The Uniformization Property in Hereditary Finite Superstructures

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Abstract

In this article, we consider admissible sets of kind $\mathrm{HF}(\mathfrak{M})$, where $\mathfrak{M}$ is a model of a regular theory. We find a criterion of uniformization in $\mathrm{HF}(\mathfrak{M})$ formulated in terms of definability of Skolem functions. We prove a corollary that reads: hereditary finite superstructures $\mathrm{HF}(\mathbf{R})$ and $\mathrm{HF}(\mathbf{Q_p})$ over reals and over pādic numbers have the uniformization property.

Keywords: hereditary finite superstructure, uniformization, admissible set.

Given an arbitrary model $\mathfrak{M}$ (for instance, the field of reals), the hereditary finite superstructure $\mathrm{HF}(\mathfrak{M})$, which is the smallest admissible set over $\mathfrak{M}$, enables us to define effective computability over $\mathfrak{M}$ by using the recursion theory for admissible sets. The uniformization problem is one of the nontrivial problems in generalized recursion theory.

In this article, we consider admissible sets of kind $\mathrm{HF}(\mathfrak{M})$, where $\mathfrak{M}$ is a model of a regular theory. We find a criterion of uniformization in $\mathrm{HF}(\mathfrak{M})$ formulated in terms of definability of Skolem functions. As a corollary, we prove that hereditary finite superstructures $\mathrm{HF}(\mathbf{R})$ and $\mathrm{HF}(\mathbf{Q_p})$ over reals and over pādic numbers have the uniformization property.

The hereditary finite superstructure $\mathrm{HF}(\mathfrak{M})$ over a model $\mathfrak{M} = \langle M, \sigma^{\mathfrak{M}} \rangle$ is a model of signature $\sigma' = \sigma \cup \{U, \in, \varnothing\}$, whose universe is $\mathrm{HF(M)} = \bigcup_{\mathrm{n}\in\omega} \mathrm{H_n(M)}$, where $\mathrm{H_0(M)} = \varnothing$, $\mathrm{H_{n+1}(M)} = \{\mathrm{a} \,|\, \mathrm{a} \subseteq \mathrm{M} \cup \mathrm{H_n(M)},\ \|\mathrm{a}\| < \omega\}$. The predicate U distinguishes the set of all elements of the model $\mathfrak{M}$ (regarded as urelements), the relation $\in$ and the constant $\varnothing$ have the usual set-theoretic meaning.

In the class of all formulas of signature σ', we define a subclass of Δ_0-formulas as the closure of the class of atomic formulas under $\wedge, \vee, \neg, \rightarrow, \exists u \in v, \forall u \in v$; the class of Σ-formulas is the closure of the class of Δ_0-formulas under $\wedge, \vee, \exists u \in v, \forall u \in v$, and the quantifier $\exists u$; the class of Π-formulas consists of the negations of Σ-formulas.

A predicate over $\mathrm{HF}(\mathfrak{M})$ is called a Σ-*predicate* (Π-*predicate*), provided it is defined by a Σ-formula (Π-formula) with parameters; it is called a Δ-*predicate* in case it is a Σ- and Π-predicate simultaneously. If the graph of a function is a Σ-predicate, we call this function a Σ-*function*.

In formulas of signature σ', we conventionally distinguish between variables with values in the set of urelements and general variables, i.e. variables whose values may be arbitrary elements of an admissible set. In what follows, given a formula of signature σ we assume all its variables, free or bounded, to be variables for urelements.

Fix a Gödel numbering of formulas of signature σ' which distinguishes variables for urelements. The Gödel number of a formula φ is denoted by $[\varphi]$. The truth predicate Σ-Sat for Σ-formulas is defined in $\mathrm{HF}(\mathfrak{M})$ as follows:

$$\Sigma\text{-Sat}\big(\mathrm{a}, \langle \mathrm{b}_0, \ldots, \mathrm{b}_\mathrm{n}\rangle\big) \iff$$
$$\big(a = [\varphi]\big) \wedge \big(\varphi(x_0, \ldots, x_n) \text{ is a } \Sigma\text{-formula}\big) \wedge \big(\mathrm{HF}(\mathfrak{M}) \models \varphi(\bar{\mathrm{b}})\big).$$

One of the most important properties of admissible sets is that Σ-Sat is a Σ-predicate [2].

