

THE LACHLAN PROBLEM

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A generic construction realizing basic characteristics of Ehrenfeucht theories, i.e. of complete first order theories with finitely many but more than one pairwise non-isomorphic countable models, is stated. On the basis of that construction as well as of the Hrushovski — Herwig generic construction, a solution of the Lachlan problem on existence of stable Ehrenfeucht theories is shown.

For persons who interest in the Mathematical Logic.

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PREFACE

The Lachlan problem is one of the basic problems of modern abstract model theory. Model Theory, formed in an independent area in 1950th years, is on a joint of Mathematical Logic and Algebra. Its subjects are syntactic objects (theories representing descriptions of real objects) and semantic objects (algebraic systems reflecting interrelations of elements of real objects), and also classifications of syntactic objects by properties of semantic objects and vice versa. Essentially various (non-isomorphic) realizations of these theories by algebraic systems (models) are possible at descriptions of complete theories (i.e., theories without consistent information in a fixed language that can be added but not added). The number of such realizations can be various in different infinite cardinalities (i.e., with different infinite number of elements) of algebraic systems. Thus, there are spectrum functions, reflecting the number of non-isomorphic models of given theory depending on cardinalities of models, and the problem of descriptions of all possible spectrum functions for the class of all theories, and also for various essential subclasses of this class.

It is surprising that the spectrum problem is solved for large (uncountable) cardinalities in the class of all theories. Here the basic achievements are connected with works by S. Shelah [22] and finally represented in the work by B. Hart, E. Hrushovski and M. S. Laskowski [74].

For the countable (minimal infinite) cardinality the situation appeared much more difficult. Firstly, it is not known till now on an existence of theories with uncountable and non-maximal number of countable models (the Vaught problem). Secondly, constructed by A. Ehrenfeucht (see [198]) initial examples of theories with a finitely many, but more than one, countable models (now such theories, in

his honour, are called *Ehrenfeucht*) had been in essence unique for a long time: all modifications had been reduced to superstructures on infinite dense linearly ordered sets. With this circumstance, the Lachlan problem on existence of essential other (i.e., not having infinite linear orders) Ehrenfeucht theories has been arisen. In a brief formulation, the *Lachlan problem* is: to define, whether there exists a stable Ehrenfeucht theory.

This problem has been partially solved by A. H. Lachlan [109], published in 1973 a proof of an absence of Ehrenfeucht theories in the class of superstable theories, that is an important subclass of the class of stable theories. It was supposed for a long time, that the statement is true for stable theories, and it was referred in the literature, alongside with the Lachlan Problem, to the *Lachlan Conjecture* (see, for example, [27], p. 202). The Lachlan Conjecture is proved to be true for many subclasses of the class of stable theories in the works by D. Lascar [111], S. Shelah [22], A. Pillay [139], [143], [145], [146], T. G. Mustafin [128], U. Saffe [160], A. Tsuboi [197], E. Hrushovski [90], A. A. Vikent'ev [202], B. Kim [105], P. Tanović [191], [192]. At the same time, structural properties of a counterexample, if it exists, has been accumulated. The following works are connected with this accumulation: by M. G. Peretyat'kin [137], [138], M. Benda [56], R. Woodrow [205], [206], A. Pillay [140], [141], B. Omarov [132], A. Tsuboi [196], S. S. Goncharov, M. Pourmahdian [70], B. Herwig [81] and by the author. A solution of the problem, namely a proof of existence of stable Ehrenfeucht theory, became possible only after an occurrence of a nice construction created by E. Hrushovski [89] in 1988 and applied for solutions of many model-theoretic problems. Now this known construction is called the generic Hrushovski construction. It allows "to collect" required models formed via classes of finite objects using amalgams.

Another important component is the theory of group polygonometries created by the author [172], [173] and generalizing classical trigonometries. The class of group polygonometries is a convenient and geometrically clear object that has allowed to realize many structural properties of stable Ehrenfeucht theories. At the same time, now, when the general mechanism of constructions of Ehrenfeucht theories became clear, the explicit description of implicitly presented polygonometrical apparatus is superfluous in the construction. Therefore, the Polygonometrical Theory and its applications are not represented in the book.

For the construction of stable Ehrenfeucht theories, we have involved a nice modification of Hrushovski construction, offered by B. Herwig [81] for a realization of a basic structural property — the infinite weight. At the same time, this modification in an original form has been insufficient, since the Hrushovski — Herwig construction is semantic and does not take into consideration a possibility of occurrence of external connections w.r.t given finite objects, which taken as “bricks” of the generic construction.

For the elimination of this lack, the theory of syntactic generic constructions [180] has been developed by the author. Syntactic constructions are based on types (not on finite objects), i.e., on descriptions (possible to be external) of finite objects, that then allow to generate models of the required theories step-by-step.

Using the aforesaid tools, we construct a series of stable Ehrenfeucht theories.

Initial my work passed during my study in Novosibirsk State University, where the first class specialists in Mathematical Logic and Algebra worked and continue to work. An occurrence of the Siberian School of Algebra and Logic, to which I regard me, became possible after the foundation of the Institute of Mathematics in Akademgorodok, Novosibirsk in 1957 and of arrival to Novosibirsk the founder of the School, Academician Anatoliy Ivanovich Mal'tsev. Now already more thirty years this School is headed by the Director of Institute of Mathematics, Academician Yuriy Leonidovich Ershov. To the statement the Problem and to successes in its solution, I am obliged in many respects to my scientific advisor, the Head of the Laboratory of Algebraic Systems, Professor Evgeniy Andreevich Palyutin. I had a lot useful and fruitful discussions with the Correspondent member of the Russian Academy of Science, the head of Department of Mathematical Logic in IM SB RAS, the Dean of Faculty of Mechanics and Mathematics of Novosibirsk State University, Professor Sergey Savost'yanovich Goncharov, with the Professor of Department of Algebra and Mathematical Logic of Novosibirsk State Technical University Aleksandr Georgievich Pinus, with participants of the seminar “Model Theory” in IM SB RAS, doc. Aleksandr Nikolaevich Ryaskin, doc. Aleksandr Aleksandrovich Vikent'ev, doc. Dmitriy Yur'evich Vlasov, the post-graduate student Mikhail Andreevich Rusaleev, with lecturers of the Department of Algebra and Mathematical Logic in NSTU. During all my scientific activity, I approved new results before publications at the seminar “Model

Theory”, supervised by Academician Yuriy Leonidovich Ershov and Professor Evgeniy Andreevich Palyutin.

I had useful direct and correspondence dialogues with many model-theoretic specialists from France, Kazakhstan, USA, Great Britain, Israel, Japan, Germany, Poland, Czechia, Serbia.

It has turned out, after my postgraduate study in NSU, since 1990, I work already 17 years in NSTU, and 15 years of them at the Department of Algebra and Mathematical Logic, founded in 1992, which since 1992 till 2006 was headed by Professor Aleksandr Georgievich Pinus, rallied the amicable and fruitful collective. The former rector of NSTU, Professor Anatoliy Sergeevich Vostrikov and the first vice-rector (nowadays the rector) of NSTU, Professor Nikolay Vasil’evich Pustovoy promoted the creation of the Department (which with such name is rather an exception than a rule in technical institutes). The benevolent scientific atmosphere in NSTU, reading of courses of Algebra, Discrete mathematics and Mathematical logic, and also an opportunity of editions of textbooks for disciplines help to successful scientific work.

Since 2005 till present, I am a Senior Researcher of the laboratory of algebraic systems in Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Science, and the final finishing of the basic results up to articles has occurred here.

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I am grateful to all colleagues mentioned above, as well as the management of the organizations, where the work, stated in the book, has become possible to carry out.

Sergey Sudoplatov

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INTRODUCTION AND HISTORICAL REVIEW

As it was already mentioned in the preface, one of the main aim of the modern model theory is a solution of *spectrum problem*, i.e., the problem of the description of functions $I(T, \lambda)$ for numbers of pairwise non-isomorphic models for theories T in cardinalities λ for various classes of theories T . An interest to this problem is caused mainly by that the substantial structural theory is required for its solution.

The problem of description of spectrum functions, and also of classes of theories depending on these functions has paid and continues to pay attention of wide group of model-theoretic specialists, forming an extensive area of researches. It is reflected in many works, among which we mention the following: books and dissertations by J. T. Baldwin [1]; O. V. Belegradek [4]; C. C. Chang, H. J. Keisler [6]; Yu. L. Ershov, S. S. Goncharov [8]; Yu. L. Ershov, E. A. Palyutin [9]; Handbook of mathematical logic [3]; W. Hodges [15]; A. Pillay [19]; B. P. Poizat [20]; G. E. Sacks [21]; S. Shelah [22]; F. O. Wagner [27]; papers by J. T. Baldwin, A. H. Lachlan [33]; M. Benda [56]; S. Buechler [59]; S. S. Goncharov, M. Pourmahdian [70]; B. Hart, S. S. Starchenko, M. Valeriote [73]; B. Hart, E. Hrushovski, M. S. Laskowski [74]; B. Herwig, J. Loveys, A. Pillay, P. Tanović, F. O. Wagner [80]; B. Herwig [81]; E. Hrushovski [90]; K. Ikeda, A. Pillay, A. Tsuboi [95]; B. Khusainov, A. Nies, R. A. Shore [103]; B. Kim [105]; A. H. Lachlan [109]; D. Lascar [111]; J. Loveys, P. Tanović [116]; L. F. Low, A. Pillay [117]; L. Mayer [120]; T. Millar [121]–[124]; M. Morley [126], [127]; T. G. Mustafin [128]; B. Omarov [132]; E. A. Palyutin [134]; E. A. Palyutin, S. S. Starchenko [135]; M. G. Peretyat'kin [137], [138]; A. Pillay [139]–[143], [145], [146]; R. Reed [156]; A. N. Ryaskin [158]; C. Ryll-

Nardzewski [159]; U. Saffe [160], [161]; S. Shelah [163]; S. Shelah, L. Harrington, M. Makkai [165]; S. Thomas [195]; A. Tsuboi [196], [197]; P. Tanović [191]; R. Vaught [198]; R. E. Woodrow [205], [206].

As already mentioned (see S. Shelah [22]; B. Hart, E. Hrushovski, M. S. Laskowski [74]), the spectrum problem is solved as a whole for countable complete theories in uncountable cardinalities λ .

The problem of the description of number $I(T, \omega)$ of pairwise non-isomorphic countable models of the theory T for the given classes of complete theories is not well-studied till present. Here, it should to mention the *Vaught Conjecture* according to which there does not exist a theory T such that $\omega < I(T, \omega) < 2^\omega$. This conjecture has been confirmed for theories of trees (J. Steel [168]), unars (L. Marcus [119]; A. Miller [125]), varieties (B. Hart, S. S. Starchenko, M. Valeriote [73]), for ω -minimal theories (L. Mayer [120]), for theories of modules over some rings (V. A. Puninskaya [152], [153]; V. A. Puninskaya, C. Toffalori [154]). In the class of stable theories, the Vaught conjecture has proved for ω -stable theories (S. Shelah [163]; S. Shelah, L. Harrington, M. Makkai [165]), for various classes of superstable theories (E. R. Baisalov [30], [31]; S. Buechler [59], [60]; L. F. Low, A. Pillay [117]; L. Newelski [129], [130], [131]), and also for 1-based theories with a nonisolated type over a finite set such that this type is orthogonal to the empty set (P. Tanović [192]). Attempts of constructions of examples denying the Vaught conjecture had taken (see R. W. Knight [107]). However, the problem is still open.

Another interesting conjecture is the *Pillay Conjecture*. It asserts that, for any countable theory T , the condition $\text{dcl}(\emptyset) \models T$ implies $I(T, \omega) \geq \omega$. A. Pillay [142] has proven this conjecture for stable theories and has shown (see [139]), that $\text{dcl}(\emptyset) \models T$ implies $I(T, \omega) \geq 4$. P. Tanović [193] has proven, that the Pillay Conjecture holds for theories without the strict order property.

C. Ryll-Nardzewski [159], in 1959, has published his well-known theorem representing a syntactic criterion of countable categoricity of a theory (i.e., the conditions $I(T, \omega) = 1$), according which the countable categoricity of a theory is equivalent to a finiteness of number of n -types of the theory for every natural n and fixed set of free variables. It means that every countable categorical theory is defined by one characteristic, namely, the *Ryll-Nardzewski function*, that puts in the correspondence to each natural n the number of types of n fixed free variables.

Many results are connected with *Ehrenfeucht theories*, i.e., theories having finitely many (> 1) countable models. R. Vaught [198] proved that there are no complete theories having exactly two countable models. Using the theory of the dense linear order, A. Ehrenfeucht [198] has constructed initial examples of theories having equally n countable models for each natural $n \geq 3$. The further researches have been connected with constructions of Ehrenfeucht theories possessing various additional properties, with findings and investigations of structural properties of Ehrenfeucht theories, and also with findings of classes of complete theories that don't contain Ehrenfeucht theories.

M. G. Peretyat'kin [137] has constructed, for every $n \geq 3$, a complete decidable theory having exactly n countable models such that the unique is constructible. B. Omarov [132], M. G. Peretyat'kin [138], T. Millar [121], [124], S. Thomas [195], R. E. Woodrow [206] have constructed examples of Ehrenfeucht theories admitting constant expansions up to theories with infinitely many countable models, and also of non-Ehrenfeucht theories such that some their constant expansions are Ehrenfeucht. R. E. Woodrow [205] has shown, that assuming the quantifier elimination and restricting the language on a binary predicate symbol and constant symbols, the countable complete theories with exactly three countable models are, in essence, the Ehrenfeucht examples. A. Pillay [141] has shown, that an infinite dense partial order is interpretable in any Ehrenfeucht theory with few links. S. S. Goncharov and M. Pourmahdian [70] have proved that every Ehrenfeucht theory has a finite rank. It is shown in the work by K. Ikeda, A. Pillay, A. Tsuboi [95], that the dense linear order is interpretable in any almost ω -categorical theory with three countable models. P. Tanović [194] has shown that the Ehrenfeucht example or the Peretyat'kin example is interpretable in any theory with three countable models having an infinite set of pairwise different constants. E. R. Baisalov [32] has described possible numbers of countable models of o -minimal theories (the class of o -minimal theories includes the classical examples of Ehrenfeucht theories). S. Lempp and T. Slaman [115] have shown, that the property of Ehrenfeuchtness is Π_1^1 -complete. W. Calvert, V. S. Harizanov, J. F. Knight, S. Miller [61] have described the complexity of index sets of classical Ehrenfeucht theories. Constructive models of Ehrenfeucht theories are considered in the works by C. J. Ash and T. Millar [29], G. A. Omarova [133], B. Khusainov, A. Nies and

R. A. Shore [103], A. N. Gavryushkin [69], decidable Ehrenfeucht theories are in the works by T. Millar [122], [123], R. Reed [156], V. S. Harizanov [13].

The *Lachlan Problem*, that is solved in the book, is known more than thirty years. As a direction to the solution of this problem for various subclasses of the class of stable theories, an absence of theories T with $1 < I(T, \omega) < \omega$ is shown. This absence has been proved for the class of uncountably categorical theories (J. T. Baldwin, A. H. Lachlan [33]), for superstable theories (A. H. Lachlan [109], D. Lascar [111], S. Shelah [163], U. Saffe [160], A. Pillay [143]), for theories with nonprincipal superstable types (T. G. Mustafin [128]), for stable theories, in which $\text{dcl}(\emptyset)$ are models (A. Pillay [142]), for normal theories (A. Pillay [143]), for weakly normal (1-based) theories (A. Pillay [145], [146]), for theories admitting finite codings (E. Hrushovski [90]), for unions of pseudo-superstable theories (A. Tsuboi [197]), for theories without dense forking chains (B. Herwig, J. Loveys, A. Pillay, P. Tanović, F. O. Wagner [80]). A. Tsuboi [196] has proved that any Ehrenfeucht theory, being a countable union of ω -categorical theories, is unstable. A. A. Vikent'ev [202] has shown a heredity of non-Ehrenfeuchtness for extensions of non-Ehrenfeucht formula restrictions. P. Tanović [191] has shown that any stable theory, interpreting an infinite set in pairwise different constants, is non-Ehrenfeucht. He also has proved [193], that if a theory T is Ehrenfeucht then the set $\text{dcl}(\emptyset)$ is finite or the theory T has the strict order property.

A development of Theory of simple theories (see. F. O. Wagner [27]; Z. Chatzidakis, A. Pillay [63]; B. Kim, A. Pillay [104]; B. Kim [105]; M. Pourmahdian [151]; S. Shelah [164]), alongside with the Lachlan problem for stable theories, has generated a similar problem for simple theories: the *Lachlan problem for simple theories*. B. Kim [105] has generalized the Lachlan theorem (see A. H. Lachlan [109]) on superstable theories and has shown, that the class of supersimple theories doesn't contain Ehrenfeucht theories.

So-called *powerful* types, that always are represented in Ehrenfeucht theories (see M. Benda [56]), play an important role for the finding of number of countable models. In essence, the proof of absence of Ehrenfeucht theories in aforesaid classes is reduced to the assertion that these classes don't contain theories with nonprincipal powerful types. Other essential properties, that Ehrenfeucht theories possess, are the non-symmetry of the semi-isolation relation on

nonempty sets of realizations of powerful types, and also the infinite weight of nonprincipal powerful types in simple theories (see A. Pillay [143]; B. Kim [105]). The principles of systematization of structural properties of Ehrenfeucht theories and their powerful types have created in the candidate dissertation by the author [23].

A. H. Lachlan [110] has proved that structures of infinite pseudoplanes are contained in models of ω -categorical stable non-superstable theories. A. Pillay [145] has obtained a similar result for stable non-1-based theories. Thus, the positive solution of the Lachlan problem is possible only in the class of theories interpreting pseudoplanes.

Interrelations of types in theories are defined in many aspects by Rudin — Keisler preorders (see M. E. Rudin [157]). These preorders have finitely many equivalence classes for Ehrenfeucht theories. D. Lascar [111]–[113] has investigated various Rudin — Keisler preorders and has shown, that any powerful type corresponds to the greatest equivalence class w.r.t. Rudin — Keisler preorder.

E. Hrushovski [93], using a modification of generic *Jonsson — Fraïssé construction* (see R. Fraïssé [68], [10]; B. Jonsson [100], [101]), has disproved the *Zilber Conjecture* constructing examples of strongly minimal not locally modular theories in which infinite groups are not interpreted. His original construction, which served as a basis for building of appropriate examples and solving other known model-theoretic problems, has given an impetus to intensive studies of both the *Hrushovski construction* together with its various (in a broad sense) modifications, capable of creating “pathological” theories with given properties (see J. T. Baldwin [2], [34], [35], [36], [46], [47]; A. Hasson [14], [75], [76], [79]; A. S. Kolesnikov [16]; J. T. Baldwin, S. Shelah [39]; J. T. Baldwin, K. Holland [41], [42], [44]; A. Baudisch [48], [49]; A. Baudisch, A. Martin-Pizarro, M. Ziegler [53]–[52]; M. J. de Bonis, A. Nesin [57]; O. Chapuis, E. Hrushovski, P. Koiran, B. P. Poizat [62]; D. M. Evans [64], [66], [67]; D. M. Evans, M. E. Pantano [65]; A. Hasson, M. Hils [77]; A. Hasson, E. Hrushovski [78]; B. Herwig [81], [82], [83]; B. Herwig, D. Lascar [84]; K. Holland [85], [86]; E. Hrushovski [89], [91], [92]; E. Hrushovski, B. I. Zilber [94]; K. Ikeda [96], [97]; A. A. Ivanov [99]; A. S. Kechris, C. Rosendal [102]; B. Kim, A. S. Kolesnikov, A. Tsuboi [106]; N. Peatfield, B. I. Zilber [136]; A. Pillay, A. Tsuboi [147]; B. P. Poizat [148], [149]; S. Shelah [166]; S. Solecki [167]; V. V. Verbovskiy [199]; V. V. Ver-

bovskiy, I. Yoneda [200]; A. M. Vershik [201]; A. Villaveces, P. Zambrano [203]; I. Yoneda [207]; M. Ziegler [208]; B. I. Zilber [209], [210], [211]), and axiomatic bases, allowing to determine applicability bounds for that construction (see A. C. J. Bonato [5]; R. Aref'ev, J. T. Baldwin, M. Mazucco [28]; J. T. Baldwin [38]–[45]; J. T. Baldwin, N. Shi [37]; A. Baudisch [50]; Z. Chatzidakis, A. Pillay [63]; D. M. Evans [64]; J. B. Goode [71]; K. Holland [87]; K. Ikeda, A. Pillay, H. Kikyo A. Tsuboi, [98]; D. W. Kueker, M. S. Laskowski [108]; M. S. Laskowski [114]; B. P. Poizat [150]; M. Pourmahdian [151]; R. Rajani [155]; F. O. Wagner [204]).

Relatively the Lachlan problem, B. Herwig [81] has shown a fruitfulness of the Hrushovski construction, realizing, using it, a small stable theory with a type having the infinite weight.

Now we pass to the statement of results of five basic Chapters of the book.

The first Chapter begins (Section 1.1) with a syntactic characterization of the class of complete theories with finitely many countable models on the basis of Rudin — Keisler preorders and distribution functions of number of limit models. The basic part of this characterization is distributed to the class of Ehrenfeucht theories. Section 1.2 is devoted to the definitions of basic cases of inessential combinations and colorings of models, used below for descriptions of intermediate constructions, and also for the solution of the Lachlan problem. We define, in Section 1.3, the concept of type reducibility, according to which a structure of type of predicate theory is invariant w.r.t. restrictions of saturated structures to the sets of realizations of the type. It is shown, that the type reducibility property doesn't hold in stable Ehrenfeucht theories. An example, realizing an absence of the type reducibility in the class of stable theories, has constructed. The concept of powerful digraph is defined in Section 1.4. These digraphs, alongside with powerful types, are always locally presented in Ehrenfeucht structures. Connections of powerful digraphs and powerful types (always presented in Ehrenfeucht structures) have shown, and properties of structures with powerful digraphs are investigated. The results, represented in Chapter 1, are published in the works [169]–[175], [177].

We describe (became already classical) semantic generic constructions in the second Chapter (Section 2.1). Used for the solution of the Lachlan problem, syntactic generic constructions, generalizing semantic constructions, are defined in Section 2.2. Properties

of various classes of syntactic generic constructions are considered in Sections 2.3 — 2.5. Various kinds of fusions of generic constructions are investigated, that also used for constructions of required Ehrenfeucht theories (Section 2.6). Basic results, described in the Chapter 2, are stated in the works [180], [185].

In the third Chapter (Section 3.1), we realize (on the basis of syntactic generic construction and of inessential ordered coloring of acyclic digraph) an example of unstable generic powerful digraph, having unbounded lengths of shortest routes and admitting an expansions till a structure of nonprincipal powerful type. Then, on the basis of generic powerful digraph, theories with powerful types (Section 3.2) and generic Ehrenfeucht theories with three countable models (Section 3.3) are constructed. A modification of generic construction is represented, allowing to realize all possible characteristics of Ehrenfeucht theories w.r.t. Rudin — Keisler preorders and distribution functions of number of limit models (Section 3.4). These characteristics are generalized in Section 3.5 for the class of all small theories with finite Rudin — Keisler preorders modulo the Vaught conjecture. The description of Rudin — Keisler preorders in the class of small theories is shown in Section 3.6. Modifications of generic construction of Ehrenfeucht theories, based on non-dense structures of powerful digraphs, and also on structures of powerful types without powerful digraphs, are described in Section 3.7. The basic results of Chapter 3 are represented in the works [176], [179], [188], [189]. The first three Chapters, excepting Section 2.6, form the first Chapter of the author's thesis for the doctor degree [24].

