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DEGREES OF PRESENTABILITY OF STRUCTURES. I

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Presentations of structures in admissible sets, as well as different relations of effective reducibility between the structures, are treated. Semilattices of degrees of Σ-definability are the main object of investigation. It is shown that the semilattice of degrees of Σ-definability of countable structures agrees well with semilattices of T- and e-degrees of subsets of natural numbers. Also an attempt is made to study properties of the structures that are inherited under various effective reducibilities and explore how degrees of presentability depend on choices of different admissible sets as domains for presentations.

In the paper we deal with presentations of structures in admissible sets, and also with different effective reducibility relations between the structures. Semilattices of degrees of Σ-definability (Ershov semilattices) are the main object of our investigation. This concept can be viewed, on the one hand, as a natural generalization of oracle definability, that is, when a complex abstract object — a structure — plays the role of an oracle and of a result of computations. (The given approach can be conceived of as a theoretical model of object-oriented programming.) On the other hand, the concept of Σ-definability of a structure in an admissible set is an effectivization of one of the main notions in model theory, that of interpretability of one structure in another, and moreover, it generalizes the concept of constructivizability of structures on natural numbers.

It will be shown that the semilattice of degrees of Σ-definability of countable structures agrees well with semilattices of T- and e-degrees of subsets of natural numbers. The concept of a structure having a degree, which is known in constructive model theory, is just a partial characteristic of complexity, since by no means all structures can have degrees. As distinct from this, degrees of Σ-definability, as well as the degrees of presentability of relatively distinct uniform and non-uniform effective reducibilities treated in the paper, are natural characteristics of complexity defined for any structure. We also make an attempt to study properties of the structures that are inherited under various effective reducibilities and look at how degrees of presentability depend on choices of different admissible sets as domains for presentations.

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1. SEMILATTICES OF DEGREES OF Σ-DEFINABILITY AND OF DEGREES OF PRESENTABILITY

The bulk of the notation as well as the terminology used in the paper are standard and are borrowed from [1, 2]. The domain of a structure $\mathfrak{M}$ is denoted by M; its signature, by $\sigma_{\mathfrak{M}}$. Hereinafter, we deal with structures having computable signatures. For any structure $\mathfrak{M}$ of the signature $\sigma_{\mathfrak{M}} = \langle P_0^{n_0}, P_1^{n_1}, \ldots\rangle$, a *hereditarily finite superstructure* $\mathbb{HF}(\mathfrak{M})$, which is the least admissible set containing M as a subset, is defined to be a structure of the signature $\sigma'_{\mathfrak{M}} = \sigma_{\mathfrak{M}} \cup \{U^1, \in^2, \mathrm{Sat}^2\}$ with domain $HF(M) = \bigcup\limits_{n\in\omega} H_n(M)$, where $H_0(M) = M$ and $H_{n+1}(M) = H_n(M) \cup \{a \mid a \subseteq H_n(M), \mathrm{card}(a) < \omega\}$; the predicate U distinguishes the set M (whose members are called *urelements*); the relation $\in$ is interpreted in a standard manner; an interpretation of the predicate Sat is the set $\{\langle k, \bar{m}\rangle \mid \mathfrak{M} \models P_k(\bar{m})\}$. We observe that the necessity of using Sat is caused by the fact that, as distinct from standard approaches [1, 2], the signature $\sigma_{\mathfrak{M}}$ may be infinite. For the case of a finite signature, this distinction is not essential.

In the class of formulas of the signature $\sigma'_{\mathfrak{M}}$, we distinguish a subclass of Δ_0-*formulas*, which is defined to be the closure of a class of atomic formulas w.r.t. $\wedge, \vee, \neg, \rightarrow$ and bounded quantifiers $\exists x \in y, \forall x \in y$; a class of Σ-*formulas* is defined to be the closure of the class of Δ_0-formulas w.r.t. $\wedge, \vee, \exists x \in y, \forall x \in y$ and the quantifier $\exists x$; a class of Π-*formulas* is defined similarly: we admit of use of the quantifier $\forall x$ instead of $\exists x$. A relation on $\mathbb{HF}(\mathfrak{M})$ is said to be Σ-*definable* (Π-*definable*) if it is defined by the relevant formula with parameters; the relation is Δ-*definable* if it is simultaneously Σ- and Π-definable.

For simplicity, let $\mathfrak{M}$ be a structure of finite predicate signature $\langle P_0^{n_0}, \ldots, P_{k-1}^{n_{k-1}}\rangle$ (this constraint is not essential) and $\mathbb{A}$ be an admissible set.

Definition 1 [1]. A structure $\mathfrak{M}$ is Σ-*definable* in $\mathbb{A}$ if there are Σ-formulas

$$\Phi(x_0, y),\ \Psi(x_0, x_1, y),\ \Psi^*(x_0, x_1, y),\ \Phi_0(x_0, \ldots, x_{n_0-1}, y),$$

$$\Phi_0^*(x_0, \ldots, x_{n_0-1}, y),\ \ldots,\ \Phi_{k-1}(x_0, \ldots, x_{n_{k-1}-1}, y),\ \Phi_{k-1}^*(x_0, \ldots, x_{n_{k-1}-1}, y)$$

of signature $\sigma_{\mathbb{A}}$ and a parameter $a \in A$ such that for $M_0 \leftrightharpoons \Phi^{\mathbb{A}}(x_0, a)$ and $\eta \leftrightharpoons \Psi^{\mathbb{A}}(x_0, x_1, a) \cap M_0^2$, the following hold: $M_0 \neq \varnothing$ and η is a congruence relation on the structure

$$\mathfrak{M}_0 \leftrightharpoons \langle M_0; P_0^{\mathfrak{M}_0}, \ldots, P_{k-1}^{\mathfrak{M}_0}\rangle,$$

where $P_i^{\mathfrak{M}_0} \leftrightharpoons \Phi_i^{\mathbb{A}}(x_0, \ldots, x_{n_i-1}) \cap M_0^{n_i}$ for all $i < k$, $\Psi^{*\mathbb{A}}(x_0, x_1, a) \cap M_0^2 = M_0^2 \setminus \Psi^{\mathbb{A}}(x_0, x_1, a)$, $\Phi_i^{*\mathbb{A}}(x_0, \ldots, x_{n_i-1}, a) \cap M_0^{n_i} = M_0^{n_i} \setminus \Phi_i^{\mathbb{A}}(x_0, \ldots, x_{n_i-1})$ for all $i < k$, and the structures $\mathfrak{M}$ and $\mathfrak{M}_0/\eta$ are isomorphic.

For structures $\mathfrak{M}$ and $\mathfrak{N}$, writing $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ signifies that $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$. It is easy to verify that the relation $\leqslant_\Sigma$ is reflexive and transitive. For any infinite cardinal α, $\mathcal{K}_\alpha$ denotes the class of structures of cardinality at most α. On $\mathcal{K}_\alpha$, an equivalence relation $\equiv_\Sigma$ is defined as follows: for $\mathfrak{M}, \mathfrak{N} \in \mathcal{K}_\alpha$, put $\mathfrak{M} \equiv_\Sigma \mathfrak{N}$ if $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ and $\mathfrak{N} \leqslant_\Sigma \mathfrak{M}$. We call the equivalence classes w.r.t. $\equiv_\Sigma$ *degrees of* Σ-*definability.*

The structure

$$\mathcal{S}_\Sigma(\alpha) = \langle \mathcal{K}_\alpha / \equiv_\Sigma, \leqslant_\Sigma\rangle$$

is an upper semilattice with a least element, which is a degree consisting of constructivizable structures, and for any $\mathfrak{M}, \mathfrak{N} \in \mathcal{K}_\alpha$, $[\mathfrak{M}]_\Sigma \vee [\mathfrak{N}]_\Sigma = [(\mathfrak{M}, \mathfrak{N})]_\Sigma$, where $(\mathfrak{M}, \mathfrak{N})$ is a model-theoretic pair of the structures $\mathfrak{M}$ and $\mathfrak{N}$. For brevity, we denote the semilattice $\mathcal{S}_\Sigma(\omega)$ by $\mathcal{S}_\Sigma$. The fact that $\mathrm{card}(\mathcal{K}_\alpha) = 2^\alpha$ and $\mathrm{card}([\mathfrak{M}]_\Sigma) \leqslant \alpha$ for any $\mathfrak{M} \in \mathcal{K}_\alpha$ implies that $\mathrm{card}(\mathcal{S}_\Sigma(\alpha)) = 2^\alpha$ for any infinite cardinal α.