Recall the definitions of regular theory [1] and theory with definable Skolem functions [3]. A theory T of signature σ is called *regular* if it is model complete and decidable. By model completeness, each its formula is T-equivalent to some $\exists$-formula and, moreover, by decidability, this formula can be found effectively (henceforth, by effectiveness we mean existence of an appropriate recursive procedure on the set of Gödel numbers).

A theory T is said to be a *theory with definable Skolem functions*, provided that, for each formula $\varphi(x_0, \ldots, x_n)$ of signature σ, there exists a formula $\psi(x_0, \ldots, x_n)$ of the same signature such that

$$T \vdash \forall x_1 \ldots \forall x_n \Big[\exists x_0\, \varphi(x_0, \ldots, x_n) \to \exists !\, x_0\, \big(\varphi(x_0, \ldots, x_n) \wedge \psi(x_0, \ldots, x_n)\big)\Big].$$

Hereinafter, let T be a regular theory of signature σ with definable Skolem functions and let $\mathfrak{M} = \langle M, \sigma^{\mathfrak{M}}\rangle$ be a model of T.

Lemma 1. *Assume that P is an n-ary definable predicate over $\mathfrak{M}$. Then, given a formula defining P, we can effectively find an $\exists$-formula with the same set of parameters which defines an $n\bar{a}ry$ predicate Q on $\mathfrak{M}$ such that*

1) if $P = \varnothing$ then $Q = \varnothing$;

2) if $P \neq \varnothing$ then $Q = \{\bar{x}\}$, $\bar{x} \in P$.

Proof. Induction on n.

Suppose that $n = 1$ and let $\Phi(x_0, \bar{y})$ be a formula of signature σ that defines the predicate P with parameters $\overline{m}$ in M, i.e., $x \in P \iff \mathfrak{M} \models \Phi(x, \overline{m})$. There exists a formula $\Psi(x_0, \bar{y})$ such that

$$T \vdash \forall \bar{y} \Big[\exists x_0 \, \Phi(x_0, \bar{y}) \to \exists ! \, x_0 \, \big(\Phi(x_0, \bar{y}) \wedge \Psi(x_0, \bar{y}) \big) \Big].$$

Since T is regular, we can effectively find an $\exists$-formula $\Theta(x_0, \bar{y})$ equivalent to $\Phi \wedge \Psi$. The predicate $Q = \big\{ x \,|\, \mathfrak{M} \models \Theta(x, \overline{m}) \big\}$ satisfies all the requirements.

Now suppose that $n > 1$ and the claim is proven for all $k < n$. The predicate P is defined by the formula $\Phi(x_0, \ldots, x_{n-1}, \bar{y})$ and parameters $\overline{m}$. By the inductive hypothesis, for the predicate

$$X = \big\{ x_0 \;\big|\; \mathfrak{M} \models \exists x_1 \ldots \exists x_{n-1} \, \Phi(x_0, \ldots, x_{n-1}, \overline{m}) \big\},$$

we can effectively find an $\exists$-formula $\Psi_1(x_0, \bar{y})$ that defines a single element in X. We can also use the inductive hypothesis to effectively find an $\exists$-formula $\Psi_2(x_1, \ldots, x_{n-1}, \bar{y})$ that defines a single element in the predicate

$$Y = \Big\{ \langle x_1, \ldots, x_{n-1} \rangle \;\Big|\; \mathfrak{M} \models \exists x_0 \big(\Phi(x_0, \ldots, x_{n-1}, \overline{m}) \wedge \Psi_1(x_0, \overline{m}) \big) \Big\}.$$

The required predicate Q is defined by the formula $\Psi_1(x_0, \bar{y}) \wedge \Psi_2(x_1, \ldots, x_n, \bar{y})$ with parameters $\overline{m}$. The lemma is proven. $\square$

