On Processes and Structures*

Alexey Stukachev

Novosibirsk State University, Russia
Sobolev Institute of Mathematics, Novosibirsk, Russia
aistu@math.nsc.ru

Abstract. We consider an approach to computability in admissible sets based on a general notion of computable process, with Σ-predicates and Σ-operators as special cases, inspired by ideas from the Ershov–Scott theory of approximation spaces. We present some results from different topics in generalized computability, including reducibilities on admissible sets and structures, general notion of a jump, and computable analysis (more exactly, computability over the reals), obtained with the help of this approach, and state some open questions.
Keywords: computability theory, admissible sets, computable model theory, approximation spaces, computable analysis.

1 Introduction

The present paper is a continuation of [15, 16, 18, 19] and especially [17]. It is motivated by three questions in generalized computability which turned out to be closely connected:
1) For a given admissible set $\mathbb{A}$, what is *computability on* $\mathbb{A}$?
2) What is a *jump* of a given computability or structure?
3) How to define a *measure (degree) of complexity* of a given structure or admissible set?

First, a usual understanding of computability theory on admissible sets as the study of Σ-definable objects (predicates, relations, subsets, etc.) is too stringent when various formalizations of computable processes are considered, like in the case of mass problems with reducibilities on them via computable operators in the style of Medvedev, Muchnik and Dyment [15]. Degrees of presentability, being a special case of mass problems on admissible sets in general, provide natural tools for measuring the constructive complexity of structures, so the task of extending this approach is quite actual. Concerning this problem, we present a framework which allows to study both Σ-predicates and Σ-operators from a

* This work was supported by the Ministry of Education and Science of Russian Federation (project 8227), the Russian Foundation for Basic Research (grants 11-01-00688-a, 13-01-91001-ANF-a), and the State Maintenance Program for the Leading Scientific Schools of the Russian Federation (grant N.Sh.-276.2012.1).

common viewpoint based on a notion of (constructive) process on a domain of computation. Computability on admissible sets is generated from this point of view by classes Σ-processes on Σ-admissible families (in the paper such pairs are called *computability components*). It turns out that these notions in case of HF-computabilities are naturally connected with another motivating problem — the study of jumps of computabilities and jumps of structures. We apply the methods constructed in the first part to formulate a general notion of jump of computability, extending the notions of Turing jump and hyperjump of a set of natural numbers, and the notion of Σ-jump of a structure, considered recently by different authors [7, 9, 12, 18].

The results obtained in the paper explain how new notions and insights can be helpful in quite different areas. We propose, formalize and study the following approaches to the questions stated above:

1) Computability on $\mathbb{A}$ is a family of its components, with each component defined as a pair: a family of objects — subsets of A, and a class of Σ-processes acting on them, with the property that every finite fragment of an output can be obtained using some finite fragments of the arguments and resources.

2) The jump of a component of computability on $\mathbb{A}$ is a structure with the domain consisting of its objects and the diagram is obtained by the *termination* of its processes.

3) The measure of (relative) complexity of a structure is given by its degrees in semilattices of Σ-degrees and degrees of presentability, while the measure of (relative) complexity of an admissible set is given by its equivalence class with the equivalence relation generated by the Morozov reducibility [8].

There are three main classes of objects considered in this paper: *admissible sets*, *computabilities*, generated by admissible sets, and structures, obtained as *jumps* of computabilities. Note that a structure, in turn, generate admissible sets like HF- or HYP-superstructures, so we can speak about, say, HF-computability over that structure.

The new results in this paper are as follows. First, concerning computabilities on admissible sets in general, we obtain a strengthening of the result of A.S.Morozov [8] which states that a certain reducibility between admissible sets implies an embedding of computable objects (i.e., Σ-predicates) on them. Namely, we prove that this reducibility implies much more general fact: there exists an embedding of computabilities on these admissible sets. We show how the theory of admissible sets and , in particular, the notion of Σ-admissible family is connected with the Ershov–Scott theory of approximation spaces.

Second, we present results connecting $0^\diamond$, the jump of the maximal component of HF-computability over 0, with the reals in both algebraical and topological settings. We show that these structures are constructible from, respectively, $(0^\diamond)'$ and $0^\diamond$. These results can be considered as a first step in the study of jump inversions in general, established for the minimal component of HF-computability over arbitrary structure in the strongest possible form [18, 19].

In the text, we use definitions and notations from [6, 2]. In particular, for an admissible set $\mathbb{A}$, $U^{\mathbb{A}}$ denotes the set of urelements from $\mathbb{A}$, and A^* denotes the set of elements of $\mathbb{A}$ which are sets (i.e., not urelements).

2 Processes and Approximation Spaces

We will use very basic definitions and facts from the Ershov–Scott theory of approximation spaces.

Definition 1 By a *p-domain* we mean a triple $\mathcal{X} = \langle X, F, \leqslant \rangle$, where X is a topological T_0 space which is a φ-space [3, 5], $F \subseteq X$ is the basic subset of *finite elements*, and $\leqslant$ is the specialization order on X.