A *presentation* of a structure $\mathfrak{M}$ in an admissible set $\mathbb{A}$ is any structure $\mathcal{C}$ which is isomorphic to $\mathfrak{M}$ and whose domain C is a subset of A ($=$ is treated as a congruence relation on $\mathcal{C}$, and it may differ from the standard equality relation on C). In what follows, we will identify the presentation $\mathcal{C}$ (more precisely, its atomic diagram) with some subset of A, fixing a Gödel numbering of atomic formulas of the signature $\sigma_{\mathfrak{M}}$.

Definition 2. The *problem of presentability* for a structure $\mathfrak{M}$ in $\mathbb{A}$ is the set $\mathrm{Pr}(\mathfrak{M}, \mathbb{A})$ consisting of all possible presentations of $\mathfrak{M}$ in $\mathbb{A}$: that is,

$$\mathrm{Pr}(\mathfrak{M}, \mathbb{A}) = \{\mathcal{C} \mid \mathcal{C} \text{ is a presentation of } \mathfrak{M} \text{ in } \mathbb{A}\}.$$

Denote by $\underline{\mathfrak{M}}$ the set $\mathrm{Pr}(\mathfrak{M}, \mathbb{HF}(\varnothing))$ of presentations of the structure $\mathfrak{M}$ in the least admissible set. It is well known that computability (i.e., effective definability) in $\mathbb{HF}(\varnothing)$ is equivalent to classical computability on natural numbers (see [1, 2]).

Let $\mathbb{A}$ be an admissible set. A mapping $F : P(A)^n \to P(A)$ ($n \in \omega$) is called a Σ-*operator* [1] if there exists a Σ-formula $\Phi(x_0, \ldots, x_{n-1}, y)$ of the signature $\sigma_{\mathbb{A}}$ such that for any $S_0, \ldots, S_{n-1} \in P(A)$,

$$F(S_0, \ldots, S_{n-1}) = \left\{ a \,\middle|\, \exists a_0, \ldots, a_{n-1} \in A \left(\bigwedge_{i<n} a_i \subseteq S_i \wedge \mathbb{A} \models \Phi(a_0, \ldots, a_{n-1}, a) \right) \right\}.$$

Now we specify a condition that is necessary for the reducibilities defined below to be transitive. An operator $F : P(A) \to P(A)$ is *strongly continuous in* $S \in P(A)$ if, for any $a \subseteq F(S)$, $a \in A$, there is $a' \subseteq S$, $a' \in A$, such that $a \subseteq F(a')$. (This definition is readily generalized to the case of operators in which the number of arguments is more than one.)

For an operator $F : P(A)^n \to P(A)$, by $\delta_c(F)$ we denote the set of elements of $P(A)^n$ in which F is strongly continuous. A set $S \in P(A)^n$ is called a Σ_*-*set* if $S \in \delta_c(F)$ for any Σ-operator $F : P(A)^n \to P(A)$. It is easy to verify that every subset of an admissible set of the form $\mathbb{HF}(\mathfrak{M})$ is Σ_*. In the general case, however, this is no longer so: in [3], for instance, Σ_*-sets in $\mathbb{HYP}(\mathbb{L})$ were examined where $\mathbb{L}$ is a dense linear order. Even in this elementary case, note, the class of Σ_*-sets is non-trivial.

Let $B, C \subseteq A$. Below are reducibilities that are direct generalizations of e- and T-reducibilities on natural numbers:

(1) $B \leqslant_{e\Sigma} C$, if there is a unary Σ-operator F for which $C \in \delta_c(F)$ and $B = F(C)$;

(2) $B \leqslant_{T\Sigma} C$, if there are binary Σ-operators F_0 and F_1 such that $\langle C, A \setminus C\rangle \in \delta_c(F_0) \cap \delta_c(F_1)$, $B = F_0(C, A \setminus C)$, and $A \setminus B = F_1(C, A \setminus C)$.

Definition 3. Let $\mathfrak{M}$ be a structure and $\mathbb{A}$ an admissible set with $\mathrm{card}(A) \geqslant \mathrm{card}(M)$. We say that $\mathfrak{M}$ has *degree* (e-*degree*) $\mathbf{d}$ in $\mathbb{A}$ if $\mathbf{d}$ is least in the set of $T\Sigma$-degrees ($e\Sigma$-degrees) of all possible presentations of $\mathfrak{M}$ in $\mathbb{A}$; that is, $\mathbf{d} = \min\{\deg_{T\Sigma}(\mathcal{C}) \mid \mathcal{C} \in \mathrm{Pr}(\mathfrak{M}, \mathbb{A})\}$ ($\mathbf{d} = \min\{\deg_{e\Sigma}(\mathcal{C}) \mid \mathcal{C} \in \mathrm{Pr}(\mathfrak{M}, \mathbb{A})\}$, resp.).

We say that a countable structure $\mathfrak{M}$ has a degree (an e-degree) if $\mathfrak{M}$ has a degree in the least admissible set $\mathbb{HF}(\varnothing)$. The concept of a structure having a degree was first introduced in [4], where only T-degrees were dealt with and by presentations were meant just those on natural numbers, with domain ω. Yet, it is easy to verify that for every structure $\mathfrak{M}$ and for any presentation $\mathcal{C} \in \underline{\mathfrak{M}}$, there is a presentation $\mathcal{C}' \in \underline{\mathfrak{M}}$ with domain ω such that $\mathcal{C}' \leqslant_{T\Sigma} \mathcal{C}$. For the presentations in $\mathbb{HF}(\varnothing)$, therefore, the definition above is equivalent to that in [4].

Below is a theorem which can be proved using the classical result saying that $\forall$-computability and $\exists$-definability are equivalent (first noted in [5] and proved in [6] and subsequently re-proved and generalized in [7]). Later, we will establish yet another generalization of this result (Thm. 6) to the case of presentations in superstructures over countable structures.

THEOREM 1. A countable structure $\mathfrak{M}$ has a degree (an e-degree) if and only if some presentation $\mathcal{C} \in \underline{\mathfrak{M}}$ is Δ-definable (Σ-definable) in $\mathbb{HF}(\mathfrak{M})$.

There are natural embeddings of the semilattice $\mathcal{D}$ of Turing degrees and of the semilattice $\mathcal{D}_e$ of degrees of enumerability in the semilattice $\mathcal{S}_\Sigma$ (and hence in any semilattice like $\mathcal{S}_\Sigma(\alpha)$). We define mappings $i : \mathcal{D} \to \mathcal{S}_\Sigma$ and $j : \mathcal{D}_e \to \mathcal{S}_\Sigma$ as follows: for every degree $\mathbf{a} \in \mathcal{D}$, put

$$i(\mathbf{a}) = [\mathfrak{M}_\mathbf{a}]_\Sigma, \text{ where } \mathfrak{M}_\mathbf{a} \text{ is any structure having degree } \mathbf{a};$$

similarly, for every e-degree $\mathbf{b} \in \mathcal{D}_e$, put

$$j(\mathbf{b}) = [\mathfrak{M}_\mathbf{b}]_\Sigma, \text{ where } \mathfrak{M}_\mathbf{b} \text{ is any structure having } e\text{-degree } \mathbf{b}.$$

LEMMA 1. The mappings i and j are well defined: for any (e-)degree $\mathbf{a}$, there are structures having (e-)degree $\mathbf{a}$. Moreover, for any countable structures $\mathfrak{M}$ and $\mathfrak{N}$, if $\mathfrak{M}$ has (e-)degree $\mathbf{a}$ and $\mathfrak{M} \equiv_\Sigma \mathfrak{N}$, then $\mathfrak{N}$ also has (e-)degree $\mathbf{a}$.

Proof. Following [4], with every set $A \subseteq \omega$ we associate an Abelian group $G_A = \bigoplus_{n \in A} \mathbb{Z}_{p_n}$. It is easy to verify that the group G_A has e-degree $[A]_e$ and the group $G_{A \oplus \overline{A}}$ has degree $[A]_T$.

For instance, let $\mathfrak{M}$ have e-degree $\mathbf{a}$. Keeping in mind that $\mathbb{HF}(\mathfrak{M}) \leqslant_\Sigma \mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ and some presentation $\mathcal{C} \in \underline{\mathfrak{M}}$ is Σ-definable in $\mathbb{HF}(\mathfrak{M})$, we conclude that $\mathcal{C}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$. In view of $\mathfrak{N} \leqslant_\Sigma \mathfrak{M}$, given the Σ-definition and presentation $\mathcal{C}$, we arrive at a presentation $\mathcal{C}' \in \underline{\mathfrak{N}}$ such that $\mathcal{C}' \leqslant_e \mathcal{C}$; hence $\mathcal{C}'$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$. Thus the structure $\mathfrak{N}$ has an e-degree, which does not exceed the e-degree of $\mathfrak{M}$. A similar argument in the opposite direction shows that the given degrees coincide. □

Note, however, that the property of having a (e-)degree is not downward closed w.r.t. $\leqslant_\Sigma$. Arguing as in Lemma 1, we prove the following:

PROPOSITION 1. The mappings $i : \mathcal{D} \to \mathcal{S}_\Sigma$ and $j : \mathcal{D}_e \to \mathcal{S}_\Sigma$ are embeddings preserving 0 and $\vee$.