In what follows, we use definitions and constructions of [1]. For all $n \in \omega$, $\varkappa \in \mathrm{HF(n)}$ $\big(n = \{0, 1, \ldots, n-1\}\big)$, and $\bar{x} \in M^n$, we define an element $\varkappa(\bar{x}) \in \mathrm{HF}(\mathfrak{M})$ as follows. Define a mapping $\lambda_{\bar{x}} \, n \to M$ as $\lambda_{\bar{x}}(i) = x_i$, where $\bar{x} = \langle x_0, \ldots, x_{n-1} \rangle$. The mapping $\lambda_{\bar{x}}$ can be uniquely extended to $\lambda^{\omega}_{\bar{x}} \, \mathrm{HF(n)} \to \mathrm{HF}(\mathfrak{M})$ so that $\lambda^{\omega}_{\bar{x}}(a_0, \ldots, a_k) \rightleftharpoons \big\{ \lambda^{\omega}_{\bar{x}}(a_0), \ldots, \lambda^{\omega}_{\bar{x}}(a_k) \big\}$ for each set $\{a_0, \ldots, a_k\} \in \mathrm{HF(n)}$. Then we put $\varkappa(\bar{x}) \rightleftharpoons \lambda^{\omega}_{\bar{x}}(\varkappa)$.

For every $\varkappa \in \mathrm{HF(n)}$, we can effectively define a term $t_{\varkappa}(x_0, \ldots, x_{n-1})$ of signature $\langle \{\}, \cup, \varnothing \rangle$ so that, for all elements $x^0_0, \ldots, x^0_{n-1} \in M$, the equality $t_{\varkappa}(x^0_0, \ldots, x^0_{n-1}) = \varkappa(\bar{x}^0)$ is valid.

Define a function $h \, \omega \to \mathrm{HF}(\omega)$. For each $n \in \omega$, we put

$$h(n) = \begin{cases} n_1, & \text{if } n = c(0, n_1) \\ \{h(n_1)\}, & \text{if } n = c(1, n_1) \\ h(n_1) \cup h(n_2), & \text{if } n = c(2, c(n_1, n_2)) \text{ and } n_1 < n_2 \\ \varnothing, & \text{otherwise,} \end{cases}$$

where $c(n,m) = \frac{(n+m)^2+3n+m}{2}$ is Cantor's bijection. It is easy to see by definition that h is a numbering of $\mathrm{HF}(\omega)$, and since ω is a Δ-subset in $\mathrm{HF}(\mathfrak{M})$, we conclude that, in terms of [1], h is an $\mathrm{HF}(\mathfrak{M})$-constructivization of $\mathrm{HF}(\omega)$. Thus, $\mathrm{HF}(\omega)$ can be effectively defined in each superstructure. We consider $\mathrm{HF}(\omega)$ as a part of $\mathrm{HF}(\mathfrak{M})$.

Lemma 2. *Suppose that $\varphi(x)$ is a Δ_0-formula of signature σ' and let $\varkappa \in \mathrm{HF(n)}$. Then we can effectively find a formula $\varphi^*(x_0,\dots,x_{n-1})$ of signature σ so that, for each valuation $\gamma\,\{x_0,\dots,x_{n-1}\} \to M$,*

$$\mathrm{HF}(\mathfrak{M}) \models \varphi(\mathrm{x})^{\mathrm{x}}_{\mathrm{t}_{\varkappa}(\bar{\mathrm{x}})}[\gamma] \iff \mathfrak{M} \models \varphi^*(\mathrm{x}_0,\dots,\mathrm{x}_{n-1})[\gamma].$$

Proof. Given a formula $\varphi(x)$ and element $\varkappa \in \mathrm{HF(n)}$, we construct a formula $\varphi^x_\varkappa(x_0,\dots,x_{n-1})$ of signature $\sigma' \cup \{\varnothing, \{\}, \cup\}$ as follows:

1) if $\varphi = \varphi_1\, q\, \varphi_2$, $q \in \{\vee, \wedge, \to\}$, then $\varphi^x_\varkappa \rightleftharpoons (\varphi_1)^x_\varkappa\, q\, (\varphi_2)^x_\varkappa$
2) if $\varphi = \neg\varphi_1$ then $\varphi^x_\varkappa \rightleftharpoons \neg(\varphi_1)^x_\varkappa$
3) if $\varphi = (t_1\, p\, t_2)$, $p \in \{\in, =\}$, then $\varphi^x_\varkappa \rightleftharpoons (t_1\, p\, t_2)^x_{t_\varkappa(\bar{x})}$
4) if $\varphi = \exists y \in x(\varphi_1)$ then $\varphi^x_\varkappa \rightleftharpoons \bigvee\limits_{\varkappa' \in \varkappa} ((\varphi_1)^y_{\varkappa'})^x_\varkappa$
5) if $\varphi = \forall y \in x(\varphi_1)$ then $\varphi^x_\varkappa \rightleftharpoons \bigwedge\limits_{\varkappa' \in \varkappa} ((\varphi_1)^y_{\varkappa'})^x_\varkappa$
6) if $\varphi = U(x)$ then $\varphi^x_\varkappa \rightleftharpoons \begin{cases} \tau, & \text{if } \varkappa \in n \\ \neg\tau, & \text{otherwise} \end{cases}$
7) if $\varphi = P(t_0,\dots,t_k)$, $P \in \sigma$, then $\varphi^x_\varkappa \rightleftharpoons \begin{cases} P(t_0,\dots t_k)^x_{t_\varkappa(\bar{x})}, & \text{if } \varkappa \in n \\ \neg\tau, & \text{otherwise} \end{cases}$

