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DEGREES OF PRESENTABILITY OF STRUCTURES. II

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We show that the property of being locally constructivizable is inherited under Muchnik reducibility, which is weakest among the effective reducibilities considered over countable structures. It is stated that local constructivizability of level higher than 1 is inherited under Σ-reducibility but is not inherited under Medvedev reducibility. An example of a structure $\mathfrak{M}$ and a relation $P \subseteq M$ is constructed for which $\underline{(\mathfrak{M}, P)} \equiv \underline{\mathfrak{M}}$ but $(\mathfrak{M}, P) \not\equiv_\Sigma \mathfrak{M}$. Also, we point out a class of structures which are effectively defined by a family of their local theories.

This paper is a continuation of [1, 2] and uses the same notation.

1. PROPERTIES OF STRUCTURES INHERITED UNDER EFFECTIVE REDUCIBILITIES

Below we show that the condition of a structure $\mathfrak{M}$ having a degree specified in [1, Thm. 7] is essential. To do this, we state necessary conditions for effective reducibilities between structures, in particular, conditions that are necessary for being Σ-definable.

Structure $\mathfrak{M}$ is said to be *locally constructivizable* [3] if $\mathrm{Th}_\exists(\mathfrak{M}, \bar{m})$ is computably enumerable (c.e.) for any $\bar{m} \in M^{<\omega}$. As noted in [3], the local constructivizability of $\mathfrak{M}$ is equivalent to the fact that for every tuple $\bar{m} \in M^{<\omega}$, there exist a constructivizable structure $\mathfrak{N}$ and a tuple $\bar{n} \in N^{<\omega}$ such that $\mathrm{Th}_\exists(\mathfrak{M}, \bar{m}) = \mathrm{Th}_\exists(\mathfrak{N}, \bar{n})$. For structures $\mathfrak{M}$ and $\mathfrak{N}$, by writing $\mathfrak{M} \leqslant_\exists \mathfrak{N}$ we mean that for every tuple $\bar{m} \in M^{<\omega}$, there is a tuple $\bar{n} \in N^{<\omega}$ for which $\mathrm{Th}_\exists(\mathfrak{M}, \bar{m}) \leqslant_e \mathrm{Th}_\exists(\mathfrak{N}, \bar{n})$. In particular, if $\mathfrak{M}$ is locally constructivizable then $\mathfrak{M} \leqslant_\exists \mathfrak{N}$ for any structure $\mathfrak{N}$.

That a structure $\mathfrak{M}$ is locally constructivizable if so is $\mathfrak{N}$ with $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ was first mentioned in [3]. A direct generalization of this fact is the following: if $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ then $\mathfrak{M} \leqslant_\exists \mathfrak{N}$. In order to state other necessary conditions of Σ-definability (which are used in proving negative results on $\leqslant_\Sigma$), we consider some relevant notions.

Definition 1. Structure $\mathfrak{M}$ is *locally constructivizable of level* n $(1 < n \leqslant \omega)$ if for every tuple $\bar{m} \in M^{<\omega}$ there exist a constructivizable structure $\mathfrak{N}$ and a tuple $\bar{n} \in N^{<\omega}$ such that $(\mathfrak{M}, \bar{m}) \equiv_n^{\mathrm{HF}} (\mathfrak{N}, \bar{n})$. A countable structure $\mathfrak{M}$ is *uniformly locally constructivizable of level* n $(1 < n \leqslant \omega)$ if there exists a constructivizable structure $\mathfrak{N}$ for which $\mathfrak{M} \preccurlyeq_n^{\mathrm{HF}} \mathfrak{N}$.

For instance, the structure $\langle \omega_1^{CK}, \leqslant \rangle$ is uniformly locally constructivizable of level ω since $\langle \omega_1^{CK}, \leqslant \rangle \preccurlyeq^{\mathrm{HF}} \langle \omega_1^{CK}(1 + \eta), \leqslant \rangle$, where the last order (Harrison order) is constructivizable.

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Let σ be an arbitrary predicate (for simplicity) signature. We say that a *partial structure* $\mathfrak{M}$ of the signature σ with domain M is given if some consistent set $D(\mathfrak{M})$ of atomic sentences and their negations in the signature σ_M is fixed. Equivalently, for some (complete) structure $\mathfrak{N}$ of the signature σ, $D(\mathfrak{M}) \subseteq D(\mathfrak{N})$, where $D(\mathfrak{N})$ is an atomic diagram of $\mathfrak{M}$. We also call $D(\mathfrak{M})$ the *atomic diagram* of a partial structure $\mathfrak{M}$. The partial structure $\mathfrak{M}$ (of a computable signature) is said to be *constructivizable* if the set $D(\mathfrak{M})$ is c.e. under some numbering of the domain M.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be any, possibly partial, structures of a signature σ. We say that $\mathfrak{M}$ is a *substructure* of $\mathfrak{N}$ (written $\mathfrak{M} \subseteq \mathfrak{N}$) if $D(\mathfrak{M}) \subseteq D(\mathfrak{N})$. We also say that $\mathfrak{M}$ is *existentially closed* in $\mathfrak{N}$ (written $\mathfrak{M} \preccurlyeq_\exists \mathfrak{N}$) if $\mathfrak{N} \models \exists \bar{x}\varphi(\bar{x}, \bar{m})$ implies $\mathfrak{M} \models \exists \bar{x}\varphi(\bar{x}, \bar{m})$ for every quantifier-free formula $\varphi(\bar{x}, \bar{y})$ of the signature σ and for any $\bar{m} \in M^{<\omega}$.

PROPOSITION 1. If $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ and a structure $\mathfrak{N}$ is (uniformly) locally constructivizable of level n, then:

(1) if $1 < n \leqslant \omega$ then $\mathfrak{M}$ is also (uniformly) locally constructivizable of level n;

(2) if $n = 1$ and $\mathfrak{N}$ is uniformly locally constructivizable then there exists a partial constructivizable structure $\mathfrak{M}'$ such that $\mathfrak{M} \preccurlyeq_\exists \mathfrak{M}'$.

Proof. For instance, let $\mathfrak{N}$ be locally constructivizable of level n and let it be Σ-definable in $\mathbb{HF}(\mathfrak{N})$ by a sequence Γ of Σ-formulas with parameters $\bar{n}_0 \in N^{<\omega}$. Assume that $\bar{m} \in M^{<\omega}$ and $\bar{n} \in N^{<\omega}$ is a tuple such that for some $\varkappa_1, \ldots, \varkappa_k \in HF(\omega)$, the tuple $\langle \varkappa_1(\bar{n}), \ldots, \varkappa_k(\bar{n})\rangle$ corresponds to $\bar{m}$ in a presentation defined by Γ. Consider a tuple $\bar{n}\bar{n}_0$. Suppose that $\mathfrak{N}'$ is a constructivizable structure and $\bar{n}'\bar{n}'_0 \in N'^{<\omega}$ are such that $(\mathfrak{N}, \bar{n}\bar{n}_0) \equiv^{\mathrm{HF}}_n (\mathfrak{N}', \bar{n}'\bar{n}'_0)$. Consider a structure $\mathfrak{M}'$ which is defined in $\mathbb{HF}(\mathfrak{N}')$ by the same sequence Γ of formulas with parameters $\bar{n}'_0$. If $n > 1$ then the (complete) structure $\mathfrak{M}'$ is well defined; if $n = 1$ then $\mathfrak{M}'$ in general can be partial. In any case $\mathfrak{M}'$ is constructivizable, and for a tuple $\bar{m}'$, corresponding to $\langle \varkappa_1(\bar{n}'), \ldots, \varkappa_k(\bar{n}')\rangle$ in this presentation, we have $(\mathfrak{M}, \bar{m}) \equiv^{\mathrm{HF}}_n (\mathfrak{M}', \bar{m}')$.