The existence of an embedding $\mathcal{D}$ in $\mathcal{S}_\Sigma$ was first noted in [8]. Furthermore, in [9], it was in essence proved that i and j also preserve the jump operation, if by a jump of $\mathfrak{M}$ is meant the structure $\mathfrak{M}' = (\mathbb{HF}(\mathfrak{M}), \Sigma\text{-Sat}^{\mathbb{HF}(\mathfrak{M})})$.

The concept of a *mass problem* was introduced in [10], where it was defined to be any set of total functions from ω to ω. Intuitively, mass problems can be thought of as sets of *solutions* (in the form of functions from ω to ω) for some *problems*. Below are some examples of mass problems corresponding to known problems in computability theory:

(1) the *problem of decidability* for a set $A \subseteq \omega$ is the mass problem $\mathcal{S}_A = \{\chi_A\}$, where χ_A is the characteristic function of the set A;

(2) the *problem of enumerability* for a set $A \subseteq \omega$ is the mass problem $\mathcal{E}_A = \{f : \omega \to \omega \mid \mathrm{rng}(f) = A\}$;

(3) the *problem of separability* for sets $A, B \subseteq \omega$ is the mass problem $\mathcal{S}ep_{A,B} = \{f : \omega \to 2 \mid f^{-1}(0) \supseteq A,\ f^{-1}(1) \supseteq B\}$.

In the present paper, we examine one more class of mass problems, namely, problems of presentability, corresponding to an important problem in (constructive) model theory, that of studying different presentations of structures on natural numbers. More exactly, the problem of presentability for a structure in $\mathbb{HF}(\varnothing)$ is equivalent to a mass problem in the sense of [10]. Actually, for $\mathfrak{M}$, we consider a set of all possible presentations of $\mathfrak{M}$ on natural numbers. The set of characteristic functions of these presentations form a mass problem $\{\chi_\mathcal{C} \mid \mathcal{C} \text{ is a presentation of } \mathfrak{M}\}$, which is equivalent to the problem of presentability for $\underline{\mathfrak{M}}$ in $\mathbb{HF}(\varnothing)$ under Medvedev reducibility, defined below.

Note that for any presentation $\mathcal{C} \in \underline{\mathfrak{M}}$, its domain C is effectively determined from (more exactly, is Turing reducible to) $\mathcal{C}$, since $c \in C$ iff $(c = c) \in \mathcal{C}$.

For $\mathfrak{M}$, we can also define the set

$$\{\chi^*_{\mathcal{C}} \mid \mathcal{C} \text{ is a presentation of } \mathfrak{M}\}$$

of partial characteristic functions of all possible presentations of $\mathfrak{M}$ on natural numbers. (Recall that for $A \subseteq \omega$ arbitrary, $\chi^*_A(n) = 0$ if $n \in A$ and $\chi^*_A(n)$ is undefined otherwise.) Sets of this kind are partial mass problems in the sense of [11], and such a partial problem is again equivalent to the problem of presentability for $\underline{\mathfrak{M}}$ in $\mathbb{HF}(\varnothing)$ under Dyment reducibility, defined below. Problems of that sort were studied in [12] (using another terminology) for classes of finite structures.

In [10], also, the concept of reducibility on the class of mass problems was introduced. If $\mathcal{A}$ and $\mathcal{B}$ are mass problems then $\mathcal{A}$ is *reducible* to $\mathcal{B}$ (written $\mathcal{A} \leqslant \mathcal{B}$) if there is a recursive operator Ψ such that $\Psi(\mathcal{B}) \subseteq \mathcal{A}$, that is, $\Psi(f) \in \mathcal{A}$ for all $f \in \mathcal{B}$. Intuitively, $\mathcal{A}$ is reducible to $\mathcal{B}$ if there is a uniform effective procedure which, given any solution for $\mathcal{B}$, yields some solution for $\mathcal{A}$.

An equivalence relation $\equiv$ on mass problems is conventionally defined by the preorder $\leqslant$ as follows: $\mathcal{A} \equiv \mathcal{B}$ if $\mathcal{A} \leqslant \mathcal{B}$ and $\mathcal{B} \leqslant \mathcal{A}$. Equivalence classes of the mass problems w.r.t. $\equiv$ (called *degrees of difficulty*), together with the reducibility relation $\leqslant$, form a distributive lattice (and, moreover, a Brauer algebra), known as the *Medvedev lattice* (see [10]).

There is yet another important reducibility relation on the class of mass problems, introduced in [13]. Namely, if $\mathcal{A}$ and $\mathcal{B}$ are mass problems, then $\mathcal{A}$ is said to be *weakly reducible* to $\mathcal{B}$ (written $\mathcal{A} \leqslant_w \mathcal{B}$) if for every $f \in \mathcal{B}$ there is a recursive operator Ψ such that $\Psi(f) \in \mathcal{A}$. Thus, weak reducibility (we also refer to it as *Muchnik reducibility*) derives from strong reducibility (Medvedev reducibility) by dropping the requirement for being uniform. An equivalence relation $\equiv_w$ on mass problems is defined w.r.t. $\leqslant_w$ in a standard way; equivalence classes w.r.t. $\equiv_w$, together with the reducibility relation $\leqslant_w$, also form a distributive lattice, known as the *Muchnik lattice* (see [13]).

We cite one more definition. If $\mathcal{A}$ and $\mathcal{B}$ are partial mass problems, then $\mathcal{A}$ is *enumerably reducible* to $\mathcal{B}$ (written $\mathcal{A} \leqslant_e \mathcal{B}$) if there is a partial recursive operator Ψ such that $\mathcal{B} \subseteq \mathrm{dom}(\Psi)$ and $\Psi(\mathcal{B}) \subseteq \mathcal{A}$. A *Dyment lattice* consists of equivalence classes of partial mass problems w.r.t. the equivalence relation $\equiv_e$ and the reducibility relation $\leqslant_e$. As with Medvedev reducibility, for the reducibility w.r.t. enumerability on partial mass problems, we can define its weak (non-uniform) version, $\leqslant_{ew}$.

There is a syntactic description of the above reducibilities on the set of problems of enumerability, which is underpinned by the known result obtained in [14] and restated in [15]: namely, for any $A, B \subseteq \omega$, the relation $A \leqslant_e B$ holds iff for any $X \subseteq \omega$ the fact that B is X-computable implies that A is X-computable. This result immediately implies that for any $A, B \subseteq \omega$,

$$\mathcal{E}_A \leqslant_w \mathcal{E}_B \iff \mathcal{E}_A \leqslant \mathcal{E}_B \iff A \leqslant_e B.$$

Along with the syntactic description, this entails coincidence of Medvedev and Muchnik reducibilities on the set of problems of enumerability (see also [16]).

Let $\mathbb{A}$ be an admissible set. We define uniform reducibilities on families of subsets of A, which are direct generalizations of Medvedev, Muchnik, and Dyment reducibilities on mass problems (see [17]). Let $\mathcal{X}, \mathcal{Y} \subseteq P(A)$. Then:

(1) $\mathcal{X}$ is *Medvedev reducible* to $\mathcal{Y}$ ($\mathcal{X} \leqslant \mathcal{Y}$) if there are binary Σ-operators F_0 and F_1 such that for all $Y \in \mathcal{Y}$, $\langle Y, A \setminus Y\rangle \in \delta_c(F_0) \cap \delta_c(F_1)$, and for some $X \in \mathcal{X}$, $X = F_0(Y, A \setminus Y)$ and $A \setminus X = F_1(Y, A \setminus Y)$.

(2) $\mathcal{X}$ is *Dyment reducible* to $\mathcal{Y}$ ($\mathcal{X} \leqslant_e \mathcal{Y}$) if there is a unary Σ-operator F such that $Y \in \delta_c(F)$ for all $Y \in \mathcal{Y}$, and $F(\mathcal{Y}) \subseteq \mathcal{X}$.

(3) $\mathcal{X}$ is *Muchnik reducible* to $\mathcal{Y}$ ($\mathcal{X} \leqslant_w \mathcal{Y}$) if for every $Y \in \mathcal{Y}$ there are binary Σ-operators F_0 and F_1 such that $\langle Y, A \setminus Y\rangle \in \delta_c(F_0) \cap \delta_c(F_1)$, and for some $X \in \mathcal{X}$, $X = F_0(Y, A \setminus Y)$ and $A \setminus X = F_1(Y, A \setminus Y)$.