In the fourth Chapter, we describe (in three steps) examples of stable generic powerful digraphs on the basis of generic Hrushovski — Herwig construction with prerank functions. At first, generic construction is transferred on bipartite digraphs with colored arcs (Section 4.1). Then it is transferred from bipartite digraphs to digraphs without furcations (Section 4.2) and, at last step, from digraphs without furcations to powerful digraphs (Section 4.3). In Section 4.4, a lack of the simplified construction of Ehrenfeucht theories from Chapter 3 is explained, that, by the specificity of construction, besides instability of structure of powerful digraph generates the formula instability from the type instability. Features of generic construction, allowing to build stable Ehrenfeucht theories, have written in Section 4.5. In Sections 4.6 — 4.9, we describe required stable Ehrenfeucht theories with all possible Rudin — Keisler pre-

orders and distribution functions of number of limit models on the basis of stable generic powerful digraphs, using Hrushovski fusions of generic constructions of powerful digraphs with generic constructions of countable family of undirected graphs. Thus, in particular, an existence of stable Ehrenfeucht theories is shown, that solves the Lachlan problem. The results of Chapter 4 are stated in the works [178], [181], [182]–[186].

In the fifth, final Chapter, the family of hypergraphs of prime models of arbitrary small theory is considered. A mechanism of the structural description of models of a theory by these families is represented. Thus it is proved, in particular, the key role of graph-theoretical constructions for examples of Ehrenfeucht theories obtained in the book. Besides, the results of previous Chapters are generalized for the class of all small theories. Chapter 5 is formed by the works [187], [190].

Below, we use without specifications:

— the model-theoretical notations and terminology in Handbook of Mathematical Logic [3] and in the book by S. V. Sudoplatov, E. V. Ovchinnikova [26] (see also Yu. L. Ershov [7]; Yu. L. Ershov, S. S. Goncharov [8]; Yu. L. Ershov, E. A. Palyutin [9]; C. C. Chang, H. J. Keisler [6]; W. Hodges [15]; A. Pillay [19]; B. P. Poizat [20]; G. E. Sacks [21]; S. Shelah [22]; F. O. Wagner [27]);

— the Graph Theory terminology in the book by S. V. Sudoplatov, E. V. Ovchinnikova [25] (see also F. Harary [12]; [17]; O. Ore [18]).

Chapter 1

CHARACTERIZATION OF EHRENFUCHTNESS. PROPERTIES OF EHRENFUCHT THEORIES

§ 1.1. Syntactic characterization of the class of complete theories with finitely many countable models

In this Section, a syntactic characterization is furnished for the class of elementary complete theories with finitely many countable models, which is the analog of a known theorem by C. Ryll-Nardzewski [159], that the countable categoricity of a theory is equivalent to a finiteness of number of n -types of the theory for every natural n and fixed set of free variables. Establishing characterization is based on classifying the theories by Rudin — Keisler preorders and distribution functions of a number of models limit over types.

Below, we denote infinite models of elementary theories by \mathcal{M} , \mathcal{N}, \dots (possibly with indexes), and their universes, by M, N, \dots . The type of tuple \bar{a} in the model \mathcal{M} will be denoted by $\text{tp}_{\mathcal{M}}(\bar{a})$ or by $\text{tp}(\bar{a})$ if the model is given. The set of all types of theory T over the empty set will be denoted by $S(T)$ or by $S(\emptyset)$. Considering the set of n -types of T , this set will be denoted by $S^n(T)$ or by $S^n(\emptyset)$.

A number of pairwise non-isomorphic models of theory T in a power λ is denoted by $I(T, \lambda)$. A theory T is *Ehrenfeucht* if $1 < I(T, \omega) < \omega$.

Unless specified otherwise, we deal with just countable complete theories. Additionally in this Section, all considered theories are countable.

DEFINITION [56]. A type $p(\bar{x}) \in S(T)$ is said to be *powerful* in a theory T if every model \mathcal{M} of T realizing p also realizes every type $q \in S(T)$, that is $\mathcal{M} \models S(T)$.

The availability of a powerful type implies that T is *small*, that is, the set $S(T)$ is countable, and hence for any type $p \in S(T)$ and its realization \bar{a} , there exists a model $\mathcal{M}_{\bar{a}}$ prime over \bar{a} . Since all prime models over realizations of p are isomorphic, we often denote such by \mathcal{M}_p .

The condition that $p(\bar{x})$ is a powerful type is equivalent to every type in $S(T)$ being realized in \mathcal{M}_p , that is, $\mathcal{M}_p \models S(T)$. Every type of ω -categorical theory T is powerful.

Lemma 1.1.1. [56]. *Every Ehrenfeucht theory T has a powerful type.*

P r o o f. Assuming on the contrary, by induction, there exists a sequence of types $p_n \in S(\emptyset)$, such that $p_n \subset p_{n+1}$ and \mathcal{M}_{p_n} omits p_{n+1} . As $\mathcal{M}_{p_m} \not\cong \mathcal{M}_{p_n}$ for $m \neq n$, then $I(T, \omega) \geq \omega$. \square

As an illustration we consider the following *Ehrenfeucht examples* [198] of theories T_n , $n \in \omega$, with $I(T_n, \omega) = n \geq 3$.

E x a m p l e 1.1.1. Let T_n be the theory of a model \mathcal{M}^n , formed from a model $\langle \mathbb{Q}; < \rangle$ by adding of constants c_k , $k \in \omega$, such that $\lim_{k \rightarrow \infty} c_k = \infty$, and by unary predicates P_0, \dots, P_{n-3} which form a partition of the set \mathbb{Q} of rationals, with

$$\models \forall x, y ((x < y) \rightarrow \exists z ((x < z) \wedge (z < y) \wedge P_i(z))),$$

$i = 0, \dots, n-3$. The theory T_n has exactly n pairwise non-isomorphic models:

a prime model \mathcal{M}^n ($\lim_{k \rightarrow \infty} c_k = \infty$);

prime models \mathcal{M}_i^n over realizations of types $p_i(x) \in S^1(\emptyset)$, isolated by sets of formulas $\{c_k < x \mid k \in \omega\} \cup \{P_i(x)\}$, $i = 0, \dots, n-3$ ($\lim_{k \rightarrow \infty} c_k \in P_i$);

a saturated model $\overline{\mathcal{M}}^n$ (the limit $\lim_{k \rightarrow \infty} c_k$ is irrational). \square

DEFINITION [143]. A tuple \bar{a} *semi-isolates* a tuple \bar{b} (over \emptyset) if there exists a formula $\varphi(\bar{x}, \bar{a}) \in \text{tp}(\bar{b}/\bar{a})$ for which $\varphi(\bar{x}, \bar{a}) \vdash \text{tp}(\bar{b})$. In this case we say that the formula $\varphi(\bar{x}, \bar{a})$ *witnesses* that \bar{b} is semi-isolated over \bar{a} .

If $p \in S(T)$ then SI_p denotes the semi-isolation relation on realizations of p :

$$\text{SI}_p = \{(\bar{a}, \bar{b}) \mid \models p(\bar{a}) \wedge p(\bar{b}) \text{ and } \bar{a} \text{ semi-isolates } \bar{b}\}.$$

Notice that the relation SI_p forms a preorder for any type $p \in S(T)$. Indeed, if for realizations \bar{a}, \bar{b} and \bar{c} of p , a formula $\varphi(\bar{x}, \bar{y})$ witnesses that \bar{b} is semi-isolated over \bar{a} and a formula $\psi(\bar{y}, \bar{z})$ witnesses that \bar{c} is semi-isolated over \bar{b} , then the formula $\exists \bar{y} (\varphi(\bar{x}, \bar{y}) \wedge \psi(\bar{y}, \bar{z}))$ witnesses that \bar{c} is semi-isolated over \bar{a} .

A preorder SI_p is called a *semi-isolation preorder* on set of realization of type p .

Lemma 1.1.2. [143]. *If $p \in S(T)$ is a nonprincipal powerful type, then nonempty relation SI_p is non-symmetric.*

P r o o f. Assuming on the contrary, SI_p is an equivalence relation. Then all realizations of the type p in a model $\mathcal{M}_{\bar{a}}$, where $\models p(\bar{a})$, are SI_p -equivalent, because \bar{a} semiisolates every tuple of elements from $M_{\bar{a}}$. On the other hand, using Compactness Theorem and the nonprincipality of p , there exists a type $q(\bar{x}, \bar{y}) \in S(T)$ such that $p(\bar{x}) \cup p(\bar{y}) \subset q(\bar{x}, \bar{y})$ and $(\bar{a}', \bar{b}') \notin \text{SI}_p$ for any realizations $\bar{a}' \wedge \bar{b}'$ of q . So the type q is omitted in \mathcal{M}_p , and we get a contradiction with the powerfulness of p . \square

Thus the availability of a non-principal powerful type $p(\bar{x})$ presumes the existence of a formula $\varphi(\bar{x}, \bar{y})$, $l(\bar{x}) = l(\bar{y})$, such that, for any (some) realization \bar{a} of p , the following conditions hold:

- (1) $\varphi(\bar{a}, \bar{y}) \vdash p(\bar{y})$;
- (2) $\varphi(\bar{x}, \bar{a}) \not\vdash p(\bar{x})$, and moreover, there exists a tuple \bar{b} which realizes type p and is such that $\models \varphi(\bar{b}, \bar{a})$ and \bar{a} does not semi-isolate \bar{b} .

Every formula $\varphi(\bar{x}, \bar{y})$, satisfying conditions 1 and 2, is called a *formula, witnessing on non-symmetry of relation SI_p* .

The formula $(x < y)$ witnesses on non-symmetry of relation SI_{p_i} in the Ehrenfeucht examples.

In what follows in this Section, unless otherwise stated, we deal with a class of small theories only.

Let p and q be types in $S(T)$. We say that the type p is *dominated by a type q* , or p *does not exceed q under the Rudin — Keisler*

preorder (written $p \leq_{RK} q$), if $\mathcal{M}_q \models p$, that is, \mathcal{M}_p is an elementary submodel of \mathcal{M}_q (written $\mathcal{M}_p \preceq \mathcal{M}_q$). Besides, we say that a model \mathcal{M}_p is dominated by a model \mathcal{M}_q , or \mathcal{M}_p does not exceed \mathcal{M}_q under the Rudin — Keisler preorder, and write $\mathcal{M}_p \leq_{RK} \mathcal{M}_q$.

Syntactically, the condition $p \leq_{RK} q$ (and hence also $\mathcal{M}_p \leq_{RK} \mathcal{M}_q$) is expressed thus: there exists a formula $\varphi(\bar{x}, \bar{y})$ such that the set $q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y})\}$ is consistent and $q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y})\} \vdash p(\bar{x})$. Since we deal with a small theory, $\varphi(\bar{x}, \bar{y})$ can be chosen so that, for any formula $\psi(\bar{x}, \bar{y})$, the set $q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}$ being consistent implies that $q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y})\} \vdash \psi(\bar{x}, \bar{y})$. In this event the formula $\varphi(\bar{x}, \bar{y})$ is said to be (q, p) -principal.

Types p and q are said to be domination-equivalent, realization-equivalent, or Rudin — Keisler equivalent (written $p \sim_{RK} q$) if $p \leq_{RK} q$ and $q \leq_{RK} p$. Models \mathcal{M}_p and \mathcal{M}_q are said to be domination-equivalent or Rudin — Keisler equivalent (written $\mathcal{M}_p \sim_{RK} \mathcal{M}_q$).

Clearly, domination relations form preorders, and domination-equivalence relations are equivalence relations.

If \mathcal{M}_p and \mathcal{M}_q are not domination-equivalent then they are non-isomorphic. Moreover, non-isomorphic models may be found among domination-equivalent ones.

In Ehrenfeucht examples, models $\mathcal{M}_{p_0}^n, \dots, \mathcal{M}_{p_{n-3}}^n$ are domination-equivalent but pairwise non-isomorphic.

A syntactic characterization for the model isomorphism between \mathcal{M}_p and \mathcal{M}_q is given by the following:

Proposition 1.1.3. *For any types $p(\bar{x})$ and $q(\bar{y})$ of a small theory T , the following conditions are equivalent:*

- (1) *models \mathcal{M}_p and \mathcal{M}_q are isomorphic;*
- (2) *there exist (p, q) - and (q, p) -principal formulas $\varphi_{p,q}(\bar{y}, \bar{x})$ and $\varphi_{q,p}(\bar{x}, \bar{y})$ respectively, such that the set*

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi_{p,q}(\bar{y}, \bar{x}), \varphi_{q,p}(\bar{x}, \bar{y})\}$$

is consistent;

- (3) *there exists a (p, q) - and (q, p) -principal formula $\varphi(\bar{x}, \bar{y})$, such that the set*

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y})\}$$

is consistent.

P r o o f. (1) \Rightarrow (2). Let $\mathcal{M}_{\bar{a}}$ and $\mathcal{M}_{\bar{b}}$ be prime models over realizations \bar{a} and \bar{b} of types $p(\bar{x})$ and $q(\bar{y})$, respectively.

If there is an isomorphism between $\mathcal{M}_{\bar{a}}$ and $\mathcal{M}_{\bar{b}'}$, the existence of (p, q) - and (q, p) -principal formulas $\varphi_{p,q}(\bar{y}, \bar{x})$ and $\varphi_{q,p}(\bar{x}, \bar{y})$, satisfying the condition that

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi_{p,q}(\bar{y}, \bar{x}), \varphi_{q,p}(\bar{x}, \bar{y})\}$$

is consistent, follows from the facts that $\mathcal{M}_{\bar{a}}$ and $\mathcal{M}_{\bar{b}}$ realize just principal types over \bar{a} and \bar{b} , respectively, and $\mathcal{M}_{\bar{a}} = \mathcal{M}_{\bar{b}'}$ for some tuple \bar{b}' realizing type $q(\bar{y})$.

(2) \Rightarrow (1). Now, assume that there exist (p, q) - and (q, p) -principal formulas $\varphi_{p,q}(\bar{y}, \bar{x})$ and $\varphi_{q,p}(\bar{x}, \bar{y})$ such that the set $p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi_{p,q}(\bar{y}, \bar{x}), \varphi_{q,p}(\bar{x}, \bar{y})\}$ is consistent. We argue to show that $\mathcal{M}_{\bar{a}}$ and $\mathcal{M}_{\bar{b}}$ are isomorphic. The existence of a (p, q) -principal formula $\varphi_{p,q}(\bar{y}, \bar{x})$ implies that $\mathcal{M}_{\bar{b}}$ can be chosen to be an elementary submodel of $\mathcal{M}_{\bar{a}}$. On the other hand, the existence of a (q, p) -principal formula $\varphi_{q,p}(\bar{x}, \bar{y})$ and the consistency of $p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi_{p,q}(\bar{y}, \bar{x}), \varphi_{q,p}(\bar{x}, \bar{y})\}$ make it possible to choose $\mathcal{M}_{\bar{b}}$ so that it is elementarily embedded in $\mathcal{M}_{\bar{a}}$ in a way that $\bar{a} \hat{=} \bar{b}$ is distinguished constantly. Since $\mathcal{M}_{\bar{a}}$ is elementarily embedded in any model constantly containing \bar{a} , $\mathcal{M}_{\bar{b}}$ too is elementarily embedded in every such model. The fact that every two prime models are isomorphic implies that $\mathcal{M}_{\bar{a}}$ and $\mathcal{M}_{\bar{b}}$ are isomorphic.

(2) \Rightarrow (3). Having (p, q) - and (q, p) -principal formulas $\varphi_{p,q}(\bar{y}, \bar{x})$ and $\varphi_{q,p}(\bar{x}, \bar{y})$, and consistent set

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi_{p,q}(\bar{y}, \bar{x}), \varphi_{q,p}(\bar{x}, \bar{y})\},$$

we get a required (p, q) - and (q, p) -principal formula $\varphi(\bar{x}, \bar{y}) \equiv \varphi_{p,q}(\bar{y}, \bar{x}) \wedge \varphi_{q,p}(\bar{x}, \bar{y})$.

The implication (3) \Rightarrow (2) is obvious. \square

Denote by $\mathbf{RK}(T)$ the set **PM** of isomorphism types of models \mathcal{M}_p , $p \in S(T)$, on which the relation of domination is induced by \leq_{RK} , a relation deciding domination among \mathcal{M}_p , that is, $\mathbf{RK}(T) = \langle \mathbf{PM}; \leq_{RK} \rangle$. We say that isomorphism types $\mathbf{M}_1, \mathbf{M}_2 \in \mathbf{PM}$ are *domination-equivalent* (written $\mathbf{M}_1 \sim_{RK} \mathbf{M}_2$) if so are their representatives.

Clearly, the preordered set $\mathbf{RK}(T)$ has a least element, which is an isomorphism type of a prime model.

Proposition 1.1.4. *If $I(T, \omega) < \omega$ then $\text{RK}(T)$ is a finite pre-ordered set whose factor set $\text{RK}(T)/\sim_{RK}$, w.r.t. domination-equivalence \sim_{RK} , forms a partially ordered set with greatest element.*

P r o o f. That **PM** is a finite set is obvious, and the fact that $\text{RK}(T)/\sim_{RK}$ contains a greatest element follows from the existence of a powerful type which dominates any type in $S(T)$. \square

Below are two obvious remarks.

Remark 1.1.5. *A theory T is ω -categorical iff $|\text{RK}(T)| = 1$.*

Remark 1.1.6. *If $|\text{RK}(T)| = 2$, then any non-principal type is powerful.*

In the above-given Ehrenfeucht examples of theories T_n with $I(T_n, \omega) = n$, each preordered set $\text{RK}(T_n)$ consists of the least element and $(n - 2)$ domination-equivalent elements corresponding to the models $\mathcal{M}_{p_0}^n, \dots, \mathcal{M}_{p_{n-3}}^n$. Thus all ordered sets $\text{RK}(T_n)/\sim_{RK}$ are two-element and linearly ordered.

Recall that a model sequence $(\mathcal{M}_n)_{n \in \omega}$ is called an *elementary chain* if \mathcal{M}_n is an elementary submodel of \mathcal{M}_{n+1} , $n \in \omega$.

An elementary chain $(\mathcal{M}_n)_{n \in \omega}$ is said to be *elementary over a type $p \in S(T)$* if $\mathcal{M}_n \simeq \mathcal{M}_p$ for any $n \in \omega$.

Proposition 1.1.7. *If $I(T, \omega) < \omega$ then for any countable model \mathcal{M} of a theory T there exists a type $p \in S(T)$ and an elementary chain $(\mathcal{M}_n)_{n \in \omega}$ over p such that $\mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_n$.*

P r o o f. Let \mathcal{M} be an arbitrary countable model of a small theory T . At first we construct an elementary chain \mathfrak{C} of prime models $\mathcal{M}_{\bar{a}_i}$ over tuples \bar{a}_i , $i \in \omega$, such that $\mathcal{M} = \bigcup_{i \in \omega} \mathcal{M}_{\bar{a}_i}$. For

this purpose, we enumerate all elements of \mathcal{M} : $M = \{b_k \mid k \in \omega\}$, and also all formulas of the form $\varphi(x, \bar{c})$, $\bar{c} \in M$: $\Phi = \{\varphi_m(x, \bar{c}_m) \mid m \in \omega\}$. We shall construct \mathfrak{C} inductively and, at any step k , some finite sequence of tuples $\bar{a}_0, \dots, \bar{a}_n$ will be defined, and also each that tuple will be connected with a finite set X_i^k , $0 \leq i \leq n$, such that unions of these sets by all k w.r.t fixed i will define universes of models $\mathcal{M}_{\bar{a}_i}$. If the tuple \bar{a}_i is not defined before the step k then sets X_i^l are supposed to be empty for any $l < k$.

At the initial step, we fix the tuple $\bar{a}_0 \equiv \langle b_0 \rangle$ and for the formula $\varphi_m(x, b_0)$ from Φ , having the minimal number and satisfying $\mathcal{M} \models \exists x \varphi_m(x, b_0)$, we find a realization d_m of a principal complete type $p(x, b_0)$ containing $\varphi_m(x, b_0)$. Now we set $X_0^0 \equiv \{b_0, d_m\}$.

Suppose that at the step k , we already have found tuples $\bar{a}_0, \dots, \bar{a}_n$ and have formed finite sets X_0^k, \dots, X_n^k satisfying the following conditions:

- 1) all elements of \bar{a}_i are contained in the set of elements of \bar{a}_{i+1} , $i < n$ and belong to X_i^k ;
- 2) $\{b_0, \dots, b_k\} \subseteq X_n^k$;
- 3) $X_i^k \subset X_{i+1}^k$, $i < n - 1$;
- 4) for chosen at the step k , a minimal w.r.t m and unconsidered before formula $\varphi_m(x, \bar{c}_m)$, containing only elements of the maximal nonempty set X_j^{k-1} and satisfying $\mathcal{M} \models \exists x \varphi_m(x, \bar{c}_m)$, a realization $d_m \in M$ of a principal complete type $p(x, X_j^{k-1} \cup \{b_k\})$, containing $\varphi_m(x, \bar{c}_m)$, is found such that for any tuple \bar{a}_i with $\bar{c}_m \in X_i^{k-1}$ and for any tuple $\bar{d} \in X_i^{k-1} \cup \{d_m\}$ the type $\text{tp}(\bar{d}/\bar{a}_i)$ is principal; this realization is added in the minimal set X_i^k w.r.t i such that $\bar{c}_m \in X_i^{k-1}$.

At the step $k + 1$, we consider the element b_{k+1} . If it belongs to X_n^k the sequence $\bar{a}_0, \dots, \bar{a}_n$ stays the same and we construct sets X_i^{k+1} adding to X_i^k an element d_m satisfying the conditions 3 and 4 for $k + 1$ instead of k .

If $b_{k+1} \notin X_n^k$ and starting with some $i_0 \leq n$, all types $\text{tp}(\bar{b}/\bar{a}_i)$, $\bar{b} \in X_i^k \cup \{b_{k+1}\}$, are principal, we again don't extend the sequence $\bar{a}_0, \dots, \bar{a}_n$ and add the element b_{k+1} to the set $X_{i_0}^k$ and to all consequent sets X_i^k , $i_0 \leq i \leq n$. Then we get sets X_i^{k+1} adding an element d_m satisfying the conditions 3 and 4 for $k + 1$ instead of k .

If some type $\text{tp}(\bar{b}/\bar{a}_n)$, $\bar{b} \in X_n^k \cup \{b_{k+1}\}$, is not principal, we add to the sequence $\bar{a}_0, \dots, \bar{a}_n$ the tuple \bar{a}_{n+1} consisting of all elements of the set $X_n^k \cup \{b_{k+1}\}$. Then we add this set to the (initially empty) set X_{n+1}^k and form sets X_i^{k+1} by adding a realization d_m of a principal complete type $p(x, X_n^k \cup \{b_{k+1}\})$ containing the minimal (w.r.t m) unconsidered before formula $\varphi_m(x, \bar{c}_m)$ containing only elements of X_n^k and satisfying $\mathcal{M} \models \exists x \varphi_m(x, \bar{c}_m)$, such that for any tuple \bar{a}_i with $\bar{c}_m \in X_i^k$ and for any tuple $\bar{d} \in X_i^k \cup \{d_m\}$, the type $\text{tp}(\bar{d}/\bar{a}_i)$ is principal. We add the element d_m to the minimal (w.r.t i) set X_i^k ,

and also to the consequent sets such that $\bar{c}_m \in X_j^k$, $i \leq j \leq n$. Now we set $X_{n+1}^{k+1} = X_n^k \cup \{b_{k+1}, d_m\}$.

By construction, the sets $X_i = \bigcup_{k \in \omega} X_i^k$ are the universes of prime models $\mathcal{M}_{\bar{a}_i}$ over tuples \bar{a}_i . Moreover, we have $\mathcal{M}_{\bar{a}_i} \prec \mathcal{M}_{\bar{a}_{i+1}}$ and $\mathcal{M} = \bigcup_i \mathcal{M}_{\bar{a}_i}$. If the number of indexes i is finite, the model \mathcal{M} is prime over the greatest tuple \bar{a}_i and we add the elementary chain of the models $\mathcal{M}_{\bar{a}_i}$ to the countable chain taking the model \mathcal{M} countably many times.