In fact, let us define an effective transformation of formulas induced by Γ. For simplicity, assume that $\sigma_{\mathfrak{M}} = \langle P_0^{n_0}, \ldots, P_{k-1}^{n_{k-1}}\rangle$ and $\mathfrak{M}$ is Δ-definable in $\mathbb{HF}(\mathfrak{N})$ via a sequence $\Gamma = \langle \Phi, \Phi^*, \Psi, \Psi^*, \Phi_0, \Phi_0^*, \ldots, \Phi_{k-1}, \Phi_{k-1}^*\rangle$ of Σ-formulas with parameter $a \in HF(N)$. Define, then, effective transformations $\Gamma_1, \Gamma_2 : \mathrm{Form}(\sigma'_{\mathfrak{M}}) \to \mathrm{Form}(\sigma'_{(\mathfrak{N},\bar{n}_0)})$ in this way. Let φ be any formula of a signature $\sigma'_{\mathfrak{M}}$ without implication and with negations only alongside its atomic subformulas (every formula is logically equivalent to a formula of this kind). For such a formula, then, we define formulas $\Gamma_1(\varphi)$ and $\Gamma_2(\varphi)$ by induction on the complexity of φ as follows (hereinafter, for a Σ-formula Φ, $\sim\Phi$ denotes a Π-formula that is logically equivalent to $\neg\Phi$):

(1) if $\varphi \rightleftharpoons P_i(t_0, \ldots, t_{n_i-1})$, then $\Gamma_1(\varphi) \rightleftharpoons \sim\Phi_i^*(t_0, \ldots, t_{n_i-1}, a) \wedge \sim\Phi(t_0, a) \wedge \ldots \wedge \sim\Phi(t_{n_i-1}, a)$, $\Gamma_2(\varphi) \rightleftharpoons \Phi_i(t_0, \ldots, t_{n_i-1}, a) \wedge \Phi(t_0, a) \wedge \ldots \wedge \Phi(t_{n_i-1}, a)$;

(2) if $\varphi \rightleftharpoons \neg P_i(t_0, \ldots, t_{n_i-1})$, then $\Gamma_1(\varphi) \rightleftharpoons \sim\Phi_i(t_0, \ldots, t_{n_i-1}, a) \wedge \sim\Phi(t_0, a) \wedge \ldots \wedge \sim\Phi(t_{n_i-1}, a)$, $\Gamma_2(\varphi) \rightleftharpoons \Phi_i^*(t_0, \ldots, t_{n_i-1}, a) \wedge \Phi(t_0, a) \wedge \ldots \wedge \Phi(t_{n_i-1}, a)$;

(3) if $\varphi \rightleftharpoons U(t)$, then $\Gamma_1(\varphi) \rightleftharpoons \sim\Phi^*(t, a)$, $\Gamma_2(\varphi) \rightleftharpoons \Phi(t, a)$;

(4) if $\varphi \rightleftharpoons \neg U(t)$, then $\Gamma_1(\varphi) \rightleftharpoons \sim\Phi(t, a)$, $\Gamma_2(\varphi) \rightleftharpoons \Phi^*(t, a)$;

(5) if $\varphi \rightleftharpoons (t_1 = t_2)$, then $\Gamma_1(\varphi) \rightleftharpoons ((\sim\Phi^*(t_1, a) \wedge \sim\Phi^*(t_2, a) \wedge \sim\Psi^*(t_1, t_2, a)) \vee (\sim\Phi(t_1, a) \wedge \sim\Phi(t_2, a) \wedge (t_1 = t_2)))$, $\Gamma_2(\varphi) \rightleftharpoons ((\Phi(t_1, a) \wedge \Phi(t_2, a) \wedge \Psi(t_1, t_2, a)) \vee (\Phi^*(t_1, a) \wedge \Phi^*(t_2, a) \wedge (t_1 = t_2)))$;

(6) if $\varphi \rightleftharpoons \neg(t_1 = t_2)$, then $\Gamma_1(\varphi) \rightleftharpoons ((\sim\Phi^*(t_1, a) \wedge \sim\Phi^*(t_2, a) \wedge \sim\Psi(t_1, t_2, a)) \vee (\sim\Phi(t_1, a) \wedge \sim\Phi(t_2, a) \wedge \neg(t_1 = t_2)))$, $\Gamma_2(\varphi) \rightleftharpoons ((\Phi(t_1, a) \wedge \Phi(t_2, a) \wedge \Psi^*(t_1, t_2, a)) \vee (\Phi^*(t_1, a) \wedge \Phi^*(t_2, a) \wedge \neg(t_1 = t_2)))$;

(7) if $\varphi \rightleftharpoons (\varphi_1 * \varphi_2)$, $* \in \{\wedge, \vee\}$, then $\Gamma_i(\varphi) \rightleftharpoons (\Gamma_i(\varphi_1) * \Gamma_i(\varphi_2))$, $i = 1, 2$;

(8) if $\varphi \rightleftharpoons (Qx \in t)\psi$, $Q \in \{\forall, \exists\}$, then $\Gamma_i(\varphi) \rightleftharpoons (Qx \in t)\Gamma_i(\psi)$, $i = 1, 2$;

(9) if $\varphi \rightleftharpoons Qx\psi$, $Q \in \{\forall, \exists\}$, then $\Gamma_i(\varphi) \rightleftharpoons Qx\Gamma_{3-i}(\psi)$, $i = 1, 2$.

Thus, for any $n > 0$ and any $\varphi \in \mathrm{Form}(\sigma'_{\mathfrak{M}})$, the formula $\Gamma_1(\varphi)$ is Σ_n if so is φ; the formula $\Gamma_2(\varphi)$ is Π_n if so is φ. □

Definition 2. Structure $\mathfrak{M}$ is HF-*categorical of level* n ($n \leqslant \omega$) if every structure $\mathfrak{M}'$ of the signature and cardinality of $\mathfrak{M}$ satisfies

$$\mathbb{HF}(\mathfrak{M}) \equiv_n \mathbb{HF}(\mathfrak{M}') \;\Rightarrow\; \mathfrak{M} \cong \mathfrak{M}'.$$

If $n = \omega$ then we say that $\mathfrak{M}$ is HF-*categorical.*

Obviously, level 1 HF-categoricity is equivalent to level 1 categoricity in first-order logic. (Such structures are exemplified, for instance, by models of model-complete theories that are categorical in a suitable cardinality.) Examples of level 2 HF-categorical structures are countable equivalence relations without infinite classes, and groups of finite orders (see Prop. 2 below). A linear order $\langle \omega_1^{CK}, \leqslant \rangle$ may serve to exemplify a structure that is not categorical in HF-logic.

