uses the result of T. Slaman [10] and S. Wehner [16]: there exists a structure with problem of presentability belonging to the least non-zero degree in the Medvedev lattice (in fact, this implies that the semilattice $\mathcal{S}$ has a least non-zero element). From a relativization of Slaman construction it also follows that in the Muchnik lattice any degree which is the least over a degree of solvability is a degree of presentability. Using the similar ideas, we can also prove

Theorem 4 ([15]). *There exist a structure $\mathfrak{M}$ and an unary relation $P \subseteq M$ such that $(\mathfrak{M}, P) \equiv \mathfrak{M}$, but $(\mathfrak{M}, P) \not\leqslant_{\Sigma} \mathfrak{M}$.*

Theorem 4 is interesting with respect to the following result from [2]: for a countable structure $\mathfrak{M}$ and relation $P \subseteq M^n$, P is Σ-definable in $\mathbb{HF}(\mathfrak{M})$ if and only if, for any $\mathcal{C} \in \underline{(\mathfrak{M}, P)}$, $P^{\mathcal{C}}$ is $\mathcal{C} \restriction \sigma_{\mathfrak{M}}$-c.e..

The next theorem gives sufficient conditions for the equality of different classes of structures, effectively connected with a countable structure $\mathfrak{M}$.

Theorem 5 ([15]). *If $\mathfrak{M}$ has a degree then $\mathcal{K}_{\Sigma}(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M}) = \mathcal{K}(\mathfrak{M}) = \mathcal{K}_w(\mathfrak{M})$. If $\mathfrak{M}$ has an e-degree then $\mathcal{K}_{\Sigma}(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M}) = \mathcal{K}_{ew}(\mathfrak{M})$.*

It seems to be a natural open question whether these sufficient conditions are also necessary or not.

For structures $\mathfrak{M}$ and $\mathfrak{N}$ s.t. $\mathrm{card}(M) \leqslant \mathrm{card}(N)$, consider the class

$$\mathcal{K}(\mathfrak{M}, \mathfrak{N}) = \{\mathfrak{M}' \mid \Pr(\mathfrak{M}', \mathbb{HF}(\mathfrak{N})) \leqslant \Pr((\mathfrak{M}, \bar{m}), \mathbb{HF}(\mathfrak{N})),\ \bar{m} \in M^{<\omega}\}.$$

In the same way, classes $\mathcal{K}_e(\mathfrak{M}, \mathfrak{N})$, $\mathcal{K}_w(\mathfrak{M}, \mathfrak{N})$ and $\mathcal{K}_{ew}(\mathfrak{M}, \mathfrak{N})$ are defined.

Proposition 4. *Let $\mathfrak{M}$ be a countable structure, and let a countable $\mathfrak{N}$ be either an infinite structure of empty signature, or a dense linear order. Then $\mathcal{K}_{\Sigma}(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M}, \mathfrak{N}) = \mathcal{K}(\mathfrak{M}, \mathfrak{N})$.*

As a consequence we get that there exist natural isomorphisms between the semilattice $\mathcal{S}_{\Sigma}$ of degrees of Σ-definability and semilattices $\mathcal{S}(\mathbb{HF}(\mathfrak{N}))$ of degrees of presentability for $\mathfrak{N}$ as in the proposition above.

Another generalization of the result connecting $\forall$-recursiveness and $\exists$-definability is the following

Theorem 6 ([15]). *For any countable structures $\mathfrak{M}$ and $\mathfrak{N}$ and any $R \subseteq \mathbb{HF}(\mathfrak{N})$, the following are equivalent:*

1) for any presentation $\mathcal{C}$ of $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{N})$, $R \leqslant_{e\Sigma} \mathcal{C}$;
2) R is Σ-definable in $\mathbb{HF}(\mathfrak{M}, \mathfrak{N})$.

Definition 5. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be a countable structures. $\mathfrak{M}$ is said to have a degree (e-degree) over $\mathfrak{N}$ if there exists a least degree in the class of $T\Sigma$-degrees ($e\Sigma$-degrees) of all possible presentations of $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{N})$.*

From Theorem 6 we get a corollary, which is a generalization of Theorem 1.

Theorem 7 ([15]). *Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable structures. The following are equivalent:*

1) $\mathfrak{M}$ has a degree (e-degree) over $\mathfrak{N}$;
2) some presentation $\mathcal{C} \subseteq HF(N)$ of $\mathfrak{M}$ is Δ-definable (Σ-definable) in $\mathbb{HF}(\mathfrak{M}, \mathfrak{N})$.