Every φ-space is an α-space [5]. In particular, we will use the property of p-domains true for α-spaces in general: every element $x \in X$ is a limit of its F-approximations:

$$x = \sup\{a \in F \mid a \leqslant x\}.$$

The set F can be viewed as the set of finite approximations for elements from X, and the specialization order $\leqslant$ is usually induced on X by a T_0 topology defined in some natural way. Typical examples we will consider in this paper are built from an admissible set $\mathbb{A}$, with $F = A$, $\leqslant\ = \subseteq_{X \setminus U^{\mathbb{A}}} \cup =_{U^{\mathbb{A}}}$, and $X \subseteq A \cup P(A)$ (see Example 1 below). To explain the role of set X we should define the notions of process, constructive process, and computability.

Informally, processes are functions on p-domains which generate an output as the limit of its approximations, using 'finite' fragments of arguments and resources (like space or time). Each process is defined by its *specification* or *presentation*, and constructive processes are defined by specifications which can be 'effectively checked'.

Definition 2 Let $m, n \in \omega$. By a (m, n)-ary *specification* on $\mathcal{X}$ we mean a total function $\alpha_0 : F^{m+n+2} \to \{0, 1\}$ which is monotone with respect to the last 2 arguments in the following sense: for any $\bar{a} \in F^m$, any $\bar{b} \in F^n$, and any $c, c', d, d' \in F$,

$$\text{if } c \leqslant c' \text{ then } \alpha_0(\bar{a}, \bar{b}, c, d) \leqslant \alpha_0(\bar{a}, \bar{b}, c', d);$$
$$\text{if } d \leqslant d' \text{ then } \alpha_0(\bar{a}, \bar{b}, c, d') \leqslant \alpha_0(\bar{a}, \bar{b}, c, d).$$

Informally, $\bar{a}$ are finite arguments, $\bar{b}$ are finite fragments of (possibly infinite) arguments, c and c' are finite fragments of the resources we can use, while d and d' are finite fragments of the result of the process defined by this specification.

Definition 3 Let $\mathcal{X} = \langle X, F, \leqslant, \rangle$ be a p-domain. For $m, n \in \omega$, (m, n)*-ary process* on $\mathcal{X}$ is a partial mapping α from (a subset of) $F^m \times X^n$ to X such that there exists a specification $\alpha_0 : F^{m+n+1} \to \{0, 1\}$ which *defines* α in the following sense: for any $\bar{a} \in F^m$, $\bar{x} \in X^n$ such that $(\bar{a}, \bar{x}) \in \mathrm{dom}(\alpha)$, and any $d \in F$, $d \leqslant \alpha(\bar{a}, \bar{x})$ iff there exist $b_0, \ldots, b_{n-1}, c \in F$ such that $b_0 \leqslant x_0, \ldots, b_{n-1} \leqslant x_{n-1}$, and $\alpha_0(\bar{a}, \bar{b}, c, d) = 1$.

Let $n \in \omega$. We use the following terminology to denote the fact that a given process is either defined only on 'finite' arguments, or on arbitrary arguments approached only via approximations. Namely, for $n \in \omega$

– n-ary *functional* is a $(n, 0)$-ary process from (a subset of) F^n to X;
– n-ary *operator* is a $(0, n)$-ary process from (a subset of) X^n to X.

We denote classes of n-ary functionals and operators on $\mathcal{X}$ by $\mathcal{F}_n(\mathcal{X})$ and $\mathcal{O}_n(\mathcal{X})$, correspondingly, and the class of all processes on $\mathcal{X}$ is denoted by $\mathcal{P}(\mathcal{X})$.

Definition 4 1) *Termination* of a (partial) functional $\alpha : F^n \to X$ is a total function $\alpha_t : F^{n+1} \to \{0, 1\}$ defined as follows: for any $\bar{a} \in F^n$, $b \in F$,

$$\alpha_t(\bar{a}, b) = 1 \text{ iff } b \leqslant \alpha(\bar{a}).$$

2) *Termination* of a (partial) operator $\beta : X^n \to X$ is a total function $\beta_t : X^{n+1} \to \{0, 1\}$ defined as follows: for any $\bar{a} \in X^n$, $b \in X$,

$$\beta_t(\bar{a}, b) = 1 \text{ iff } b = \beta(\bar{a}).$$

3) *Termination* of a (partial) (m, n)-ary process $\gamma : F^m \times X^n \to X$, $n > 0$, is a total function $\beta_t : F^m \times X^{n+1} \to \{0, 1\}$ defined as follows: for any $\bar{a} \in F^m$, $\bar{b} \in X^n$, $c \in X$,