where τ denotes the statement $\exists x(x = x)$ (without loss of generality we may assume that σ does not contain functional symbols).

Next, for any pair of terms t_0, t_1 of signature $\langle \varnothing, \{\}, \cup \rangle$ over variables for urelements $x_0,\dots,x_{n-1}$, we can effectively define formulas Φ_{t_0,t_1} and Ψ_{t_0,t_1} of empty signature so that $\mathrm{FV}(\Phi_{\mathrm{t_0,t_1}}) = \mathrm{FV}(\Psi_{\mathrm{t_0,t_1}}) = \mathrm{FV}(\mathrm{t_0}) \cup \mathrm{FV}(\mathrm{t_1})$ and, for each valuation $\gamma\,\mathrm{FV}(\mathrm{t_0} = \mathrm{t_1}) \to \mathrm{M}$, the following statements be true:

$$t_0^{\langle HF(\mathfrak{M}),\{\},\cup\rangle}[\gamma] \in t_1^{\langle HF(\mathfrak{M}),\{\},\cup\rangle}[\gamma] \iff \mathfrak{M} \models \Phi_{t_0,t_1}[\gamma]$$
$$t_0^{\langle HF(\mathfrak{M}),\{\},\cup\rangle}[\gamma] \subseteq t_1^{\langle HF(\mathfrak{M}),\{\},\cup\rangle}[\gamma] \iff \mathfrak{M} \models \Psi_{t_0,t_1}[\gamma]$$

(see [1] for a proof. The formula $\varphi^*(\bar{x})$ is obtained from $\varphi^x_\varkappa(\bar{x})$ by replacing the subformulas of kind $t_0 \in t_1$ by Φ_{t_0,t_1} and the subformulas of kind $t_0 = t_1$ by $\Psi_{t_0,t_1} \wedge \Psi_{t_1,t_0}$. The lemma is proven. □

Lemma 2 can be easily extended to formulas with several variables. This lemma also implies that we can restrict our consideration to formulas with parameters in M only.

Assume that $\Phi(x,\overline{m})$ is a Δ_0-formula of signature σ' with parameters $\overline{m}$ in M. For each $n\in\omega$, we define the set

$$\mathrm{H}_n \rightleftharpoons \{\varkappa\in \mathrm{HF}(n) \mid \mathrm{HF}(\mathfrak{M}) \models \exists x_0 \ldots \exists x_{n-1}(\Phi(x,\bar{m}))^x_{t_\varkappa(\bar{x})}\}$$

and put $\mathrm{H} \rightleftharpoons \bigcup_{\mathrm{n}\in\omega}\mathrm{H_n}$. The following lemma is valid:

Lemma 3. *The set* H *is a* Δ*-subset of* $\mathrm{HF}(\mathfrak{M})$*.*

Proof. Let $\overline{\mathrm{H}}_n \rightleftharpoons \mathrm{HF(n)}\setminus\mathrm{H_n}$, $\overline{\mathrm{H}} \rightleftharpoons \mathrm{HF}(\omega)\setminus\mathrm{H}$; then $\overline{\mathrm{H}} = \bigcup_{n\in\omega}\overline{\mathrm{H}}_n$. So, it suffices to prove that H_n is a Δ-subset of $\mathrm{HF}(\mathfrak{M})$.