Since $I(T, \omega) < \omega$, we can choose, from the constructed sequence $(\mathcal{M}_i)_{i \in \omega}$, an infinite subsequence of models $(\mathcal{M}_{i_j})_{j \in \omega}$ such that all its elements are isomorphic to a model \mathcal{M}_p . This sequence is required. \square

A model \mathcal{M} is said to be *limit over a type p* if $\mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_n$ for some elementary chain $(\mathcal{M}_n)_{n \in \omega}$ over p and $\mathcal{M} \not\cong \mathcal{M}_p$.

Proposition 1.1.8. *A limit model over a type p exists iff, for any (some) realization \bar{a} of type p , there are a realization \bar{b} of p in $\mathcal{M}_{\bar{a}}$ and a tuple $\bar{c} \in M_{\bar{a}}$ such that $\text{tp}(\bar{c}/\bar{b})$ is a non-principal type.*

P r o o f. Suppose that there exists a limit model $\mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_n$ over p , where $\mathcal{M}_n \simeq \mathcal{M}_p$, $\mathcal{M}_0 = \mathcal{M}_{\bar{a}}$, $\models p(\bar{a})$, and there are no $\bar{b} \in p(\mathcal{M}_0)$ and $\bar{c} \in M_0$ such that $\text{tp}(\bar{c}/\bar{b})$ is a non-principal type. Then models \mathcal{M}_n (and hence also \mathcal{M}) realize just principal types over any realizations of type p lying in \mathcal{M}_n (in \mathcal{M}). Hence the model \mathcal{M} is prime over a realization of p , which contradicts the assumption that \mathcal{M} is limit.

Conversely, assume that for some $\bar{a} \models p^1$ there are tuples $\bar{b} \in p(\mathcal{M}_0)$ and $\bar{c} \in M_0$ such that $q(\bar{x}, \bar{b}) = \text{tp}(\bar{c}/\bar{b})$ is a nonprincipal type. Our goal is to construct an elementary chain $(\mathcal{M}_{\bar{a}_n})_{n \in \omega}$ over p satisfying the following conditions: $\bar{a}_0 = \bar{b}$, $\bar{a}_1 = \bar{a}$, and $\text{tp}(\bar{a}_{n+1}/\bar{a}_n) = \text{tp}(\bar{a}/\bar{b})$. We argue to show that $\mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_{\bar{a}_n}$ and \mathcal{M}_p are non-isomorphic. By way of contradiction, find a tuple $\bar{d} \in p(\mathcal{M}_{\bar{a}_n})$ such that $\mathcal{M} = \mathcal{M}_{\bar{d}}$. By the construction of \mathcal{M} , however, the type $q(\bar{x}, \bar{a}_n)$ is omitted in the model $\mathcal{M}_{\bar{d}}$ but is realized in the model \mathcal{M} , a contradiction. \square

¹Here and below, we write $\bar{a} \models p$ as a re-notation of $\models p(\bar{a})$.

Corollary 1.1.9. *If the semi-isolation relation SI_p on realizations of type p in a model \mathcal{M}_p is nonsymmetric then there exists a model limit over p .*

P r o o f. By Proposition 1.1.8, it suffices to notice that if \bar{a} is a realization of p then there exists a tuple $\bar{b} \in p(\mathcal{M}_{\bar{a}})$ such that \bar{b} does not semi-isolate \bar{a} , and hence $\text{tp}(\bar{a}/\bar{b})$ is a non-principal type. \square

Corollary 1.1.10. *If \mathcal{M}_p and \mathcal{M}_q are domination-equivalent non-isomorphic models then there exist models that are limit over the type p and over the type q .*

P r o o f. Let \bar{a} and \bar{b} be realizations of p and \bar{c} be a realization of q such that $\mathcal{M}_{\bar{b}} \prec \mathcal{M}_{\bar{c}} \prec \mathcal{M}_{\bar{a}}$ (which exist in view of $\mathcal{M}_p \sim_{RK} \mathcal{M}_q$). Since $\mathcal{M}_p \not\cong \mathcal{M}_q$, the type $\text{tp}(\bar{a}/\bar{c})$ is non-principal by Proposition 1.1.3. Hence a limit model over p exists by Proposition 1.1.8. The existence of a limit model over q is proved analogously. \square

Proposition 1.1.11. *If types p_1 and p_2 are domination-equivalent, and there exists a limit model over p_1 , then there exists a model that is limit over p_1 and over p_2 .*

P r o o f. We construct inductively an elementary chain $(\mathcal{M}_{\bar{a}_n})_{n \in \omega}$ of models such that:

- (a) models $\mathcal{M}_{\bar{a}_n}$ are prime over p_1 for even n , and over p_2 for odd n ;
- (b) the model $\bigcup_{n \in \omega} \mathcal{M}_{\bar{a}_n}$ is prime neither over p_1 nor over p_2 .

By Proposition 1.1.8, there exists a type $q(\bar{x}, \bar{a}_0)$, $\bar{a}_0 \models p_1$, which is not realizable in the model $\mathcal{M}_{\bar{a}_0}$ but is realizable in some $\mathcal{M}_{\bar{b}} \succ \mathcal{M}_{\bar{a}_0}$, $\bar{b} \models p_1$. Denote by $\mathcal{M}_{\bar{a}_1}$ the prime model over a realization \bar{a}_1 of type p_2 , which is an elementary extension of $\mathcal{M}_{\bar{b}}$ (such exists since p_1 and p_2 are domination-equivalent). At even steps $2n + 2$, we extend the model $\mathcal{M}_{\bar{a}_{2n+1}}$, $\bar{a}_{2n+1} \models p_2$, to a model $\mathcal{M}_{\bar{a}_{2n+2}}$, $\bar{a}_{2n+2} \models p_1$, which realizes type $q(\bar{x}, \bar{a}_{2n})$. At odd steps, we extend $\mathcal{M}_{\bar{a}_{2n+2}}$ to $\mathcal{M}_{\bar{a}_{2n+3}}$, $\bar{a}_{2n+3} \models p_2$. Clearly, the model $\bigcup_{n \in \omega} \mathcal{M}_{\bar{a}_n}$ is limit over p_1 and over p_2 . \square

Limit models \mathcal{M} and \mathcal{N} over a type p are said to be *equivalent* (written $\mathcal{M} \sim \mathcal{N}$) if there exist elementary chains $(\mathcal{M}_n)_{n \in \omega}$ and $(\mathcal{N}_n)_{n \in \omega}$ over p satisfying the following conditions:

$$1) \mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_n, \mathcal{N} = \bigcup_{n \in \omega} \mathcal{N}_n;$$

2) there exist constant expansions $\mathcal{M}'_{n+1} = \langle \mathcal{M}_{n+1}, c \rangle_{c \in M'_n}$ and $\mathcal{N}'_{n+1} = \langle \mathcal{N}_{n+1}, c \rangle_{c \in N'_n}$, $n \in \omega$, $\mathcal{M}'_0 = \mathcal{M}_0$, $\mathcal{N}'_0 = \mathcal{N}_0$, such that $\mathcal{M}'_{n+1} \simeq \mathcal{N}'_{n+1}$, $n \in \omega$.

The following proposition is obvious.

Proposition 1.1.12. *If \mathcal{M} and \mathcal{N} are limit models over a type p then $\mathcal{M} \simeq \mathcal{N}$ iff $\mathcal{M} \sim \mathcal{N}$.*

Let $\widetilde{\mathbf{M}} \in \text{RK}(T)/\sim_{RK}$ be the class consisting of isomorphism types of domination-equivalent models $\mathcal{M}_{p_1}, \dots, \mathcal{M}_{p_n}$. Denote by $\text{IL}(\widetilde{\mathbf{M}})$ the number of equivalence classes of models each of which is limit over some type p_i .

Propositions 1.1.4, 1.1.7, 1.1.12 and Corollary 1.1.10 can be combined to yield

Theorem 1.1.13. *For any countable complete theory T , the following conditions are equivalent:*

(1) $I(T, \omega) < \omega$;

(2) T is small, $|\text{RK}(T)| < \omega$ and $\text{IL}(\widetilde{\mathbf{M}}) < \omega$ for any $\widetilde{\mathbf{M}} \in \text{RK}(T)/\sim_{RK}$.

If (1) or (2) holds then T possesses the following properties:

(a) $\text{RK}(T)$ has the least element $\widetilde{\mathbf{M}}_0$ (an isomorphism type of a prime model) and $\text{IL}(\widetilde{\mathbf{M}}_0) = 0$;

(b) $\text{RK}(T)$ has the greatest \sim_{RK} -class $\widetilde{\mathbf{M}}_1$ (a class of isomorphism types of all prime models over realizations of powerful types) and $|\text{RK}(T)| > 1$ implies $\text{IL}(\widetilde{\mathbf{M}}_1) \geq 1$;

(c) if $|\widetilde{\mathbf{M}}| > 1$ then $\text{IL}(\widetilde{\mathbf{M}}) \geq 1$.

Moreover, the following decomposition formula holds: :

$$I(T, \omega) = |\text{RK}(T)| + \sum_{i=0}^{|\text{RK}(T)/\sim_{RK}|-1} \text{IL}(\widetilde{\mathbf{M}}_i),$$

where $\widetilde{\mathbf{M}}_0, \dots, \widetilde{\mathbf{M}}_{|\text{RK}(T)/\sim_{RK}|-1}$ are all elements of the partially ordered set $\text{RK}(T)/\sim_{RK}$.

Notice that, by Propositions 1.1.3 and 1.1.12, the conditions specified in item (2) of Theorem 1.1.13 admit of a syntactic representation; so, this theorem is an analog of the Ryll-Nardzewski theorem providing a syntactic characterization of ω -categoricity.

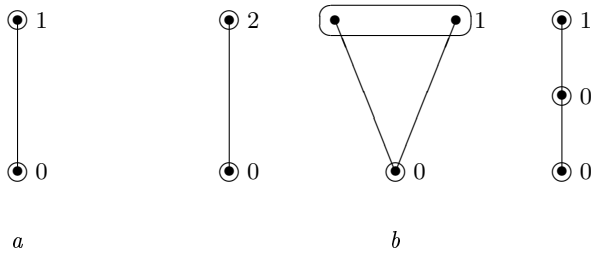


Figure 1.1

In figure 1.1, a and b , possible variants of Hasse diagrams of Rudin — Keisler preorders \leq_{RK} and values of distribution functions IL of numbers of limit models on \sim_{RK} -equivalence classes are represented for the cases $I(T, \omega) = 3$ and $I(T, \omega) = 4$. In figure 1.2, correspondent configurations for $I(T, \omega) = 5$ have shown.

Let $p_1, \dots, p_n \in S(T)$ be types the prime models over which are representatives of all isomorphism types in a finite preordered set $RK(T)$ of T . We say that the theory T possesses the *consistent extension property of chains of models prime over tuples* (CEP) if, for any type p_i , every two limit models over p_i are equivalent.

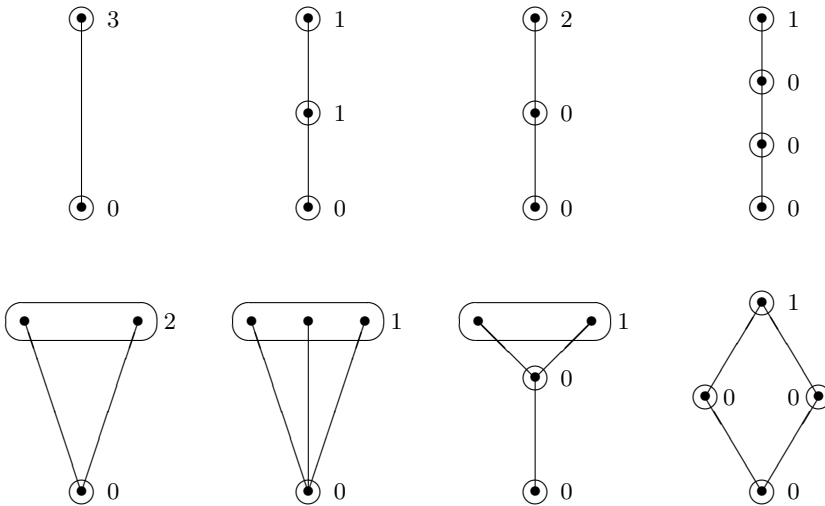


Figure 1.2

Proposition 1.1.11 implies that if T insists on (CEP) then $\text{IL}(\widetilde{\mathbf{M}}) \leq 1$ for any $\widetilde{\mathbf{M}} \in \text{RK}(T)/\sim_{RK}$. Since there exists no model that is limit over a principal type, (CEP) , for $|\text{RK}(T)/\sim_{RK}| = 2$, implies the existence of a unique (up to isomorphism) model \mathcal{M} the isomorphism type of which does not lie in $\text{RK}(T)$ (and \mathcal{M} , in this event, is saturated).

Thus the following theorem, based on Theorem 1.1.13, is valid.

Theorem 1.1.14. *Let T satisfy (CEP) . Then the following conditions are equivalent:*

- (1) $I(T, \omega) < \omega$;
- (2) T is small and $|\text{RK}(T)| < \omega$.

In this event, we have the inequality

$$I(T, \omega) \leq |\text{RK}(T)| + |\text{RK}(T)/\sim_{RK}| - 1,$$

which turns into equality for $|\text{RK}(T)/\sim_{RK}| \leq 2$.

Theorem 1.1.14 implies

Corollary 1.1.15. *For any complete theory T , the following are equivalent:*

- (1) $I(T, \omega) = 3$;
- (2) T is small, possesses (CEP) and $|\text{RK}(T)| = 2$. \square

§ 1.2. Inessential combinations and colorings of models

In the first paragraph of this Section, we define operations of inessential and almost inessential combinations of models, and also of theories. A bases of (almost) inessential combinations of theories are states, and also preservations of properties of smallness and λ -stability at transformation to (almost) inessential combinations of theories are shown. A sufficient condition is resulted for the inessentiality of combinations of theories in assumption of inessentiality of combinations of their models.

In the second paragraph, concepts of coloring of model, of colored model and of colored theory are defined, and the results of the first item are transferred for (almost) inessential colorings. An example is given, showing that inessential colorings of models don't imply that corresponding theories have an inessential colorings. Also

there is an example, showing a separability of the class of theories with almost inessential colorings from the class of theories with inessential colorings.

In the third paragraph, a concept of ordered coloring is defined, the role of such colorings in constructions of Ehrenfeucht theories is investigated, and an example is given of ω -stable theory with an ordered coloring, inducing a continuum of pairwise non-isomorphic limit models over a type.

1. Combinations of models and theories. Recall (see U. Saffe, E. A. Palyutin, S. S. Starchenko [162]) that a theory T is said to be Δ -based, where Δ is some set of formulas without parameters if any formula of T is equivalent in T to a Boolean combination of formulas of Δ .

A theory T is said to be *almost Δ -based*, where Δ is a set of formulas without parameters if there exists a function $f : \omega \rightarrow \omega$ such that any formula $\varphi(x_1, \dots, x_n)$ of T is equivalent in T to a formula of form

$$\exists y_1 \dots \exists y_{f(n)} \psi(x_1, \dots, x_n, y_1, \dots, y_{f(n)}),$$

where $\psi(x_1, \dots, x_n, y_1, \dots, y_{f(n)})$ is a Boolean combination of formulas of Δ .

Recall, that the set of all (complete and uncomplete) types over a set A is denoted by ${}^{\subseteq}S(A)$.

A type $q(\bar{x}) \in {}^{\subseteq}S(A)$ is *isolated* or *is defined* by a set $\Phi(\bar{x}, A)$ of formulas of q if $\Phi(\bar{x}, A) \vdash q(\bar{x})$.

The following two assertions are clear.

Lemma 1.2.1. *If a type $q(\bar{x}) \in {}^{\subseteq}S(A)$ is isolated by a set $\Phi(\bar{x}, A)$ and the type $\Phi(\bar{x}, A)$ is isolated by a set $\Psi(\bar{x}, A)$ then $q(\bar{x})$ is isolated by $\Psi(\bar{x}, A)$.*

Lemma 1.2.2. *If $\models \Phi(\bar{a}, \bar{b})$ then the type $\text{tp}(\bar{a}/\bar{b})$ is isolated by $\Phi(\bar{x}, \bar{y})$ iff the type $\text{tp}(\bar{b}/\bar{a})$ is isolated by $\Phi(\bar{a}, \bar{y})$ and the type $\text{tp}(\bar{a})$ is isolated by $\left\{ \exists \bar{y} \left(\bigwedge_i \varphi_i(\bar{x}, \bar{y}) \mid \varphi_i(\bar{x}, \bar{y}) \in \Phi(\bar{x}, \bar{y}) \right) \right\}$.*

Recall that a countable model \mathcal{M} of T is *weakly ω -universal* if any type over the empty set is realizable in \mathcal{M} that is $\mathcal{M} \models S(T)$.

Let Δ be a set of formulas of a theory T , $p(\bar{x})$ be a type of T , lying in $S(T)$. The type $p(\bar{x})$ is said to be Δ -based if $p(\bar{x})$ is isolated by a set of formulas $\varphi^\delta \in p$, where $\varphi \in \Delta$, $\delta \in \{0, 1\}$.

The following lemma, being a corollary of Compactness Theorem, noticed in U. Saffe, E. A. Palyutin, S. S. Starchenko [162].

Lemma 1.2.3. *A theory T is Δ -based iff for any tuple \bar{a} of weakly ω -universal model of T , the $\text{tp}(\bar{a})$ is Δ -based.*

Lemma 1.2.4. *A theory T is almost Δ -based iff for any tuple \bar{a} of weakly ω -universal model \mathcal{M} of T there exists a tuple $\bar{b} \in M$, containing all coordinates of \bar{a} and such that $\text{tp}(\bar{b})$ is Δ -based.*

P r o o f. Suppose a theory T is almost Δ -based and \bar{a} is a tuple of weakly ω -universal model \mathcal{M} of T . So $\text{tp}(\bar{a})$ is isolated by some set

$$\left\{ \exists \bar{y} \left(\bigwedge_i \varphi_i(\bar{x}, \bar{y}) \right) \mid \varphi_i^{\delta_i}(\bar{x}, \bar{y}) \in \Delta \right\}.$$

Now using Compactness Theorem and weak ω -universality of \mathcal{M} , there exists a tuple $\bar{b} \in M$, extending \bar{a} and satisfying all formulas $\varphi_i^{\delta_i}(\bar{x}, \bar{y})$. Since, by the condition, the set of formulas $\varphi_i^{\delta_i}(\bar{a}, \bar{y})$ isolates $\text{tp}(\bar{b}/\bar{a})$, using Lemma 1.2.2, we get the Δ -baseness of $\text{tp}(\bar{b})$.

Suppose now that for any tuple \bar{a} of weakly ω -universal model \mathcal{M} of T , there exists a tuple $\bar{b} \in M$, extending \bar{a} and such that $\text{tp}(\bar{b})$ is Δ -based. Then by Compactness Theorem, there exists a function $f : \omega \rightarrow \omega$, bounding minimal lengths of tuples \bar{b} depending of lengths of tuples \bar{a} . On the other hand, we have that any consistent formula $\varphi(\bar{x})$ is deducible from some consistent formula of form $\exists \bar{y} \psi(\bar{x}, \bar{y})$, where $\psi(\bar{x}, \bar{y})$ is a conjunction of formulas and negations of formulas from Δ , $l(\bar{y}) \leq f(l(\bar{x}))$. By Compactness Theorem we get that the formula $\varphi(\bar{x})$ is equivalent to a disjunction of formulas of form $\exists \bar{y} \psi(\bar{x}, \bar{y})$, and so to a formula of form $\exists \bar{y} \psi(\bar{x}, \bar{y})$, where $\psi(\bar{x}, \bar{y})$ is a Boolean combination of formulas of Δ . Thus the theory T is almost Δ -based. \square

Let \mathcal{M}_1 and \mathcal{M}_2 be models of disjoint languages Σ_1 and Σ_2 respectively such that $M_1 = M_2$. A model \mathcal{M} of the language $\Sigma_1 \cup \Sigma_2$ is said to be a *combination of models* \mathcal{M}_1 and \mathcal{M}_2 if $M = M_1$ and interpretations of language symbols of \mathcal{M} coincide with correspondent interpretations in \mathcal{M}_1 and \mathcal{M}_2 . We denote \mathcal{M} by $\text{Comb}(\mathcal{M}_1, \mathcal{M}_2)$.

A theory T is said to be a *combination of theories* T_1 and T_2 over models $\mathcal{M}_i \models T_i$, $i = 1, 2$, if $T = \text{Th}(\text{Comb}(\mathcal{M}_1, \mathcal{M}_2))$.

Let \bar{a} be a tuple in a model $\text{Comb}(\mathcal{M}_1, \mathcal{M}_2)$. a type $\text{tp}_{\mathcal{M}}(\bar{a})$ is said to be an *inessential combination of types* $\text{tp}_{\mathcal{M}_1}(\bar{a})$ and $\text{tp}_{\mathcal{M}_2}(\bar{a})$

if $\text{tp}_{\mathcal{M}}(\bar{a})$ is isolated by $\text{tp}_{\mathcal{M}_1}(\bar{a}) \cup \text{tp}_{\mathcal{M}_2}(\bar{a})$. The set of tuples $\bar{a} \in M$, such that $\text{tp}_{\mathcal{M}}(\bar{a})$ is an inessential combination of $\text{tp}_{\mathcal{M}_1}(\bar{a})$ and $\text{tp}_{\mathcal{M}_2}(\bar{a})$, will be denoted by $\text{IECT}_{\mathcal{M}}$.

A combination of models $\mathcal{M} = \text{Comb}(\mathcal{M}_1, \mathcal{M}_2)$ is said to be *inessential* (written $\mathcal{M} = \text{IEC}(\mathcal{M}_1, \mathcal{M}_2)$) if $\text{IECT}_{\mathcal{M}}$ consists of all tuples in \mathcal{M} . A combination of models $\mathcal{M} = \text{Comb}(\mathcal{M}_1, \mathcal{M}_2)$ is said to be *almost inessential* (written $\mathcal{M} = \text{AIEC}(\mathcal{M}_1, \mathcal{M}_2)$) if for any tuple $\bar{a} \in M$, there exists a tuple $\bar{b} \in \text{IECT}_{\mathcal{M}}$, extending \bar{a} .

A combination T of theories T_1 and T_2 is said to be (*almost*) *inessential* if $\mathcal{M} = \text{IEC}(\mathcal{M}_1, \mathcal{M}_2)$ (accordingly $\mathcal{M} = \text{AIEC}(\mathcal{M}_1, \mathcal{M}_2)$) for any model $\mathcal{M} \models T$, where \mathcal{M}_i is a restriction of \mathcal{M} to the language $\Sigma(T_i)$, $i = 1, 2$.

Clearly, any inessential combination of theories is almost inessential.

Lemma 1.2.5. *Let T be a combination of theories T_1 and T_2 , \mathcal{M} be a weakly ω -universal model of T . Then the following conditions are equivalent:*

- (1) *the combination T is (almost) inessential;*
- (2) *$\mathcal{M} = \text{IEC}(\mathcal{M}_1, \mathcal{M}_2)$ ($\mathcal{M} = \text{AIEC}(\mathcal{M}_1, \mathcal{M}_2)$), where \mathcal{M}_i is a restriction of \mathcal{M} to the language $\Sigma(T_i)$, $i = 1, 2$.*

P r o o f is obvious.