(4) $\mathcal{X}$ is *weakly Dyment reducible* to $\mathcal{Y}$ ($\mathcal{X} \leqslant_e \mathcal{Y}$) if there is a unary Σ-operator F such that $Y \in \delta_c(F)$ for every $Y \in \mathcal{Y}$, and $F(Y) \in \mathcal{X}$.

For an admissible set $\mathbb{A}$ and for $* \in \{e,\ , w, ew\}$, $\mathcal{M}_*(\mathbb{A})$ denotes the structure of degrees, $\langle P(P(A))/\equiv_*, \leqslant_*\rangle$. We will write $\mathcal{M}_*$ instead of $\mathcal{M}_*(\mathbb{HF}(\varnothing))$ for brevity. All structures of the form $\mathcal{M}_*(\mathbb{A})$ are lattices with 0 and 1, and $\mathcal{M}$, $\mathcal{M}_e$, and $\mathcal{M}_w$ are isomorphic to, respectively, Medvedev, Dyment, and Muchnik lattices.

For a countable structure $\mathfrak{M}$, we consider the following classes consisting of structures that are effectively reducible to $\mathfrak{M}$:

$$\begin{aligned}
\mathcal{K}_\Sigma(\mathfrak{M}) &= \{\mathfrak{N} \mid \mathfrak{N} \leqslant_\Sigma \mathfrak{M}\},\\
\mathcal{K}_e(\mathfrak{M}) &= \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant_e \underline{(\mathfrak{M}, \bar{m})} \text{ for some } \bar{m} \in M^{<\omega}\},\\
\mathcal{K}(\mathfrak{M}) &= \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant \underline{(\mathfrak{M}, \bar{m})} \text{ for some } \bar{m} \in M^{<\omega}\},\\
\mathcal{K}_{ew}(\mathfrak{M}) &= \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant_{ew} \underline{\mathfrak{M}}\},\\
\mathcal{K}_w(\mathfrak{M}) &= \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant_w \underline{\mathfrak{M}}\}.
\end{aligned}$$

THEOREM 2. For any structure $\mathfrak{M}$, the following inclusions hold:

$$\mathcal{K}_\Sigma(\mathfrak{M}) \subseteq \mathcal{K}_e(\mathfrak{M}) \subseteq \mathcal{K}(\mathfrak{M}) \subseteq \mathcal{K}_w(\mathfrak{M}),$$
$$\mathcal{K}_e(\mathfrak{M}) \subseteq \mathcal{K}_{ew}(\mathfrak{M}) \subseteq \mathcal{K}_w(\mathfrak{M}).$$

Proof. In order to verify that $\mathcal{K}_\Sigma(\mathfrak{M}) \subseteq \mathcal{K}_e(\mathfrak{M})$, we suppose that some structure $\mathfrak{N}$ is Δ-definable in $\mathbb{HF}(\mathfrak{M})$ via a computable sequence Γ of Σ-formulas with parameters $\bar{m} \in M^{<\omega}$. (There is no loss of generality in assuming that all the parameters are elements of M.) Then a Σ-operator effecting the reducibility $\underline{\mathfrak{N}} \leqslant_e \underline{(\mathfrak{M}, \bar{m})}$ can be constructed from Γ, since checking for truth of a Σ-formula in $\mathbb{HF}(\mathfrak{M}, \bar{m})$ requires only that we use some finite subset of the atomic diagram of $(\mathfrak{M}, \bar{m})$ and some natural number.

We verify that $\mathcal{K}(\mathfrak{M}) \subseteq \mathcal{K}_w(\mathfrak{M})$. Note that if $\underline{\mathfrak{N}} \leqslant \underline{(\mathfrak{M}, \bar{m})}$ then in any presentation of the structure $\mathfrak{M}$, we can distinguish an arbitrary set presentation for $\bar{m}$ and apply an s-m-n theorem to the Σ-operator effecting the given reducibility, and then obtain a Σ-operator transforming that presentation into one for the structure $\mathfrak{N}$.

We are left to check $\mathcal{K}_e(\mathfrak{M}) \subseteq \mathcal{K}(\mathfrak{M})$. For instance, let $\underline{\mathfrak{N}} \leqslant_e \underline{(\mathfrak{M}, \bar{m})}$ via a unary Σ-operator Ψ. Using Ψ, we construct binary Σ-operators Ψ_1 and Ψ_2 so that for any $\mathcal{C} \in \underline{(\mathfrak{M}, \bar{m})}$, $\Psi_1(\mathcal{C}, \overline{\mathcal{C}}) = \mathcal{C}'$ and $\Psi_1(\mathcal{C}, \overline{\mathcal{C}}) = \mathcal{C}'$ for some $\mathcal{C}' \in \underline{\mathfrak{N}}$. To do this, we describe effective procedures which transform every presentation $\mathcal{C} \in \underline{(\mathfrak{M}, \bar{m})}$ with domain ω into a presentation $\mathcal{C}'$ and its complement for the structure $\mathfrak{N}$.

Step by step we define a domain of $\mathcal{C}'$ together with a bijection π mapping this domain into one for the presentation $\Psi(\mathcal{C})$. Namely, at step s, we define a subset $C_s \supseteq C_{s-1}$ of the domain of $\mathcal{C}'$ (as usual, we assume that $C_{-1} = \varnothing$) as follows: search through all numbers from 0 to s not in $\pi(C_{s-1})$; place the number s in C_s and place the pair $\langle s, c\rangle$ in π iff $c \leqslant s$, $c \notin \pi(C_{s-1})$, is the least number for which there exists a finite set $D_k \subseteq f$ with number $k \leqslant s$ in the standard numbering of finite sets such that $(c = c) \in \Psi(D_k)$. We have thus constructed the domain $C = \bigcup\limits_{s \in \omega} C_s$ (whose complement $\overline{C}$ has also been effectively defined via this process) and the bijection π. From these, we can effectively determine a presentation $\mathcal{C}' \in \underline{\mathfrak{N}}$ such that π is an isomorphism between the structures $\Psi(\mathcal{C})$ and $\mathcal{C}'$. □

For every symbol $* \in \{e, \ , w, ew\}$, we define a relation $\leqslant_*$ on $\mathcal{K}_\omega$ by setting $\mathfrak{M} \leqslant_* \mathfrak{N}$ iff $\mathcal{K}_*(\mathfrak{M}) \subseteq \mathcal{K}_*(\mathfrak{N})$ and letting $\mathcal{S}_* = \langle \mathcal{K}_\omega / \equiv_*, \leqslant_* \rangle$ be the structure of degrees of presentability corresponding to this relation.

THEOREM 3. For every $* \in \{e, \ , w, ew\}$, the structure $\mathcal{S}_*$ is an upper semilattice with 0, and the following embeddings ($\hookrightarrow$) and homomorphisms ($\rightarrow$) hold:

$$\mathcal{D} \hookrightarrow \mathcal{D}_e \hookrightarrow \mathcal{S}_\Sigma \rightarrow \mathcal{S}_e \rightarrow \mathcal{S} \hookrightarrow \mathcal{M}.$$

Proof. Indeed, for any structures $\mathfrak{M}$ and $\mathfrak{N}$ and any $* \in \{e, \ , w, ew\}$, we have $[\mathfrak{M}]_* \vee [\mathfrak{N}]_* = [(\mathfrak{M}, \mathfrak{N})]_*$, where $(\mathfrak{M}, \mathfrak{N})$ is a model-theoretic pair of the structures $\mathfrak{M}$ and $\mathfrak{N}$. An embedding $\mathcal{D}_e \hookrightarrow \mathcal{S}_\Sigma$ was established in Prop. 1. That homomorphisms $f : \mathcal{S}_\Sigma \rightarrow \mathcal{S}_e$ and $g : \mathcal{S}_e \rightarrow \mathcal{S}$ exist follows from Theorem 2, if we put $f([\mathfrak{M}]_\Sigma) = [\mathfrak{M}]_e$ and $g([\mathfrak{M}]_e) = [\mathfrak{M}]$ for any structure $\mathfrak{M}$. Note that $\mathcal{S} \hookrightarrow \mathcal{M}$ is the identity embedding. □

Obviously, (strong) reducibility in the sense of Medvedev always implies (weak) reducibility in the sense of Muchnik, that is, for any mass problems $\mathcal{A}$ and $\mathcal{B}$,

$$\mathcal{A} \leqslant \mathcal{B} \Rightarrow \mathcal{A} \leqslant_w \mathcal{B}.$$

In [13], a sufficient condition is specified under which the two reducibilities coincide. We recall its formulation. Below, by finite functions are meant those of the form $\tilde{f} : n \rightarrow \omega$, where $n < \omega$. An *interval* is a mass problem like

$$\mathcal{I}_{\tilde{f}} = \{f : \omega \rightarrow \omega \mid \tilde{f} \subseteq f\},$$

where $\tilde{f}$ is a finite function. A Baire topology on the set ω^ω is defined by choosing a set of all intervals to be the basis of open sets. A mass problem is said to be *closed* if it is a closed subset of ω^ω in the Baire topology. A mass problem $\mathcal{A}$ is *uniform* if $\mathcal{A} \cap \mathcal{I}_{\tilde{f}} \leqslant \mathcal{A}$ for any interval $\mathcal{I}_{\tilde{f}}$ with $\mathcal{A} \cap \mathcal{I}_{\tilde{f}} \neq \varnothing$.