Making use of Lemma 2, given a formula Φ and an element $\varkappa$, we effectively find a formula $\Psi_\varkappa(\bar{x},\overline{m})$ of signature σ such that

$$\varkappa\in\mathrm{H}_n \iff \mathfrak{M}\models \exists x_0\ldots\exists x_{n-1}\,\Psi_\varkappa(\bar{x},\overline{m}).$$

By regularity, given the formula $\exists\bar{x}\,\Psi_\varkappa(\bar{x},\bar{y})$, we can effectively find an $\exists$-formula $\Theta_\varkappa(\bar{y})$ equivalent to it. Thus,

$$\varkappa\in\mathrm{H}_n \iff \mathrm{HF}(\mathfrak{M})\models \Sigma\text{-Sat}\big([\Theta_\varkappa],\overline{\mathrm{m}}\big).$$

The case $\varkappa\in\overline{\mathrm{H}}_n$ is handled similarly. The lemma is proven. □

Now, let $\mathfrak{M}$ be a model of a regular theory with definable Skolem functions. We formulate and prove the uniformization theorem for $\mathrm{HF}(\mathfrak{M})$, the main statement.

Theorem 1. *Assume that* $E\subseteq \mathrm{HF}(\mathfrak{M})\times\mathrm{HF}(\mathfrak{M})$ *is a* Σ*-predicate. Then there exists a* Σ*-function* F *such that the following assertions are valid:*

1) $\mathrm{dom}(F) = \mathrm{pr}_1(E)$*,*

2) $\mathrm{graph}(F)\subseteq E$*,*

where $\mathrm{dom}(F) = \{x \mid F(x)\downarrow\}$, $\mathrm{graph}(F) = \{\langle x,y\rangle \mid F(x)=y\}$*, and* $\mathrm{pr}_1(E) = \{x \mid \exists\, y(\langle x,y\rangle\in E)\}$*.*

Proof. Without loss of generality, we may assume that the predicate $E(x,y)$ is defined by a formula $\exists z\,\Phi(x,y,z,\overline{m})$, where $\Phi(x,y,z,\overline{m})$ is a Δ_0-formula with parameters $\overline{m}$ in M.

It is evident that $\mathrm{pr}_1(E)$ is a Σ-predicate. Indeed, consider the Δ_0-formula

$$\Psi(x,t,\overline{m}) \rightleftharpoons \exists u\in t\;\exists v\in t\;\exists y\in u\;\exists z\in v\;\big(t=\langle y,z\rangle\wedge\Phi(x,y,z,\overline{m})\big),$$

where $\langle a,b\rangle \rightleftharpoons \big\{\{a\},\{a,b\}\big\}$ by definition. It is clear that $x \in \mathrm{pr}_1(E) \iff \mathrm{HF}(\mathfrak{M}) \models \exists \mathrm{t}\, \Psi(\mathrm{x},\mathrm{t},\overline{\mathrm{m}})$.

For each $a \in \mathrm{HF}(\mathfrak{M})$, there exist $n \in \omega$, $\varkappa \in \mathrm{HF}(\mathrm{n})$, and $a_0,\dots,a_{n-1} \in M$ such that $a = \varkappa(\bar{a})$. Let $x^* \in \mathrm{HF}(\mathfrak{M})$, $x^* = \varkappa_0(\bar{x})$, where $\varkappa_0 \in \mathrm{HF}(\mathrm{l})$, $\bar{x} = \langle x_0,\dots,x_{l-1}\rangle \in M^l$. In the same way as in Lemma 3, we define the sets

$$H_n \rightleftharpoons \{\ \varkappa \in \mathrm{HF}(n) \mid \mathrm{HF}(\mathfrak{M}) \models \exists t_0 \dots \exists t_{n-1} (\Psi(x^*,t,\bar{m}))^t_{t_\varkappa(\bar{t})}\}$$

for all $n \in \omega$ and put $\mathrm{H} \rightleftharpoons \bigcup_{n\in\omega} \mathrm{H}_n$.

If $x^* \in \mathrm{pr}_1(E)$ then the set $\big\{t \bigm| \mathrm{HF}(\mathfrak{M}) \models \Psi(\mathrm{x}^*,\mathrm{t},\overline{\mathrm{m}})\big\}$ is nonempty; hence, the set H is nonempty too. In this case, the element $\varkappa_1 \in \mathrm{H}$ minimal in the sense of the enumeration h above is uniquely defined. In other words, $\varkappa_1$ is taken so as to satisfy the following conditions:

$$\exists k\Big((k \in \omega) \wedge \big(\varkappa_1 = h(k)\big) \wedge (\varkappa_1 \in \mathrm{H}) \wedge \forall k' < k\big(h(k') \notin \mathrm{H}\big)\Big).$$

By virtue of Lemma 3, this condition is expressed in $\mathrm{HF}(\mathfrak{M})$ by some Σ-formula $\Psi_1(\varkappa_1, x^*, \overline{m})$.