Theorem 1.2.6. *Let T be a combination of Δ_i -based theories T_i , $i = 1, 2$. Then the following conditions are equivalent:*

- (1) *the combination T is inessential;*
- (2) *T is $(\Delta_1 \cup \Delta_2)$ -based.*

P r o o f. (1) \Rightarrow (2). Suppose that T is an inessential combination of theories T_1 and T_2 , \mathcal{M} be a weakly ω -universal model of T . By Lemma 1.2.3, it suffices to show that for any tuple $\bar{a} \in M$, its type $\text{tp}_{\mathcal{M}}(\bar{a})$ is $(\Delta_1 \cup \Delta_2)$ -based. Denote by \mathcal{M}_i a restriction of \mathcal{M} to the language $\Sigma(T_i)$, $i = 1, 2$. Since by Lemma 1.2.5 we have $\mathcal{M} = \text{IEC}(\mathcal{M}_1, \mathcal{M}_2)$, the type $\text{tp}_{\mathcal{M}}(\bar{a})$ is isolated by $\text{tp}_{\mathcal{M}_1}(\bar{a}) \cup \text{tp}_{\mathcal{M}_2}(\bar{a})$. As T_i is Δ_i -based then, by Lemma 1.2.3, the type $\text{tp}_{\mathcal{M}_i}(\bar{a})$ is isolated by some set $\Phi_i(\bar{x})$ of formulas and negations of formulas of Δ_i , $i = 1, 2$. Then by Lemma 1.2.1, the type $\text{tp}_{\mathcal{M}}(\bar{a})$ is isolated by $\Phi_1(\bar{x}) \cup \Phi_2(\bar{x})$. Thus, the type $\text{tp}_{\mathcal{M}}(\bar{a})$ is $(\Delta_1 \cup \Delta_2)$ -based.

(2) \Rightarrow (1). Let T be a $(\Delta_1 \cup \Delta_2)$ -based theory, i.e., any type $\text{tp}_{\mathcal{M}}(\bar{a})$ of tuple \bar{a} in a weakly ω -universal model \mathcal{M} is isolated by

some set $\Phi(\bar{x})$ of formulas and negations of formulas of $\Delta_1 \cup \Delta_2$. Since $\Sigma(T_1) \cap \Sigma(T_2) = \emptyset$ then $\Phi(\bar{x}) = \Phi_1(\bar{x}) \cup \Phi_2(\bar{x})$, where $\Phi_i(\bar{x})$ is the set of formulas in $\Phi(\bar{x})$ of the language $\Sigma(T_i)$. Here, by Lemma 1.2.3, the set $\Phi_i(\bar{x})$ isolates a type $\text{tp}_{\mathcal{M}_i}(\bar{a})$, where \mathcal{M}_i is a restriction of \mathcal{M} to the language $\Sigma(T_i)$. Since $\Phi_i(\bar{x}) \subset \text{tp}_{\mathcal{M}_i}(\bar{a})$, then $\text{tp}_{\mathcal{M}_1}(\bar{a}) \cup \text{tp}_{\mathcal{M}_2}(\bar{a})$ isolates $\text{tp}_{\mathcal{M}}(\bar{a})$. As \mathcal{M} is a weakly ω -universal model, then by Lemma 1.2.5, T is an inessential combination of T_1 and T_2 . \square

Theorem 1.2.7. *Let T be a combination of Δ_i -based theories T_i , $i = 1, 2$. Then the following conditions are equivalent:*

- (1) *the combination T is almost inessential;*
- (2) *T is almost $(\Delta_1 \cup \Delta_2)$ -based.*

Proof is similar to the proof of Theorem 1.2.6 using Lemma 1.2.4 instead of Lemma 1.2.3. \square

Recall, that a theory T is said to be λ -stable, where λ is an infinite cardinality if for any set A of power λ , the number of types over A is not more than λ , that is $|S(A)| \leq \lambda$.

Theorem 1.2.8. *If T is an almost inessential combination of theories T_1 and T_2 then T λ -stable (small) iff T_1 and T_2 are λ -stable (small).*

Proof. As T_1 and T_2 are restrictions of T , then the λ -stability (smallness) of T implies λ -stability (smallness) of T_1 and T_2 .

Suppose that T_1 and T_2 are λ -stable. Consider a model \mathcal{M} of T , having the cardinality λ , and its elementary extension \mathcal{M}' of cardinality λ , including with any tuple \bar{a} its extending tuple $\bar{b} \in \text{IECT}_{\mathcal{M}'}$. Denote by \mathcal{M}'_i a restriction of \mathcal{M}' to the language $\Sigma(T_i)$, $i = 1, 2$. As T_i are λ -stable, we have $|S(\mathcal{M}'_i)| \leq \lambda$. On the other hand, an almost inessentiality of the combination of T_1 and T_2 implies

$$|S(\mathcal{M})| \leq |S(\mathcal{M}')| \leq |S(\mathcal{M}'_1)| \cdot |S(\mathcal{M}'_2)| \leq \lambda \cdot \lambda = \lambda.$$

As considered model \mathcal{M} is arbitrary, the theory T is λ -stable.

Suppose now, that T_1 and T_2 are small, i.e., $|S(T_1)| = |S(T_2)| = \omega$. Then an almost inessentiality of the combination of T_1 and T_2 implies $|S(T)| \leq |S(T_1)| \cdot |S(T_2)| = \omega \cdot \omega = \omega$, that is the theory T is small too. \square

Let $p(\bar{x})$ be a type in $S(T)$, $\Phi(\bar{x})$ be a set of formulas $\varphi_n(\bar{x})$, $n \in \omega$ such that $\vdash \varphi_{n+1}(\bar{x}) \rightarrow \varphi_n(\bar{x})$ and $p(\bar{x})$ is isolated by $\Phi(\bar{x})$.

Consider a model \mathcal{M} of T . A sequence $(\bar{a}_n)_{n \in \omega}$ of tuples in \mathcal{M} is said to be *defining* for $p(\bar{x})$ (over $\Phi(\bar{x})$) if $\models \varphi_n(\bar{a}_n)$ for any $n \in \omega$. A defining sequence of type $p(\bar{x})$ is said to be *regular* in \mathcal{M} if the type $p(\bar{x})$ has a realization in \mathcal{M} . Otherwise a defining sequence is said to be *irregular* in \mathcal{M} .

Clearly, any sequence of tuples may be defining for unique type. So it is possible not to indicate types correspondent to regular sequences.

If $(\bar{a}_n)_{n \in \omega}$ is a regular in a model \mathcal{M} , defining sequence of type $p(\bar{x})$ and $\mathcal{M} \models p(\bar{a})$, then we say that \bar{a} is a *limit of sequence* $(\bar{a}_n)_{n \in \omega}$ in \mathcal{M} and it will be denoted by $\bar{a} \in \left(\lim_{n \rightarrow \infty} \bar{a}_n \right)_{\mathcal{M}}$. Here, the set $\left(\lim_{n \rightarrow \infty} \bar{a}_n \right)_{\mathcal{M}}$ coincides with the set $p(\mathcal{M})$ of realizations of $p(\bar{x})$ in \mathcal{M} .

Proposition 1.2.9. *Let \mathcal{M} be an inessential combination of weakly ω -universal models \mathcal{M}_1 and \mathcal{M}_2 such that any regular sequence $(\bar{a}_n)_{n \in \omega}$ in \mathcal{M}_i is regular in \mathcal{M}_{2-i} , $i = 0, 1$, and moreover*

$$\left(\lim_{n \rightarrow \infty} \bar{a}_n \right)_{\mathcal{M}_1} \cap \left(\lim_{n \rightarrow \infty} \bar{a}_n \right)_{\mathcal{M}_2} \neq \emptyset.$$

Then \mathcal{M} is a weakly ω -universal model and $\text{Th}(\mathcal{M})$ is an inessential combination of theories $\text{Th}(\mathcal{M}_1)$ and $\text{Th}(\mathcal{M}_2)$.

P r o o f. Consider an arbitrary type $p(\bar{x}) \in S(\emptyset)$ of theory $\text{Th}(\mathcal{M})$ and show that $p(\bar{x})$ is realizable in \mathcal{M} . Indeed, let $(\bar{a}_n)_{n \in \omega}$ be a defining sequence of type $p(\bar{x})$ in \mathcal{M} . Then $(\bar{a}_n)_{n \in \omega}$ is defining in \mathcal{M}_1 and in \mathcal{M}_2 . A weak ω -universality of \mathcal{M}_1 and \mathcal{M}_2 implies a regularity of this sequence in \mathcal{M}_1 , and also in \mathcal{M}_2 . Moreover, there exists a tuple \bar{a} such that $\bar{a} \in \left(\lim_{n \rightarrow \infty} \bar{a}_n \right)_{\mathcal{M}_1} \cap \left(\lim_{n \rightarrow \infty} \bar{a}_n \right)_{\mathcal{M}_2}$. As the combination of models is inessential, the set $\text{tp}_{\mathcal{M}_1}(\bar{a}) \cup \text{tp}_{\mathcal{M}_2}(\bar{a})$ isolates the type $\text{tp}_{\mathcal{M}}(\bar{a})$. But $\text{tp}_{\mathcal{M}_1}(\bar{a}) \cup \text{tp}_{\mathcal{M}_2}(\bar{a}) \subset p(\bar{x})$, so $p(\bar{x}) = \text{tp}_{\mathcal{M}}(\bar{a})$ and \bar{a} is a realization of type $p(\bar{x})$ in \mathcal{M} . Thus, \mathcal{M} is a weakly ω -universal model.

An inessentiality of the combination of theories $\text{Th}(\mathcal{M}_1)$ and $\text{Th}(\mathcal{M}_2)$ follows from the weak ω -universality of \mathcal{M} by Lemma 1.2.5. \square

2. Colored models. Let \mathcal{M} be a model. Any function $\text{Col} : M \rightarrow \lambda \cup \{\infty\}$, where λ is a power and ∞ is a symbol of infinity, is said to be a *coloring of model* \mathcal{M} . Here, for any $a \in M$, a value $\text{Col}(a)$ is said to be a *color of element* a . A pair $\langle \mathcal{M}, \text{Col} \rangle$ is said to be a *colored model*.

Below, colored models $\langle \mathcal{M}, \text{Col} \rangle$ will be identified with expansions of \mathcal{M} by unary predicates $\text{Col}_\mu = \{a \in M \mid \text{Col}(a) = \mu\}$, $\mu < \lambda$. Clearly, a colored model $\langle \mathcal{M}, \text{Col} \rangle$ is a combination of \mathcal{M} with a *coloring of its universe*, i.e., with a model $\langle M, \text{Col} \rangle = \langle M; \text{Col}_\mu \rangle_{\mu < \lambda}$: $\langle \mathcal{M}, \text{Col} \rangle = \text{Comb}(\mathcal{M}, \langle M, \text{Col} \rangle)$.

A coloring Col of model \mathcal{M} is said to be *innerly inessential* if, for any tuple $\bar{a} \in M$, a type $\text{tp}_{\langle \mathcal{M}, \text{Col} \rangle}(\bar{a})$ is isolated by types of \bar{a} in \mathcal{M} , and also by colors of elements of \bar{a} .

Clearly, an inner inessentiality of coloring Col of model \mathcal{M} is equivalent to equality $\langle \mathcal{M}, \text{Col} \rangle = \text{IEC}(\mathcal{M}, \langle M, \text{Col} \rangle)$.

A coloring Col of model \mathcal{M} is said to be *innerly almost inessential* if, for any tuple $\bar{a} \in M$, there exists a tuple $\bar{b} \in M$ extending \bar{a} and such that $\text{tp}_{\langle \mathcal{M}, \text{Col} \rangle}(\bar{b})$ is isolated by type of \bar{b} in \mathcal{M} , and also by colors of elements of \bar{b} .

An inner almost inessentiality of coloring Col of models \mathcal{M} is characterized by an equality $\langle \mathcal{M}, \text{Col} \rangle = \text{AIEC}(\mathcal{M}, \langle M, \text{Col} \rangle)$.

For any model $\mathcal{M}' \models \text{Th}(\langle \mathcal{M}, \text{Col} \rangle)$ a coloring $\text{Col}' : M' \rightarrow \lambda \cup \{\infty\}$ is defined naturally by the following rules:

- 1) $\text{Col}'(a) = \mu$ if $\mathcal{M}' \models \text{Col}_\mu(a)$;
- 2) $\text{Col}'(a) = \infty$ if $\mathcal{M}' \not\models \text{Col}_\mu(a)$ for any $\mu < \lambda$.

Below, models \mathcal{M}' will be denoted by $\langle \mathcal{M}', \text{Col}' \rangle$, and by \mathcal{M}' we shall denote a restriction of model $\langle \mathcal{M}', \text{Col}' \rangle$ to language $\Sigma(\mathcal{M})$.

Any expansion T' of theory T by pairwise inconsistent unary predicates Col_μ , $\mu < \lambda$, is said to be a *colored theory*. Clearly, any colored theory is a theory of some colored model $\langle \mathcal{M}, \text{Col} \rangle$, where $\mathcal{M} \models T$.

A coloring Col of model \mathcal{M} is said to be *(almost) inessential* if, for any model $\langle \mathcal{M}', \text{Col}' \rangle$ of colored theory $\text{Th}(\langle \mathcal{M}, \text{Col} \rangle)$, a correspondent coloring Col' is innerly (almost) inessential.

The following example shows that an inner inessentiality of coloring of model doesn't imply an inessentiality of the coloring.

Example 1.2.1. Let \mathcal{M} be a model consisting of constants $\{c_n^i \mid n \in \omega\}$, $i \in \{0, 1, 2\}$, and expanded by substitution f acting by $f(c_n^0) = c_{2n}^2$, $f(c_{2n}^2) = c_n^0$, $f(c_n^1) = c_{2n+1}^2$, $f(c_{2n+1}^2) = c_n^1$, $n \in \omega$. Consider a coloring $\text{Col} : M \rightarrow \{0, 1, 2\}$, defined by $\text{Col}(c_n^i) = i$,

$n \in \omega$, $i \in \{0, 1, 2\}$. Clearly, any coloring of \mathcal{M} is innerly inessential, because a type $p(\bar{x})$ of any tuple $\bar{a} \in M$ is isolated by some set of formulas $\{(x_j \approx c_{n_j}^{i_j}) \mid 1 \leq j \leq l(\bar{x})\}$. But a coloring Col' of weakly ω -universal model $\langle \mathcal{M}', \text{Col}' \rangle \models \text{Th}(\langle \mathcal{M}, \text{Col} \rangle)$ is not inessential.

Indeed, consider elements $a_k \in M'$ such that

$$\models \text{Col}_2(a_k) \wedge \neg(a_k \approx c_n^2) \wedge \exists x_k (\text{Col}_k(x_k) \wedge (f(x_k) \approx a_k)),$$

$n \in \omega$, $k = 0, 1$. Clearly, $\text{tp}_{\mathcal{M}'}(a_0) = \text{tp}_{\mathcal{M}'}(a_1)$ and $\text{tp}_{\langle \mathcal{M}', \text{Col}' \rangle}(a_0) = \text{tp}_{\langle \mathcal{M}', \text{Col}' \rangle}(a_1)$, but $\text{tp}_{\langle \mathcal{M}', \text{Col}' \rangle}(a_0) \neq \text{tp}_{\langle \mathcal{M}', \text{Col}' \rangle}(a_1)$, i.e.,

$$\langle \mathcal{M}', \text{Col}' \rangle \neq \text{IEC}(\mathcal{M}', \langle \mathcal{M}', \text{Col}' \rangle). \quad \square$$

Recall, that for any set A of T the union of sets of solutions of formulas $\varphi(x, \bar{a})$, $\bar{a} \in A$ such that $\models \exists^{=n} x \varphi(x, \bar{a})$ for some $n \in \omega$ (accordingly $\models \exists^{=1} x \varphi(x, \bar{a})$) is said to be an *algebraic (definable) closure* of A . An algebraic closure of A is denoted by $\text{acl}(A)$ and its definable closure, by $\text{dcl}(A)$.

Using the following assertion, it is easy to construct examples of innerly almost inessential colorings, being not innerly inessential.

Proposition 1.2.10. *If a colored model $\langle \mathcal{M}, \text{Col} \rangle$ contains tuples \bar{a} , \bar{b} and elements c , d such that $\text{tp}_{\mathcal{M}}(\bar{a} \hat{\ } c) = \text{tp}_{\mathcal{M}}(\bar{b} \hat{\ } d)$, $\text{tp}_{\langle \mathcal{M}, \text{Col} \rangle}(\bar{a}) = \text{tp}_{\langle \mathcal{M}, \text{Col} \rangle}(\bar{b})$, $c \in \text{dcl}(\bar{a})$, $d \in \text{dcl}(\bar{b})$ and $\text{Col}(c) \neq \text{Col}(d)$, then the coloring Col is not innerly inessential.*

P r o o f. Notice, that $\text{tp}_{\langle \mathcal{M}, \text{Col} \rangle}(\bar{a}) \neq \text{tp}_{\langle \mathcal{M}, \text{Col} \rangle}(\bar{b})$, because an existence of automorphism f of homogeneous extension of model $\langle \mathcal{M}, \text{Col} \rangle$, transforming \bar{a} to \bar{b} , implies $f(c) = d$, that is impossible for $\text{Col}(c) \neq \text{Col}(d)$. Since $\text{tp}_{\mathcal{M}}(\bar{a}) = \text{tp}_{\mathcal{M}}(\bar{b})$ and $\text{tp}_{\langle \mathcal{M}, \text{Col} \rangle}(\bar{a}) = \text{tp}_{\langle \mathcal{M}, \text{Col} \rangle}(\bar{b})$, then $\bar{a}, \bar{b} \notin \text{IECT}_{\mathcal{M}}$. Thus, the coloring Col is not innerly inessential. \square

E x a m p l e 1.2.2. Consider a system $\Gamma = \langle \{a, b, c, d\}; \{(a, b), (b, a), (c, d), (d, c)\} \rangle$ and its innerly almost inessential coloring Col defined by equalities $\text{Col}(a) = \text{Col}(b) = \text{Col}(c) = 0$, $\text{Col}(d) = 1$. The coloring Col is not innerly inessential, as $\text{tp}_{\Gamma}(a) = \text{tp}_{\Gamma}(c)$ and $\text{tp}_{\langle \Gamma, \text{Col} \rangle}(a) \neq \text{tp}_{\langle \Gamma, \text{Col} \rangle}(c)$. \square

Notice, that for any colored model $\langle \mathcal{M}', \text{Col}' \rangle$, the theory $\text{Th}(\langle \mathcal{M}', \text{Col}' \rangle)$ is totally transcendental and Δ_{Col} -based, where Δ_{Col} is a closure of $\{(x \approx y)\} \cup \{\text{Col}_{\mu}(x) \mid \mu < \lambda\}$ w.r.t. substitutions of variables.

The following Theorems hold in view of Theorems 1.2.6–1.2.8.

Theorem 1.2.11. *Let Col be a coloring of model \mathcal{M} of Δ -based theory T . The following conditions are equivalent:*

- (1) *Col is an (almost) inessential coloring;*
- (2) *$\text{Th}(\langle \mathcal{M}, \text{Col} \rangle)$ is an (almost) $(\Delta \cup \Delta_{\text{Col}})$ -based theory.*

Theorem 1.2.12. *If Col is an almost inessential coloring of model \mathcal{M} of cardinality $|\Sigma(\mathcal{M})| + \omega$, then $\text{Th}(\langle \mathcal{M}, \text{Col} \rangle)$ is λ -stable (small) iff $\text{Th}(\mathcal{M})$ is λ -stable (small).*

3. Ordered colorings. Let \mathcal{M} be a model of theory T , $\varphi(x, y)$ be a formula of T . A coloring $\text{Col} : M \rightarrow \lambda \cup \{\infty\}$ (where λ is an infinite cardinality) is said to be φ -ordered if the following conditions hold:

- (a) for any $\mu \leq \nu < \lambda$ there exist elements $a, b \in M$ such that $\models \text{Col}_\mu(a) \wedge \text{Col}_\nu(b) \wedge \varphi(a, b)$;
- (b) if $\mu < \nu < \lambda$ then there are no elements $c, d \in M$ such that $\models \text{Col}_\mu(c) \wedge \text{Col}_\nu(d) \wedge \varphi(d, c)$.

Recall, that a theory T is said to be *transitive* if T has unique 1-type over the empty set.

A coloring Col of model \mathcal{M} is said to be *n-inessential*, $n \in \omega \setminus \{0\}$, if $(M')^n \subseteq \text{IECT}_{\langle \mathcal{M}', \text{Col}' \rangle}$ for any model $\langle \mathcal{M}', \text{Col}' \rangle \models \text{Th}(\langle \mathcal{M}, \text{Col} \rangle)$.

Clearly, any inessential coloring is n -inessential for any $n \geq 1$.

Notice, that if $\text{Col} : M \rightarrow \lambda \cup \{\infty\}$ is a surjective 1-inessential coloring of model \mathcal{M} of transitive theory T , then the set of 1-types of $\text{Th}(\langle \mathcal{M}, \text{Col} \rangle)$ over \emptyset consists of types $p_\mu(x)$, $\mu \in \lambda \cup \{\infty\}$, where $p_\mu(x)$ is a type isolated by formula $\text{Col}_\mu(x)$, $\mu \in \lambda$, $p_\infty(x)$ is a non-principal type isolated by set of formulas $\{\neg \text{Col}_\mu(x) \mid \mu < \lambda\}$.

In the theory T_3 with three countable models, being the Ehrenfeucht example, an expansion of a model of transitive theory $\text{Th}(\langle \mathbb{Q}; < \rangle)$ by constants c_k , $k \in \omega$, can be interpretable as an inessential coloring Col given by the following conditions:

$$\text{Col}(a) = \begin{cases} 0 & \text{if } a < c_0, \\ 2k + 1 & \text{if } a = c_k, \\ 2k + 2 & \text{if } c_k < a < c_{k+1}. \end{cases}$$

It's easy to see that the coloring Col is φ -ordered, where $\varphi(x, y) \equiv x < y$. Moreover, a relation SI_{p_∞} on the set of realizations of powerful type p_∞ is non-symmetric, that is witnessed by the formula φ . In Ehrenfeucht examples, $n \geq 4$, constant expansions of models $\langle \mathbb{Q}; <, P_0, \dots, P_{n-3} \rangle$ can be also considered as colored models with inessential ordered colorings.

Consider a sufficient conditions for a φ -ordered 1-inessential coloring to imply a non-symmetry of relation SI_{p_∞} witnessed by φ .

Proposition 1.2.13. *Let $\varphi(x, y)$ be a principal (i.e., isolating a complete type) formula of transitive theory T , Col be an 1-inessential φ -ordered coloring of model \mathcal{M} of theory T such that $\langle \mathcal{M}', \text{Col}' \rangle \models \text{Th}(\langle \mathcal{M}, \text{Col} \rangle)$ and $\langle \mathcal{M}', \text{Col}' \rangle \models \varphi(a, b)$ imply $(a, b) \in \text{IECT}_{\langle \mathcal{M}', \text{Col}' \rangle}$. Then for any (i.e., some) realization a of type $p_\infty(x)$ the following conditions hold:*

- 1) if $\models \varphi(a, b)$ then $\models p_\infty(b)$ and a semi-isolates b ;
- 2) if $\models \varphi(a, b)$ then b doesn't semi-isolate a .

P r o o f. 1. Assume on the contrary that $\models p_\infty(a)$, $\models \varphi(a, b)$ and $\not\models p_\infty(b)$. Then for some μ the set $\{\neg \text{Col}_\nu(x) \mid \nu < \lambda\} \cup \{\varphi(x, y), \text{Col}_\mu(y)\}$ is consistent and, in particular, the set $\{\neg \text{Col}_\nu(x) \mid \nu \leq \mu\} \cup \{\varphi(x, y), \text{Col}_\mu(y)\}$ is consistent too. So there exists $\alpha > \mu$ such that

$$\models \exists x, y (\text{Col}_\mu(y) \wedge \text{Col}_\alpha(x) \wedge \varphi(x, y)).$$

It contradicts to point *b*) of the definition of φ -ordering of coloring Col . Thus, $\models \varphi(a, b)$ implies $\models p_\infty(b)$ and a semi-isolates b .