Let $\mathcal{E}$ be a countable equivalence relation, that is, a countable structure with a signature consisting of one binary predicate whose interpretation in the structure is an equivalence relation. The *characteristic* of $\mathcal{E}$ is a set $\chi(\mathcal{E}) \subseteq \omega^2$ defined as follows:

$$\chi(\mathcal{E}) = \{\langle m, n \rangle \mid \mathcal{E} \text{ contains at least } m \text{ equivalence classes of size } n\}.$$

We also define a *weak characteristic* of $\mathcal{E}$, setting

$$\chi^*(\mathcal{E}) = \{\langle m, n \rangle \mid \mathcal{E} \text{ contains at least } m \text{ classes of size at least } n\}.$$

It is not hard to verify that $\chi^*(\mathcal{E}) \equiv_e \mathrm{Th}_\exists(\mathcal{E})$. It is also clear that if finite classes of $\mathcal{E}$ are of bounded size then $\mathcal{E}$ is constructivizable. If not, we have $\chi^*(\mathcal{E}) = \omega^2$. In any case every equivalence relation is locally constructivizable. Note also that the characteristic $\chi(\mathcal{E})$ defines the relation $\mathcal{E}$ up to number of infinite equivalence classes. Furthermore, we have

PROPOSITION 2. Let $\mathcal{E}$ and $\mathcal{E}'$ be countable equivalence relations. Then:
(1) $\mathcal{E} \equiv_1 \mathcal{E}' \iff \chi^*(\mathcal{E}) = \chi^*(\mathcal{E})$;
(2) $\mathcal{E} \equiv_2^{\mathrm{HF}} \mathcal{E}' \iff \chi(\mathcal{E}) = \chi(\mathcal{E}')$.

Proof. Item (1) being obvious, we only consider (2). In fact, $\langle m, n \rangle \in \chi(\mathcal{E})$ iff there exist pairwise distinct elements $a_1^1, \ldots, a_n^1, \ldots, a_1^m, \ldots, a_n^m$ in E satisfying the following: $a_k^i \sim a_l^i$ for all i, k, l; $a_k^i \not\sim a_l^j$ for all i, j, k with $i \neq j$; for any $a \in E$, $a \sim a_k^i$ implies $a = a_l^i$ for some l. This condition may be written in the form of an $\exists\forall$-formula in the signature of equivalence relations. □

COROLLARY 1. If $\mathfrak{M}$ is a structure, which is constructivizable of level 2, then every countable equivalence relation, which is Σ-definable in $\mathbb{HF}(\mathfrak{M})$, is constructivizable.

Proof. By Proposition 1, if $\mathcal{E}$ is a countable equivalence relation for which $\mathcal{E} \leqslant_\Sigma \mathfrak{M}$, then $\mathcal{E}$ is also locally constructivizable of level 2. This, in view of Proposition 2, implies that $\mathcal{E}$ is constructivizable. □

The next proposition holds that a class of locally constructivizable (of level 1) countable structures is closed downward w.r.t. $\leqslant_w$, which is weakest among the reducibilities under consideration; this result follows immediately from Proposition 1 and from [1].

PROPOSITION 3. Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable structures. Then $\mathfrak{N} \leqslant_\exists \mathfrak{M}$ if $\mathfrak{N} \in \mathcal{K}_w(\mathfrak{M})$. In particular, if $\mathfrak{M}$ is locally constructivizable, then every structure $\mathfrak{N} \in \mathcal{K}_w(\mathfrak{M})$ is also locally constructivizable.

A pair $(\mathfrak{M}, \mathfrak{N})$ is locally constructivizable iff so are $\mathfrak{M}$ and $\mathfrak{N}$; therefore, a set of degrees generated by locally constructivizable structures is an ideal in semilattices $\mathcal{S}_*$, $* \in \{\Sigma, e, \;, w, ew\}$. Classes of locally constructivizable structures of level n, $n > 1$, however, are downward closed w.r.t. $\leqslant_\Sigma$ only (so they form initial segments in $\mathcal{S}_\Sigma$). For weaker reducibilities, this is not the case. For example, we have

THEOREM 1. There exists a countable structure $\mathfrak{M}_0$ which is locally constructivizable of level 1 (strictly) and is such that $\mathfrak{M}_0 \leqslant \mathfrak{M}$ for every nonconstructivizable countable structure $\mathfrak{M}$. Specifically, if $\mathfrak{M}$ is locally constructivizable of level $n > 1$ but is not constructivizable, then $\mathcal{K}_\Sigma(\mathfrak{M}) \subsetneq \mathcal{K}(\mathfrak{M})$.

The **proof** makes use of the result (obtained in [4], and independently, in [5]) which holds that there exists a structure whose problem of presentability belongs to a least nonzero degree of the Medvedev lattice (which, in particular, means that a semilattice $\mathcal{S}$ of degrees of presentability has a least nonzero element). Every such structure is locally constructivizable. Namely, we have

PROPOSITION 4. Let $\mathfrak{M}$ be a structure such that $\underline{\mathfrak{M}} \leqslant_w \mathcal{A}$ for every noncomputable mass problem $\mathcal{A}$. Then $\mathfrak{M}$ is locally constructivizable.

Proof. Obviously, $\mathrm{Th}_\exists(\mathfrak{M}, \bar{m}) \leqslant_e \mathcal{C}$ for every presentation $\mathcal{C}$ of an arbitrary system $\mathfrak{M}$ and for any $\bar{m} \in M^{<\omega}$. Therefore, if $\mathfrak{M}$ satisfies the hypotheses of the proposition then, in particular, for every $\bar{m} \in M^{<\omega}$, we have $\mathrm{Th}_\exists(\mathfrak{M}, \bar{m}) \leqslant_e X$ with any $X \subseteq \omega$ which is not c.e. This immediately implies that $\mathfrak{M}$ is locally constructivizable. □

Based on the construction in [4] of a structure $\mathfrak{M}_S$ with the above property, we furnish an example of a structure $\mathfrak{M}$ and a relation $P \subseteq M$ such that $\underline{(\mathfrak{M}, P)} \equiv \underline{\mathfrak{M}}$ but $(\mathfrak{M}, P)$ is not Σ-definable in $\mathbb{HF}(\mathfrak{M})$. Let $\mathfrak{M}'_S$ be a constructivizable structure for which $\mathfrak{M}_S \subseteq \mathfrak{M}'_S$. The structure $\mathfrak{M}'_S$ is obtained by adding to $\mathfrak{M}_S$ infinitely many labels for every (not necessarily maximal) path in $\mathfrak{M}_S$.

For any structure $\mathfrak{A}$ that is not constructivizable but possesses a constructivizable extension which is 2-elementary in HF-logic, we consider a structure $(\mathfrak{A}, \mathfrak{M}'_S)$, a model-theoretic pair of the structures $\mathfrak{A}$ and $\mathfrak{M}'_S$. Define a unary relation $P \subseteq M'_S$ as consisting of labels for the paths in $\mathfrak{M}_S \subseteq \mathfrak{M}'_S$: that is, for every such path, P should contain infinitely many such labels, and we must be left with infinitely many labels of such paths not in P.