$$\gamma_t(\bar{a}, \bar{b}, c) = 1 \text{ iff } c = \gamma(\bar{a}, \bar{b}).$$

3 Computabilities and Reducibilities on Admissible Sets

Example 1. Let $\mathbb{A}$ be an admissible set. A p-domain $\mathcal{X}$ on $\mathbb{A}$ can be constructed in the following way. Let $X_0 \subseteq P(A)$ be an arbitrary Σ-admissible family in sense of [6], $X = A \cup X_0$, $F = A$, and $\leqslant \, = \, \subseteq_{X \setminus U^{\mathbb{A}}} \cup =_{U^{\mathbb{A}}}$. So, the topology on $X \cap P(A)$ is a strong topology from [6], and the topology on $U^{\mathbb{A}}$ is trivial. A class of processes can be taken as a suitable subclass of $\mathcal{P}(\mathcal{X})$ — the set of operators on $\mathbb{A}$ which are strongly continuous [6]. The set of all Σ-predicates and Σ-operators on $\mathbb{A}$ is, in fact, a subset of $\mathcal{P}(\mathcal{X})$ (and form a subclass of 'computable processes'), by the axiom of Δ_0-Collection and the definition of Σ-admissible family, respectively. Note that in case when $\mathbb{A} = \mathbb{HF}(\mathfrak{M})$ it is possible to take $X_0 = P(\mathrm{HF}(\mathrm{M}))$ [6].

On the other hand, we can first fix a subclass of computable processes $\mathcal{C} \subseteq \mathcal{P}(\mathcal{X})$ and then take $X_0 \subseteq P(A)$ to be a Σ-admissible family relative to $\mathcal{C}$. We will use the following technical modifications of some basic notions from [2, 6]. Let $\mathbb{A}$ be an admissible set.

1) A mapping $\alpha : A^n \to P(A)$ is called a *Σ-predicate on* $\mathbb{A}$ if there is a Σ-formula $\varphi_\alpha(x_1, \ldots, x_n, y)$ of signature $\sigma_{\mathbb{A}}$ (with no parameters from A) such that, for all $a_1, \ldots, a_n, b \in A$, $b \in \alpha(a_1, \ldots, a_n)$ iff $\mathbb{A} \models \varphi_\alpha(a_1, \ldots, a_n, b)$ (φ_α is called *Σ-specification*, or *Σ-presentation*, of α).

2) A mapping $\beta : P(A)^n \to P(A)$ is called a *Σ-operator on* $\mathbb{A}$ if there is a Σ-formula $\varphi_\beta(x_1,\dots,x_n,y)$ of signature $\sigma_\mathbb{A}$ (with no parameters from A) such that, for all $S_1,\dots,S_n \in P(A)$, $b \in A$

$$b \in \beta(S_1,\dots,S_n) \text{ iff } \exists a_1 \subseteq S_1,\dots,\exists a_n \subseteq S_n \text{ s.t. } \mathbb{A} \models \varphi_\beta(a_1,\dots,a_n,b)$$

(here it is assumed that $a_1,\dots,a_n \in A^*$). Again, φ_β is called *Σ-specification*, or *Σ-presentation*, of β.

We assume that if $\mathbb{A}$ is fixed, $\leqslant$ denotes $\subseteq_{P(A)} \cup =_{U^\mathbb{A}}$.

Definition 5 Let $\mathbb{A}$ be an admissible set, and let $m,n \in \omega$. Mapping γ from $A^m \times (A \cup P(A))^n$ to $P(A)$ is called a ((m,n)-ary) *Σ-process on* $\mathbb{A}$ if there is a Σ-formula $\varphi_\gamma(x_1,\dots,x_m,y_1,\dots,y_n,z)$ of signature $\sigma_\mathbb{A}$ (with no parameters from A) such that, for all $a_1,\dots,a_m \in A$, $x_1,\dots,x_m \in A \cup P(A)$, $c \in A$,

$$c \in \gamma(\bar a,\bar x) \text{ iff } \exists b_1 \leqslant x_1,\dots,\exists b_n \leqslant x_n \text{ s.t. } \mathbb{A} \models \varphi_\gamma(\bar a,\bar b,c).$$

Formula φ_γ is called *Σ-specification*, or *Σ-presentation*, of γ. The set of all Σ-presentations of a given process γ is denoted by $Pres_\Sigma(\gamma)$.

Any Σ-presentation $\varphi_\gamma(\bar x,\bar y,c)$ of a process γ can be transformed into the Δ_0-formula $\theta_\gamma(\bar x,\bar y,c,d)$ which is a specification of γ in the sense of Definitions 2, 3: take

$$\theta_\gamma(\bar x,\bar y,c,d) \leftrightharpoons (\forall c' \in c)\varphi_\gamma(\bar x,\bar y,c')^{(d)},$$

where $\varphi_\gamma(\bar x,\bar y,c')^{(d)}$ is the relativization of φ_γ to d [2].

We denote by $\mathcal{F}_\Sigma(\mathbb{A})$ the class of all Σ-predicates on $\mathbb{A}$, by $\mathcal{O}_\Sigma(\mathbb{A})$ the class of all Σ-operators on $\mathbb{A}$, and by $\mathcal{P}_\Sigma(\mathbb{A})$ the class of all Σ-processes on $\mathbb{A}$ (hence, $\mathcal{P}_\Sigma(\mathbb{A}) \supseteq \mathcal{F}_\Sigma(\mathbb{A}) \cup \mathcal{O}_\Sigma(\mathbb{A})$).