Suppose that $\varkappa_1 \in \mathrm{HF}(\mathrm{n})$. Consider the set

$$T = \Big\{\langle t_0,\dots,t_{n-1}\rangle \in M^n \Bigm| \mathrm{HF}(\mathfrak{M}) \models \Psi(\mathrm{x}^*,\mathrm{t},\overline{\mathrm{m}})^{\mathrm{t}}_{\varkappa_1(\bar{\mathrm{t}})}\Big\}.$$

By Lemma 2 we can effectively construct a formula $\Theta(\bar{x},\bar{t},\bar{y})$ of signature σ so that, for each valuation $\gamma\ \{\bar{x},\bar{t}\} \to M$, the following be true:

$$\mathrm{HF}(\mathfrak{M}) \models \Psi(\mathrm{x},\mathrm{t},\overline{\mathrm{m}})^{\mathrm{x},\ \ \ \mathrm{t}}_{\varkappa_0(\bar{\mathrm{x}}),\ \varkappa_1(\bar{\mathrm{t}})}[\gamma] \iff \mathfrak{M} \models \Theta(\bar{\mathrm{x}},\bar{\mathrm{t}},\overline{\mathrm{m}})[\gamma].$$

By Lemma 1, given the formula Θ, we can effectively find an $\exists$-formula $\Theta^*(\bar{x},\bar{t},\overline{m})$ that defines a unique element $\overline{t^*}$ in the set T.

The element $t^* \rightleftharpoons \varkappa_1(\overline{t^*})$ satisfies the formula $\Psi(x^*,t^*,\overline{m})$; hence, it has the form $t^* = \langle y^*, z^*\rangle$ and, moreover, $\langle x^*, y^*\rangle \in E$. We put $F(x^*) \rightleftharpoons y^*$ by definition.

The required Σ-function F is defined as follows:

$F(x^*) = y^* \iff \exists t^* \exists z^* \exists \varkappa_0 \exists \varkappa_1 \big((t^* = \langle y^*, z^*\rangle) \wedge \Psi_1(\varkappa_1, x^*, \bar{m}) \wedge$
$\wedge\ \Sigma\text{-}Sat([\exists x_0 \dots \exists x_{l-1} \exists t_0 \dots \exists t_{n-1} (x^* = \varkappa_0(\bar{x}) \wedge t^* = \varkappa_1(\bar{t}) \wedge \Theta^*(\bar{x},\bar{t},\bar{y}))], \bar{m})\big)$. The theorem is proven. □

In [3] it was proven that the theory of real-closed fields and the theory of p-adic normed fields are theories with definable Skolem functions. Therefore, we have:

Corollary 1. *The structures* $\mathrm{HF}(\mathbf{R})$ *and* $\mathrm{HF}(\mathbf{Q_p})$ *have the uniformization property.*

Actually, the requirement of definability of Skolem functions is too stringent. Below, we prove this theorem to be true under a weaker condition that is necessary and sufficient.

We call a model $\mathfrak{M}$ of signature σ a *model with Σ-definable Skolem functions*, provided that, given any formula $\varphi(x_0,\ldots,x_n)$ of signature σ, we can effectively find a Σ-formula $\psi(x_0,\ldots,x_n)$ of signature σ' such that

$$\mathrm{HF}(\mathfrak{M})\models\forall \mathrm{x}_1\ldots\forall \mathrm{x}_n\Big[\exists \mathrm{x}_0\varphi(\mathrm{x}_0,\ldots,\mathrm{x}_n)\to\exists!\,\mathrm{x}_0\big(\varphi(\mathrm{x}_0,\ldots,\mathrm{x}_n)\wedge\psi(\mathrm{x}_0,\ldots,\mathrm{x}_n)\big)\Big]$$

(recall that $x_0,\ldots,x_n$ together with all bounded variables in φ are variables for urelements).