2. Assume on the contrary that $\models p_\infty(a)$, $\models \varphi(a, b)$ and b semi-isolates a . By the condition of Proposition, the formula $\varphi(x, b)$ can not witness to a semi-isolation of a over b . On the other hand, there exists a formula $\psi(x, y)$ such that $\models \psi(a, b)$ and $\psi(x, b) \vdash p_\infty(x)$. Moreover, the set $p_\infty(x) \cup p_\infty(y) \cup \{\varphi(x, y) \wedge \psi(x, y)\}$ is consistent. By Compactness Theorem, a non-principality of $p_\infty(x)$ implies a consistency of $p_\infty(x) \cup p_\infty(y) \cup \{\varphi(x, y) \wedge \neg \psi(x, y)\}$. It means that the set $\{\neg \text{Col}_\mu(x) \wedge \neg \text{Col}_\mu(y) \mid \mu < \lambda\} \cup \{\varphi(x, y)\}$ doesn't semi-isolate a complete type. This one is a contradiction to the conditions that the formula $\varphi(x, y)$ is principal in T and $(a, b) \in \text{IECT}_{\langle \mathcal{M}', \text{Col}' \rangle}$ for any (a, b) with $\models \varphi(a, b)$. Thus, $\models \varphi(a, b)$ and $\models p_\infty(a)$ imply that b doesn't semi-isolate a . \square

Notice, that the conclusion of Proposition 1.2.13 is true if we assume that $\varphi(x, y)$ is a disjunction of principal formulas.

Recall several notions of Graph Theory. An algebraic system $\Gamma = \langle X; Q \rangle$ with a (nonsymmetric, symmetric) binary relation Q is said to be a *graph* (accordingly a *directed graph* or, in abbreviated form, a *digraph*, an *undirected graph* or, in abbreviated form, an *undigraph*). Furthermore, the set X is said to be a set of *vertices* and the relation Q is a set of *arcs* of graph Γ . Any nonempty sequence

$S = (a_0, \dots, a_n)$ of vertices in a graph Γ , such that $\Gamma \models Q(a_i, a_{i+1})$, $i = 0, \dots, n-1$, is said to be a *route* in Γ . Here, the route S is also said to be a (a_0, a_n) -*route* and the number n is the *length* of S . A *cycle* in a digraph Γ is an arbitrary (a, a) -route of nonzero length. If digraph doesn't have cycles it is said to be *acyclic*. A graph $\langle X; Q \rangle$ is said to be *connected* if any two different vertices $a, b \in X$ are connected by a (a, b) -route in graph $\langle X; Q \cup Q^{-1} \rangle$. Any (a, a) -route (a_0, \dots, a_n) of nonzero length in an undirected graph Γ , such that there are no arcs (a_i, a_{i+1}) that are repeated or coincide with (a_{j+1}, a_j) for $a_i \neq a_{i+1}$, is said to be a *cycle* in Γ . An undigraph Γ is *acyclic* Γ doesn't have cycles.

The following example, constructed on the base of free directed pseudoplane, found independently by A. Pillay [146] and by the author [169], shows, that different elementary chains over a same type may generate non-isomorphic limit models forming continuum pairwise non-isomorphic limit models.

E x a m p l e 1.2.3. Consider a countable model \mathcal{M}_0 of a connected acyclic digraph $\langle M_0; Q \rangle$ with acyclic undigraph $\langle M_0; Q \cup Q^{-1} \rangle$ such that every element has infinitely many images and infinitely many preimages. The system \mathcal{M}_0 has constructed independently by A. Pillay [146] and by the author [169] for a realization of non-symmetric semiisolation relation in the class of stable theories. The system \mathcal{M}_0 is said to be a *free directed pseudoplane*.

Expand the signature by new binary predicates Q_0 and Q_1 , forming a partition of the predicate Q with the following condition: for any element $a \in M_0$ there exist infinitely many images and infinitely many preimages with respect to Q_0 and to Q_1 . Define then a 1-inessential Q -ordered coloring $\text{Col}: M_0 \rightarrow \omega \cup \{\infty\}$ of the resulting model so that every element of color n has the following:

- (1) infinitely many images of color μ relative to Q_0 and to Q_1 for any $\mu \geq n$ (including ∞);
- (2) infinitely many preimages of color m relative to Q_0 and to Q_1 for any $m \leq n$.

The ω -stability of $\text{Th}(\langle \langle M_0; Q, Q_0, Q_1 \rangle, \text{Col} \rangle)$ is followed from its Δ -baseness (implied by an acyclicity of structure), where Δ is the least closed, relative substitutions of variables, set of formulas without more than two free variables such that this set contains a formula $(x \approx y)$ and satisfies the following condition: if $\varphi(x, y) \in \Delta$ then $\exists z (\varphi(x, z) \wedge Q_i^{\delta_1}(z, y) \wedge \text{Col}_n^{\delta_2}(z)) \in \Delta$, where $\delta_1, \in \{-1, 1\}$, $i, \delta_2 \in \{0, 1\}$, $Q_i^1(x, y) = Q(x, y)$, $Q_i^{-1}(x, y) = Q(y, x)$, $\text{Col}_n^1(z) =$

$\text{Col}_n(z)$, $\text{Col}_n^0(z) = \neg\text{Col}_n(z)$. Here, the countable number of 1-types over every countable set A is guaranteed by the countable number of possibilities of distance distributions from elements of A to realizations of types.

The ω -stability of $\text{Th}(\langle\langle M_0; Q, Q_0, Q_1 \rangle, \text{Col} \rangle)$ implies an existence of prime model \mathcal{M}_{p_∞} over a realization of type $p_\infty(x)$. By Proposition 1.2.13, the relation SI_{p_∞} is nonsymmetric and it is witnessed by formulas $Q_0(x, y)$ and $Q_1(x, y)$.

We argue to show that there exist 2^ω pairwise non-isomorphic limit models over p_∞ . For this goal to be met, we construct inductively elementary chains $(\mathcal{M}_{\alpha \upharpoonright n})_{n \in \omega}$, $\alpha \in 2^\omega$, over p_∞ . We take by $\mathcal{M}_{\alpha \upharpoonright 0}$ an arbitrary prime model over a realization $a_{\alpha \upharpoonright 0}$ of p_∞ . If models $\mathcal{M}_{\alpha \upharpoonright 0}, \dots, \mathcal{M}_{\alpha \upharpoonright n}$ are already constructed, and $\mathcal{M}_{\alpha \upharpoonright n}$ is a prime model over a realization $a_{\alpha \upharpoonright n}$ of type p_∞ , then as $\mathcal{M}_{\alpha \upharpoonright n+1}$ we take a prime model over a realization $a_{\alpha \upharpoonright n+1}$ of p_∞ , where $\mathcal{M}_{\alpha \upharpoonright n} \prec \mathcal{M}_{\alpha \upharpoonright n+1}$ and $\models Q_{\alpha(n)}(a_{\alpha \upharpoonright n+1}, a_{\alpha \upharpoonright n})$. Denote the model $\bigcup_{n \in \omega} \mathcal{M}_{\alpha \upharpoonright n}$ by \mathcal{M}_α . Sequences α and β in 2^ω are said to be *equivalent* if there exist $k, m \in \omega$ such that $\alpha(k+n) = \beta(m+n)$ for all $n \in \omega$. Clearly, models \mathcal{M}_α and \mathcal{M}_β iff α and β are equivalent. Since every equivalence class is countable, there are 2^ω equivalence classes. Choosing one model in each class yields 2^ω pairwise non-isomorphic limit models over p_∞ . \square

§ 1.3. Type reducibility and powerful types

In this Section, we define concepts of p -principal p -type and of reducibility of theory over a type and prove, on the one hand, that, for a theory T without the strict order property and with a non-principal powerful type p , there is a non- p -principal p -type and T is not reducible over p . On the other hand, we show an example of ω -stable theory, having a non- p -principal p -type such that this type is realized in models \mathcal{M}_p .

Let $p(\bar{x})$ be a type in $S(T)$. A type $q(\bar{x}_1, \dots, \bar{x}_n) \in S(T)$ is said to be a (n, p) -type if $q(\bar{x}_1, \dots, \bar{x}_n) \supseteq \bigcup_{i=1}^n p(\bar{x}_i)$. The set of all (n, p) -types of T is denoted by $S_{n,p}(T)$ and elements of $S_p(T) = \bigcup_{n \in \omega \setminus \{0\}} S_{n,p}(T)$ are p -types.

Without loss of generality of the results, we will consider in this Section for simplicity that p is a type in $S^1(\emptyset)$.

A type $q(\bar{y})$ in $S(T)$ is said to be *p-principal* if there is a formula $\varphi(\bar{y}) \in q(\bar{y})$ such that $\cup\{p(y_i) \mid y_i \in \bar{y}\} \cup \{\varphi(\bar{y})\} \vdash q(\bar{y})$.

The following lemma is obvious.

Lemma 1.3.1. *For any type p and natural $n \geq 1$, the following conditions are equivalent:*

- (1) *the set of (n, p) -types with free variables x_1, \dots, x_n is infinite;*
- (2) *there is a non- p -principal (n, p) -type.*

Let \mathcal{M} be a countable saturated model of a theory T having a predicate language. Consider, induced by \mathcal{M} , a subsystem $p(\mathcal{M}) = \langle p(M); \Sigma(T) \rangle$ of language $\Sigma(T)$ of T with the universe $p(M) = \{a \in M \mid \models p(a)\}$ and relations $R(p(\mathcal{M})) = R(\mathcal{M}) \cap (p(M))^{\mu(R)}$, $R \in \Sigma(T)$. Denote the theory $\text{Th}(p(\mathcal{M}))$ by T_p .

A theory is said to be *reduced over a type p* if T and T_p admits the quantifier elimination.

Proposition 1.3.2. *If a small theory T is reduced over a type p , then, in a model \mathcal{M}_p , any non- p -principal p -type is omitted.*

P r o o f. Notice, that by the quantifier elimination of T and T_p , there exists a bijection $\cdot_p : S_p(T) \rightarrow S(T_p)$ such that $q_p(p(M)) = q(M)$, $q \in S_p(T)$, and the restriction of this bijection on the set of p -principal types implements a one-to-one correspondence with the set of principal types of T_p . Denote a prime model of T_p by \mathcal{M}_0 , that exists by the smallness of T . Assume that, in \mathcal{M}_p , a non- p -principal type $q(\bar{y})$ in $S_p^n(T)$ is realized. Then there is a quantifier-free formula $\psi(x, \bar{y})$ of T such that

$$T \vdash \exists x (\varphi(x) \wedge \exists \bar{y} \psi(x, \bar{y}) \wedge \forall \bar{y} (\psi(x, \bar{y}) \rightarrow \chi(\bar{y})))$$

for any $\varphi(x) \in p(x)$ and quantifier-free formulas $\chi(\bar{y}) \in q(\bar{y})$. It hence follows that

$$T_p \vdash \exists x (\exists \bar{y} \psi(x, \bar{y}) \wedge \forall \bar{y} (\psi(x, \bar{y}) \rightarrow \chi(\bar{y}))),$$

where $\chi(\bar{y})$ is a quantifier-free formula of type $q_p(\bar{y})$. Since T_p is a transitive theory admitting the quantifier elimination, there is an element $a \in M_0$ such that

$$\mathcal{M}_0 \models \exists \bar{y} \psi(a, \bar{y}) \wedge \forall \bar{y} (\psi(a, \bar{y}) \rightarrow q_p(\bar{y})).$$

It means that the type $q_p(\bar{y}) \in S^n(T_p)$ is realized in \mathcal{M}_0 . But q_p is a nonprincipal type, since the correspondent p -type q is non- p -principal. Consequently, a nonprincipal type is realized in the prime model \mathcal{M}_0 , a contradiction. \square

Fix a theory T and consider its *Morleyzation*, i.e., an expansion to a complete theory T' of language $\Sigma(T) \cup \{R_\varphi \mid \varphi \text{ is a formula of } T\}$ such that R_φ is a $l(\bar{y})$ -ary predicate symbol with $T' \vdash R_\varphi(\bar{y}) \leftrightarrow \varphi(\bar{y})$, where $\varphi(\bar{y})$ is a formula of T . A restriction of T' to the complete theory of language $\{R_\varphi \mid \varphi \text{ is a formula of } T\}$ is denoted by T^* . Thus, we get an operation $\cdot^* : T \rightarrow T^*$. It's easy to see an existence of a one-to-one correspondence $\cdot^* : S(T) \rightarrow S(T^*)$ for which a complete type $q^*(\bar{y}) \equiv \{\psi(\bar{y}) \mid T^* \vdash R_\varphi(\bar{y}) \rightarrow \psi(\bar{y}) \text{ for some formula } \varphi(\bar{y}) \in q\}$ corresponds to the type $q(\bar{y}) \in S(T)$. For this correspondence, the λ -stability, the *simpleness* (see [27]) and the smallness of theories, and also the properties of isolation and of being powerful for types are preserved. So considering questions on existence of nonprincipal powerful types in aforesaid classes of theories, it suffices to take theories of form T^* .

Then Lemma 1.3.1 and Proposition 1.3.2 imply

Corollary 1.3.3. *If $|S_{n,p}(T)| = \omega$ and the small theory T^* is reduced over the type p^* , then some (n,p) -type is omitted in the model \mathcal{M}_p .*

Recall, that a theory T has the *strict order property* if there exists a formula $\varphi(\bar{x}, \bar{y})$ of T and tuples \bar{a}_i , $i \in \omega$ such that the following equivalence holds:

$$\vdash \varphi(\bar{a}_i, \bar{y}) \rightarrow \varphi(\bar{a}_j, \bar{y}) \Leftrightarrow i \leq j.$$

Notice, that theories with formula-definable infinite linear orders have the strict order property. In particular, the Ehrenfeucht examples have this property (see example 1.1.1). Here, for any powerful type p and for any natural n , the number of (n,p) -types with free variables $\bar{x}_1, \dots, \bar{x}_n$ is finite and, consequently, all p -types are p -principal.

The following proposition, contained implicitly in R. E. Woodrow [205], clarifies that the described situation is impossible for theories without the strict order property.

Proposition 1.3.4. *If $p(\bar{x})$ is a nonprincipal powerful type of theory T and T doesn't have the strict order property, then $|S_{2,p}(T)| = \omega$.*

P r o o f. Consider a formula $\varphi(\bar{x}, \bar{y})$ of T witnessing the non-symmetry of relation SI_p (such a formula exists by Lemma 1.1.2). We set $\varphi^0(\bar{x}, \bar{y}) \equiv (\bar{x} \approx \bar{y})$, $\varphi^1(\bar{x}, \bar{y}) \equiv \varphi(\bar{x}, \bar{y})$, $\varphi^{n+1}(\bar{x}, \bar{y}) \equiv \exists \bar{z}(\varphi^n(\bar{x}, \bar{z}) \wedge \varphi(\bar{z}, \bar{y}))$, $n \in \omega \setminus \{0\}$. It suffices to show, that in a system $p(\mathcal{M}^*)$ (where \mathcal{M}^* is a countable saturated model of T^*) for any $\bar{a} \in p(\mathcal{M})$, inequalities $R_{\varphi^{n+1}}(\bar{a}, p(\mathcal{M})) \setminus R_{\varphi^n}(\bar{a}, p(\mathcal{M})) \neq \emptyset$ hold for every $n \in \omega$. Assume on the contrary, that for some n an inclusion $R_{\varphi^{n+1}}(\bar{a}, p(\mathcal{M})) \subseteq R_{\varphi^n}(\bar{a}, p(\mathcal{M}))$ is true. Consider a formula $\psi(\bar{x}, \bar{y}) \equiv \bigvee_{i=0}^n \varphi^i(\bar{x}, \bar{y})$. Then, by assumption, for any

$\bar{a}, \bar{b} \in p(\mathcal{M})$, satisfying $\models \varphi(\bar{a}, \bar{b})$ and $(\bar{b}, \bar{a}) \notin \text{SI}_p$, we get $\vdash \psi(\bar{b}, \bar{y}) \rightarrow \psi(\bar{a}, \bar{y})$. Since the formula $\psi(\bar{b}, \bar{y})$ witnesses on the semi-isolation over \bar{b} of every its realization, $\models \psi(\bar{a}, \bar{a})$ and $(\bar{b}, \bar{a}) \notin \text{SI}_p$, then $\models \exists \bar{y}(\psi(\bar{a}, \bar{y}) \wedge \neg \psi(\bar{b}, \bar{y}))$. Since tuples \bar{a} and \bar{b} realizes the same type p , there exists a sequence $(\bar{c}_n)_{n \in \omega}$ of realizations of p such that $\vdash \psi(\bar{c}_i, \bar{y}) \rightarrow \psi(\bar{c}_j, \bar{y})$ and $\models \exists \bar{y}(\psi(\bar{c}_j, \bar{y}) \wedge \neg \psi(\bar{c}_i, \bar{y}))$, $i < j < \omega$. This relations contradict an absence of the strict order property in T . \square

In view of Corollary 1.3.3 and Proposition 1.3.4, we have

Theorem 1.3.5. *If T a theory without the strict order property, $p(x)$ is a nonprincipal powerful type of T , then T^* is not reduced over p^* .*

The following Proposition shows that a realizability on non- p -principal p -types in a model \mathcal{M}_p implies the non-symmetry of relation SI_p .

Proposition 1.3.6. *If a non- p -principal p -type q is realized in a model \mathcal{M}_a , where a is a realization of p , then, for every element b_i of a realization \bar{b} of q in \mathcal{M}_a , the pair (a, b_i) belongs to SI_p and (b_i, a) doesn't belong to SI_p .*

P r o o f. Let a be a realization of type p , $\varphi(a, \bar{y})$ be a formula, isolating a non- p -principal p -type $q(\bar{y})$. Assume, that some element b_i of a realization \bar{b} of $q(\bar{y})$ in \mathcal{M}_a semi-isolates the element a . Consider a formula $\psi(y_i, x)$ witnessing on the semi-isolation of a over b_i . Then the type $q(\bar{y})$ is isolated by set $\cup\{p(y_i) \mid y_i \in \bar{y}\} \cup \{\exists x(\varphi(x, \bar{y}) \wedge \psi(y_i, x))\}$. This is impossible, since the p -type $q(\bar{y})$ is not p -principal. \square

Recall, that any maximal, w.r.t. inclusion, connected subgraph of graph $\Gamma = \langle X; Q \rangle$ is said to be a *connected component* of Γ . Each connected component \mathcal{C} of Γ is uniquely defined by every its element $a \in \mathcal{C}$ and is denoted by $\mathcal{C}(a, \Gamma)$, or by $\mathcal{C}(a, Q)$ if the universe X is given.

For a graph $\Gamma = \langle X; Q \rangle$, we define inductively the following relations Q^n , $n \in \mathbf{Z}$: $Q^0 \doteq \text{id}_X$, $Q^1 \doteq Q$, $Q^{n+1} \doteq Q^n \circ Q$, $Q^{-n} \doteq (Q^n)^{-1}$, $n \in \omega$.

Example 1.3.1. We are going to construct an ω -stable theory, having a type p such that some non- p -principal p -type is realizable in an elementary submodel \mathcal{M}_p of model \mathcal{M} .

The language Σ will consist of unary predicate symbols Col_n , $n \in \omega$, binary predicate symbols Q, R_1, R_2 and 3-ary predicate symbol S .

The predicate Q defines on the universe M a free directed pseudoplane, as in Example 1.2.3, with a *transitive* (that is connecting any two elements) automorphism group, with infinitely many connected components $C(a, Q)$ and with an 1-inessential Q -ordered coloring Col , correspondent to symbols Col_n , $n \in \omega$, satisfying the following conditions:

- (1) infinitely many images of color μ relative to Q for any $\mu \geq n$ (including ∞);
- (2) infinitely many preimages of color m relative to Q for any $m \leq n$.

The predicate R_1 connects only elements a and b of same colors, for which $\models \exists x (Q(x, a) \wedge Q(x, b))$ holds, and defines, in a set of solutions of every formula $Q(a, y)$, a *sequence function* with unique image c_1 , unique preimage $c_2 \neq c_1$ for each element b , such that $\models Q(a, b)$, and without cycles.

The predicate R_2 , as well as Q , defines in M a free directed pseudoplane with a transitive automorphism group, with infinitely many connected components $C(a, R_2)$ and with an 1-inessential R_2 -ordered coloring Col , correspondent to symbols Col_n , $n \in \omega$, satisfying the following conditions:

- (1) infinitely many images of color μ relative to R_2 for any $\mu \geq n$ (including ∞);
- (2) infinitely many preimages of color m relative to R_2 for any $m \leq n$.

Furthermore, every two different preimages of every element, w.r.t. $\bigcup_{n \in \omega} (R_2)^n$, lie in different connected components w.r.t. Q , and exactly two elements, b and c , lying in a same connected component w.r.t. Q , are R_2 -connected with an image of each element a w.r.t. R_2 . These elements satisfies $\models \exists x (Q(x, b) \wedge Q(x, c))$, have the same color, and if $\text{Col}(a) = n$ then, in the graph with relation $R_1 \cup (R_1)^{-1}$,

the length of shortest (b, c) -route is not less than n . Moreover, we consider, that for each pair of elements b, c of same color and with $\models \exists x (Q(x, b) \wedge Q(x, c))$, $(b, c) \in R_1^n \cup (R_1)^{-n}$ and for any color $m \leq n$, there exists a common preimage of b and c w.r.t. R_2 , having the color m .

The predicate S connects all possible triples of elements a, b, c such that $\models R_2(a, b) \wedge R_2(a, c)$.

As in Example 1.3.1 we state, that all requirements can be realized such that the theory T_0 of described model is Δ -based, where Δ is the least closed (w.r.t. substitutions of variables) set of formulas without more than two free variables, containing the formula $(x \approx y)$ and satisfying the following condition: if $\varphi(x, y) \in \Delta$ then $\exists z (\varphi(x, z) \wedge \mathfrak{R}^{\delta_1}(z, y) \wedge \text{Col}_n^{\delta_2}(z)) \in \Delta$, where $\delta_1 \in \{-1, 1\}$, $\delta_2 \in \{0, 1\}$, $\mathfrak{R}^1(x, y) = \mathfrak{R}(x, y)$, $\mathfrak{R}^{-1}(x, y) = \mathfrak{R}(y, x)$, $\mathfrak{R} \in \{Q, R_1, R_2\}$, $\text{Col}_n^1(z) = \text{Col}_n(z)$, $\text{Col}_n^0(z) = \neg \text{Col}_n(z)$. Using Δ -baseness, a routine consideration of cases of connections of elements of tuples shows the ω -stability of T_0 .

The set of formulas $\{\neg \text{Col}_n(x) \mid n \in \omega\}$ isolates unique, being in T_0 , nonprincipal 1-type. This type, denoted by $p_\infty(x)$, is realized by elements of infinite color.

For any element a of infinite color, the formula $S(a, x, y)$ isolates a non- p_∞ -principal $(2, p_\infty)$ -type $q(x, y) \in S(T_0)$, that defined by set of formulas

$$\{\exists z (Q(z, x) \wedge Q(z, y) \wedge \neg \text{Col}_n(z)) \wedge \neg R_1^n(x, y) \mid n \in \omega\}.$$

Here for elements a_n of color $n \in \omega$, formulas $S(a_n, x, y)$ isolate types, approximating a description of type $q(x, y)$. \square

§ 1.4. Powerful digraphs

In this Section, we introduce the concept of a powerful digraph and establish its “local” presence in the structure of any nonprincipal powerful type p . We also show, that, having the $(2, p)$ -invariance property of theory, a structure of powerful digraph is contained in a restriction of saturated structure to a structure of realizations of nonprincipal powerful type, having the global pairwise intersection property. We describe structures of transitive closures of saturated powerful digraphs, formed in models of theories with nonprincipal

powerful 1-types, when the number of nonprincipal 1-types is finite. Moreover, we prove, that the structure of powerful digraph, considered in a model of *simple* theory [27], induces infinite weight. It means, that there are no powerful graphs in structures of known classes of simple theories (such as supersimple or finitely based theories), that don't contain Ehrenfeucht theories.