Definition 6 Let $\mathbb{A}$ be an admissible set and let $\mathcal{C} \subseteq \mathcal{P}_\Sigma(\mathbb{A})$ be a class of Σ-processes on $\mathbb{A}$. A family $\mathcal{S} \subseteq A \cup P(A)$ is called *Σ-admissible relative to $\mathcal{C}$* if

1) $\mathcal{S}$ is closed relative to processes from $\mathcal{C}$: for any (m,n)-ary process $\alpha \in \mathcal{C}$,

$$\forall a_1,\dots,a_m \in A \forall x_1,\dots,x_n \in \mathcal{S}\ \alpha(a_1,\dots,a_m,x_1,\dots,x_n) \in \mathcal{S};$$

2) processes from $\mathcal{C}$ are strongly continuous on elements from $\mathcal{S}$: for any (m,n)-process $\alpha \in \mathcal{C}$,

$$\forall a_1,\dots,a_m \in A \forall x_1,\dots,x_n \in \mathcal{S} \forall c \in A(c \leqslant \alpha(\bar a,\bar x) \to$$

$$\to \exists b_1 \in A \dots \exists b_n \in A(b_1 \leqslant x_1 \wedge \dots \wedge b_n \leqslant x_n \wedge c \leqslant \alpha(\bar a,\bar b))).$$

Definition 7 Let $\mathbb{A}$ be an admissible set. By a *computability component on* $\mathbb{A}$ we mean a pair $(\mathcal{S},\mathcal{C})$, where

1) $A \subseteq \mathcal{S} \subseteq P(A)$ is a Σ-admissible family relative to $\mathcal{C}$, and

2) $\mathcal{F}_\Sigma(\mathbb{A}) \subseteq \mathcal{C} \subseteq \mathcal{P}_\Sigma(\mathbb{A})$ is a class of Σ-processes on $\mathbb{A}$ which is closed under superposition.

For an admissible set $\mathbb{A}$, by *computability on* $\mathbb{A}$ we mean the family $\mathrm{Com}(\mathbb{A})$ of all computability components on $\mathbb{A}$:

$$\mathrm{Com}(\mathbb{A}) = \{(\mathcal{S},\mathcal{C}) \mid (\mathcal{S},\mathcal{C}) \text{ is a computability component on } \mathbb{A}\}.$$

Note that if, for a computability component $(\mathcal{S},\mathcal{C})$, it is true that $\mathcal{O}_\Sigma(\mathbb{A}) \subseteq \mathcal{C}$, then $\mathcal{S}$ should be a Σ-admissible family on $\mathbb{A}$ in the sense of [6].

To demonstrate the usefulness of these new notions, we prove a strengthening of the result of A.S.Morozov [8] which states that a certain reducibility on admissible sets implies an embedding of computable objects (i.e., Σ-predicates) on them.

The reducibility on admissible sets was defined by A.S.Morozov [8] as a modification of the notion of *Σ-definability* of a structure in an admissible set, introduced by Yu.L.Ershov.

Definition 8 (Yu.L.Ershov [4, 6]) Let $\mathfrak{M}$ be a structure of computable predicate signature $\langle P_0^{n_0}, P_1^{n_1}, \ldots\rangle$ and let $\mathbb{A}$ be an admissible set. $\mathfrak{M}$ is *Σ-definable in* $\mathbb{A}$ if there exist a computable sequence of Σ-formulas

$$\Phi(x_0, y), \Phi_=(x_0, x_1, y), \Psi_=(x_0, x_1, y), \Phi_0(x_0, \ldots, x_{n_0-1}, y),$$

$$\Psi_0(x_0, \ldots, x_{n_0-1}, y), \Phi_1(x_0, \ldots, x_{n_1-1}, y), \Psi_1(x_0, \ldots, x_{n_1-1}, y), \ldots,$$

such that for some parameter $a \in A$, and letting

$$M_0 \leftrightharpoons \Phi^{\mathbb{A}}(x_0, a), \ \ \eta \leftrightharpoons \Phi_=^{\mathbb{A}}(x_0, x_1, a) \cap M_0^2$$

one has that $M_0 \neq \varnothing$ and η is a congruence relation on the structure

$$\mathfrak{M}_0 \leftrightharpoons \langle M_0, P_0^{\mathfrak{M}_0}, P_1^{\mathfrak{M}_0}, \ldots\rangle,$$

where $P_i^{\mathfrak{M}_0} \leftrightharpoons \Phi_i^{\mathbb{A}}(x_0, \ldots, x_{n_i-1}) \cap M_0^{n_i}$ for all i,

$$\Psi_=^{\mathbb{A}}(x_0, x_1, a) \cap M_0^2 = M_0^2 \setminus \Phi_=^{\mathbb{A}}(x_0, x_1, a),$$
$$\Psi_i^{\mathbb{A}}(x_0, \ldots, x_{n_i-1}, a) \cap M_0^{n_i} = M_0^{n_i} \setminus \Phi_i^{\mathbb{A}}(x_0, \ldots, x_{n_i-1})$$

for all i, and the structure $\mathfrak{M}$ is isomorphic to the quotient structure $\mathfrak{M}_0 / \eta$.