Let $\mathfrak{M}$ be a model of a regular theory with Σ-definable Skolem functions. In this case, an analog of Lemma 1 holds, namely:

Lemma 4. *Suppose that P is a definable n-ary predicate over $\mathfrak{M}$. Then each formula defining P can be effectively transformed into a Σ-formula that defines a predicate Q on* $\mathrm{HF}(\mathfrak{M})$ *with the same parameters, such that*

1) if $P=\varnothing$ then $Q=\varnothing$,

2) if $P\neq\varnothing$ then $Q=\{\bar{x}\}$, $\bar{x}\in P$.

Proof. The case $n=1$ is evident; so, assume that $n>1$ and that the statement is true for all $k<n$. Suppose that the predicate P is defined by a formula $\varphi(x_0,\ldots,x_{n-1},\bar{y})$ of signature σ with parameters $\overline{m}$. Given the predicate

$$X=\big\{x_0\;\big|\;\mathfrak{M}\models\exists x_1\ldots\exists x_{n-1}\,\varphi(x_0,\ldots,x_{n-1},\overline{m})\big\},$$

we can effectively find by induction a Σ-formula $\Phi(x,\bar{y})$ that defines a single element in X. Consider the predicate

$$Y=\Big\{\langle x_1,\ldots,x_{n-1}\rangle\in M^{n-1}\;\Big|\;\mathrm{HF}(\mathfrak{M})\models\exists \mathrm{x}_0\big(\varphi(\bar{\mathrm{x}},\overline{\mathrm{m}})\wedge\Phi(\mathrm{x}_0,\overline{\mathrm{m}})\big)\Big\}.$$

By Lemma 2, $\mathrm{HF}(\mathfrak{M})\models\Phi(\mathrm{x}_0,\overline{\mathrm{m}})\iff\mathfrak{M}\models\bigvee_{\mathrm{i}\in\omega}\varphi_{\mathrm{i}}(\mathrm{x}_0,\overline{\mathrm{m}})$, where φ_i are formulas of signature σ and the set $\big\{[\varphi_i]\;\big|\;i\in\omega\big\}$ is recursively enumerable. Whence

$$Y=\Big\{\langle x_1,\ldots,x_{n-1}\rangle\in M^{n-1}\;\Big|\;\mathfrak{M}\models\bigvee_{i\in\omega}\exists x_0\big(\varphi(\bar{x},\overline{m})\wedge\varphi_i(x_0,\overline{m})\big)\Big\}.$$

Assume that $i_0 = \mu i\Big(\mathfrak{M} \models \exists\bar{x}\big(\varphi(\bar{x},\overline{m}) \wedge \varphi_i(x_0,\overline{m})\big)\Big)$; by regularity, i_0 is defined by a Σ-formula in $\mathrm{HF}(\mathfrak{M})$. Since the formula $\Phi(x_0,\overline{m})$ is true for at most one element, we have

$$Y = \Big\{\langle x_1,\ldots,x_{n-1}\rangle \in M^{n-1} \;\Big|\; \mathfrak{M} \models \exists x_0\big(\varphi(\bar{x},\overline{m}) \wedge \varphi_{i_0}(x_0,\overline{m})\big)\Big\}.$$

We find by induction a Σ-formula $\Psi(x_1,\ldots,x_{n-1},\bar{y})$ that defines a single element in Y. The required predicate Q is defined by the Σ-formula $\Phi(x_0,\bar{y}) \wedge \Psi(x_1,\ldots,x_{n-1},\bar{y})$ with the same parameters. The lemma is proven. □

Thus, if $\mathfrak{M}$ is a model of a regular theory then the following is true:

Theorem 2. *The uniformization is true in* $\mathrm{HF}(\mathfrak{M})$ *if and only if* $\mathfrak{M}$ *is a model with* Σ*-definable Skolem functions.*

Proof. Sufficiency follows immediately from Lemma 4 and the proof of Theorem 1. Prove necessity. Define a binary Σ-predicate G as follows:

$$\langle a,x\rangle \in G \iff \Big(a = \big\langle[\varphi],x_1,\ldots,x_{n-1}\big\rangle\Big) \wedge \Sigma\text{-Sat}\big([\varphi],\langle \mathrm{x},\mathrm{x}_1,\ldots,\mathrm{x}_{n-1}\rangle\big).$$