We have $\underline{\mathfrak{M}_S} \leqslant \underline{(\mathfrak{A}, \mathfrak{M}'_S)}$, and hence $\underline{((\mathfrak{A}, \mathfrak{M}'_S), P)} \leqslant \underline{(\mathfrak{A}, \mathfrak{M}'_S)}$ (new labels for the relation P are constructed using $\mathfrak{M}_S$ as a model). Reducibility in the opposite direction is obvious. Moreover, the structure $((\mathfrak{A}, \mathfrak{M}'_S), P)$ is not Σ-definable in $\mathbb{HF}(\mathfrak{A}, \mathfrak{M}'_S)$, since in this instance $\mathfrak{M}_S$ would be Σ-definable in $\mathbb{HF}(\mathfrak{A}, \mathfrak{M}'_S)$, which it is not by Prop. 1. In fact, we have

LEMMA 1. Let $\mathfrak{M}$, $\mathfrak{M}'$, and $\mathfrak{N}$ be countable structures such that $\mathfrak{N}$ is locally constructivizable and $\mathfrak{M} \preccurlyeq_2^{\mathrm{HF}} \mathfrak{M}'$. Then $(\mathfrak{M}, \mathfrak{N}) \preccurlyeq_2^{\mathrm{HF}} (\mathfrak{M}', \mathfrak{N})$.

Proof. Let

$$(\mathfrak{M}, \mathfrak{N}) \models \bigwedge_{i\in\omega} \forall \bar{a}_i \forall \bar{b}_i \bigvee_{j\in\omega} (\varphi_{ij}(\bar{a}_i) \wedge \psi_{ij}(\bar{b}_i)).$$

For every $i \in \omega$ and every $\bar{b} \in N^{<\omega}$, set $J_i(\bar{b}) = \{j \in \omega \mid \mathfrak{N} \models \psi_j(\bar{b})\}$. By the hypothesis, for all $i \in \omega$ and all $\bar{b} \in N^{<\omega}$,

$$\mathfrak{M} \models \forall \bar{a}_i \bigvee_{j\in J_i(\bar{b})} \varphi_{ij}(\bar{a}).$$

Since $\mathfrak{M} \preccurlyeq_n^{\mathrm{HF}} \mathfrak{M}'$, the same formulas are valid also in $\mathfrak{M}'$, that is,

$$(\mathfrak{M}', \mathfrak{N}) \models \forall \bar{a}_i \bigvee_{j\in\omega} (\varphi_{ij}(\bar{a}) \wedge \psi_{ij}(\bar{b}))$$

for every $i \in \omega$ and every $\bar{b} \in N^{<\omega}$, as desired. □

Thus we have in fact proved the following:

THEOREM 2. There exist a countable structure $\mathfrak{M}$ and a unary relation $P \subseteq M$ for which $\underline{(\mathfrak{M}, P)} \equiv \underline{\mathfrak{M}}$ but $(\mathfrak{M}, P) \nleqslant_\Sigma \mathfrak{M}$.

Theorem 2 is of interest in connection with the following result in [6]: for any countable structure $\mathfrak{M}$, a relation $P \subseteq M^n$, $n \in \omega$, is Σ-definable in $\mathbb{HF}(\mathfrak{M})$ iff $P^{\mathcal{C}}$ is $\mathcal{C} \upharpoonright \sigma_{\mathfrak{M}}$-c.e. for every $\mathcal{C} \in \underline{(\mathfrak{M}, P)}$ (see [6]).

For structures $\mathfrak{M}$ and $\mathfrak{N}$ with $\mathrm{card}(M) \leqslant \mathrm{card}(N)$, consider the class

$$\mathcal{K}(\mathfrak{M}, \mathfrak{N}) = \{\mathfrak{M}' \mid \Pr(\mathfrak{M}', \mathbb{HF}(\mathfrak{N})) \leqslant \Pr((\mathfrak{M}, \bar{m}), \mathbb{HF}(\mathfrak{N})),\ \bar{m} \in M^{<\omega}\}.$$

Classes $\mathcal{K}_e(\mathfrak{M}, \mathfrak{N})$, $\mathcal{K}_w(\mathfrak{M}, \mathfrak{N})$, and $\mathcal{K}_{ew}(\mathfrak{M}, \mathfrak{N})$ are defined similarly.

PROPOSITION 5. Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable structures and let $\mathfrak{N}$ be a structure of the empty signature, or dense linear order. Then $\mathcal{K}_\Sigma(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M}, \mathfrak{N}) = \mathcal{K}(\mathfrak{M}, \mathfrak{N})$.

Proof. For $\mathfrak{N} = \mathbb{S}$, where $\mathbb{S}$ is an infinite structure of the empty signature, the argument is trivial. Consider a presentation of $\mathfrak{M}$ with a subset S as domain. For $\mathbb{A} = \mathbb{L}$, where $\mathbb{L}$ is a dense linear order, consider a presentation of $\mathfrak{M}$ with a domain consisting of subsets of L that are "mutually dense," that is, between any representatives of any distinct elements, and also on the left and right, there are infinitely many representatives of any other element. □

As a consequence, there exist natural isomorphisms between a semilattice $\mathcal{S}_\Sigma$ of degrees of Σ-definability and semilattices $\mathcal{S}(\mathbb{HF}(\mathfrak{N}))$ of degrees of presentability, where $\mathfrak{N}$ is a countable structure of the empty signature, or dense linear order.

Definition 3. Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable structures. Structure $\mathfrak{M}$ *has a degree* (an *e-degree*) *over structure* $\mathfrak{N}$ if there exists a least degree among all $T\Sigma$-degrees ($e\Sigma$-degrees) of all possible presentations of $\mathfrak{M}$ in $\mathbb{HF}(\mathfrak{N})$.

An immediate consequence of [1, Thm. 6] and a generalization of [1, Thm. 1] is the following:

THEOREM 3. Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable structures. Then the conditions below are equivalent:

(1) $\mathfrak{M}$ has a degree (an e-degree) over $\mathfrak{N}$;

(2) some presentation $\mathcal{C} \subseteq HF(N)$ of $\mathfrak{M}$ is Δ-definable (Σ-definable) in $\mathbb{HF}(\mathfrak{M}, \mathfrak{N})$.

Obviously, for $\mathfrak{M} \leqslant_\exists \mathfrak{N}$, the structure $\mathfrak{M}$ has a degree (an e-degree) over $\mathfrak{N}$ iff $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$. It is also clear that if $\mathfrak{M}$ has a degree (an e-degree) over $\mathfrak{N}$, and $\mathfrak{N} \leqslant_\Sigma \mathfrak{N}'$, then $\mathfrak{M}$ has a degree (an e-degree) over $\mathfrak{N}'$. Furthermore, we have

PROPOSITION 6. For any countable structure $\mathfrak{A}$, there exists a structure $\mathfrak{M}$ which has a degree but is not Σ-definable in $\mathbb{HF}(\mathfrak{A})$.