It is clear that, if $\mathfrak{M} \leqslant_\exists \mathfrak{N}$, then $\mathfrak{M}$ has a degree (e-degree) over $\mathfrak{N}$ if and only if $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$. It is also clear that, if $\mathfrak{M}$ has a degree (e-degree) over $\mathfrak{N}$ and $\mathfrak{N} \leqslant_\Sigma \mathfrak{N}'$, then $\mathfrak{M}$ has a degree (e-degree) over $\mathfrak{N}'$. It is easy to verify that for any countable structure $\mathfrak{A}$ there exists a structure $\mathfrak{M}$ which has a degree but is not Σ-definable in $\mathbb{HF}(\mathfrak{A})$. As in the nonrelativised case, we also have

Theorem 8 ([15]). *Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable structures. If $\mathfrak{M}$ has a degree over $\mathfrak{N}$ then $\mathcal{K}_\Sigma(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M}, \mathfrak{N}) = \mathcal{K}(\mathfrak{M}, \mathfrak{N})$. If $\mathfrak{M}$ has an e-degree over $\mathfrak{N}$ then $\mathcal{K}_\Sigma(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M}, \mathfrak{N})$.*

Another necessary condition of Σ-definability of one structure in another is the existence of some effective uniform reducibility between local HF-theories of these structures. Further on, by a *family* we mean any set of subsets of ω. We define effective operators on families, by expanding domains of recursive operators to the set $P(P(\omega))$ in the following way: for any family $\mathcal{X} \subseteq P(\omega)$ and any enumeration operator $\Phi : P(\omega) \to P(\omega)$, let

$$\Phi(\mathcal{X}) = \{\Phi(D) | D \in \mathcal{X}^{<\omega} \text{ is a finite tuple}\}.$$

Here by $\Phi(D)$ for $D = \langle X_1, \dots, X_n \rangle$ we mean the set $\Phi(X_1 \oplus \dots \oplus X_n)$.

Let $\mathcal{A}, \mathcal{B} \subseteq P(P(\omega))$ be arbitrary collections of families. We say that $\mathcal{A}$ is *Dyment reducible* to $\mathcal{B}$ (denoted by $\mathcal{A} \leqslant_e \mathcal{B}$) if $\Phi(\mathcal{B}) \subseteq \mathcal{A}$ for some enumeration operator Φ.

On the set of families we define mappings $i : P(P(\omega)) \to P(P(P(\omega)))$ and $j : P(P(\omega)) \to P(P(P(\omega)))$ in the following way: for a family $\mathcal{X}$, let $i(\mathcal{X}) = \{\mathcal{X}\}$ and $j(\mathcal{X}) = \{\{X\} | X \in \mathcal{X}\}$. It is clear that, for any families $\mathcal{X}, \mathcal{Y} \subseteq P(\omega)$, $j(\mathcal{X}) \leqslant_e j(\mathcal{Y})$ if and only if $\mathcal{X} \leqslant_e \mathcal{Y}$ (in the latter case by $\leqslant_e$ we mean the Dyment reducibility on families). Further on, by $\mathcal{X} \leqslant_e^o \mathcal{Y}$ we denote the fact that $i(\mathcal{X}) \leqslant_e i(\mathcal{Y})$.

Let $\mathcal{X} \subseteq P(P(\omega))$, and let $\bar{X}^0 = \langle X_1^0, \dots, X_k^0 \rangle$, $X_1^0, \dots, X_k^0 \subseteq \omega$. The *shift* of $\mathcal{X}$ by $\bar{X}^0$ is the family $\bar{X}^0 * \mathcal{X} = \{X_1^0 \oplus \dots \oplus X_k^0 \oplus X | X \in \mathcal{X}\}$.

The Dyment reducibility on the set $P(P(P(\omega)))$ of collections of families is reflexive and transitive. Equivalence relation $\equiv_e$ is defined as usual: $\mathcal{A} \equiv_e \mathcal{B}$ if $\mathcal{A} \leqslant_e \mathcal{B}$ and $\mathcal{B} \leqslant_e \mathcal{A}$. We denote the poset of degrees $\langle P(P(P(\omega)))/\equiv_e, \leqslant_e \rangle$ by $\mathcal{M}_e'$, having in mind the Dyment lattice $\mathcal{M}_e$.