A countable acyclic digraph $\Gamma = \langle X; Q \rangle$ is said to be *powerful* if the following conditions hold:

- (a) the automorphism group of Γ is *transitive*, that is any two vertices are connected by an automorphism;
- (b) the formula $Q(x, y)$ is equivalent in the theory $\text{Th}(\Gamma)$ to a disjunction of principal formulas;
- (c) $\text{acl}(\{a\}) \cap \bigcup_{n \in \omega} Q^n(\Gamma, a) = \{a\}$ for each vertex $a \in X$;
- (d) $\Gamma \models \forall x, y \exists z (Q(z, x) \wedge Q(z, y))$ (the *pairwise intersection property*).

Clearly, in the classical examples of Ehrenfeucht theories, the countable graph with the relation $x < y$ of dense linear order is powerful.

The following example, represented in the works by A. I. Mal'tsev [118] and E. E. Bouscaren, B. P. Poizat [58], defines a stable theory of *acyclic pairing function* with a structure of powerful digraph.

Example 1.4.1. Let M be a set with two functions $f_1, f_2 : M \rightarrow M$ such that $(f_1, f_2) : M \rightarrow M \times M$ is a bijection, for which there are no nonempty sequences i_1, \dots, i_n and elements $a \in M$ such that $f_{i_n}(\dots f_{i_1}(a)\dots) = a$. The theory $T \equiv \text{Th}(\langle M; f_1, f_2 \rangle)$, being a theory of a locally free algebra², is stable. Consider the *free* 1-type $p \in S^1(T)$, i.e., the type of elements a such that

$$f_{i_n}(\dots f_{i_1}(a)\dots) = f_{j_m}(\dots f_{j_1}(a)\dots) \Leftrightarrow i_1 \dots i_n = j_1 \dots j_m.$$

It's easy to see, that some countable set of realizations of p with the relation, defined by formula $(y \approx f_1(x)) \vee (y \approx f_2(x))$, forms a powerful digraph. \square

In the following example of a theory with three countable models, which is similar to an example in the work by M. G. Peretyat'kin [137], the powerful digraph also includes the relation $<$.

²The stability of theories of locally free algebras is proved by O. V. Belegadek [55].

Example 1.4.2. Let $\mathcal{M} = \langle M; \leq \rangle$ be a lower semilattice without the least and greatest elements such that:

(a) for each pair of incomparable elements, their join does not exist;

(b) for each pair of distinct comparable elements, there is an element between them;

(c) for each element a there exist infinitely many pairwise incomparable elements greater than a , whose infimum is equal to a .

Expand the system \mathcal{M} by constants c_n , $n \in \omega$, such that $c_n < c_{n+1}$, $n \in \omega$. The theory T of this expansion has exactly three countable models: the prime model; the saturated model; the prime model over the realization of the powerful type $p_\infty(x)$, isolated by the set $\{c_n < x \mid n \in \omega\}$ of formulas. \square

Apart from these examples, a rather rich class of powerful digraphs is formed by the acyclic digraphs $\langle P; Q \rangle = \langle P; \{(p, p') \mid p' = pg_0 \text{ on some line}\} \rangle$, corresponding to polygonometries $\text{pm}(G, \langle P, L, \in \rangle, g_0)$ on projective planes [172].

Let \mathcal{M} be a model of a theory T , $p(\bar{x})$ be a complete type of T over the empty set, $\psi(\bar{x}, \bar{y})$ be a formula of T , where $l(\bar{x}) = l(\bar{y})$. Denote by $p(M)$ the set of realizations of $p(\bar{x})$ in \mathcal{M} , and by $R_\psi^p(\mathcal{M})$, the binary relation $\{(\bar{a}, \bar{b}) \in (p(M))^2 \mid \mathcal{M} \models \psi(\bar{a}, \bar{b})\}$.

The following statement shows that the powerful digraphs reside “locally” in the structure of each nonprincipal powerful type.

Proposition 1.4.1. *If $p(\bar{x})$ is a nonprincipal powerful type of some theory T and \mathcal{M} is a countable saturated model of T then for each formula $\varphi(\bar{x}) \in p(\bar{x})$, there exists a formula $\psi(\bar{x}, \bar{y})$ of T (where $l(\bar{x}) = l(\bar{y})$), satisfying the following conditions:*

(1) *for each $\bar{a} \in p(M)$ the formula $\psi(\bar{a}, \bar{x})$ is equivalent to a disjunction of principal formulas $\psi_i(\bar{a}, \bar{x})$, $i \leq n$, such that $\psi_i(\bar{a}, \bar{x}) \vdash p(\bar{x})$, and $\models \psi_i(\bar{a}, \bar{b})$ implies, that \bar{b} doesn't semi-isolate \bar{a} ;*

(2) *for every $\bar{a}, \bar{b} \in p(M)$ there exists a tuple \bar{c} such that $\models \varphi(\bar{c}) \wedge \psi(\bar{c}, \bar{a}) \wedge \psi(\bar{c}, \bar{b})$.*

Proof. By the condition and Lemma 1.1.2, there are realizations \bar{a} and \bar{b} of $p(\bar{x})$ in the model $\mathcal{M}_{\bar{a}}$ such that \bar{b} doesn't semi-isolate \bar{a} . Since $\mathcal{M}_{\bar{b}} \prec \mathcal{M}_{\bar{a}}$, for each realization $\bar{c} \in M_{\bar{b}}$ there is a principal formula $\chi_{\bar{c}}(\bar{a}, \bar{y})$ such that $\models \chi_{\bar{c}}(\bar{a}, \bar{c})$. Enumerate all realizations of $p(\bar{x})$ in $\mathcal{M}_{\bar{b}}$: $p(M_{\bar{b}}) = \{\bar{c}_n \mid n \in \omega\}$. Put $\chi_n(\bar{a}, \bar{y}) \equiv \chi_{\bar{c}_n}(\bar{a}, \bar{y})$ for $n \in \omega$.

Fix some formula $\varphi(\bar{x}) \in p(\bar{x})$ and show that some formula $\bigvee_{i=0}^m \chi_i(\bar{x}, \bar{y})$ can be taken as $\psi(\bar{x}, \bar{y})$. Clearly, each of those formulas satisfies condition 1. Assuming that none of them satisfies condition 2, by Compactness Theorem, we get the consistency of the set

$$r(\bar{x}, \bar{y}) \Rightarrow p(\bar{x}) \cup p(\bar{x}) \cup \left\{ \neg \exists \bar{z} \left(\left(\bigvee_{i=0}^m \chi_i(\bar{z}, \bar{x}) \right) \wedge \left(\bigvee_{i=0}^m \chi_i(\bar{z}, \bar{y}) \right) \wedge \varphi(\bar{z}) \right) \mid m \in \omega \right\}.$$

Since $p(\bar{x})$ is powerful, the type $r(\bar{x}, \bar{y})$ is realized in $\mathcal{M}_{\bar{b}}$ by some tuples \bar{d}_1 and \bar{d}_2 : $\mathcal{M}_{\bar{b}} \models r(\bar{d}_1, \bar{d}_2)$. Thus, there exist formulas $\chi_{\bar{d}_1}(\bar{x}, \bar{y})$ and $\chi_{\bar{d}_2}(\bar{x}, \bar{y})$ such that $\models \chi_{\bar{d}_1}(\bar{a}, \bar{d}_1) \wedge \chi_{\bar{d}_2}(\bar{a}, \bar{d}_2)$, which contradicts the consistency of $r(\bar{x}, \bar{y})$. Therefore, $r(\bar{x}, \bar{y})$ is inconsistent; hence, for some m_0 , we have the inconsistency of the set

$$p(\bar{x}) \cup p(\bar{y}) \cup \left\{ \neg \exists \bar{z} \left(\left(\bigvee_{i=0}^{m_0} \chi_i(\bar{z}, \bar{x}) \right) \wedge \left(\bigvee_{i=0}^{m_0} \chi_i(\bar{z}, \bar{y}) \right) \wedge \psi_n(\bar{z}) \right) \mid n \in \omega \right\}.$$

Putting $\psi(\bar{x}, \bar{y}) \Leftarrow \bigvee_{i=0}^{m_0} \chi_i(\bar{x}, \bar{y})$, we deduce the claim. \square

Call property 2 of Proposition 1.4.1, the *local pairwise intersection property* and denote it by (LPIP). If for a formula $\psi(\bar{x}, \bar{y})$ the stronger property is true:

2') for every $\bar{a}, \bar{b} \in p(M)$ there exists a tuple $\bar{c} \in p(M)$ such that $\models \psi(\bar{c}, \bar{a}) \wedge \psi(\bar{c}, \bar{b})$,

we then call it the *global pairwise intersection property* for $p(\bar{x})$ with respect to $\psi(\bar{x}, \bar{y})$ and it will be denoted by (GPIP).

Whenever a formula $\psi(\bar{x}, \bar{y})$ with properties 1 and 2' exist, call the digraph $\langle p(M); R_{\psi}^p(\mathcal{M}) \rangle$ *prepowerful*.

Recall, that theories T_0 and T_1 of languages Σ_0 and Σ_1 respectively are said to be *similar* if for any models $M_i \models T_i$, $i = 0, 1$, there are formulas of T_i , defining in M_i predicates, functions and constants of language Σ_{1-i} such that the corresponding algebraic system of Σ_{1-i} is a model of T_{1-i} .

A theory T is said to be (n, p) -*invariant* if for each formula $\psi(\bar{x})$ (where $l(\bar{x}) = n$) of T_p the restriction of the Morleyzation of T_p to

the language $\{R_\psi\}$ is similar to the restriction of the Morleyzation of the theory of the structure of some formula-definable set $\varphi(\mathcal{M})$ (where $\varphi \in p$) to the same language.

Show that the existence of a prepowerful structure on the set of realizations of a nonprincipal powerful type p of some $(2, p)$ -invariant theory implies the existence of a formula-defining a powerful digraph structure on this set.

Proposition 1.4.2. *If $p(x)$ is a nonprincipal powerful type of some $(2, p)$ -invariant theory T and $\langle p(M); R_\psi^p(\mathcal{M}) \rangle$ is a prepowerful digraph then for some formula $\theta(x, y)$ with $T \vdash \theta(x, y) \rightarrow \psi(x, y)$, the digraph $\langle p(M); R_\theta^p(\mathcal{M}) \rangle$ is powerful.*

P r o o f. The $(2, p)$ -invariance of T implies that we can choose to make principal the formulas $R_{\chi_i}^p(x, y)$ corresponding in T to $\chi_i(x, y)$ such that $\psi(a, x) = \bigvee_{i=0}^m \chi_i(a, x)$, where $\chi_i(a, x)$ are principal formulas for each $a \in p(M)$ and all $i = 1, \dots, m$. Indeed, the element a belongs to a prime model \mathcal{M}_0 of the restriction of the Morleyzation of T_p to the language $\{R_\psi\}$. Some realizations d_i of principal formulas, corresponding to the formulas $\chi_i(a, y)$ $i = 0, \dots, m$, belong to the same model. Consider formulas $\chi'_i(x, y)$ of language $\{R_\psi\}$, corresponding to complete formulas of the types $\text{tp}(a \hat{\ } d_i)$, $i = 0, \dots, m_0$. Take as $\theta(x, y)$ some formula of T satisfying the following conditions:

(1) the restriction of the Morleyzation of the theory of the structure of some formula-definable set $\varphi(\mathcal{M})$ (where $\varphi \in p$) to the language $\{R_\theta\}$ is similar to the restriction of the Morleyzation of T_p to the same language;

$$(2) \vdash ((\varphi(x) \wedge \varphi(y)) \rightarrow (\theta(x, y) \leftrightarrow \bigvee_{i=0}^m \chi'_i(x, y)));$$

$$(3) \vdash \theta(\bar{x}, \bar{y}) \rightarrow \psi(\bar{x}, \bar{y}).$$

Since the model \mathcal{M} is saturated, the automorphism group of the digraph $\Gamma = \langle p(M); R_\theta^p(\mathcal{M}) \rangle$ is transitive.

Notice, that for each $a \in p(M)$ it follows from $\models \theta(a, b)$ that $b \in p(M)$ and b does not semi-isolate a . Thus, since the semi-isolation relation is transitive, the digraph Γ is acyclic.

The non-symmetry of the semi-isolation relation SI_p , witnessed by the formula θ , implies the equality

$$\text{acl}(\{b\}) \cap \bigcup_{n \in \omega} (R_\theta^p(\mathcal{M}))^n(\Gamma, b) = \{b\}$$

for each $b \in p(\mathcal{M})$. Indeed, assuming that there exists

$$d \in \text{acl}(\{b\}) \cap \bigcup_{n \in \omega} (R_\theta^p(\mathcal{M}))^n(\Gamma, b) \setminus \{b\},$$

we deduce that b semi-isolates d in \mathcal{M} . Thus, because the semi-isolation relation is transitive, the element b will semi-isolate a , where $\models \theta(a, b)$ and $a \in p(M)$; this is a contradiction.

The (GPIP) for $p(\bar{x})$ with respect to $\psi(x, y)$ implies the same property with respect to $\theta(x, y)$, and so, we have the pairwise intersection property for Γ . Therefore, Γ is a powerful digraph. \square

Notice, that under the assumptions of Proposition 1.4.2 the finiteness of number of nonprincipal 1-types implies the relation

$$\text{acl}(\{a\}) \cap \bigcup_{n \in \omega} (R_\theta^p(\mathcal{M}))^n(a, \Gamma) = \{a\} \quad (1.1)$$

for each vertex a of the digraph $\Gamma = \langle p(M); R_\theta^p(\mathcal{M}) \rangle$.

Indeed, assume that the type $\text{tp}(b/a)$ is algebraic for some $b \in (R_\theta^p(\mathcal{M}))^n(\bar{a}, \Gamma)$, $b \neq a$. Then the original theory contains a semi-isolating formula $\theta(a, y)$ such that $\models \theta(a, b) \wedge \exists^{=k} \bar{y} \theta(a, y)$ for some $k \in \omega$. Since b does not semi-isolate a and the number of nonprincipal 1-types is finite, it follows that there exists a tuple c realizing a principal type such that $\models \theta(c, b) \wedge \exists^{=k} \bar{y} \theta(c, y)$. This means that the nonprincipal type $p(x)$ is realized in the prime model; this is a contradiction.

If the number of nonprincipal 1-x)-types is infinite then (1.1) need not hold. To illustrate that, consider the following example of an ω -stable theory with a nonprincipal 1-type $p_0(x)$, having a nonsymmetric semiisolation relation via a formula $Q(x, y)$ such that $\text{acl}(\{a\}) = \bigcup_{n \in \omega} Q^n(a, \mathcal{M})$ for each realization a of $p_0(x)$.

E x a m p l e 1.4.3. Denote by Ω the set of nonempty finite sequences $\bar{\alpha} = \langle \alpha_0, \alpha_1, \dots, \alpha_n \rangle$ with $\alpha_i \in \omega$ for $i \leq n$ and $l(\bar{\alpha}) = \alpha_0 + 2$.

Let T_0 be a theory of language $\langle P_{\bar{\alpha}}^{(1)}, Q^{(2)} \rangle_{\bar{\alpha} \in \Omega}$ with the following axioms:

1) if $\bar{\alpha} = \bar{\alpha}' \wedge m \in \Omega$ then

$$\vdash (P_{\bar{\alpha}' \wedge (m+1)}(x) \rightarrow P_{\bar{\alpha}' \wedge m}(x)) \wedge \exists^{\geq \omega} x (P_{\bar{\alpha}' \wedge m}(x) \wedge \neg P_{\bar{\alpha}' \wedge (m+1)}(x));$$

2) if $\bar{\alpha}_1 = \bar{\alpha}'_1 \wedge 0$ and $\bar{\alpha}_2 = \bar{\alpha}'_2 \wedge 0$ are tuples in Ω and $\bar{\alpha}'_1 \neq \bar{\alpha}'_2$ then $\vdash \neg \exists x (P_{\bar{\alpha}_1}(x) \wedge P_{\bar{\alpha}_2}(x))$;

3) the relation Q forms the graph of a free (acyclic) unar with infinitely many preimages of each element;

$$4) \vdash \forall x, y ((P_{\langle 0, m \rangle}(x) \wedge \neg P_{\langle 0, m+1 \rangle}(x) \wedge Q(x, y)) \rightarrow (P_{\langle 0, m \rangle}(y) \wedge \neg P_{\langle 0, m+1 \rangle}(y))) \text{ for } m \in \omega;$$

5) if $\models P_{\langle 0, m \rangle}(a) \wedge \neg P_{\langle 0, m+1 \rangle}(a)$ then the set of realizations of the formula $Q(x, a)$ consists of infinitely many realizations of the formula $P_{\langle 0, m \rangle}(x) \wedge \neg P_{\langle 0, m+1 \rangle}(x)$, and infinitely many realizations of the formulas $P_{\langle 1, k, m \rangle}(x) \wedge \neg P_{\langle 1, k, m+1 \rangle}(x)$ for each $k \in \omega$;

6) if $\bar{\alpha} = k \wedge \bar{\alpha}' \wedge l \wedge m$ is a tuple in Ω , $k \geq 1$, then

$$\vdash \forall x, y \left((P_{k \wedge \bar{\alpha}' \wedge l \wedge m}(x) \wedge \neg P_{k \wedge \bar{\alpha}' \wedge l \wedge (m+1)}(x) \wedge Q(x, y)) \rightarrow \right. \\ \left. (P_{(k-1) \wedge \bar{\alpha}' \wedge m}(y) \wedge \neg P_{(k-1) \wedge \bar{\alpha}' \wedge (m+1)}(y)) \right),$$

$m \in \omega$;

7) if $k \neq 0$ and $\models P_{k \wedge \bar{\alpha} \wedge m}(a) \wedge \neg P_{k \wedge \bar{\alpha} \wedge (m+1)}(a)$, then the set of realizations of the formula $Q(x, a)$ consists of infinitely many realizations of the formulas $P_{(k+1) \wedge \bar{\alpha} \wedge l \wedge m}(x) \wedge \neg P_{(k+1) \wedge \bar{\alpha} \wedge l \wedge (m+1)}(x)$ for each $l \in \omega$.

The construction of a saturated model satisfying axioms 1–7, enables us to verify the completeness of T_0 . The ω -stability of T_0 follows because each formula without parameters is equivalent to a Boolean combination of formulas of the form $P_{\bar{\alpha}}(x)$, $\bar{\alpha} \in \Omega$, and $\exists z (Q^{n_1}(x, z) \wedge Q^{n_2}(y, z))$, $n_1, n_2 \in \omega$. Moreover, like Example 1.2.3, the countable number of 1-types over each countable set A is followed by the countable number of possibilities of distance distributions from elements of A to realizations of types.

For the type $p_0(x) \in S^1(\emptyset)$, isolated by the set $\{P_{\langle 0, m \rangle} \mid m \in \omega\}$ of formulas, the semi-isolation relation is not symmetric via the formula $Q(x, y)$. For each $a \models p_0$, the set of realizations of the

formula $Q(x, a)$ is exhausted by the realizations of the type p_0 and of the nonprincipal types $p_{\langle 1, k \rangle}(x) \in S^1(\emptyset)$, isolated by the sets $\{P_{\langle 1, k, m \rangle} \mid m \in \omega\}$, $k \in \omega$. Since the relation Q forms the graph of a free unar with infinitely many preimages of each element, $\text{acl}(\{a\}) = \text{dcl}(\{a\}) = \bigcup_{n \in \omega} Q^n(a, \mathcal{M})$ for each element a of a model $\mathcal{M} \models T_0$. \square

In connection with the argument above, the problem seems interesting of describing the powerful digraphs that can be expanded to the structures of powerful 1-types both in the case of finitely many and infinitely many nonprincipal 1-types.

Recall that a partially ordered set $\langle X; \leq \rangle$ is said to be *directed downward* (*upward*) if for each $x, y \in X$, there exists $z \in X$ such that $z \leq x$ and $z \leq y$ ($x \leq z$ and $y \leq z$).

Let us list the main possibilities that exhaust the structures of the transitive closures of powerful digraphs obtained from the structures of nonprincipal powerful types $p(\bar{x})$ for which the number of nonprincipal $l(\bar{x})$ -types is finite.

Theorem 1.4.3. *Given a saturated powerful digraph $\Gamma = \langle X; Q \rangle$ in which $\text{acl}(\{a\}) \cap \bigcup_{n \in \omega} Q^n(a, \Gamma) = \{a\}$ for each $a \in X$, the transitive closure $\text{TC}(\Gamma) = \left\langle X; \bigcup_{n \in \omega} Q^n \right\rangle$ is isomorphic to a downward directed set with a transitive automorphism group and one of the following orders:*

- (1 $_\alpha$) a dense partial order with maximal antichains containing α elements $\alpha \in (\omega + 1) \setminus \{0\}$;
- (2) a partial order with infinitely many covering elements for each element.

P r o o f. The reflexivity and transitivity of the relation \leq of the digraph $\text{TC}(\Gamma) = \langle X; \leq \rangle$ are obvious. The antisymmetry of \leq follows from the acyclicity of Γ . The existence in the partially ordered set $\text{TC}(\Gamma)$ of the meet of two arbitrary elements follows from the pairwise intersection property. If the order \leq is not dense then the existence of infinitely many covering elements for each $a \in X$ follows from the relation $\text{acl}(\{a\}) \cap \bigcup_{n \in \omega} Q^n(a, \Gamma) = \{a\}$, since in this situation the consistent formula $a < x \wedge \neg \exists y (a < y \wedge y < x)$ does not belong to an algebraic type over a . \square

Notice, that the dense partial orders with maximal antichains of cardinality α , that we mentioned, are realized by replacing each

element in a dense linear order without endpoints by an equivalence class containing α pairwise incomparable elements.

Note that a partial order with infinitely many covering elements for each element comes only from the powerful digraphs for which the formula $Q(x, y)$ is not principal.

Indeed, if $Q(x, y)$ is a principal formula then the truth of $\models Q(a, b) \wedge Q(a, c) \wedge Q(c, b)$ for each a and some b, c and the existence of automorphisms, fixing a and connecting arbitrary elements in $Q(a, \Gamma)$, imply that for each b in $Q(a, \Gamma)$ there exists c in $Q(a, \Gamma) \cap Q(\Gamma, b)$. Consequently, in the graph $\text{TC}(\Gamma)$, between each pair of distinct elements there is another element. Therefore, we have

Corollary 1.4.4. *If $\Gamma = \langle X; Q \rangle$ is a powerful digraph with a principal formula $Q(x, y)$ then the relation $\bigcup_{n \in \omega} Q^n$ is a dense partial order.*

Notice, that if the relation \leq in the transitive closure of a saturated powerful digraph $\Gamma = \langle X, Q \rangle$ is not definable by formulas in the language of Γ (i.e., if the lengths of the shortest paths are unbounded) then by Compactness Theorem, in $\text{TC}(\Gamma)$, over each a , there is an infinite antichain belonging to $Q(a, \Gamma)$. Therefore, the theorem 1.4.3 implies

Corollary 1.4.5. *Given a saturated powerful digraph $\Gamma = \langle X; Q \rangle$ with unbounded lengths of shortest routes such that*

$$\text{acl}(\{a\}) \cap \bigcup_{n \in \omega} Q^n(a, \Gamma) = \{a\}$$

for each $a \in X$, its transitive closure $\text{TC}(\Gamma) = \left\langle X; \bigcup_{n \in \omega} Q^n \right\rangle$ is isomorphic to a downward directed set with a transitive automorphism group that has one of the following orders:

- (1) a dense partial order with infinite antichains;
- (2) a partial order with infinitely many covering elements of each element.