A structure Σ-definable in $\mathbb{A}$ is called *$\mathbb{A}$-constructivizable*. The relation of *Σ-reducibility* $\leqslant_\Sigma$ on (types of isomorphism of) structures was also defined using this notion. Namely, for structures $\mathfrak{M}$ and $\mathfrak{N}$, we denote by $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ the fact that $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$. This relation is reflexive and transitive, and the corresponding notion of Σ-degree gives a natural measure of complexity for structures of arbitrary cardinalities, see [16–19].

We will also need a ‘positive’ version of Σ-definability: for a structure $\mathfrak{M}$ and an admissible set $\mathbb{A}$, $\mathfrak{M}$ is *Σ^+-definable in* $\mathbb{A}$ if there exist a computable sequence of Σ-formulas $\Phi(x_0, y), \Phi_0(x_0, \dots, x_{n_0-1}, y), \Phi_1(x_0, \dots, x_{n_1-1}, y), \dots$ such that, for some parameter $a \in A$ and an onto mapping $\nu : \Phi^{\mathbb{A}}(x_0, a) \twoheadrightarrow M$, for every $i \in \omega$ and every $a_0, \dots, a_{n_i-1} \in \Phi^{\mathbb{A}}(x_0, a)$,

$$\mathbb{A} \models \Phi_i(a_0, \dots, a_{n_i-1}, a) \iff \mathfrak{M} \models P_i(\nu(a_0), \dots, \nu(a_{n_i-1})).$$

Again, for structures $\mathfrak{M}$ and $\mathfrak{N}$, we denote by $\mathfrak{M} \leqslant^+_\Sigma \mathfrak{N}$ the fact that $\mathfrak{M}$ is Σ^+-definable in $\mathbb{HF}(\mathfrak{N})$. It should be noted, however, that $\leqslant^+_\Sigma$ is transitive only in case when all structures are treated 'positively' in the sense that their atomic diagrams are not necessarily closed under negations.

For a structure with an infinite computable signature, we assume that some Gödel numbering of formulas of this signature is fixed. We assume that the signature of $\mathbb{HF}(\mathfrak{B})$ contains a predicate symbol Sat^2 interpreted by the satisfiability relation for atomic formulas in $\mathfrak{B}$, with respect to a fixed Gödel numbering. In the case of structures with a finite signature this assumption is not essential.

The next definition is a technical modification of the original one (it was used in this form in [9]).

Definition 9 (A.S. Morozov [8]) Let $\mathbb{A}$ and $\mathbb{B}$ be admissible sets. $\mathbb{A}$ *is Σ-reducible to* $\mathbb{B}$ (denoted $\mathbb{A} \sqsubseteq_\Sigma \mathbb{B}$) if there is an onto mapping $\nu : B \twoheadrightarrow A$ such that

1) ν is a $\mathbb{B}$-constructivization of $\mathbb{A}$ as a structure in sense of [4, 6];
2) there is a binary Σ-predicate E on $\mathbb{B}$ s.t. $pr_1(E) = B$ and, for all $b, c \in B$,

$$\langle b, c\rangle \in E \text{ implies } \nu(b) = \{\nu(z) | z \in c\}.$$

Definition 10 If, for admissible sets $\mathbb{A}, \mathbb{B}$, there exist mappings $\nu : B \twoheadrightarrow A$ and $\mu : Pres_\Sigma(\mathcal{P}_\Sigma(\mathbb{A})) \to Pres_\Sigma(\mathcal{P}_\Sigma(\mathbb{B}))$ such that μ is computable and, for every $(\mathcal{S}, \mathcal{C}) \in \mathrm{Com}(\mathbb{A})$, there exists $(\mathcal{S}', \mathcal{C}') \in \mathrm{Com}(\mathbb{B})$ such that

$$(\nu^{-1}(\mathcal{S}), \mu(Pres(\mathcal{C}))) \text{ is isomorphic to } (\mathcal{S}', \mathcal{C}'),$$

we say that $\mathrm{Com}(\mathbb{A})$ *is Σ-embeddable into* $\mathrm{Com}(\mathbb{B})$.

Theorem 1 Let $\mathbb{A}, \mathbb{B}$ be admissible sets. If $\mathbb{A} \sqsubseteq_\Sigma \mathbb{B}$ then $\mathrm{Com}(\mathbb{A})$ is Σ-embeddable into $\mathrm{Com}(\mathbb{B})$.

Proof. We prove that if $\mathbb{A}$ is Σ-reducible to $\mathbb{B}$ then Σ-processes on $\mathbb{A}$ are represented, in an effective and uniform way, by Σ-processes on $\mathbb{B}$ working with the names of elements from A. So, the result is somethat similar to one on relationship between Σ-degrees and degrees of presentability of structures.