Proposition 5. *$\mathcal{M}_e'$ is a lattice with 0 and 1, and $j : \mathcal{M}_e \to \mathcal{M}_e'$ is an embedding preserving $\wedge, \vee, 0, 1$.*

Proof. The greatest element of $\mathcal{M}_e'$ is, evidently, $[\varnothing]_e$, while the least is $[\{\{\varnothing\}\}]_e$, and $j(0_{\mathcal{M}_e}) = i(0_{\mathcal{M}_e}) = \{\{\varnothing\}\}$, $j(1_{\mathcal{M}_e}) = \varnothing$, $i(1_{\mathcal{M}_e}) = \{\varnothing\}$, $\varnothing \equiv_e \{\varnothing\}$. Join and meet operations on $\mathcal{M}_e'$ are defined as follows: for collections $\mathcal{A}, \mathcal{B} \subseteq P(P(\omega))$, $\mathcal{A} \vee \mathcal{B} = \{\mathcal{X} \vee \mathcal{Y} | \mathcal{X} \in \mathcal{A}, \mathcal{Y} \in \mathcal{B}\}$, where $\mathcal{X} \vee \mathcal{Y} = \{X \oplus Y | X \in \mathcal{X}, Y \in \mathcal{Y}\}$; and $\mathcal{A} \wedge \mathcal{B} = \{0\} * \mathcal{A} \cup \{1\} * \mathcal{B}$, where $\{0\} * \mathcal{A} = \{\{0\} * \mathcal{X} | \mathcal{X} \in \mathcal{A}\}$, $\{1\} * \mathcal{B} = \{\{1\} * \mathcal{Y} | \mathcal{Y} \in \mathcal{B}\}$.

In the same way as the nonuniform analog $\mathcal{M}_{ew}$ of the Dyment lattice is defined, we can define a nonuniform analog $\mathcal{M}'_{ew}$ of $\mathcal{M}'_e$. Besides, in the same way as Medvedev and Muchnik lattices are defined, we can define the lattices $\mathcal{M}'$ and $\mathcal{M}'_w$, generated by collections of families of total sets (or functions).

Proposition 6. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be a structures of arbitrary cardinality, and let $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$. Then, for some $\bar{n}_0 \in N^{<\omega}$, for any $1 < n \leqslant \omega$ holds*

$$\begin{array}{l}
\{\mathrm{Th}^{\mathrm{HF}}_{\Sigma_n}(\mathfrak{M},\bar{m})|\bar{m} \in M^{<\omega}\} \leqslant^o_e \{\mathrm{Th}^{\mathrm{HF}}_{\Sigma_n}(\mathfrak{N},\bar{n}_0,\bar{n})|\bar{n} \in N^{<\omega}\},\\
\{\mathrm{Th}^{\mathrm{HF}}_{\Pi_n}(\mathfrak{M},\bar{m})|\bar{m} \in M^{<\omega}\} \leqslant^o_e \{\mathrm{Th}^{\mathrm{HF}}_{\Pi_n}(\mathfrak{N},\bar{n}_0,\bar{n})|\bar{n} \in N^{<\omega}\},\\
\{\mathrm{Th}^{\mathrm{HF}}_{n}(\mathfrak{M},\bar{m})|\bar{m} \in M^{<\omega}\} \leqslant^o_e \{\mathrm{Th}^{\mathrm{HF}}_{n}(\mathfrak{N},\bar{n}_0,\bar{n})|\bar{n} \in N^{<\omega}\}.
\end{array}$$

For a structure $\mathfrak{M}$, consider the family $\mathcal{E}(\mathfrak{M}) = \{\mathrm{Th}_\exists(\mathfrak{M},\bar{m})|\bar{m} \in M^{<\omega}\}$ consisting of the existential theories of arbitrary expansions of $\mathfrak{M}$ by a finite number of constants. From the previous proposition we get a

Corollary 1. *For any structures $\mathfrak{M}$ and $\mathfrak{N}$,*

$$\mathfrak{M} \leqslant_\Sigma \mathfrak{N} \Rightarrow \mathcal{E}(\mathfrak{M}) \leqslant^o_e \mathcal{E}(\mathfrak{N},\bar{n}_0) \textit{ for some } \bar{n}_0 \in N^{<\omega}.$$