In order to formulate the next sufficient condition, we need some preliminary definitions. Let $\mathcal{A}$ be a mass problem. We specify conditions of a game in which two players take part to resolve $\mathcal{A}$. At a first step, the first player chooses an interval $\mathcal{I}_{\tilde{f}_1}$ such that $\mathcal{A} \cap \mathcal{I}_{\tilde{f}_1} \neq \varnothing$. At a second step, the second player chooses an interval $\mathcal{I}_{\tilde{f}_2}$ for which $\mathcal{A} \cap \mathcal{I}_{\tilde{f}_1} \cap \mathcal{I}_{\tilde{f}_2} \neq \varnothing$. At a third step, the first player chooses $\mathcal{I}_{\tilde{f}_3}$ so that $\mathcal{A} \cap \mathcal{I}_{\tilde{f}_1} \cap \mathcal{I}_{\tilde{f}_2} \cap \mathcal{I}_{\tilde{f}_3} \neq \varnothing$, etc. The second player wins the game if intersection of the intervals $\mathcal{I}_{\tilde{f}_1}, \mathcal{I}_{\tilde{f}_2}, \mathcal{I}_{\tilde{f}_3}, \ldots$ is a point (function) in $\mathcal{A}$. A mass problem $\mathcal{A}$ is said to be *winning* [13] if the second player always has a winning strategy. Now we are in a position to state the following:

THEOREM 4 [13]. Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems. If $\mathcal{A}$ is closed and $\mathcal{B}$ is uniform and winning, then

$$\mathcal{A} \leqslant \mathcal{B} \iff \mathcal{A} \leqslant_w \mathcal{B}.$$

Of course the conditions of the theorem — due to the generality of the situation — are rather strong. The strongest is the condition of being closed, which hinders its application in many particular cases. For problems of enumerability in [13], for instance, it was shown that for any $A \subseteq \omega$, the problem $\mathcal{E}_A$ is uniform and winning, but is closed only if $\mathrm{card}(A) \leqslant 1$. On the set of problems of enumerability, however, Medvedev reducibility coincides with Muchnik's.

For problems of presentability in $\mathbb{HF}(\varnothing)$, too, we can naturally define a topology on a set of subsets, which is similar to the Baire topology on ω^ω, fixing some computable numbering of elements of $\mathbb{HF}(\varnothing)$ by natural numbers and identifying the subsets with characteristic functions of their preimages under that numbering.

LEMMA 2. Every problem of presentability is uniform.

Proof. Let $\mathfrak{M}$ be a structure and $\tilde{f}$ a finite function such that $\mathcal{I}_{\tilde{f}} \cap \underline{\mathfrak{M}} \neq \varnothing$. Then $\tilde{f}$ presents some finite part of the atomic diagram of $\mathfrak{M}$. We describe an effective procedure which uniformly transforms every presentation $\mathcal{C} \in \underline{\mathfrak{M}}$ into one of $\underline{\mathfrak{M}} \cap \mathcal{I}_{\tilde{f}}$. We search through all finite subsets of the atomic diagram of $\mathcal{C}$ until we find one that is isomorphic to a subset presented by $\tilde{f}$, and then apply a finite permutation establishing this isomorphism to the domain of the presentation $\mathcal{C}$. □

Obviously, no problem of presentability is open. As for the property of being closed, we have

LEMMA 3. Let $\mathfrak{M}$ be a countable structure of predicate signature. The problem $\underline{\mathfrak{M}}$ is closed if and only if for any countable structure $\mathfrak{N}$ of the same signature as is $\mathfrak{M}$, with $\mathfrak{N} \not\cong \mathfrak{M}$, there is an $\exists$-sentence φ in this signature such that:

(1) $\mathfrak{N} \models \varphi$;

(2) for any structure $\mathfrak{N}'$ with the signature of $\mathfrak{M}$, $\mathfrak{N}' \models \varphi$ implies $\mathfrak{N}' \not\cong \mathfrak{M}$.

This lemma readily yields the following:

THEOREM 5. Let $\mathfrak{M}$ be a countable structure of predicate signature. The problem $\underline{\mathfrak{M}}$ is closed if and only if $\mathrm{card}(M) = 1$.

Proof. It suffices to show that $\underline{\mathfrak{M}}$ is not closed whenever $\mathrm{card}(M) \geqslant 2$. Consider a proper finite substructure $\mathfrak{M}' \subsetneq \mathfrak{M}$ (which exists since the signature has no function symbols). Then $\mathfrak{M}' \not\cong \mathfrak{M}$; but every $\exists$-sentence true in $\mathfrak{M}'$ is also true in $\mathfrak{M}$. Lemma 3 implies that $\underline{\mathfrak{M}}$ is not closed. □

There are structures the problems of presentability for which are winning. For instance:

LEMMA 4. Let $\mathbb{L}$ be a countable dense linear order. The problem $\underline{\mathbb{L}}$ of presentability is winning if and only if $\mathbb{L}$ has neither least nor greatest elements.

Proof. For example, if $\mathbb{L}$ has a least element, then the winning strategy for the second player is the following: at each step, a new element is added which is smaller than are all the constructed ones. If, however, $\mathbb{L}$ has neither least nor greatest elements then the first player has a winning strategy: at each step, new elements are added to all empty intervals, and also at the left and right. □

We may furnish a complete description of countable equivalence relations which have winning problems of presentability.

LEMMA 5. Let $\mathcal{E}$ be a countable equivalence relation. The problem $\underline{\mathcal{E}}$ of presentability is winning if and only if there is $m_0 \leqslant \omega$ such that:

(1) in $\mathcal{E}$, the number of classes consisting of more than m_0 elements is finite;

(2) for every $m \leqslant m_0$, the number of classes in $\mathcal{E}$ consisting of m elements is finite, except possibly classes in a least dimension.

Proof. Let $\underline{\mathcal{E}}$ be a winning problem. Suppose that for any $m_0 \leqslant \omega$, one of (1), (2) fails. If there is m_0 such that $\mathcal{E}$ has infinitely many classes consisting of m_0 elements, and there are classes with a smaller number of elements, then there exists a winning strategy for the second player: at each step, we must see to it that the constructed piece of the diagram does not contain classes of less than m_0 elements. We have arrived at a contradiction with the problem $\underline{\mathcal{E}}$ being winning. Therefore, (2) should be satisfied for all m_0. For any $m_0 < \omega$, the number of classes consisting of more than m_0 elements will be infinite by assumption. In this event the second player, too, has a winning strategy: at step s, we must see to it that in the constructed piece of the diagram, all equivalence classes have s distinct elements; again we have arrived at a contradiction with $\underline{\mathcal{E}}$ being winning.

Now let (1) and (2) be simultaneously satisfied for some $m_0 \leqslant \omega$. In this case the first player has a winning strategy. Indeed, the number of classes of finite dimension is finite, except possibly classes in a least dimension. Therefore, if, on the very first move, these classes, together with sufficiently big parts of infinite classes, are included in the diagram, the first player wins independently of actions of the second. □

LEMMA 6. No problem of presentability is discrete.

The **proof** is similar to that of Lemma 2: Every interval either lacks presentations of a given system, or there are infinitely many such presentations. □

Note also that the class of problems of presentability has the following property: for any non-empty mass problem $\mathcal{A}$, there is a problem of presentability, $\underline{\mathfrak{M}}$, such that $\mathcal{A} \leqslant \underline{\mathfrak{M}}$. In fact, this property is obviously shared by the class of problems of decidability, and every degree of solvability is a degree of presentability.

2. ∀-COMPUTABILITY AND ∃-DEFINABILITY

Below is a theorem which generalizes the known result in [5, 6] which says that ∀-computability is equivalent to ∃-definability.

THEOREM 6. For any countable structures $\mathfrak{M}$ and $\mathfrak{N}$ and any relation $R \subseteq \mathbb{HF}(\mathfrak{N})$, the following conditions are equivalent:

(1) $R \leqslant_{e\Sigma} \mathfrak{C}$ for every presentation $\mathfrak{C}$ of $\mathfrak{M}$ in the admissible set $\mathbb{HF}(\mathfrak{N})$;

(2) R is Σ-definable in $\mathbb{HF}(\mathfrak{M}, \mathfrak{N})$.