Notice, that while the necessity of the local presence of powerful digraphs in the structures of nonprincipal powerful types is proved, the question remains open of sufficiency, i.e., the possibility of expansion of each powerful digraph to the structure of a powerful type.

Recall several notions of Stability Theory related to the class of simple theories [3], [22], [27]. A formula $\varphi(\bar{x}, \bar{a})$ in a theory T is said to be *copied over a set A* if there are a natural m and tuples \bar{a}^n , $n \in \omega$, such that the following conditions hold:

(1) $\text{tp}(\bar{a}/A) = \text{tp}(\bar{a}^n/A)$, $n \in \omega$;

(2) the set $\{\varphi(\bar{x}, \bar{a}^n) \mid n \in \omega\}$ of formulas is *m -inconsistent*, i.e., for every $w \subset \omega$ of the cardinality m the formula $\bigwedge_{n \in w} \varphi(\bar{x}, \bar{a}^n)$ is inconsistent in T .

tuples \bar{a} and \bar{b} are said to be *dependent over A* if there exists a formula $\varphi(\bar{x}, \bar{a})$ in T that is copied over A and satisfies $\models \varphi(\bar{b}, \bar{a})$. If tuples \bar{a} and \bar{b} are not dependent over A they are said to be *independent over A* . Tuples, being dependent (independent) over \emptyset , are called simply *dependent (independent)*. A sequence of tuples is said to be *independent* if each tuple of this sequence is independent with every tuple, formed by coordinates of other elements of the sequence.

Recall, that a type $p(\bar{x})$ has *infinite own weight* if there exists a realization \bar{a} of $p(\bar{x})$ and an infinite independent sequence $(\bar{a}_n)_{n \in \omega}$ of realizations of $p(\bar{x})$ such that the tuples \bar{a} and \bar{a}_n are dependent for each $n \in \omega$.

The following Proposition shows that the powerful digraphs do not occur in the structures of known classes of simple theories that do not include Ehrenfeucht theories.

Proposition 1.4.6. *If $T = \text{Th}(\Gamma)$ is a simple theory of some powerful digraph $\Gamma = \langle X; Q \rangle$ then the (unique) type $p \in S^1(\emptyset)$ has infinite own weight.*

P r o o f. Show first that if $\models Q^k(a, b)$ for some $k > 0$, then a and b are dependent. For that it suffices to establish that the formula $Q^k(a, x)$ is copied over \emptyset . Indeed, there exists a number $m \in \omega$ such that for each element a_0 there is at most m elements a_1, \dots, a_m satisfying the conditions $Q^k(a_i, a_j)$ for all $1 \leq i < j \leq m$, because otherwise by the Compactness Theorem and the acyclicity of the digraph Γ there is an infinite sequence $(a_n)_{n \in \omega}$ with the condition

$$\models Q^k(a_i, a_j) \Leftrightarrow i < j,$$

which contradicts the simplicity of T .

Define a sequence $(a_n)_{n \in \omega}$ inductively. Pick an arbitrary element a_0 in X . If a_0, \dots, a_{n-1} have been chosen then pick a_n satisfying the condition $\Gamma \models Q^k(a_{n-1}, a_n)$, and belonging to the maximal number of sets $Q^k(a_i, \Gamma)$, $i < n$. The remarks above imply that the set $\{Q^k(a_n, x) \mid n \in \omega\}$ is m -inconsistent. Since every two elements are connected by an automorphism, the formula $Q^k(a, x)$ is copied over \emptyset .

Notice now that, by the pairwise intersection property, for all elements $a_1, \dots, a_n \in X$ there exists

$$a \in Q(\Gamma, a_1) \cap Q^2(\Gamma, a_2) \cap \dots \cap Q^n(\Gamma, a_n),$$

and, in particular, every n elements comprising an independent sequence depend on some a . Since every two elements are connected by an automorphism and the integer n is unbounded, there exist infinitely many elements that form an independent sequence and depend on a . \square

Chapter 2

GENERIC CONSTRUCTIONS

§ 2.1. Semantic generic constructions

The construction of a generic structure with requisite properties begins by defining a class \mathbf{K}_0 of finite structures of a countable predicate language. The class \mathbf{K}_0 is endowed with a partial order relation \leq which is invariant under the transition to isomorphic structures, connoting the property of being a self-sufficient structure, or strong substructure, and satisfying the following axioms:

- (1) if $\mathcal{A} \leq \mathcal{B}$, then $\mathcal{A} \subseteq \mathcal{B}$;
- (2) if $\mathcal{A} \leq \mathcal{C}$, $\mathcal{B} \in \mathbf{K}_0$, and $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, then $\mathcal{A} \leq \mathcal{B}$;
- (3) \emptyset is the least element of the system $(\mathbf{K}_0; \leq)$;
- (4) (the *amalgamation property*) for any structures $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}_0$, having embeddings $f_0 : \mathcal{A} \rightarrow \mathcal{B}$ and $g_0 : \mathcal{A} \rightarrow \mathcal{C}$ such that $f_0(\mathcal{A}) \leq \mathcal{B}$ and $g_0(\mathcal{A}) \leq \mathcal{C}$, there are a structure $\mathcal{D} \in \mathbf{K}_0$ and embeddings $f_1 : \mathcal{B} \rightarrow \mathcal{D}$ and $g_1 : \mathcal{C} \rightarrow \mathcal{D}$ for which $f_1(\mathcal{B}) \leq \mathcal{D}$, $g_1(\mathcal{C}) \leq \mathcal{D}$ and $f_0 \circ f_1 = g_0 \circ g_1$.

With the class \mathbf{K}_0 determined from finite structures of \mathbf{K}_0 using *amalgamation* (i.e., embedding the structures \mathcal{B} and \mathcal{C} over \mathcal{A} in structures \mathcal{D} so as to comply with the amalgamation property), we construct a countable $(\mathbf{K}_0; \leq)$ -*generic model* \mathcal{M} step by step so as to satisfy the following:

- (a) for any finite substructure $\mathcal{A} \subseteq \mathcal{M}$, there is a structure $\mathcal{B} \in \mathbf{K}_0$, $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M}$, for which $\mathcal{B} \leq \mathcal{M}$, i.e., $\mathcal{B} \leq \mathcal{B}'$ for any structure $\mathcal{B}' \in \mathbf{K}_0$ with $\mathcal{B} \subseteq \mathcal{B}' \subseteq \mathcal{M}$;
- (b) for any finite substructure $\mathcal{A} \subseteq \mathcal{M}$ and any structure $\mathcal{B} \in \mathbf{K}_0$ such that $\mathcal{A} \leq \mathcal{B}$, there is a structure $\mathcal{B}' \leq \mathcal{M}$ for which $\mathcal{B} \simeq_{\mathcal{A}} \mathcal{B}'$.

Thus, the following Theorem holds (Theorem 2.12 in work by J. T. Baldwin and N. Shi [37]).

Theorem 2.1.1. *For any partially ordered class $(\mathbf{K}_0; \leq)$, satisfying conditions 1-4, there exists a $(\mathbf{K}_0; \leq)$ -generic model.*

The scheme above represents a *semantic approach* to constructing a generic model \mathcal{M} and the corresponding *generic theory* $\text{Th}(\mathcal{M})$.

The utility of the semantic approach for realizations of desired model-theoretic properties has confirmed by numerous examples (see the bibliography, reflected in the historical survey) for the cases when each predicate is not definable via other ones.

§ 2.2. Syntactic generic constructions

In constructing generic models in which some predicates are a priori definable via other ones, it is more preferable (and sometimes inevitable) to use a *syntactic approach*, within which complete or incomplete types over finite sets containing some external information on elements are treated rather than finite structures.

The syntactic approach for creating generic theories, written in this Section, generalizes the semantic approach, described above, and also leads to creating generic models. This approach will be used below, in third and fourth Chapters for creating generic theories, representing all possible stable, and also unstable Ehrenfeucht theories w.r.t. Rudin — Keisler preorders and distribution functions of number of limit models. Examples of this and following Sections show, that the syntactic approach forms a proper generalization of semantic one for constructions of generic models.

Below in this Section, we write X, Y, Z, \dots for finite sets of variables, and denote by A, B, C, \dots finite sets of elements, as well as finite sets in algebraic systems, or else the algebraic systems with finite universes themselves; $\Phi(A), \Psi(B), X(C), \dots$ stand for *diagrams*, that is, complete or incomplete types over corresponding sets having no free variables.

For a type $\Phi(A)$, we denote by $[\Phi(A)]_X^A$ the type $\Phi(X)$, obtained via some bijective substitution into $\Phi(A)$ of variables in X for constants in A , and denote by $[\Phi(A)]_B^A$ the type $\Phi(B)$ obtained via a bijective substitution into $\Phi(A)$ of constants in B for constants in A .

We fix an at most countable language Σ and consider a class \mathbf{T}_0 of (complete or incomplete) types $\Phi(A)$ over finite sets A such that $\varphi(\bar{a}) \in \Phi(A)$ or $\neg\varphi(\bar{a}) \in \Phi(A)$ for any quantifier-free formula $\varphi(\bar{x})$ and any tuple $\bar{a} \in A$. Suppose that the class \mathbf{T}_0 is equipped with a partial order \leq , closed under bijective substitutions $[\Phi(A)]_{A'}^A$ of pairwise distinct constants in A' for constants in A into types $\Phi(A) \in \mathbf{T}_0$. Furthermore, we assume that results of bijective substitutions $[\Phi(A)]_X^A$ of sets X of variables for constants in A into types $\Phi(A) \in \mathbf{T}_0$ (over all sets A) form a countable set.

A partially ordered class $(\mathbf{T}_0; \leq)$ is said to be *generic* if \mathbf{T}_0 is closed under intersections and satisfies the following:

(i) if $\Phi \leq \Psi$, then $\Phi \subseteq \Psi$;

(ii) if $\Phi \leq X$, $\Psi \in \mathbf{T}_0$, and $\Phi \subseteq \Psi \subseteq X$, then $\Phi \leq \Psi$;

(iii) some type $\Phi_0(\emptyset)$ is the least element of the system $(\mathbf{T}_0; \leq)$;

(iv) (the *t-amalgamation property*) for any types $\Phi(A)$, $\Psi(B)$, $X(C) \in \mathbf{T}_0$, if there exist injections $f_0 : A \rightarrow B$ and $g_0 : A \rightarrow C$ with $[\Phi(A)]_{f_0(A)}^A \leq \Psi(B)$ and $[\Phi(A)]_{g_0(A)}^A \leq X(C)$, then there are a type $\Theta(D) \in \mathbf{T}_0$ and injections $f_1 : B \rightarrow D$ and $g_1 : C \rightarrow D$ for which $[\Psi(B)]_{f_1(B)}^B \leq \Theta(D)$, $[X(C)]_{g_1(C)}^C \leq \Theta(D)$ and $f_0 \circ g_1 = g_0 \circ f_1$;

(v) (the *local realizability property*) if $\Phi(A) \in \mathbf{T}_0$ and $\Phi(A) \vdash \exists x \varphi(x)$ (respectively, t is a term of language $\Sigma \cup A$ containing no free variables), then there are a type $\Psi(B) \in \mathbf{T}_0$, $\Phi(A) \leq \Psi(B)$, and an element $b \in B$ for which $\Psi(B) \vdash \varphi(b)$ ($(t \approx b) \in \Psi(B)$);

(vi) (the *t-uniqueness property*) for any types $\Phi(A)$, $\Psi(A) \in \mathbf{T}_0$ if the set $\Phi(A) \cup \Psi(A)$ is consistent then $\Phi(A) = \Psi(A)$.

A type Φ is called a *strong subtype* of a type Ψ if $\Phi \leq \Psi$.

A type $\Phi(A)$ is said to be (*strongly*) *embeddable* in a type $\Psi(B)$ if there is an injection $f : A \rightarrow B$ such that $[\Phi(A)]_{f(A)}^A \subseteq \Psi(B)$ ($[\Phi(A)]_{f(A)}^A \leq \Psi(B)$). The injection f , in this instance, is called a (*strong*) *embedding* of type $\Phi(A)$ in type $\Psi(B)$ and is denoted by $f : \Phi(A) \rightarrow \Psi(B)$.

A type $\Phi(A)$ is said to be (*strongly*) *embeddable* in a model \mathcal{M} if $\Phi(A)$ is (strongly) embeddable in some type $\Psi(B)$, where $\mathcal{M} \models \Psi(B)$. The corresponding embedding $f : \Phi(A) \rightarrow \Psi(B)$, in this event, is called a (*strong*) *embedding* of type $\Phi(A)$ in model \mathcal{M} and is denoted by $f : \Phi(A) \rightarrow \mathcal{M}$.

Let \mathbf{T}_0 be a class of types, \mathbf{P} be a class of models, and \mathcal{M} be a model in \mathbf{P} . The class \mathbf{T}_0 is *cofinal* in the model \mathcal{M} if, for each finite set $A \subseteq M$, there are a finite set B , $A \subseteq B \subseteq M$, and a type $\Phi(B) \in \mathbf{T}_0$ such that $\mathcal{M} \models \Phi(B)$. The class \mathbf{T}_0 is *cofinal* in \mathbf{P} if \mathbf{T}_0 is cofinal in every model of \mathbf{P} . We denote by $\overline{\mathbf{T}_0}$ the class of all models \mathcal{M} with the condition that \mathbf{T}_0 is cofinal in \mathcal{M} , and by \mathbf{P} a subclass of $\overline{\mathbf{T}_0}$ such that each type $\Phi \in \mathbf{T}_0$ is true for some model in \mathbf{P} .

Now we extend the relation \leq from the generic class (\mathbf{T}_0, \leq) to a class of subsets of models in the class $\overline{\mathbf{T}_0}$.

Let \mathcal{M} be a model in $\overline{\mathbf{T}_0}$, A and B be finite sets in \mathcal{M} with $A \subseteq B$. We call A a *strong subset* of the set B (in the model \mathcal{M}), and write $A \leq B$, if there exist types $\Phi(A), \Psi(B) \in \mathbf{T}_0$, for which $\Phi(A) \leq \Psi(B)$ and $\mathcal{M} \models \Psi(B)$.

A finite set A is called a *strong subset* of a set $M_0 \subseteq M$ (in the model \mathcal{M}), where $A \subseteq M_0$, if $A \leq B$ for any finite set B such that $A \subseteq B \subseteq M_0$ and $\Phi(A) \leq \Psi(B)$ for some types $\Phi(A), \Psi(B) \in \mathbf{T}_0$ with $\mathcal{M} \models \Psi(B)$. If A is a strong subset of M_0 then, as above, we write $A \leq M_0$. If $A \leq M$ in \mathcal{M} then we refer to A as a *self-sufficient set* (in \mathcal{M}).

Notice, that by the t -uniqueness property, the types $\Phi(A)$ and $\Psi(B)$ specified in the definition of strong subsets are defined uniquely. A type $\Phi(A) \in \mathbf{T}_0$, corresponding to a self-sufficient set A in \mathcal{M} , is said to be a *self-sufficient type* (in \mathcal{M}).

The following statement, generalizing Lemma 2.8 in the work by J. T. Baldwin and N. Shi [37], shows that, for finite sets corresponding to generic classes whose types are generated by uniformly finite types, the condition of being self-sufficient is type definable.

Proposition 2.2.1. *Let \mathbf{T}_0 be a generic class consisting of types $\Phi(A)$, which are deduced from finite subtypes whose cardinalities are uniformly bounded in terms of cardinalities $|A|$, and let \mathcal{M} be a model in the class $\overline{\mathbf{T}_0}$. Then for each finite set $A \leq M$, there exists a type $\Gamma_A(X)$ such that $\mathcal{M} \models \Gamma_A(A)$, and for each set $A' \subseteq M$, $\mathcal{M} \models \Gamma_A(A')$ implies $A' \leq M$.*

P r o o f. Let A be a self-sufficient set in \mathcal{M} , $\Phi(A)$ be a type in \mathbf{T}_0 , $\mathcal{M} \models \Phi(A)$. For any set B , $A \subseteq B \subseteq M$, and any types $\Psi(B) \in \mathbf{T}_0$ with $\Phi(A) \not\leq \Psi(B)$, we denote by $\Psi_0(X, Y)$ the minimal (by inclusion) finite type from which a type $[[\Psi(B)]_{X|Y}^A]_{X|Y}^{B \setminus A}$, where $X \cap Y = \emptyset$, is deducible. The required type is

$$\Gamma_A(X) = \Phi(X) \cup \{\forall Y \neg \wedge \Psi(X, Y) \mid A \subseteq B, \Phi(A) \not\leq \Psi(B)\}. \quad \square$$

A class $(\mathbf{T}_0; \leq)$ possesses the *joint embedding property* (JEP) if for any types $\Phi(A), \Psi(B) \in \mathbf{T}_0$, there is a type $X(C) \in \mathbf{T}_0$ such that $\Phi(A)$ and $\Psi(B)$ are strongly embeddable in $X(C)$. Clearly, every generic class has JEP.

A model $\mathcal{M} \in \mathbf{P}$ has *finite closures* with respect to the class $(\mathbf{T}_0; \leq)$ if any finite set $A \subseteq M$ is contained in some self-sufficient set in \mathcal{M} . A class \mathbf{P} has *finite closures* with respect to the class $(\mathbf{T}_0; \leq)$ if each model in \mathbf{P} has finite closures.

Clearly, a countable model \mathcal{M} has finite closures with respect to $(\mathbf{T}_0; \leq)$ iff $M = \bigcup_{i \in \omega} A_i$ for some self-sufficient sets A_i with $A_i \leq A_{i+1}$, $i \in \omega$.

A countable model $\mathcal{M} \in \overline{\mathbf{T}}_0$ is $(\mathbf{T}_0; \leq)$ -*generic* if it satisfies the following conditions:

(a) \mathcal{M} has finite closures;

(b) if $A \leq M$, $\Phi(A), \Psi(B) \in \mathbf{T}_0$, $\mathcal{M} \models \Phi(A)$ and $\Phi(A) \leq \Psi(B)$, then there exists a set $B' \leq M$ such that $A \subseteq B'$ and $\mathcal{M} \models \Psi(B')$.

Similarly to the construction of a $(\mathbf{K}_0; \leq)$ -generic model, given any generic class $(\mathbf{T}_0; \leq)$, we can embark on a step-by-step construction of a $(\mathbf{T}_0; \leq)$ -generic model \mathcal{M} using t -amalgamations and local realizabilities.

Thus we have following:

Theorem 2.2.2. *For any generic class $(\mathbf{T}_0; \leq)$, there exists a $(\mathbf{T}_0; \leq)$ -generic model.*

A theory $\text{Th}(\mathcal{M})$ of a $(\mathbf{T}_0; \leq)$ -generic model \mathcal{M} is said to be $(\mathbf{T}_0; \leq)$ -*generic*. A theory T is *generic* if T is $(\mathbf{T}_0; \leq)$ -generic, for some generic class $(\mathbf{T}_0; \leq)$.

A model \mathcal{M} is $(\mathbf{T}_0; \leq)$ -*universal* if each type in \mathbf{T}_0 is strongly embeddable in \mathcal{M} .

A model \mathcal{M} is $(\mathbf{T}_0; \leq)$ -*homogeneous* if for any self-sufficient sets A and B in \mathcal{M} such that, for corresponding types $\Phi(A)$ and $\Psi(B)$ witnessing self-sufficiency, the equality $[\Phi(A)]_B^A = \Psi(B)$ and the existence of a bijective mapping f realizing a substitution $[\Phi(A)]_B^A$, equal to $\Psi(B)$, insists on there being an automorphism of \mathcal{M} , containing f .

Clearly, a $(\mathbf{T}_0; \leq)$ -generic model is $(\mathbf{T}_0; \leq)$ -universal and $(\mathbf{T}_0; \leq)$ -homogeneous.

A generic class $(\mathbf{T}_0; \leq)$ consisting of quantifier-free types is said to be *quantifier free*.

The next theorem shows that constructing any $(\mathbf{K}_0; \leq)$ -generic model can be reduced to constructing some $(\mathbf{T}_0; \leq)$ -generic one.

Theorem 2.2.3. *For any $(\mathbf{K}_0; \leq)$ -generic model \mathcal{M} , there exists a quantifier-free class $(\mathbf{T}_0; \leq')$ such that \mathcal{M} is $(\mathbf{T}_0; \leq')$ -generic.*

P r o o f. The required class is the quantifier-free generic class $(\mathbf{T}_0; \leq')$, where $\mathbf{T}_0 = \{\Phi(A) \mid \Phi(X) \text{ is the quantifier-free type } \text{tp}^{qf}(A), A \in \mathbf{K}_0\}$, $\Phi(A) \leq' \Psi(B) \Leftrightarrow \mathcal{A} \leq \mathcal{B}$. \square

Any generic class $(\mathbf{T}_0; \leq)$, which consists of types $\Phi(A)$, corresponding to finite structures \mathcal{A} with universes A , allows us to construct a class \mathbf{K}_0 , which consists of all finite structures isomorphic to structures \mathcal{A} satisfying quantifier-free subtypes $\Phi(A)^{qf}$ of types $\Phi(A) \in \mathbf{T}_0$. Having defined the class \mathbf{K}_0 we specify a relation \leq' with the following condition: $\mathcal{A} \leq' \mathcal{B} \Leftrightarrow \Phi(A) \leq \Psi(B)$ for some $\Phi(A), \Psi(B) \in \mathbf{T}_0$ such that $\mathcal{A} \models \Phi(A)^{qf}$ and $\mathcal{B} \models \Psi(B)^{qf}$.

The class $(\mathbf{K}_0; \leq')$ obtained meets the above conditions (1)–(4), which are imposed on a class generating a $(\mathbf{K}_0; \leq)$ -generic model. Moreover, the $(\mathbf{K}_0; \leq')$ -generic model may fail to be isomorphic to the $(\mathbf{T}_0; \leq)$ -generic model. The fact that the $(\mathbf{T}_0; \leq)$ -generic model is uniquely reconstructed from the class $(\mathbf{K}_0; \leq')$ implies that for every structure $\mathcal{A} \in \mathbf{K}_0$ and for corresponding types $\Phi(A) \in \mathbf{T}_0$, the types will bear information on a number of all possible extensions \mathcal{B} of the structure \mathcal{A} with $\mathcal{A} \leq' \mathcal{B}$, and on interrelations of elements of those extensions. If such information is available and is used, then the $(\mathbf{T}_0; \leq)$ -generic model is defined uniquely up to isomorphism.

§ 2.3. Self-sufficient classes

A generic class $(\mathbf{T}_0; \leq)$ is *self-sufficient* if the following axiom holds:

(vii) if $\Phi, \Psi, X \in \mathbf{T}_0$, $\Phi \leq \Psi$, and $X \subseteq \Psi$, then $\Phi \cap X \leq X$.

Throughout this and two below Sections, we denote by $(\mathbf{T}_0; \leq)$ a self-sufficient generic class, by $\overline{\mathcal{M}}$ a $(\mathbf{T}_0; \leq)$ -generic model, by T a theory $\text{Th}(\overline{\mathcal{M}})$, and by \mathbf{K} a subclass of \mathbf{T}_0 consisting of all models of the theory T .

In order to illustrate the last condition $(\text{Mod}(T) \subseteq \overline{\mathbf{T}_0})$, we present two classes of examples.