Proof. Let $A \subseteq \omega$ be a subset of natural numbers which is not Δ-definable in $\mathbb{HF}(\mathfrak{A})$ (such exists by reason of the fact that every countable admissible set has countably many Σ-subsets). An Abelian group $G_{A \oplus \bar{A}}$ is not Σ-definable in $\mathbb{HF}(\mathfrak{A})$, since otherwise A would be a Δ-subset in $\mathbb{HF}(\mathfrak{A})$. At the same time, G_A has a degree in $\mathbb{HF}(\varnothing)$ and hence in $\mathbb{HF}(\mathfrak{A})$. □

As in a nonrelativized case, we have

THEOREM 4. Let $\mathfrak{M}$ and $\mathfrak{N}$ be countable structures. If $\mathfrak{M}$ has a degree over $\mathfrak{N}$, then $\mathcal{K}_\Sigma(\mathfrak{M}, \mathfrak{N}) = \mathcal{K}_e(\mathfrak{M}, \mathfrak{N}) = \mathcal{K}(\mathfrak{M}, \mathfrak{N})$. If $\mathfrak{M}$ has an e-degree over $\mathfrak{N}$, then $\mathcal{K}_\Sigma(\mathfrak{M}, \mathfrak{N}) = \mathcal{K}_e(\mathfrak{M}, \mathfrak{N})$.

The **proof** is a direct generalization of the argument in [1, Thm. 7] combined with Theorem 3. □

2. UNIFORM REDUCIBILITIES OF LOCAL HF-THEORIES

Another necessary condition for the relation $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ to hold between structures $\mathfrak{M}$ and $\mathfrak{N}$ is the existence of uniform effective reducibility between families of local HF-theories for these structures. An exact definition of such reducibility will be given below.

Hereinafter, by a *family* we mean an arbitrary family of subsets of ω. We define the action of effective operators on families by extending the scope of action of classical enumeration operators by a set $P(P(\omega))$ of families, setting, for a family $\mathcal{X} \subseteq P(\omega)$ and an enumeration operator $\Phi : P(\omega) \to P(\omega)$,

$$\Phi(\mathcal{X}) = \{\Phi(D) \mid D \in \mathcal{X}^{<\omega} \text{ is a finite collection of sets}\},$$

where $\Phi(D) = \Phi(X_1 \oplus \ldots \oplus X_n)$ for $D = \langle X_1, \ldots, X_n \rangle$.

Let $\mathcal{A}, \mathcal{B} \subseteq P(P(\omega))$ be arbitrary classes of families. We say that $\mathcal{A}$ is *Dyment reducible* to $\mathcal{B}$ (written $\mathcal{A} \leqslant_e \mathcal{B}$) if $\Phi(\mathcal{B}) \subseteq \mathcal{A}$ for some enumeration operator Φ. Then we define maps $i : P(P(\omega)) \to P(P(P(\omega)))$ and $j : P(P(\omega)) \to P(P(P(\omega)))$ on families by setting $i(\mathcal{X}) = \{\mathcal{X}\}$ and $j(\mathcal{X}) = \{\{X\} \mid X \in \mathcal{X}\}$ for the family $\mathcal{X}$. Obviously, for any $\mathcal{X}, \mathcal{Y} \subseteq P(\omega)$, the reducibility $j(\mathcal{X}) \leqslant_e j(\mathcal{Y})$ holds iff $\mathcal{X} \leqslant_e \mathcal{Y}$ (in the latter case by $\leqslant_e$ we mean Dyment reducibility on families). We write $\mathcal{X} \leqslant_e^o \mathcal{Y}$ to signify the fact that $i(\mathcal{X}) \leqslant_e i(\mathcal{Y})$, and write $\mathcal{X} \leqslant^o \mathcal{Y}$ for the fact that $i(T(\mathcal{X})) \leqslant_e i(T(\mathcal{Y}))$, where $T(\mathcal{X}) = \{X \oplus \overline{X} \mid X \in \mathcal{X}\}$.

Let $\mathcal{X} \subseteq P(P(\omega))$ and $\bar{X}^0 = \langle X_1^0, \ldots, X_k^0 \rangle$, $X_1^0, \ldots, X_k^0 \subseteq \omega$. A *shift* of the family $\mathcal{X}$ by the set $\bar{X}^0$ is a family $\bar{X}^0 * \mathcal{X} = \{X_1^0 \oplus \ldots \oplus X_k^0 \oplus X \mid X \in \mathcal{X}\}$. Dyment reducibility on the set $P(P(P(\omega)))$ is obviously reflexive and transitive. An equivalence relation $\equiv_e$ is defined in a regular way: $\mathcal{A} \equiv_e \mathcal{B}$ iff $\mathcal{A} \leqslant_e \mathcal{B}$ and $\mathcal{B} \leqslant_e \mathcal{A}$. By analogy with the Dyment lattice $\mathcal{M}_e$, the degree structure $\langle P(P(P(\omega)))/\equiv_e, \leqslant_e \rangle$ is denoted by $\mathcal{M}'_e$.

PROPOSITION 7. $\mathcal{M}'_e$ is a lattice with 0 and 1, and $j : \mathcal{M}_e \to \mathcal{M}'_e$ is an embedding preserving $\wedge, \vee, 0, 1$.

Proof. The greatest element of $\mathcal{M}'_e$ is obviously $[\varnothing]_e$, and the least one is $[\{\{\varnothing\}\}]_e$; moreover, $j(0_{\mathcal{M}_e}) = i(0_{\mathcal{M}_e}) = \{\{\varnothing\}\}$, $j(1_{\mathcal{M}_e}) = \varnothing$, $i(1_{\mathcal{M}_e}) = \{\varnothing\}$, and $\varnothing \equiv_e \{\varnothing\}$. The operations $\vee$ and $\wedge$ on $\mathcal{M}'_e$ are defined as follows: for classes $\mathcal{A}, \mathcal{B} \subseteq P(P(\omega))$, set

(1) $\mathcal{A} \vee \mathcal{B} = \{\mathcal{X} \vee \mathcal{Y} \mid \mathcal{X} \in \mathcal{A}, \mathcal{Y} \in \mathcal{B}\}$, where $\mathcal{X} \vee \mathcal{Y} = \{X \oplus Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}$;

(2) $\mathcal{A} \wedge \mathcal{B} = \{0\} * \mathcal{A} \cup \{1\} * \mathcal{B}$, where $\{0\} * \mathcal{A} = \{\{0\} * \mathcal{X} \mid \mathcal{X} \in \mathcal{A}\}$ and $\{1\} * \mathcal{B} = \{\{1\} * \mathcal{Y} \mid \mathcal{Y} \in \mathcal{B}\}$. □

By analogy with how the nonuniform analog $\mathcal{M}_{ew}$ of a Dyment lattice is defined, we can define a nonuniform analog $\mathcal{M}'_{ew}$ of the lattice $\mathcal{M}'_e$. Furthermore, lattices $\mathcal{M}'$ and $\mathcal{M}'_w$ generated by classes of families of total sets (or functions) may be defined similarly to Medvedev and Muchnik lattices.