Proof. (2) ⇒ (1). Suppose $R \subseteq \mathbb{HF}(\mathfrak{N})$ is Σ-definable in $\mathbb{HF}(\mathfrak{M}, \mathfrak{N})$ via some Σ-formula $\Phi(x, \bar{m}, \bar{n})$ with parameters $\bar{m} \in M^{<\omega}$ and $\bar{n} \in N^{<\omega}$ (without loss of generality, we may assume that these parameters are urelements). Hence, in view of [1], there is a sequence $\{\varphi_{k,i}(\bar{x}, \bar{y}, \bar{z}_k) \mid k, i \in \omega\}$ of ∃-formulas of the signature of $(\mathfrak{M}, \mathfrak{N})$, which is computable uniformly in k and in i and is such that for all $k \in \omega$ and $\bar{r} \in N^{<\omega}$, $\varkappa(k)(\bar{r}) \in R$ iff $(\mathfrak{M}, \mathfrak{N}) \models \varphi_{k,i}(\bar{m}, \bar{n}, \bar{r})$ for some $i \in \omega$. Now let $\mathfrak{C}$ be an arbitrary presentation of $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{N})$ and $\bar{c} \in C^{<\omega}$ be a tuple such that $(\mathfrak{M}, \bar{m}) \cong (\mathfrak{C}, \bar{c})$. In $\mathbb{HF}(\mathfrak{N})$, we can easily define a Σ-operator F (using $\bar{n}$ and $\bar{c}$ as parameters) for which $F(\mathfrak{C}) = R$.

(1) ⇒ (2). We outline a general method for constructing presentations of structures, which makes use of forcing in hereditarily finite superstructures (see [18, 19]). Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable structures with computable signatures σ_1 and σ_2, respectively, and let σ^* be obtained from a (disjoint) union $\sigma_1 \cup \sigma_2$ by adding new predicate symbols $R^1, U^1, \in^2$, new function symbols $\{\}^1, \cup^2$, and a new constant symbol $\varnothing$. We also fix a binary predicate symbol P, not in σ^*. Denote by $\sigma_i(P)$ and $\sigma^*(P)$ the signatures obtained by adding this symbol to σ_i, $i \in \{1, 2\}$, and σ^*, respectively. For any structure $\mathfrak{A}$ of signature σ, σ_A denotes the signature obtained by adding to σ new constant symbols, for all elements of A.

Along with presentation, we will use another concept, which is similar, but is more densely linked, to the structure under consideration. Let $\mathfrak{M}$ be any structure and $\mathbb{A}$ an admissible set. A *copy* of $\mathfrak{M}$ in $\mathbb{A}$ is any surjective mapping $\pi : C \to M$, where $C \subseteq A$ is the set of *designations* for elements of $\mathfrak{M}$. Every copy of $\mathfrak{M}$ in $\mathbb{A}$ defines a corresponding presentation uniquely, but not vice versa. In particular, if $C \subseteq \omega$ then the copy π defines a presentation of $\mathfrak{M}$ on natural numbers.

In what follows, we handle only the case where $\mathbb{A} = \mathbb{HF}(\mathfrak{N})$. In this instance every copy of $\mathfrak{M}$ is a partial function on the admissible set $\mathbb{HF}(\mathfrak{M}, \mathfrak{N})$, with M a set of ranges. It seems more convenient to use as relations graphs of partial functions rather than the partial functions themselves. Therefore, we will assume that a copy of $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{N})$ is a binary relation P in $\mathbb{HF}(\mathfrak{M}, \mathfrak{N})$ for which $\mathrm{Pr}_1(P) \subseteq \mathbb{HF}(N)$ and $\mathrm{Pr}_2(P) = M$, treating every element $a \in \mathbb{HF}(N)$, for which $\langle a, m \rangle \in P$, as a designation for $m \in M$. Let π be a function with a graph P. The relation P assigns a non-empty set of designations to every element of M.

At the moment, we fix some Gödel numbering $\ulcorner\ \urcorner$ of terms and formulas of the signature $\sigma^*(P)$. The *atomic diagram* of a copy π is the set

$$D(\pi) = \{\langle \ulcorner\varphi\urcorner, \bar{a}\rangle \mid \varphi \text{ is a literal of signature } \sigma_1,\ \bar{a} \in HF(N)^{<\omega},\ \mathfrak{M} \models \varphi(\pi(\bar{a}))\},$$

where $\pi(\bar{a})$ denotes the tuple $\langle \pi(a_0), \dots, \pi(a_k)\rangle$, for $\bar{a} = \langle a_0, \dots, a_k\rangle$. A copy π is *computable* in $\mathbb{HF}(\mathfrak{N})$ if $D(\pi)$ is a Δ-definable subset of $\mathbb{HF}(\mathfrak{N})$.

Now, we fix an arbitrary relation $R \subseteq HF(N)$. The idea behind our reasoning is to construct a copy π of a structure $\mathfrak{M}$ for which the structure $\langle \mathbb{HF}(\mathfrak{M}, \mathfrak{N}), R, P\rangle$ would be generic in the sense of [18]. We construct π as a union of the sequence $p_0 \subseteq p_1 \subseteq \dots$ of finite functions, that is, $\pi = \bigcup_{n\in\omega} p_n$. Every finite function, which can be extended to a copy of $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{N})$, is called a *forcing condition*, and we denote the set of all such conditions by $\mathcal{P}(\mathfrak{M}, \mathfrak{N})$. A *forcing relation* between elements of $\mathcal{P}(\mathfrak{M}, \mathfrak{N})$ and sentences of the signature $\sigma^*(P)_{HF(M,N)}$, which admit bounded quantifiers, are conventionally defined as, for instance, in [18]. Namely, for a forcing condition p and a sentence Φ, we define the relation 'p forces Φ' (written $p \Vdash \Phi$) by induction on the complexity of the sentence Φ as follows:

(1) if Φ is an atomic sentence of the signature $\sigma^*(P)_{HF(M,N)}$ then $p \Vdash \Phi$ iff $\langle \mathbb{HF}(\mathfrak{N}), R, p\rangle \models \Phi$;
(2) $p \Vdash (\Phi_1 \vee \Phi_2)$ iff $p \Vdash \Phi_1$ or $p \Vdash \Phi_2$;
(3) $p \Vdash \exists x \Psi(x)$ iff $p \Vdash \Psi(a)$ for some $a \in HF(M, N)$;
(4) $p \Vdash \neg\Phi$ iff there is no forcing condition $q \supseteq p$ for which $q \Vdash \Phi$.

Other logical connectives, as well as the universal quantifier and bounded quantifiers, are treated as abbreviations. Therefore, we have

(5) $p \Vdash (\Phi_1 \wedge \Phi_2)$ iff $p \Vdash \neg(\neg\Phi_1 \vee \neg\Phi_2)$, i.e., for any condition $q \supseteq p$, there are conditions $r_1, r_2 \supseteq q$ such that $r_1 \Vdash \Phi_1$ and $r_2 \Vdash \Phi_2$;
(6) $p \Vdash (\Phi_1 \to \Phi_2)$ iff $p \Vdash (\neg\Phi_1 \vee \Phi_2)$;
(7) $p \Vdash \forall x \Psi(x)$ iff $p \Vdash \neg\exists x \neg\Psi(x)$, i.e., for any condition $q \supseteq p$ and any $a \in HF(M, N)$, there is a condition $r \supseteq q$ such that $r \Vdash \Phi(a)$;
(8) $p \Vdash (\exists x \in a)\Phi(x)$ iff $p \vdash \exists x((x \in a) \wedge \Phi(x))$;
(9) $p \Vdash (\forall x \in a)\Phi(x)$ iff $p \Vdash \forall x((x \in a) \to \Phi(x))$.

We will need several statements, which are standard for any construction via forcing.

LEMMA 7. For every sentence Φ of the signature $\sigma^*(P)_{HF(M,N)}$ and for any $p, q \in \mathcal{P}(\mathfrak{M}, \mathfrak{N})$, $p \subseteq q$ and $p \Vdash \Phi$ imply $q \Vdash \Phi$.

The **proof** is by induction on the complexity of Φ. □

LEMMA 8. For every sentence Φ of the signature $\sigma^*(P)_{HF(M,N)}$ and for any $p \in \mathcal{P}(\mathfrak{M}, \mathfrak{N})$, there is $q \supseteq p$ for which $q \Vdash \Phi$ or $q \Vdash \neg\Phi$.