There exist structures for which the necessary condition of Σ-definability from Corollary 1 is also sufficient. With any family $\mathcal{X} \subseteq P(\omega)$ we connect a structure $\mathfrak{A}_\mathcal{X}$ defined in the following way: the domain of this structure is the set $\omega \cup S$, where S is the set of cardinality 2^ω. The signature consists of unary functional symbol s, defined in the usual way on ω ($s(n) = n+1$) and identical on S, and a binary relation symbol R, which is interpreted as follows: $R \subseteq S \times \omega$, $\mathcal{X} = \{\{n \in \omega|R(s,n)\}|s \in S\}$, and to any member of $\mathcal{X}$ correspond uncountably many elements from S, and there are also uncountably many elements from S which are not connected by R with elements from ω, and uncountably many connected with every element from ω.

We also define a structure $\mathfrak{B}_\mathcal{X}$, with domain $C \cup S$, where C and S are disjoint sets of cardinality 2^ω, and the signature consists of an unary relation symbol P separating the set S, and a binary relation symbol R, forming finite cycles (loops are possible only on elements from S), in which only one element is from S, for different cycles the sets of involved in them elements of C are disjoint, $\mathcal{X} = \{\{n \in \omega|\exists c_0 \ldots \exists c_n(R(s,c_0) \wedge R(c_0,c_1) \ldots \wedge R(c_n,s))\}|s \in S\}$, for any member of $\mathcal{X}$ there are uncountably many elements from S corresponding to it, there are uncountably many elements from S involved in cycles of all possible sizes, there are also uncountably many elements from S which are not connected by R with any other element, and there are uncountably many elements of C which are not involved in cycles.

Immediately from the definitions of $\mathfrak{A}_\mathcal{X}, \mathfrak{B}_\mathcal{X}$ and Corollary 1 we get

Proposition 7. *For any families $\mathcal{X}, \mathcal{Y} \subseteq P(\omega)$,*

$$\begin{array}{l}
\mathfrak{A}_\mathcal{X} \leqslant_\Sigma \mathfrak{A}_\mathcal{Y} \iff \mathcal{X} \cup \{\varnothing,\omega\} \leqslant^o \bar{Y}_0 * (\mathcal{Y} \cup \{\varnothing,\omega\}) \textit{ for some } \bar{Y}_0 \in \mathcal{Y}^{<\omega},\\
\mathfrak{B}_\mathcal{X} \leqslant_\Sigma \mathfrak{B}_\mathcal{Y} \iff \mathcal{X} \cup \{\varnothing,\omega\} \leqslant^o_e \bar{Y}_0 * (\mathcal{Y} \cup \{\varnothing,\omega\}) \textit{ for some } \bar{Y}_0 \in \mathcal{Y}^{<\omega}.
\end{array}$$

In particular, if structures $\mathfrak{M}$, $\mathfrak{N}$ are both of kind $\mathfrak{A}_{\mathcal{X}}$ or $\mathfrak{B}_{\mathcal{X}}$, then $\mathfrak{M} \leqslant_{\Sigma} \mathfrak{N}$ if and only if $\mathcal{E}(\mathfrak{M}) \leqslant_e^o \mathcal{E}(\mathfrak{N}, \bar{n}_0)$ for some $\bar{n}_0 \in N^{<\omega}$.

We say that a structure $\mathfrak{M}$ is *locally n-low* ($n \in \omega$) if $\mathrm{Th}^{\mathrm{HF}}_{\Sigma_n}(\mathfrak{M}, \bar{m}) \in \Sigma^0_n$ for all $\bar{m} \in M^{<\omega}$. It is clear that if $\mathfrak{M}$ is localy constructivizable of level n then $\mathfrak{M}$ is locally n-low (for $n = 1$ the converse is also true). From Proposition 6 it follows that, for any n, the property of being locally n-low is also closed downwards with respect to $\leqslant_{\Sigma}$.

As a corollary of Proposition 1 we have that $\mathrm{card}(\mathcal{S}_{\Sigma}) = 2^{\omega}$. Consider the question about the cardinality of the semilattice $\mathcal{S}_{\Sigma}(2^{\omega})$. It is also maximal, as follows from

Theorem 9. *There exists an antichain in $\mathcal{S}_{\Sigma}(2^{\omega})$ of cardinality $2^{2^{\omega}}$. In particular,* $\mathrm{card}(\mathcal{S}_{\Sigma}(2^{\omega})) = 2^{2^{\omega}}$.

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