We present a uniform effective procedure which transform any Σ-specification of a Σ-process on $\mathbb{A}$ into some Σ-specification of Σ-process on $\mathbb{B}$ representing the first one. In a standard way, we define an effective uniform transformation $\Phi(\bar{x}, \bar{a}) \mapsto \Phi^*(\bar{x}, \bar{b})$ of Σ-formulas of signature $\sigma_{\mathbb{A}}$ with parameters $\bar{a}$ from A, to Σ-formulas of signature $\sigma_{\mathbb{B}}$ with parameters $\bar{b}$ from B, by induction on the complexity:

- $(P(\bar{x}))^* = (\Phi(x_0) \wedge \ldots \wedge \Phi(x_{n-1}) \wedge \Phi_P(\bar{x}))$, $P^n \in \sigma_{\mathbb{A}}$;
- $(\neg P(\bar{x}))^* = (\Phi(x_0) \wedge \ldots \wedge \Phi(x_{n-1}) \wedge \Psi_P(\bar{x}))$, $P^n \in \sigma_{\mathbb{A}}$;
- $((\exists x \in y)\Theta)^* = (\Phi(y) \wedge \exists x(\Phi_{\in}(x, y) \wedge \Theta^*))$;
- $((\forall x \in y)\Theta)^* = (\Phi(y) \wedge \exists z((\langle y, z\rangle \in E) \wedge ((\forall v \in z)\Theta^*))$;
- $(\Theta_1 \circ \Theta_2)^* = (\Theta_1^* \circ \Theta_2^*)$, $\circ \in \{\wedge, \vee\}$,

and so on. Now, any Σ-specification Φ_α of (m, n)-ary Σ-process α on $\mathbb{A}$ is mapped to a Σ-formula $(\Phi_\alpha)^*$ which is a Σ-specification of (m, n)-ary Σ-process α^* on $\mathbb{B}$ such that, if $\langle S_0, \ldots, S_{n-1}\rangle \in \delta_c(\alpha)$ (δ_c denotes the domain of strong continuity), then $\langle \nu^{-1}(S_0), \ldots, \nu^{-1}(S_{n-1})\rangle \in \delta_c(\alpha^*)$ and

$$\nu^{-1}(\alpha(S_0, \ldots, S_{n-1})) = \alpha^*(\nu^{-1}(S_0), \ldots, \nu^{-1}(S_{n-1})).$$

Hence, defining mapping μ on Σ-processes as $\mu(p) = (p)^*$, the pair (ν^{-1}, μ) establish the desired isomorphism.

4 Jumps of Computabilities: Σ-Jump of a Structure as the Jump of the Minimal Component of HF-Computability

Definition 11 Let $\mathbb{A}$ be an admissible set, and let $(\mathcal{S}, \mathcal{C})$ be a computability component on $\mathbb{A}$. The *jump of* $(\mathcal{S}, \mathcal{C})$ is the structure $J_{\mathbb{A}}(\mathcal{S}, \mathcal{C})$ with $\mathcal{S}$ as the domain and the atomic diagram consisting of unary predicate distinguishing the set A of finite objects, and the predicates distinguishing terminations $t(\mathcal{C})$ of processes from $\mathcal{C}$ (given by their Σ-presentations in some fixed Gødel numbering).

This extends in a natural way all existing definitions of jump operations defined on subsets of natural numbers or on structures. Indeed, in the last case, we use the fact that every structure generates the least admissible set containing it — its HF-superstructure. If we take the least computability component on that HF-superstructure and terminate all its processes (i.e., all Σ-predicates), we get the structure which is called *Σ-jump* of the original one. The formal definition is as follows:

Definition 12 Let $\mathfrak{A}$ be a structure. By *Σ-jump*, or *minimal Σ-jump*, of $\mathfrak{A}$, we mean the structure

$$\mathfrak{A}' = (X; F, \mathcal{T}),$$

with the domain $X = HF(A)$, and relations $F = HF(A)$ (domain consists of finite objects only, so the unary relation F is trivial in this case and usually skipped), and $\mathcal{T} = t(\mathcal{F}_\Sigma(\mathbb{HF}(\mathfrak{A})))$ as the termination of all Σ-predicates on $\mathbb{HF}(\mathfrak{A})$ (denoted here, as in [18, 19], by $\Sigma\text{-Sat}_{\mathbb{HF}(\mathfrak{A})}$).

In a similar way the jump operation was introduced in [1] for the semilattice of s-degrees of countable structures. Also, in the same way a notion of the jump of an admissible set with respect to various effective reducibilities was introduced in [8, 9].

It should be noted, however, that, to make Definition 12 a special case of Definition 11, one should formally extend the domain X of $\mathfrak{A}'$ by adding all Σ-definable relations (given as elements). But it is easy to prove that this formalization is Σ-equivalent to the one in Definition 12.