PROPOSITION 8. Let $\mathfrak{M}$ and $\mathfrak{N}$ be structures of arbitrary cardinality and $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$. For some $\bar{n}_0 \in N^{<\omega}$ and some $1 < n \leqslant \omega$, we then have the following reducibilities:

$$\{\mathrm{Th}^{\mathrm{HF}}_{\Sigma_n}(\mathfrak{M}, \bar{m}) \mid \bar{m} \in M^{<\omega}\} \leqslant_e^o \{\mathrm{Th}^{\mathrm{HF}}_{\Sigma_n}(\mathfrak{N}, \bar{n}_0, \bar{n}) \mid \bar{n} \in N^{<\omega}\},$$
$$\{\mathrm{Th}^{\mathrm{HF}}_{\Pi_n}(\mathfrak{M}, \bar{m}) \mid \bar{m} \in M^{<\omega}\} \leqslant_e^o \{\mathrm{Th}^{\mathrm{HF}}_{\Pi_n}(\mathfrak{N}, \bar{n}_0, \bar{n}) \mid \bar{n} \in N^{<\omega}\},$$
$$\{\mathrm{Th}^{\mathrm{HF}}_{n}(\mathfrak{M}, \bar{m}) \mid \bar{m} \in M^{<\omega}\} \leqslant_e^o \{\mathrm{Th}^{\mathrm{HF}}_{n}(\mathfrak{N}, \bar{n}_0, \bar{n}) \mid \bar{n} \in N^{<\omega}\}.$$

Proof. Let $\varphi \in \mathrm{Form}(\sigma'_{\mathfrak{M}})$. Then $\varphi \in \mathrm{Th}^{\mathrm{HF}}_{\Sigma_n}(\mathfrak{M}, \bar{m})$ for some tuple $\bar{m} \in M^{<\omega}$ iff there are a tuple $\bar{n} \in N^{<\omega}$ and elements $\varkappa_1, \ldots, \varkappa_s \in HF(\omega)$ for which $\langle \varkappa_1(\bar{n}), \ldots, \varkappa_s(\bar{n}) \rangle$ corresponds to $\bar{m}$ in a presentation defined by Γ, and $\mathbb{HF}(\mathfrak{N}) \models \Gamma_1(\varphi)(\varkappa_1(\bar{n}), \ldots, \varkappa_s(\bar{n}))$, where Γ_1 is as in the proof of Prop. 1. Thus, for any $\bar{m} \in M^{<\omega}$ and any $\varphi \in \mathrm{Form}(\sigma'_{\mathfrak{M}})$, the inclusion $\varphi \in \mathrm{Th}^{\mathrm{HF}}_{\Sigma_n}(\mathfrak{M}, \bar{m})$ holds iff there exist elements $\varkappa_1, \ldots, \varkappa_s \in HF(\omega)$ such that $\mathbb{HF}(\mathfrak{N}) \models \Phi(\varkappa_i(\bar{n}), a)$ for all $1 \leqslant i \leqslant s$, and

$$\mathbb{HF}(\mathfrak{N}) \models \Gamma_1(\varphi)(\varkappa_1(\bar{n}), \ldots, \varkappa_s(\bar{n})).$$

Note that the tuple $\langle \varkappa_1, \ldots, \varkappa_s \rangle$ can be coded, for instance, by a tuple $\langle n, \ldots, n, n' \rangle \in N^{<\omega}$ of length $\gamma^{-1}(\bar{\varkappa}) + 1$, where $n \neq n'$ and γ is some Gödel numbering of the set $HF(\omega)^{<\omega}$. $\square$

With $\mathfrak{M}$ we associate a family $\mathcal{E}(\mathfrak{M}) = \{\mathrm{Th}_\exists(\mathfrak{M}, \bar{m}) \mid \bar{m} \in M^{<\omega}\}$ of $\exists$-types of finite tuples of elements in $\mathfrak{M}$. A consequence of Theorem 4 is the following:

COROLLARY 2. For any structures $\mathfrak{M}$ and $\mathfrak{N}$, $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ implies $\mathcal{E}(\mathfrak{M}) \leqslant^o_e \mathcal{E}(\mathfrak{N}, \bar{n}_0)$ for some $\bar{n}_0 \in N^{<\omega}$.

There are examples of structures for which the above-mentioned necessary condition $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ is also sufficient. With each family $\mathcal{X} \subseteq P(\omega)$ we associate a structure $\mathfrak{A}_\mathcal{X}$ defined as follows. As a domain of the structure we take the set $\omega \cup S$, where S is a set of cardinality 2^ω. The signature of $\mathfrak{A}_\mathcal{X}$ consists of a unary function symbol s, which is interpreted on ω in a regular manner ($s(n) = n+1$) and is identical on S, and a binary predicate symbol R, which is interpreted by the rule $R \subseteq S \times \omega$, $\mathcal{X} = \{\{n \in \omega \mid R(s,n)\} \mid s \in S\}$. In this case, in correspondence with every element of $\mathcal{X}$ are 2^ω distinct elements (labels) of S. Moreover, there are 2^ω elements of S that are not related via R to any element of ω, and there are 2^ω elements of S that are related via R to all elements of ω.

We also define a structure $\mathfrak{B}_\mathcal{X}$ as follows. As a domain of the structure we take the set $C \cup S$, where C and S are disjoint sets of cardinality 2^ω. The signature of $\mathfrak{B}_\mathcal{X}$ consists of a unary predicate symbol P, distinguishing the set S, and a binary predicate symbol R, forming finite cycles (cycles of length 1 are admitted on just members of S). These cycles each contains exactly one element of S, and for different cycles, the sets of elements of C occurring in the cycles are disjoint, with $\mathcal{X} = \{\{n \in \omega \mid \exists c_0 \ldots \exists c_n (R(s, c_0) \wedge R(c_0, c_1) \ldots \wedge R(c_n, s))\} \mid s \in S\}$. In this case, in correspondence with every element of $\mathcal{X}$ are 2^ω elements (labels) of S, and there are 2^ω elements of S each of which occurs in a cycle of any finite length. Moreover, there are 2^ω elements of S which are not related via R to any elements of the domain, and there are 2^ω elements of C which do not occur in any one of the cycles.

PROPOSITION 9. For any families $\mathcal{X}, \mathcal{Y} \subseteq P(\omega)$, the following hold:

$$\mathfrak{A}_\mathcal{X} \leqslant_\Sigma \mathfrak{A}_\mathcal{Y} \iff \mathcal{X} \cup \{\varnothing, \omega\} \leqslant^o \bar{Y}_0 * (\mathcal{Y} \cup \{\varnothing, \omega\}) \text{ for some } \bar{Y}_0 \subseteq \mathcal{Y},$$
$$\mathfrak{B}_\mathcal{X} \leqslant_\Sigma \mathfrak{B}_\mathcal{Y} \iff \mathcal{X} \cup \{\varnothing, \omega\} \leqslant^o_e \bar{Y}_0 * (\mathcal{Y} \cup \{\varnothing, \omega\}) \text{ for some } \bar{Y}_0 \subseteq \mathcal{Y}.$$

As a consequence, if both structures $\mathfrak{M}$ and $\mathfrak{N}$ are of the form $\mathfrak{A}_\mathcal{X}$ or $\mathfrak{B}_\mathcal{X}$, then

$$\mathfrak{M} \leqslant_\Sigma \mathfrak{N} \iff \mathcal{E}(\mathfrak{M}) \leqslant^o_e \mathcal{E}(\mathfrak{N}, \bar{n}_0) \text{ for some } \bar{n}_0 \in N^{<\omega}.$$

Structure $\mathfrak{M}$ is said to be *locally n-low* ($n \in \omega$) if $\mathrm{Th}^{\mathrm{HF}}_{\Sigma_n}(\mathfrak{M}, \bar{m}) \in \Sigma^0_n$ for all $\bar{m} \in M^{<\omega}$. Clearly, if $\mathfrak{M}$ is locally constructivizable of level n, then $\mathfrak{M}$ is a locally n-low structure (for $n = 1$, the converse is also true). Proposition 8 implies that for any n, the property of a structure being n-low, as well as of being constructivizable of level n, is inherited under Σ-definability. In other words, if $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ and $\mathfrak{N}$ is a locally n-low structure then $\mathfrak{M}$ likewise is locally n-low.