By hypothesis, there exists a Σ-function F that uniformizes G. Take an arbitrary formula $\varphi(x_0,\ldots,x_{n-1})$ of signature σ. By regularity, we can effectively find an equivalent $\exists$-formula $\psi(x_0,\ldots,x_{n-1})$. The function

$$f(x_1,\ldots,x_{n-1}) = \lambda x_1.\ \ldots\ \lambda x_{n-1}.\, F\Big(\big\langle[\psi],x_1,\ldots,x_{n-1}\big\rangle\Big)$$

is a Skolem function for the formula φ. The theorem is proven. □

A dense linear order L without endpoints serves as an example of a model of a regular theory such that HF(L) has no uniformization property (this easily follows from the fact that any two tuples in L that are ordered in the same way cannot be distinguished by a formula). On the other hand, definable subsets of the fields $\mathbf{R}$ and $\mathbf{Q_p}$ have rather many common properties. In particular, the following statement holds (see [4,5]):

Proposition 1. *Let* K *be equal to* $\mathbf{R}$ *or* $\mathbf{Q_p}$*. Each nonempty definable subset in* K *either is finite or has nonempty interior.*

With the use of this property, we can suggest a common way of defining the Skolem functions for $\mathbf{R}$ and $\mathbf{Q_p}$; the way is intuitively effective and, consequently, can be expressed in terms of Σ-definability of Skolem functions in appropriate superstructures. We restrict our consideration to the field $\mathbf{Q_p}$ (for $\mathbf{R}$ the arguments are similar).

Lemma 5. *The field* $\mathbf{Q_p}$ *is a model with* Σ*-definable Skolem functions.*

Proof. Assume that $\Phi(x, \bar{y})$ is an arbitrary formula of signature $\sigma =$ $= \langle +, \cdot, ^{-1}, 0, 1, p\rangle$ and let P be a predicate defined by this formula with parameters $\bar{a}$, i.e., $x \in P \iff \mathbf{Q}_p \models \Phi(x, \bar{a})$.

Consider a system of neighborhoods about rational points, $\{C_{q,k} \mid q \in \mathbf{Q}, \mathbf{k} \in \mathbf{N}\}$, which covers $\mathbf{Q}_p$, where $C_{q,k} = \{q + p^k z \mid z \in \mathbf{Z}_p\}$. By Proposition 1, if $P \neq \varnothing$ then there exist $q \in \mathbf{Q}$ and $k \in \mathbf{N}$ such that $C_{q,k}$ satisfies one of the following conditions:

1) $\exists\,!\, x \in C_{q,k}\, \Phi(x, \bar{a})$,

2) $\forall x \in C_{q,k}\, \Phi(x, \bar{a})$.

The set $C_{q,k}$ is defined by numbers $q = (-1)^s m/n$ and k, where $s, m, n, k \in \mathbf{N}$. Hence, the neighborhood C_{q_0,k_0}, that satisfies one of the conditions and is such that the corresponding numbers s_0, m_0, n_0, k_0 are minimal, is uniquely defined. Define the Skolem function f for the formula Φ as follows: put $f(\bar{a}) \rightleftharpoons x_0$, where, in the first case, x_0 is taken as the unique point in $C_{q_0,k_0} \cap P$ and, in the second case, $x_0 \rightleftharpoons q_0$.

The function f so defined is a Σ-function on $\mathrm{HF}(\mathbf{Q}_\mathrm{p})$. Indeed, since $\mathbf{Z_p}$ is definable, we have

$$x \in C_{q,k} \iff \mathbf{Q}_p \models \exists z\big((z \in \mathbf{Z_p}) \wedge (\mathbf{x} = \mathbf{q} + \mathbf{p^k z})\big),$$

where p^k denotes the term $\underbrace{p \cdot \cdots \cdot p}_{k}$ and q denotes the term $(\underbrace{1 + \cdots + 1}_{m})/(\underbrace{1 + \cdots + 1}_{n})$; hence, the assertion "$C_{q,k}$ satisfies one of the two conditions" is expressed by the formula $\Psi_{s,m,n,k}(\bar{a})$ of signature σ, which effectively depends on s, m, n, k. From here, by regularity, the numbers s_0, m_0, n_0, k_0 (regarded as ordinals), along with x_0, are defined by a Σformula in $\mathrm{HF}(\mathbf{Q_p})$. The lemma is proven. □

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