The operation of Σ-jump agrees with the jump operations for Turing and enumeration degrees w.r.t. the natural embeddings i and j: if a structure $\mathfrak{A}$ has a (e-)degree $\mathbf{a}$, then the structure $\mathfrak{A}'$ has (e-)degree $\mathbf{a}'$. In fact, it is true that the mappings $i : \mathcal{D} \to \mathcal{S}_\Sigma$ and $j : \mathcal{D}_e \to \mathcal{S}_\Sigma$ are embeddings preserving 0, $\vee$ and the jump operation (see [16, 19] for details). Hence, the operation of Σ-jump is a natural extension of Turing and enumeration jumps. One of the important facts about the minimal HF-computability is that the jump inversion theorem from classical computability theory is still true in this general setting.

Theorem 2 ([18, 19]) Let $\mathfrak{A}$ be a structure such that $\mathbf{0}' \leqslant_\Sigma \mathfrak{A}$. Then there exists a structure $\mathfrak{B}$ such that

$$\mathfrak{B}' \equiv_\Sigma \mathfrak{A}.$$

5 Jumps of Maximal Components of HF-Computabilities: $P\Sigma$-Jump of 0 and the Reals

Definition 13 Let $\mathfrak{A}$ be a structure. By *$P\Sigma$-jump*, or *maximal Σ-jump*, of $\mathfrak{A}$, we mean the structure

$$\mathfrak{A}^\diamond = (X; F, \mathcal{T}),$$

with the domain $X = HF(A) \cup P(HF(A))$, and the atomic diagram consisting of relations $F = HF(A)$ distinguishing finite objects, and $\mathcal{T}$ as the termination of all Σ-processes on $\mathbb{HF}(\mathfrak{A})$.

Lemma 1 *Relations $\in$ and $\subseteq$ between elements of the sets F and X are obtained as terminations of Σ-processes which act on X and depend on a in an effective and uniform way.*

Proof. Indeed, for arbitrary $a \in A$, consider the following unary Σ-operators $F_a^\in$ and $F_a^\subseteq$ with a as parameter: for every $S \in X$, let

$$F_a^\in(S) = \{1 \mid (\mathbb{A}, S) \models (\exists b \subseteq S)(b = \{a\})\};$$

$$F_a^\subseteq(S) = \{1 \mid (\mathbb{A}, S) \models (\exists b \subseteq S)(b = a)\}$$

It is easy to note that $P\Sigma$-jump is indeed a jump with respect to $\leqslant_\Sigma$, because immediately from cardinality reasons we get that, for any structure $\mathfrak{A}$,

$$\mathfrak{A} <_\Sigma \mathfrak{A}^\diamond.$$

A natural question is an analogue of Jump Inversion Theorem for $P\Sigma$-jump. We start from investigating the Σ-degree of $0^\diamond$.

Definition 14 Let $\mathbb{R}$ denote the set of real numbers. We consider the following structures:

1) algebraical field of reals $\mathcal{R} = (\mathbb{R}, +, \times, 0, 1, =)$;
2) topological field of reals

$$\mathcal{R}_o = (\mathbb{R}, \Gamma_+^A, \Gamma_+^B, \Gamma_\times^A, \Gamma_\times^B, 0, 1, <),$$

where $\Gamma_+^A = \{\langle x, y, z\rangle \in \mathbb{R}^3 | x + y < z\}$, $\Gamma_+^B = \{\langle x, y, z\rangle \in \mathbb{R}^3 | z < x + y\}$ (similar definitions for $\Gamma_\times^A$, $\Gamma_\times^B$).

Theorem 3 $\mathcal{R} \leqslant_\Sigma (0^\diamond)'$.

Proof. We define a Σ-presentation of $\mathcal{R}$ in $(0^\diamond)' = (\mathbb{HF}(0^\diamond), \Sigma\text{-Sat}_{\mathbb{HF}(0^\diamond)})$ as follows. Take as the domain the set

$$R = \{\langle k, m, \alpha\rangle | k \in \{-1, 0, 1\}, m \in \omega, \alpha \in Fun(\omega, 2)\},$$

where $Fun(\omega, 2)$ is the set of total functions from ω to $2 = \{0, 1\}$, and each triple $x = \langle k, m, \alpha\rangle$ represents the real number

$$r_x = k(m + \sum_{n\in\omega} \frac{\alpha(n)}{2^{n+1}}).$$

Lemma 2 R is Σ-definable in $(\mathbb{HF}(0^\diamond), \Sigma\text{-Sat}_{\mathbb{HF}(0^\diamond)})$.

Proof. For arbitrary $S \in HF(0^\diamond)$, $S \in Fun(\omega, 2)$ if and only if $\exists X(S = F(X)) \wedge \Phi(S)$, where Σ-operator F on $\mathbb{HF}(\varnothing)$ is defined as follows: for any $X \subseteq HF(\varnothing)$,

$$F(X) = \{y | \exists a \subseteq X \exists n \exists k[(a = \{y\}) \wedge (y = \langle n, k\rangle) \wedge Nat(n) \wedge (k \in 2)]\},$$

and

$$\Phi(S) = \forall n(Nat(n) \to (((\langle n, 0\rangle \in S) \wedge (\langle n, 1\rangle \notin S)) \vee ((\langle n, 1\rangle \in S) \wedge (\langle n, 0\rangle \notin S))).$$

Since $Fn(S)$ is a Π-formula in $\mathbb{HF}(0^\diamond)$, R is Σ-definable in $(\mathbb{HF}(0^\diamond), \Sigma\text{-Sat}_{\mathbb{HF}(0^\diamond)})$.