As noted in [1], $\mathrm{card}(\mathcal{S}_\Sigma) = 2^\alpha$ for every infinite cardinal α. Now we consider the question whether maximal antichains exist in $\mathcal{S}_\Sigma(2^\omega)$.

THEOREM 5. A semilattice $\mathcal{S}_\Sigma(2^\omega)$ contains an antichain of cardinality 2^{2^ω}.

Proof. We modify the construction in [7] showing that $\mathrm{card}(\mathcal{M}) = 2^{2^\omega}$. Let $\mathcal{X} = \{X_i \subseteq \omega \mid i \in I\}$ be a family for which $\mathrm{card}(I) = 2^\omega$, and $\{[X_i]_T \mid i \in I\}$ be an antichain in $\mathcal{D}$ such that $[X_i]_T \mid [X_{j_1} \oplus \ldots \oplus X_{j_k}]_T$ for $i \notin \{j_1, \ldots, j_k\}$. For every $P \subseteq I$, set $\mathcal{X}_P = \{X_i \mid i \in P\}$. Obviously, there exists a set $\mathcal{A} \subseteq P(I)$ such that $\mathrm{card}(\mathcal{A}) = 2^{2^\omega}$ and any distinct $P, Q \in \mathcal{A}$ are incomparable w.r.t. $\subseteq$. Hence, for all distinct $P, Q \in \mathcal{A}$, the structures $\mathfrak{A}_{\mathcal{X}_P}$ and $\mathfrak{A}_{\mathcal{X}_Q}$, where $\mathcal{X}_P = \{X_i \mid i \in P\}$, are incomparable w.r.t. $\leqslant_\Sigma$. □

3. ∗-HOMOGENEOUS STRUCTURES

For an arbitrary structure $\mathfrak{M}$, we consider the following classes:

$$\mathcal{K}^*_\Sigma(\mathfrak{M}) = \{\mathfrak{N} \mid \mathfrak{N} \text{ is } \Sigma\text{-definable without parameters in } \mathbb{HF}(\mathfrak{M})\},$$
$$\mathcal{K}^*_\Delta(\mathfrak{M}) = \{\mathfrak{N} \mid \mathfrak{N} \text{ is } \Delta\text{-definable without parameters in } \mathbb{HF}(\mathfrak{M})\}.$$

If $\mathfrak{M}$ is countable then we also define the classes

$$\mathcal{K}^*_e(\mathfrak{M}) = \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant_e \underline{\mathfrak{M}}\}, \;\; \mathcal{K}^*(\mathfrak{M}) = \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant \underline{\mathfrak{M}}\}.$$

Clearly, for any countable structure $\mathfrak{M}$, $\mathcal{K}^*_\Delta(\mathfrak{M}) \subseteq \mathcal{K}^*_\Sigma(\mathfrak{M}) \subseteq \mathcal{K}_\Sigma(\mathfrak{M})$, $\mathcal{K}^*_e(\mathfrak{M}) \subseteq \mathcal{K}_e(\mathfrak{M})$, and $\mathcal{K}^*(\mathfrak{M}) \subseteq \mathcal{K}(\mathfrak{M})$.

Definition 4. Structure $\mathfrak{M}$ is *∗-homogeneous* if $\mathcal{K}^*_\Delta(\mathfrak{M}) = \mathcal{K}_\Delta(\mathfrak{M})$, and is *weakly ∗-homogeneous* if $\mathcal{K}^*_\Sigma(\mathfrak{M}) = \mathcal{K}_\Sigma(\mathfrak{M})$.

It follows immediately from the definition that the property of $\mathfrak{M}$ being (weakly) ∗-homogeneous is equivalent to $(\mathfrak{M}, \bar{m})$ being (Σ-)Δ-definable without parameters in $\mathbb{HF}(\mathfrak{M})$, for all $\bar{m} \in M^{<\omega}$. It is also obvious that if a countable structure $\mathfrak{M}$ is (weakly) ∗-homogeneous then $\mathcal{K}^*(\mathfrak{M}) = \mathcal{K}(\mathfrak{M})$ ($\mathcal{K}^*_e(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M})$, resp.). In other words, for all $\bar{m} \in M^{<\omega}$, $\underline{(\mathfrak{M}, \bar{m})} \equiv \underline{\mathfrak{M}}$ ($\underline{(\mathfrak{M}, \bar{m})} \equiv_e \underline{\mathfrak{M}}$, resp.).

PROPOSITION 10. If a structure $\mathfrak{M}$ has a degree, then there exists $\bar{m} \in M^{<\omega}$ such that the structure $(\mathfrak{M}, \bar{m})$ is ∗-homogeneous. If $\mathfrak{M}$ has an e-degree, then there exists $\bar{m} \in M^{<\omega}$ such that $(\mathfrak{M}, \bar{m})$ is weakly ∗-homogeneous.

Proof. For instance, let $\mathfrak{M}$ have a degree. Then there are a tuple $\bar{m} \in M^{<\omega}$ and a presentation $\mathcal{C}_0 \in \underline{\mathfrak{M}}$ for which $\mathcal{C}_0 \oplus \overline{\mathcal{C}_0} \leqslant_e \mathrm{Th}_e(\mathfrak{M}, \bar{m})$. Thus an s-expansion of the structure $\mathfrak{M}$ defined by the presentation $\mathcal{C}_0$, and hence the structure $(\mathfrak{M}, \bar{m})$, will be Δ-definable without parameters in $\mathbb{HF}(\mathfrak{M})$. □

An immediate consequence of [1, Thm. 7] is the following:

COROLLARY 3. If a structure $\mathfrak{N}$ is ∗-homogeneous and has a degree, then for every structure $\mathfrak{M}$ we have

$$\underline{\mathfrak{M}} \leqslant \underline{\mathfrak{N}} \iff \underline{\mathfrak{M}} \leqslant_w \underline{\mathfrak{N}}.$$

The next definition is well known in model theory (see [8]).

Definition 5. Structure $\mathfrak{M}$ is *ultrahomogeneous* if every isomorphism between finitely generated substructures of $\mathfrak{M}$ extends to an automorphism of the entire structure $\mathfrak{M}$.

It is easy to verify that if a structure of a predicate signature is ultrahomogeneous then it is ∗-homogeneous. It is also clear that a constructivizable structure (i.e., one having a computable presentation) likewise is ∗-homogeneous.

We give an example of a nonultrahomogeneous and nonconstructivizable structure which is ∗-homogeneous.

LEMMA 2. If $\alpha_1, \dots, \alpha_n$ are constructive ordinals then an expansion $\langle \omega_1^{CK}; \leqslant, \alpha_1, \dots, \alpha_n \rangle$ is Δ-definable without parameters in $\mathbb{HF}(\langle \omega_1^{CK}, \leqslant \rangle)$.

Proof. We make use of the fact that $\alpha + \omega_1^{CK} = \omega_1^{CK}$ for every constructive ordinal α. Indeed, if α is a constructive ordinal then so is $\alpha \cdot \omega$; hence $\alpha \cdot \omega < \omega_1^{CK}$. In view of $\alpha + \alpha \cdot \omega = \alpha \cdot \omega$, we have $\alpha + \omega_1^{CK} = \omega_1^{CK}$.