The **proof** follows from item (4) in the definition of a forcing relation. Indeed, if there is no $q \supseteq p$ such that $q \Vdash \Phi$ then $p \Vdash \neg\Phi$ by definition. □

LEMMA 9. Let Φ be a sentence in the signature $\sigma^*(P)_{HF(M,N)}$. There is no condition $p \in \mathcal{P}(\mathfrak{M}, \mathfrak{N})$ for which $p \Vdash \Phi$ and $p \Vdash \neg\Phi$.

The **proof** is by induction on the complexity of the sentence Φ. If Φ is an atomic sentence, the statement is obvious. Assume, for instance, that $\Phi = (\Phi_1 \vee \Phi_2)$. If $p \Vdash \Phi$ and $p \Vdash \neg\Phi$ then $p \Vdash \Phi_i$ for some $i \in \{1, 2\}$. At the same time, $q \nVdash \Phi_1$ and $q \nVdash \Phi_1$ for any $q \supseteq p$. Contradiction. Other cases can be treated similarly. □

LEMMA 10. There exists a (so-called generic) copy π of a structure $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{N})$ such that for any sentence Φ in the signature $\sigma^*(P)_{HF(M,N)}$,

$$(\mathbb{HF}(\mathfrak{M},\mathfrak{N}), R, \pi) \models \Phi \iff \exists p \in \mathcal{P}(\mathfrak{M},\mathfrak{N}) \restriction \pi \ (p \Vdash \Phi).$$

Here $\mathcal{P}(\mathfrak{M},\mathfrak{N}) \restriction \pi$ is the set of all forcing conditions, which are subsets of π.

Proof. Consider an arbitrary numbering $\Phi_0, \Phi_1, \ldots, \Phi_k, \ldots$ of sentences in the signature $\sigma^*(P)_{HF(M,N)}$, and also an arbitrary numbering $m_0, m_1, \ldots, m_k, \ldots$ of the domain of $\mathfrak{M}$. Let p_k, $k \in \omega$, be some forcing condition for which $m_k \in \mathrm{rng}(p_k)$, and let $p_k \Vdash \Phi_k$ or $p_k \Vdash \neg\Phi_k$ (such p_k exists in view of the previous lemmas). Define a copy π of $\mathfrak{M}$ to be the union $\bigcup\limits_{k\in\omega} p_k$. It remains to use induction on the complexity of the sentence Φ. □

LEMMA 11. For every formula $\Phi(x)$ of the signature $\sigma^*(P)$, there is a formula $\Phi^*(y, x)$ of the signature σ^* such that for any $a \in HF(M,N)$ and any forcing condition p,

$$p \Vdash \Phi(a) \iff \langle \mathbb{HF}(\mathfrak{M},\mathfrak{N}), R\rangle \models \Phi^*(p, a).$$

The formula Φ^* is constructed from Φ via a uniform effective procedure; moreover, Φ^* is $\Delta_0(\Sigma)$ if so is Φ.

The **proof** is by induction on the complexity of the formula Φ.

(1) Let Φ be an atomic or negated atomic formula. If $\Phi = P(m, a)$ then $p \Vdash \Phi$ iff $\langle m, a\rangle \in p$; if $\Phi = \neg P(n, a)$ then $p \Vdash \Phi$ iff $\langle m, a'\rangle \in p$ for some $a' \neq a$; in all other cases, $p \Vdash \Phi$ iff $\langle \mathbb{HF}(\mathfrak{M},\mathfrak{N}), R\rangle \models \Phi$.

(2) Let $\Phi = (\Phi_1 \vee \Phi_2)$. Then $\Phi^* = ((\Phi_1)^* \vee (\Phi_2)^*)$.

(3) Let $\Phi = \neg\Phi_0$. By the definition of a forcing relation, $p \Vdash \Phi(a)$ iff $\forall q$(“q is a forcing condition” $\wedge\, [q \supseteq p \to \neg(\Phi_0)^*(p, a)]$).

(4) Let $\Phi = \exists y \Phi_0(a, y)$. Again, by definition, $p \Vdash \Phi(a)$ iff $\exists b (\Phi_0')^*(q, \langle a, b\rangle))$, where $\Phi_0'(z) = \Phi_0(\mathrm{Pr}_1(z), \mathrm{Pr}_2(z))$. □

We appeal to the proof of (1) $\Rightarrow$ (2) in Theorem 6. Let $R \leqslant_{e\Sigma} \mathcal{C}$ for any presentation $\mathcal{C}$ of $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{N})$. Assume also that R is not Σ-definable in $\mathbb{HF}(\mathfrak{M},\mathfrak{N})$. Let $\{\Phi_i(x) \mid i \in \omega\}$ be some numbering of the set of Σ-formulas of the signature $\sigma'_{HF(M,N)}$. Consider a generic copy $\pi = \bigcup\limits_{n\in\omega} p_n$ of $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{N})$ possessing the following extra property: p_n satisfies $p_n \Vdash (R \neq \Phi_n(x))$ for any $n \in \omega$. Such a copy exists, since otherwise $p_{n-1} \Vdash (R = \Phi_n(x))$, and by Lemma 11, R would be Σ-definable in $\mathbb{HF}(\mathfrak{M},\mathfrak{N})$. Thus, for a presentation $\mathcal{C}_\pi$ corresponding to the generic copy π, we have $R \nleqslant_{e\Sigma} \mathcal{C}_\pi$, for otherwise R would be definable in $\langle \mathbb{HF}(\mathfrak{M},\mathfrak{N}), \pi\rangle$ by some Σ-formula Φ_n. In view of π being generic, this would imply forcing of a corresponding statement, which leads us to a contradiction with the initial assumption. □

At the moment, we consider reducibilities between problems of presentability and some other types of mass problems. For the problems of enumerability in Theorem 6, the following result (sort of analogous to the Selman–Rozinas theorem) is straightforward.

COROLLARY 1. Let $\mathfrak{M}$ be a countable structure, $A \subseteq \omega$, and $A \neq \varnothing$. The conditions below are equivalent:

(1) $\mathcal{E}_A \leqslant_w \underline{\mathfrak{M}}$;

(2) $\mathcal{E}_A \leqslant \underline{(\mathfrak{M}, \bar{m})}$ for some $\bar{m} \in M^{<\omega}$;

(3) A is Σ-definable in $\mathbb{HF}(\mathfrak{M})$.

A similar result for problems of decidability follows immediately from the previous.

COROLLARY 2. Let $\mathfrak{M}$ be a countable structure and $A \subseteq \omega$. The conditions below are equivalent:

(1) $\mathcal{S}_A \leqslant_w \underline{\mathfrak{M}}$;

(2) $\mathcal{S}_A \leqslant \underline{(\mathfrak{M}, \bar{m})}$ for some $\bar{m} \in M^{<\omega}$;

(3) A is Δ-definable in $\mathbb{HF}(\mathfrak{M})$.

Proof. Indeed, for any mass problem $\mathcal{B}$, $\mathcal{S}_A \leqslant_w \mathcal{B}$ iff $\mathcal{E}_A \leqslant_w \mathcal{B}$ and $\mathcal{E}_{\overline{A}} \leqslant_w \mathcal{B}$ (similarly for the relation $\leqslant$). □

Now we embark on the formulation and proof of a result which, on the one hand, gives a syntactic description of Muchnik reducibilities on the set of problems of presentability for models of special type in terms of effective definability in hereditarily finite superstructures, and on the other hand, it reveals relationship between Medvedev and Muchnik reducibilities in this instance. Let $\mathfrak{M}$ be a countable structure of computable predicate signature $\langle P_0^{n_0}, \ldots, P_k^{n_k}, \ldots\rangle$ and $\mathbb{A}$ be an admissible set.

THEOREM 7. Let a countable structure $\mathfrak{M}$ have a degree. Then

$$\mathcal{K}_\Sigma(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M}) = \mathcal{K}(\mathfrak{M}) = \mathcal{K}_w(\mathfrak{M}).$$

In view of Proposition 2, it suffices to state that $\mathcal{K}_w(\mathfrak{M}) \subseteq \mathcal{K}_\Sigma(\mathfrak{M})$. To do this, we make use of the result showing how to describe structures having a degree in terms of being definable in hereditarily finite superstructures. For any countable structure $\mathfrak{M}$ of signature σ, an *s-expansion* of $\mathfrak{M}$ is any structure $\mathfrak{M}'$ of signature $\sigma \cup \{s^1; 0\}$, where s is a unary function and 0 is a constant symbol, such that $\mathfrak{M}' \upharpoonright \sigma = \mathfrak{M}$ and $\langle M, s^{\mathfrak{M}'}, 0^{\mathfrak{M}'}\rangle \cong \langle \omega, s, 0\rangle$.