Theorem 4 $\mathcal{R}_o \leqslant_\Sigma^+ 0^\diamond$.

Proof. We take the set

$$R_o = \{\langle k, m, S\rangle | k \in \{-1, 0, 1\}, m \in \omega, S \subseteq \mathbb{HF}(\varnothing)\},$$

as the domain of the presentation. It is easy to note that the cardinality of the presentation of $\mathcal{R}_0$ is the same as the cardinality of $\mathcal{R}_0$, which is not necessary for structures with no equality. The proof follows from the following lemmas.

Lemma 3 For any set $S \subseteq \mathbb{HF}(\varnothing)$, the following sets could be obtained as the results of some Σ-operators acting on S:

1) $S \cap \omega$;
2) $S \cap n \,(= S \cap \{0, \ldots, n-1\})$.

In particular, in case 2 the corresponding Σ-operator depends on n in uniform and effective way.

Proof. 1) It is enough to note that $S \cap \omega = F_0(S)$, where

$$F_0(S) = \{y \in \mathrm{HF}(\varnothing) \,|\, (\mathbb{HF}(\varnothing), \mathrm{S}) \models (\exists \mathrm{a} \subseteq \mathrm{S})(\exists \mathrm{x})(\mathrm{a} = \{\mathrm{x}\}) \wedge \mathrm{Nat}(\mathrm{x})\}.$$

2) In the same way, for any $n > 0$, $S \cap n = F_n(S)$, where

$$F_n(S) = \{y \in \mathrm{HF}(\varnothing) \,|\, (\mathbb{HF}(\varnothing), \mathrm{S}) \models (\exists \mathrm{a} \subseteq \mathrm{S})(\exists \mathrm{x})(\mathrm{a} = \{\mathrm{x}\}) \wedge (\mathrm{x} \in \mathrm{n})\}.$$

Lemma 4 The strict order relation $\{\langle x_1, x_2 \rangle \in R_o^2 \,|\, r_{x_1} < r_{x_2}\}$ is Σ-definable in $\mathbb{HF}(0^\diamond)$.

Proof. 1) Consider, for example, the case $x_1 = \langle 0, 1, S_1 \rangle$, $x_2 = \langle 0, 1, S_2 \rangle$. By Lemma 3, functions $\alpha_i : \omega \to \{0, 1\}$ such that $S_i = \{n \in \omega | \alpha_i = 1\}$, are Σ-definable in $\mathbb{HF}(0^\diamond)$. Since $r_{x_1} < r_{x_2}$ mean

$$\sum_{n \in \omega} \frac{\alpha_1(n)}{2^{n+1}} < \sum_{n \in \omega} \frac{\alpha_2(n)}{2^{n+1}},$$

it is equivalent to say there exists $n_0 \in \omega$ such that

$$\sum_{n < n_0} \frac{\alpha_1(n)}{2^{n+1}} - \sum_{n < n_0} \frac{\alpha_2(n)}{2^{n+1}} > \frac{1}{2^{n_0 - 1}}.$$

Again by Lemma 3, this condition can be defined in $\mathbb{HF}(0^\diamond)$ by a Σ-formula.

2) General case $x_1 = \langle k_1, m_1, S_1 \rangle$, $x_2 = \langle k, m_1, S_2 \rangle$ is considered in the same way: $r_{x_1} < r_{x_2}$ means that

$$(k_2 m_2 - k_1 m_1) + (k_2 \sum_{n \in \omega} \frac{\alpha_2(n)}{2^{n+1}} - k_1 \sum_{n \in \omega} \frac{\alpha_1(n)}{2^{n+1}}) > 0.$$

Depending on k_i and m_i, this condition is either trivially checked or reduced to a strict inequality between series as in the first case.

In the same way it can be proved that relations $\Gamma_+^A, \Gamma_+^B, \Gamma_\times^A, \Gamma_\times^B$ are all Σ-definable in $\mathbb{HF}(0^\diamond)$.

6 Open Questions

1. What is an analogue of Jump Inversion for a given computability component of HF-computability over 0 or any given structure?
2. What is an analogue of Jump Inversion for a given computability component of $\mathbb{A}$-computability? This question is especially interesting for the least computability component of $\mathbb{HYP}(\mathfrak{M})$-computability.
3. Is $0^\diamond \leqslant_\Sigma^+ \mathcal{R}_0$? This would mean that in the maximal component of HF-computability over 0 holds an analogue of the Matijasevich Theorem. Also, is it natural to ask, whether or not $(0^\diamond)' \leqslant_\Sigma \mathcal{R}$.

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