For simplicity, we now assume that $\alpha_1 < \ldots < \alpha_n$. Since these ordinals are all constructive, the structure $\langle \alpha_n; \leqslant, \alpha_1, \dots, \alpha_{n-1} \rangle$ is Δ-definable (without parameters of course) in $\mathbb{HF}(\varnothing)$. Therefore, the sum $\langle \alpha_n + \omega_1^{CK}; \leqslant, \alpha_1, \dots, \alpha_n \rangle$ is also Δ-definable without parameters in $\mathbb{HF}(\langle \omega_1^{CK}; \leqslant \rangle)$. □

COROLLARY 4. Let $\alpha_1, \dots, \alpha_n \in \omega_1^{CK}$ be constructive ordinals. Then $\underline{\langle \omega_1^{CK}; \leqslant, \bar{\alpha} \rangle} \leqslant \underline{\langle \omega_1^{CK}, \leqslant \rangle}$.

The **proof** follows immediately from Lemma 2. □

COROLLARY 5. Structure $\langle \omega_1^{CK}; \leqslant \rangle$ is $*$-homogeneous.

The **proof** follows immediately from Lemma 2, since the fact that $\alpha_1, \dots, \alpha_n \in \omega_1^{CK}$ implies that $\alpha_1, \dots, \alpha_n$ are constructive ordinals. □

From cardinality considerations, we conclude that no uncountable ordinal is $*$-homogeneous.

4. PRESENTABILITY DIMENSIONS

For a problem of presentability consisting of all possible presentations of some structure, it seems natural to try to find a smallest subset of the structure that would have the same properties under Medvedev (Muchnik) reducibility.

Definition 6. A countable structure $\mathfrak{M}$ has (*strong*) *presentability dimension* α (written Pr-dim$(\mathfrak{M}) = \alpha$), where α is a cardinal, if $\underline{\mathfrak{M}} \equiv \mathcal{B}$ for some $\mathcal{B} \subseteq \underline{\mathfrak{M}}$, card$(\mathcal{B}) = \alpha$, and α is the least cardinal satisfying these conditions.

Similarly, we can define the concept of *weak presentability dimension* Pr-dim$_w(\mathfrak{M})$, replacing $\equiv$ by $\equiv_w$ in the previous definition. Obviously, for every structure $\mathfrak{M}$, Pr-dim$_w(\mathfrak{M}) = 1$ iff $\mathfrak{M}$ has a degree. It is also clear that for every (countable) structure $\mathfrak{M}$,

$$1 \leqslant \text{Pr-dim}_w(\mathfrak{M}) \leqslant \text{Pr-dim}(\mathfrak{M}) \leqslant 2^\omega.$$

PROPOSITION 11. Let $\mathfrak{M}$ be a countable structure. Then the following conditions are equivalent:

(1) Pr-dim$_w(\mathfrak{M}) = 1$;

(2) Pr-dim$(\mathfrak{M}, \bar{m}) = 1$ for some $\bar{m} \in M^{<\omega}$.

The **proof** follows immediately from [1, Thm. 7]. □

COROLLARY 6. If $\mathfrak{M}$ is $*$-homogeneous, then

$$\text{Pr-dim}(\mathfrak{M}) = 1 \iff \text{Pr-dim}_w(\mathfrak{M}) = 1.$$

Verification of the next result is a simple matter.

PROPOSITION 12. Suppose that $\mathfrak{M}$ is a countable structure, and for some presentation $\mathcal{C}$ of $\mathfrak{M}$, $\mathcal{C} \leqslant_e \text{Th}_\exists(\mathfrak{M})$ (or, which is equivalent, some presentation of $\mathfrak{M}$ is Δ-definable without parameters in $\mathbb{HF}(\mathfrak{M})$). Then Pr-dim$(\mathfrak{M}) = 1$.

We are unaware as to whether this sufficient condition is also necessary.

It seems natural to ask the following: Does there exist a structure $\mathfrak{M}$ such that $1 <$ Pr-dim$(\mathfrak{M}) \leqslant \omega$? In this case Pr-dim$_w(\mathfrak{M}) = 1$. Indeed, the equality derives from Pr-dim$_w(\mathfrak{M}) \leqslant$ Pr-dim$(\mathfrak{M})$ and the following important assertion (obtained in [9], and independently, in [10]): for any structure $\mathfrak{M}$, $\text{Pr-dim}_w(\mathfrak{M})$ either equals 1 or is uncountable. This immediately implies that for every structure $\mathfrak{M}$, $\text{Pr-dim}(\mathfrak{M})$ either equals 1 or is infinite.

By analogy with Definition 6, for a countable structure $\mathfrak{M}$ we can introduce dimensions $\text{Pr-dim}_e(\mathfrak{M})$ and $\text{Pr-dim}_{ew}(\mathfrak{M})$. In this case it is also true that $1 \leqslant \text{Pr-dim}_w(\mathfrak{M}) \leqslant \text{Pr-dim}(\mathfrak{M}) \leqslant 2^\omega$, and $\mathfrak{M}$ has an e-degree iff $\text{Pr-dim}_{ew}(\mathfrak{M}) = 1$. Again, from $\text{Pr-dim}_w(\mathfrak{M}) = 1$ it follows that $\text{Pr-dim}_{ew}(\mathfrak{M}) = 1$, and $\text{Pr-dim}_{ew}(\mathfrak{M}) = 1$ implies $\text{Pr-dim}_e(\mathfrak{M}, \bar{m}) = 1$ for some tuple $\bar{m} \in M^{<\omega}$.

For any $A \subseteq \omega$, $[\mathcal{S}'_A]_w$ denotes a degree of the Muchnik lattice which is least among all the degrees greater than $[\mathcal{S}_A]_w$. It turns out that every such degree is a degree of presentability.

PROPOSITION 13. For any $A \subseteq \omega$, there exists a structure $\mathfrak{M}_A$ such that $[\underline{\mathfrak{M}_A}]_w = [\mathcal{S}'_A]_w$.

Proof. For every $A \subseteq \omega$, $\mathfrak{M}_A$ denotes a structure obtained by relativizing the construction in [4] with respect to A. Let $\mathfrak{D}_A$ be an arbitrary structure having degree $[A]_T$ (e.g., an Abelian group $G_{A\oplus\overline{A}}$). Then $A <_T \mathcal{C}$ for every $\mathcal{C} \in \underline{(\mathfrak{M}_A, \mathfrak{D}_A)}$, and for any $X \subseteq \omega$ with $A <_T X$, there exists an X-computable presentation of the structure $(\mathfrak{M}_A, \mathfrak{D}_A)$. Therefore, the spectrum of this structure is an "open cone" $\{\mathbf{x} \mid \mathbf{x} > \mathbf{a}\}$, where $\mathbf{a} = [A]_T$. The problem of presentability for $(\mathfrak{M}_A, \mathfrak{D}_A)$ belongs to a least degree of difficulty of the Muchnik lattice which is greater than $[\mathcal{S}_A]_w$. Finally, we note that nonuniformness of the above-described construction is brought about by the necessity to use the set A as an oracle, having *a priory* arbitrary X with $A <_T X$. $\square$

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