We denote by $\mathcal{S}_{\mathfrak{M}}$ a mass problem, which is the union of problems of presentability of $\mathfrak{M}'$ for all s-expansions $\mathfrak{M}'$ of the structure $\mathfrak{M}$. It follows immediately from the definition that for any structure $\mathfrak{M}$, $\mathcal{S}_{\mathfrak{M}} \leqslant \underline{\mathfrak{M}}$. In fact, we outline an effective procedure which transforms every presentation $\mathcal{C} \in \underline{\mathfrak{M}}$ into some presentation of $\mathcal{S}_{\mathfrak{M}}$. Assume that a value for the constant 0 is a least element (in the sense of ordering on natural numbers) of the domain of $\mathcal{C}$ (more exactly, it is an equivalence class w.r.t. = in $\mathcal{C}$ corresponding to that element); as a value for $s(0)$, then, we choose a least element (i.e., its equivalence class) of the domain which does not lie in the equivalence class for 0, etc. As a consequence we also see that $\mathcal{S}_{\mathfrak{M}} \leqslant_w \underline{\mathfrak{M}}$.

THEOREM 8. Let $\mathfrak{M}$ be a countable structure. The following conditions are equivalent:

(1) $\mathfrak{M}$ has a degree;

(2) there exists a presentation $\mathcal{C} \in \underline{\mathfrak{M}}$ which (being a subset of ω) is Δ-definable in $\mathbb{HF}(\mathfrak{M})$;

(3) some s-expansion of $\mathfrak{M}$ is Δ-definable in $\mathbb{HF}(\mathfrak{M})$;

(4) $\underline{\mathfrak{M}} \equiv \mathcal{S}_A$ for some $A \subseteq \omega$.

Proof. (2) $\Rightarrow$ (3). Let $\mathcal{C} \in \underline{\mathfrak{M}}$ be such that $\mathcal{C}$ is Δ-definable in $\mathbb{HF}(\mathfrak{M})$. To transform $\mathcal{C}$ into a presentation of $\underline{\mathfrak{M}'}$, we can use the same effective procedure as is described before the theorem. Hence $\mathfrak{M}'$ is Δ-definable in $\mathbb{HF}(\mathfrak{M})$.

(3) $\Rightarrow$ (2). Let $\mathfrak{M}'$ be Δ-definable in $\mathbb{HF}(\mathfrak{M})$. We claim that in this case some presentation $\mathcal{C} \in \underline{\mathfrak{M}}$ is Δ-definable in $\mathbb{HF}(\mathfrak{M})$, and moreover, as the domain of $\mathcal{C}$ we can take ω. We establish a one-to-one mapping f between the domain of $\mathfrak{M}'$ (more exactly, its presentation in $\mathbb{HF}(\mathfrak{M})$) and ω, which is Δ-definable in $\mathbb{HF}(\mathfrak{M})$, as follows: for all $a \in HF(M)$ and $n \in \omega$, put $f(a) = n$ iff there are $a_0, \ldots, a_n \in HF(M)$ such that $a_0 = 0^{\mathfrak{M}'}$, $a_1 = s^{\mathfrak{M}'}(a_0)$, ..., and $a = a_n = s^{\mathfrak{M}'}(a_{n-1})$ under this presentation of $\mathfrak{M}'$.

(2) $\Rightarrow$ (1). Suppose that for some presentation $\mathcal{C} \in \underline{\mathfrak{M}}$, its atomic diagram is Δ-definable in $\mathbb{HF}(\mathfrak{N})$ with parameters $\bar{n} \in N^{<\omega}$ (again we may assume that all the parameters are elements of N). This readily implies that $\mathcal{C} \leqslant_T \mathcal{C}'$ for any $\mathcal{C}' \in \underline{\mathfrak{M}}$. Indeed, computable operators effecting these reducibilities are constructed from Σ-formulas defining $\mathcal{C}$.

(1) $\Rightarrow$ (2). Assume that there exists a presentation $\mathcal{C} \in \underline{\mathfrak{M}}$ such that $\mathcal{C} \leqslant_T \mathcal{C}'$ for any $\mathcal{C}' \in \underline{\mathfrak{M}}$. This, in terms of mass problems, is equivalent to $\mathcal{S}_{\mathcal{C}} \leqslant_w \underline{\mathfrak{M}}$. In view of Theorem 2, $\mathcal{C}$ is therefore Δ-definable (as a

subset of ω) in $\mathbb{HF}(\mathfrak{M})$. $\Box$

Lastly, we argue for the inclusion $\mathcal{K}_w(\mathfrak{M}) \subseteq \mathcal{K}_\Sigma(\mathfrak{M})$ in Theorem 7. Suppose $\mathfrak{N}$ is a structure for which $\underline{\mathfrak{N}} \leqslant_w \underline{\mathfrak{M}}$. We also fix some presentation $\mathcal{C}_0 \in \underline{\mathfrak{M}}$ such that $\mathcal{C}_0$ is a Δ-definable subset of $\mathbb{HF}(\mathfrak{M})$. In view of $\underline{\mathfrak{N}} \leqslant_w \underline{\mathfrak{M}}$, there exists a presentation $\mathcal{C} \in \underline{\mathfrak{N}}$ such that $\mathcal{C} \leqslant_{T\Sigma} \mathcal{C}_0$. Since $\mathcal{C}_0$ is Δ-definable in $\mathbb{HF}(\mathfrak{N})$, the same is true of $\mathcal{C}$; hence $\mathfrak{N}$ will be Δ-definable in $\mathbb{HF}(\mathfrak{M})$ via $\mathcal{C}$, proving Theorem 7.

THEOREM 9. Let $\mathfrak{M}$ be a countable structure. The following conditions are equivalent:

(1) $\mathfrak{M}$ has an e-degree;

(2) there is a presentation $\mathcal{C}$ of $\mathfrak{M}$ which (being a subset of ω) is Σ-definable in $\mathbb{HF}(\mathfrak{M})$;

(3) $\underline{\mathfrak{M}} \equiv \mathcal{E}_A$ for some $A \subseteq \omega$.

The **proof** is similar to that of Theorem 8. $\Box$

A consequence of Theorems 8 and 9 is the following:

PROPOSITION 2. If $\mathfrak{M}$ has a degree then it has an e-degree.

In [4] are implicit examples of structures having e-degrees but not having degrees — namely, an example of the set $A \subseteq \omega$ for which the mass problem $\mathcal{E}_A$ has no least element under Turing reducibility; there, also, it is shown how to connect an arbitrary set $A \subseteq \omega$ with an Abelian group G_A for which $\underline{G_A} \equiv \mathcal{E}_A$.

An analog of Theorem 7 for systems with e-degrees is

THEOREM 10. Let a countable structure $\mathfrak{M}$ have an e-degree. Then

$$\mathcal{K}_\Sigma(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M}) = \mathcal{K}_{ew}(\mathfrak{M}).$$

The **proof** is similar to that of Theorem 7. $\Box$

Yet another characterization of structures with degrees is

THEOREM 11. Let $\mathfrak{M}$ be a countable structure. The following conditions are equivalent:

(1) $\mathfrak{M}$ has a degree;

(2) for some s-expansion $\mathfrak{M}'$ of $\mathfrak{M}$, $\mathfrak{M}' \in \mathcal{K}_\Sigma(\mathfrak{M})$;

(3) for some s-expansion $\mathfrak{M}'$ of $\mathfrak{M}$, $\mathfrak{M}' \in \mathcal{K}(\mathfrak{M})$;

(4) for some s-expansion $\mathfrak{M}'$ of $\mathfrak{M}$, $\mathfrak{M}' \in \mathcal{K}_w(\mathfrak{M})$.

An admissible set $\mathbb{A}$ is *recursively listed* [2] if in $\mathbb{A}$ there exists a surjective Σ-function $f : o(\mathbb{A}) \to A$; $\mathbb{A}$ is *partial recursively listed* if in $\mathbb{A}$ there exists a partial surjective Σ-function $f : o(\mathbb{A}) \to A$. In the former case, note, the function f may be chosen to be bijective.

PROPOSITION 3. Let $\mathfrak{M}$ be a countable structure. Then:

(1) some copy of $\mathfrak{M}$ in $\mathbb{HF}(\varnothing)$ is Δ-definable in $\mathbb{HF}(\mathfrak{M})$ iff $\mathbb{HF}(\mathfrak{M})$ is recursively listed;

(2) some copy of $\mathfrak{M}$ in $\mathbb{HF}(\varnothing)$ is Σ-definable in $\mathbb{HF}(\mathfrak{M})$ iff $\mathbb{HF}(\mathfrak{M})$ is partial recursively listed.

COROLLARY 3. If $\mathfrak{M}$ is recursively listed (partial recursively listed) then $\mathfrak{M}$ has a degree (an e-degree).

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