Clearly, any quantifier-free generic class $(\mathbf{T}_0; \leq)$ of finite language, which is closed under restrictions of types $\Phi(A) \in \mathbf{T}_0$ to any subset of A , satisfies $\text{Mod}(T) \subseteq \overline{\mathbf{T}_0}$. The last-mentioned condition is also met for any self-sufficient class $(\mathbf{T}_0; \leq)$, having the following *t-covering property*:

(viii) each type $\Phi(X)$ of theory T is deduced from some type $[\Psi_\Phi(B)]_{XUY}^B$, where $\Psi_\Phi(B) \in \mathbf{T}_0$.

Let Φ and Ψ be types in the self-sufficient class $(\mathbf{T}_0; \leq)$. A pair (Φ, Ψ) is *minimal* if $\Phi \subseteq \Psi$, $\Phi \not\leq \Psi$, and for any type $\Psi' \in \mathbf{T}_0$, the condition $\Phi \subseteq \Psi' \subsetneq \Psi$ implies $\Phi \leq \Psi'$.

Let \mathcal{M} be a model in the class $\overline{\mathbf{T}_0}$ and S be a set in the model \mathcal{M} . We say that the set S is *closed* in \mathcal{M} and write $S \leq M$ if, for any minimal pair $(\Phi(A), \Psi(B))$ with $\mathcal{M} \models \Psi(B)$, $A \subseteq S$ implies $B \subseteq S$.

Now we show that the conditions $S \leq M$ and $A \leq M$ are compatible for the case where S is a finite set and some type $\Phi(S)$ belongs to the class \mathbf{T}_0 with $\mathcal{M} \models \Phi(S)$.

Proposition 2.3.1. *Let \mathcal{M} be a model in the class $\overline{\mathbf{T}_0}$ and $\Phi(A)$ be a type in the class \mathbf{T}_0 with $\mathcal{M} \models \Phi(A)$. The following conditions are equivalent:*

- (1) A is a self-sufficient set in \mathcal{M} ;
- (2) for any minimal pair $(\Phi'(A'), \Psi(B))$ with $\mathcal{M} \models \Psi(B)$, $A' \subseteq A$ implies $B \subseteq A$.

Proof. (1) \Rightarrow (2). Let A be a self-sufficient set in \mathcal{M} , $(\Phi'(A'), \Psi(B))$ be a minimal pair with $\mathcal{M} \models \Psi(B)$, and $A' \subseteq A$. By axiom (vii), we have $\Phi(A) \cap \Psi(B) \leq \Psi(B)$. By the minimality of $(\Phi'(A'), \Psi(B))$, we obtain $A' \subseteq A \cap B$ and $B = A \cap B$, that is, $B \subseteq A$.

(2) \Rightarrow (1). Assume that $A \not\leq M$, that is, there is a finite set $B \subseteq M$ such that $A \subset B$, and for corresponding types $\Phi(A), \Psi(B) \in \mathbf{T}_0$, we have $\Phi(A) \not\leq \Psi(B)$. Now the finiteness of B and axiom (vii) imply that there exists a minimal pair $(\Phi(A), \Psi_0(B_0))$ satisfying $\mathcal{M} \models \Psi_0(B_0)$, with $B_0 \not\subseteq A$. The last fact clashes with condition (2). \square

Notice, that axiom (vii) gives rise to the following property of models (similar to this axiom): if $\mathcal{M}, \mathcal{N}, \mathcal{N}' \in \overline{\mathbf{T}_0}$, $M \leq N$, and $\mathcal{N}' \subseteq \mathcal{N}$, then $M \cap \mathcal{N}' \leq \mathcal{N}'$.

The next statement generalizes Lemma 2.18 in the work by J. T. Baldwin and N. Shi [37].

Proposition 2.3.2. *The following conditions are equivalent:*

- (1) class $(\mathbf{T}_0; \leq)$ does not contain an infinite ascending chain of minimal pairs;
- (2) class \mathbf{K} has finite closures;
- (3) every ω -saturated model in \mathbf{K} has finite closures;
- (4) some ω -saturated model in \mathbf{K} has finite closures.

Proof. (1) \Rightarrow (2). If the class \mathbf{K} has no finite closures, that is, if some finite set A of some model $\mathcal{M} \in \mathbf{K}$ cannot be extended to a self-sufficient set, then (in view of $\mathbf{K} \subseteq \overline{\mathbf{T}}_0$) there exists a finite set B such that $A \subseteq B \subseteq M$, $\mathcal{M} \models \Phi(B)$, $\Phi(B) \in \mathbf{T}_0$, and nor can B be extended to a self-sufficient set. Then, by induction, we may construct an infinite ascending chain of minimal pairs in \mathbf{T}_0 beginning with some pair $(\Phi(B), \Psi(C))$.

(2) \Rightarrow (3) are (3) \Rightarrow (4) obvious.

(4) \Rightarrow (1). The fact that there exists an infinite ascending chain of minimal pairs in $(\mathbf{T}_0; \leq)$ implies that such is embeddable in any ω -implies that such is embeddable in any \mathbf{K} , which is impossible for models with finite closures. \square

A consequence of Proposition 2.3.2 is

Corollary 2.3.3. *If a generic model $\overline{\mathcal{M}}$ is saturated, then the class \mathbf{K} has finite closures.*

Let \mathcal{M} and \mathcal{N} be some models in the class $\overline{\mathbf{T}}_0$, S be a closed set in the model \mathcal{M} . An injection $f : S \rightarrow N$ is called a *strong embedding* of S in \mathcal{N} if $f(S)$ is a closed set in \mathcal{N} , and for any type $\Phi(A) \in \mathbf{T}_0$ such that $\mathcal{M} \models \Phi(A)$ and $A \subseteq S$, we have $\mathcal{N} \models \Phi(f(A))$.

We say that a generic class $(\mathbf{T}_0; \leq)$ has *amalgamation over closed (self-sufficient) sets* if, for any models $\mathcal{M}_0, \mathcal{M}_1 \in \mathbf{K}$ and any closed (self-sufficient) set S in some model of the class \mathbf{K} , the existence of strong embeddings $f : S \rightarrow M_0$ and $g : S \rightarrow M_1$ implies that there exist a model $\mathcal{N} \models T$ and elementary embeddings $f' : \mathcal{M}_0 \rightarrow \mathcal{N}$ and $g' : \mathcal{M}_1 \rightarrow \mathcal{N}$ such that $f \circ f' = g \circ g'$. In this instance we also say that the $(\mathbf{T}_0; \leq)$ -generic theory T has *amalgamation over closed (self-sufficient) sets*.

The next theorem generalizes Lemma 2.21 in the work by J. T. Baldwin and N. Shi [37].

Theorem 2.3.4. *Let $(\mathbf{T}_0; \leq)$ be a self-sufficient class, $\overline{\mathcal{M}}$ be a $(\mathbf{T}_0; \leq)$ -generic model, and \mathbf{K} be the class of all models of $T = \text{Th}(\overline{\mathcal{M}})$ which has finite closures. The following conditions are equivalent:*

- (1) *theory T has amalgamation over closed sets;*
- (2) *theory T has amalgamation over self-sufficient sets;*
- (3) *$\overline{\mathcal{M}}$ is an ω_1 -universal model;*
- (4) *$\overline{\mathcal{M}}$ is an ω -saturated model.*

P r o o f. (1) \Rightarrow (2) Is obvious. (2) \Rightarrow (1) Follows from the compactness theorem and the condition that the class \mathbf{K} has finite closures.

(2) \Rightarrow (3). Let \mathcal{N} be a countable model of T . We show that \mathcal{N} is elementary embeddable in $\overline{\mathcal{M}}$. We represent \mathcal{N} as a union $\bigcup_{i \in \omega} A_i$ of an ascending \leq -chain of self-sufficient sets A_i , $i \in \omega$, in \mathcal{N} . Since $\overline{\mathcal{M}}$ is a generic model, the sets A_i are strongly embeddable in $\overline{\mathcal{M}}$ via some strong embeddings $f_i : A_i \rightarrow \overline{\mathcal{M}}$, so that $f_i \subseteq f_{i+1}$, $i \in \omega$. Let f be the embedding $\bigcup_{i \in \omega} f_i$ and N' be the image $f(N)$. It is easy

to see that N' is a universe of a submodel \mathcal{N}' of $\overline{\mathcal{M}}$. We claim that $\mathcal{N}' \preceq \overline{\mathcal{M}}$. We claim that $(\mathcal{N}', \bar{a}) \equiv (\overline{\mathcal{M}}, \bar{a})$ for any tuple $\bar{a} \in N'$. Since the class \mathbf{K} has finite closures, the tuple \bar{a} can be extended to some tuple \bar{b} enumerating the image $f(A_i)$ of some set A_i . The self-sufficiency of A_i implies the self-sufficiency of $f(A_i)$ in both of the models \mathcal{N}' and $\overline{\mathcal{M}}$. Since T enjoys amalgamation over self-sufficient sets, we have $(\mathcal{N}', \bar{b}) \equiv (\overline{\mathcal{M}}, \bar{b})$. Consequently $(\mathcal{N}', \bar{a}) \equiv (\overline{\mathcal{M}}, \bar{a})$.

(3) \Rightarrow (4). Let $\overline{\mathcal{M}}$ be an ω_1 -universal model. Since \mathbf{K} has finite closures, it is enough to show that all 1-types over self-sufficient sets in $\overline{\mathcal{M}}$ are realized in $\overline{\mathcal{M}}$. Let A be a self-sufficient set in $\overline{\mathcal{M}}$, $\Phi(A)$ be a type in \mathbf{T}_0 for which $\overline{\mathcal{M}} \models \Phi(A)$, and p be a type in $S^1(A)$. Consider a countable elementary extension \mathcal{N} of $\overline{\mathcal{M}}$ in which the type p is realized by some element a , letting f be an elementary embedding of model \mathcal{N} in model $\overline{\mathcal{M}}$. Since $f(A)$ is self-sufficient in $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}} \models \Phi(f(A))$, there exists an automorphism g of $\overline{\mathcal{M}}$, mapping $f(A)$ to A . Consequently the element $g(f(a))$ is the required realization of p in $\overline{\mathcal{M}}$.

(4) \Rightarrow (2). We fix strong embeddings $f : A \rightarrow M_0$ and $g : A \rightarrow M_1$, and also a type $\Phi(A) \in \mathbf{T}_0$ for which $\mathcal{M}_0 \models \Phi(f(A))$ and $\mathcal{M}_1 \models \Phi(g(A))$. By Compactness Theorem, we may assume that \mathcal{M}_0 and \mathcal{M}_1 are countable models. Since $\overline{\mathcal{M}}$ is saturated, there are elementary embeddings $f_1 : \mathcal{M}_0 \rightarrow \overline{\mathcal{M}}$ and $g_1 : \mathcal{M}_1 \rightarrow \overline{\mathcal{M}}$. Hence $f_1(f(A))$ and $g_1(g(A))$ are self-sufficient sets in $\overline{\mathcal{M}}$, and we have $\overline{\mathcal{M}} \models \Phi(f_1(f(A)))$ and $\overline{\mathcal{M}} \models \Phi(g_1(g(A)))$. Therefore there is an automorphism h of $\overline{\mathcal{M}}$ mapping $f_1(f(A))$ to $g_1(g(A))$. The maps g_1 and $f_1 \circ h$ witness that $\overline{\mathcal{M}}$ is the required amalgam of the models \mathcal{M}_0 and \mathcal{M}_1 . \square

Let \mathbf{K} be a class having finite closures, \mathcal{M} be a model in \mathbf{K} , and S be a set in \mathcal{M} . The least (by inclusion) closed set in \mathcal{M} , containing S , is called an *intrinsic closure* of S in \mathcal{M} and is denoted by $\text{icl}_{\mathcal{M}}(S)$, or by \overline{S} , if it is clear from the context which of the models \mathcal{M} is in point. If the set \overline{S} is finite then it is referred to as a *self-sufficient closure* of the set S . A type in the class \mathbf{T}_0 , corresponding to the self-sufficient closure \overline{A} of a set A , is denoted by $\overline{\Phi(A)}$. If $\Phi(A) \in \mathbf{T}_0$ and $\mathcal{M} \models \Phi(A)$, then the type $\overline{\Phi(A)}$ is called a *self-sufficient closure* of the type $\Phi(A)$.

Theorem 2.3.5. *If the class \mathbf{K} has finite closures then for any model $\mathcal{M} \in \mathbf{K}$ and any finite set $A \subseteq M$ there exists a self-sufficient closure \overline{A} of A . Moreover, $\overline{A} \subseteq \text{acl}_{\mathcal{M}}(A)$.*

P r o o f. Let A_1 and A_2 be self-sufficient sets in \mathcal{M} containing A . By axiom (vii), their intersection $A_1 \cap A_2$ is also a self-sufficient set in \mathcal{M} containing A . Cardinalities of self-sufficient sets are finite; so there is a unique self-sufficient set in \mathcal{M} containing A and having the least cardinality.

We show that $\overline{A} \subseteq \text{acl}_{\mathcal{M}}(A)$. Let \mathcal{N} be an ω -saturated elementary extension of \mathcal{M} . Assume that $p \equiv \text{tp}(\overline{A}/A)$ is a non-algebraic type. Then there exists a realization A' of p in \mathcal{N} which is distinct from \overline{A} . However $\text{icl}_{\mathcal{M}}(A) = \text{icl}_{\mathcal{N}}(A) = \overline{A}$. Hence the existence of A' contradicts the uniqueness of $\text{icl}_{\mathcal{N}}(A)$. \square

Corollary 2.3.6. *If the class \mathbf{K} has finite closures then the generic model $\overline{\mathcal{M}}$ is homogeneous.*

P r o o f. Let \overline{a} and \overline{b} be two tuples of the same type in $\overline{\mathcal{M}}$, and let A and B be sets consisting of elements of \overline{a} and \overline{b} respectively. Then the conditions $\overline{A} \subseteq \text{acl}_{\overline{\mathcal{M}}}(A)$ and $\overline{B} \subseteq \text{acl}_{\overline{\mathcal{M}}}(B)$ imply that

$\bar{\Phi}(\bar{B}) = \bar{\Psi}(\bar{B})$ for types $\bar{\Phi}(\bar{A}), \bar{\Psi}(\bar{B}) \in \mathbf{T}_0$, where $\bar{\mathcal{M}} \models \bar{\Phi}(\bar{A})$ and $\bar{\mathcal{M}} \models \bar{\Psi}(\bar{B})$. Since $\bar{\mathcal{M}}$ is a generic model, there exists an automorphism $f \in \text{Aut}(\bar{\mathcal{M}})$ mapping \bar{A} to \bar{B} so that $f(A) = B$. \square

§ 2.4. Genericity of countable homogeneous models

A generic class $(\mathbf{T}_0; \leq)$ is (*minimal hereditary*) if \mathbf{T}_0 consists of (minimal by inclusion) types $\Phi(A)$ containing all possible formulas describing a number of copies of a system of elements of a set B over a system of elements of a set A , and interrelations of elements of copies for each set $B \supseteq A$, where a respective type $\Psi(B)$ belongs to \mathbf{T}_0 and satisfies $\Phi(A) \leq \Psi(B)$.

Theorem 2.4.1. *Every at most countable homogeneous (saturated) algebraic system \mathcal{M} is a $(\mathbf{T}_0; \leq)$ -generic model for some hereditary generic class $(\mathbf{T}_0; \leq)$ (with the t -covering property).*

P r o o f. Let \mathcal{M} be a countable homogeneous algebraic system. The required class is the hereditary class $(\mathbf{T}_0; \leq)$, where \mathbf{T}_0 consists of all copies of complete types $\Phi(A)$ for every finite set $A \subseteq M$, and \leq is an inclusion relation. If \mathcal{M} is a saturated model, then the class $(\mathbf{T}_0; \leq)$ possesses the t -covering property, since the theory $\text{Th}(\mathcal{M})$ is small. \square

By virtue of the fact that each complete countable theory has an at most countable homogeneous model, Theorem 2.4.1 yields the following:

Corollary 2.4.2. *Every complete countable theory is generic.*

We know that the property of T being small implies that there exists a countable saturated model of T . By Theorem 2.4.1, therefore, the theory T is $(\mathbf{T}_0; \leq)$ -generic for some generic class $(\mathbf{T}_0; \leq)$ with the t -covering property.

Conversely, if T is a $(\mathbf{T}_0; \leq)$ -generic theory for some generic class $(\mathbf{T}_0; \leq)$ with the t -covering property, then the countability of a number of types $[\Phi(A)]_X^A$, where $\Phi(A) \in \mathbf{T}_0$, and the t -covering property imply that the set $S(T)$ of types for T , is countable, that is, T is small.

We thus arrive at

Theorem 2.4.3. *For any complete countable theory T , the following conditions are equivalent:*

- (1) T is small;
- (2) T is $(\mathbf{T}_0; \leq)$ -generic for some generic class $(\mathbf{T}_0; \leq)$ with the t -covering property.

A generic class $(\mathbf{T}_0; \leq)$ is *complete* if there exists a type $\Phi(A) \in \mathbf{T}_0$ containing some complete theory of language Σ .

The property of $(\mathbf{T}_0; \leq)$ being complete implies that the set of formulas occurring in types of \mathbf{T}_0 generate a $(\mathbf{T}_0; \leq)$ -generic theory. At the same time, the condition of being hereditary for $(\mathbf{T}_0; \leq)$ cannot guarantee generation of a complete theory. For example, in the generic construction of an infinite linearly ordered set via a minimal hereditary class $(\mathbf{T}_0; \leq)$, the formula $\forall x, y ((x \leq y) \vee (y \leq x))$ is not deduced from a set of formulas occurring in types $\Phi(X)$, where $\Phi(A) \in \mathbf{T}_0$ for some A .

In the proof of Theorem 2.4.1 (on representability of any countable homogeneous model as a generic one), use was made of complete generic classes. At the same time, generic classes, in solving various problems, are defined for constructing an *a priori* unknown theory. For a required theory to possess requisite properties, therefore, it is preferable that our generic classes involve types containing a minimum of relevant information.

Let $(\mathbf{T}_0; \leq)$ and $(\mathbf{T}'_0; \leq')$ — be generic classes of languages Σ and Σ' , respectively, with $\Sigma \subseteq \Sigma'$. We say that the class $(\mathbf{T}'_0; \leq')$ *dominates* the class $(\mathbf{T}_0; \leq)$, and write $\mathbf{T}_0 \trianglelefteq \mathbf{T}'_0$, if for any type $\Phi(A) \in \mathbf{T}_0$ there is a type $\Phi'(A') \in \mathbf{T}'_0$ such that $\Phi(A) \subseteq \Phi'(A')$, and the condition of there being some finite systems, which are extensions over A , together with available information on interrelations of elements in these extensions written in the type $\Phi(A)$, implies that the same extensions exist over A , and that similar information is available on interrelations of elements in those extensions written in the type $\Phi'(A')$.

Obviously, the relation \trianglelefteq is a preorder on the class of generic classes.

It is easy to see that if a model \mathcal{M} is isomorphically embeddable in a model $\mathcal{M}' \upharpoonright \Sigma$, then the minimal hereditary class $(\mathbf{T}'_0; \leq')$, which is equal to the closure of a set of types $\Phi'(B)$ for all possible finite sets $B \in M'$ w.r.t. bijective substitutions of constants on which the inclusion relation \leq' is defined, dominates the minimal hereditary class $(\mathbf{T}_0; \leq)$, which is equal to the closure of a set of types $\Phi(A)$ for all possible finite sets $A \in M$ w.r.t. bijective substitutions of constants on which the inclusion relation \leq is defined.

At the same time, the condition $\mathbf{T}_0 \trianglelefteq \mathbf{T}'_0$ implies that the $(\mathbf{T}_0; \leq)$ -generic model is isomorphically embeddable in the restriction of the $(\mathbf{T}'_0; \leq')$ -generic model to the language Σ .

Thus, we have

Theorem 2.4.4. *Let \mathcal{M} and \mathcal{M}' be countable homogeneous models of languages Σ and Σ' , respectively. The following conditions are equivalent:*

- (1) *model \mathcal{M} is isomorphically embeddable in model $\mathcal{M}' \upharpoonright \Sigma$;*
- (2) *there are generic classes $(\mathbf{T}_0; \leq)$ and $(\mathbf{T}'_0; \leq')$ such that \mathcal{M} and \mathcal{M}' are, respectively, $(\mathbf{T}_0; \leq)$ - and $(\mathbf{T}'_0; \leq')$ -generic, and $\mathbf{T}_0 \leq \mathbf{T}'_0$.*

§ 2.5. The uniform t -amalgamation property and saturated generic models

Let $(\mathbf{T}_0; \leq)$ be a self-sufficient class satisfying the following conditions:

- a) for any type $\Phi(A) \in \mathbf{T}_0$, the type $\overline{\Phi}(\overline{A})$ yields a formula $\chi_{\overline{\Phi}}(\overline{A})$ describing the self-sufficient condition for the closure $\overline{\Phi}(\overline{A})$; moreover, $\chi_{\overline{\Phi}}(\overline{A})$ also contains a formula which is deducible from $\Phi(A)$ and describes an upper bound for the cardinality of the set \overline{A} ;
- b) for any self-sufficient types $\overline{\Phi}(\overline{A})$ and $\overline{\Psi}(\overline{B})$, where $\overline{\Phi}(\overline{A}) \leq \overline{\Psi}(\overline{B})$, and for any formula $\psi(X, Y)$ in $\overline{\Psi}(X \cup Y)$ (here, X and Y are disjoint sets of variables, bijective with sets \overline{A} and $\overline{B} \setminus \overline{A}$ respectively), there exists a formula $\varphi(X)$ which is deducible from $\overline{\Phi}(X)$ and is such that the following formula holds true in $\overline{\mathcal{M}}$:

$$\forall X ((\chi_{\overline{\Phi}}(X) \wedge \varphi(X)) \rightarrow \exists Y (\chi_{\overline{\Psi}}(X, Y) \wedge \psi(X, Y))).$$

If the above conditions are satisfied then we say that the class $(\mathbf{T}_0; \leq)$ has the *uniform t -amalgamation property*.

Notice, that the concept of uniform t -amalgamation generalizes *uniform amalgamation* as defined in work by J. T. Baldwin and N. Shi [37].

As distinct from the uniform amalgamation property, the uniform t -amalgamation property, if satisfied, is not supposed to involve a function $f \in \omega^\omega$, representing uniform upper bounds $f(n)$ for the cardinalities of self-sufficient closures \overline{A} depending only on cardinalities n of given sets A .

As an example of a generic class that possesses the uniform t -amalgamation property but lacks in the above-mentioned upper bound, we may take a class of types corresponding to finite acyclic

undigraphs with unbounded degrees of all elements. In fact, taking elements a_n and b_n , that are connected by shortest routes of length n , we have unbounded cardinalities of self-sufficient closures, that formed by adding of elements of shortest routes: $|\overline{\{a_n, b_n\}}| = n + 1$.

The following theorem generalizes Theorem 2.28 in work by J. T. Baldwin and N. Shi [37].

Theorem 2.5.1. *If $(\mathbf{T}_0; \leq)$ is a self-sufficient class having the uniform t -amalgamation property and the class \mathbf{K} has finite closures, then the $(\mathbf{T}_0; \leq)$ -generic model $\overline{\mathcal{M}}$ is ω -saturated. Moreover, any finite set $A \subseteq \overline{\mathcal{M}}$ is extendable to its self-sufficient closure $\overline{A} \subseteq \overline{\mathcal{M}}$, the type $\text{tp}(\overline{A})$ contains the type $\overline{\Phi}(Y)$ for a self-sufficient type $\overline{\Phi}(A)$, and $\overline{\Phi}(Y) \vdash \text{tp}(\overline{A})$.*

P r o o f. Let $\overline{\mathcal{M}}$ be a $(\mathbf{T}_0; \leq)$ -generic model and \mathcal{N} be an ω -saturated model of $\text{Th}(\overline{\mathcal{M}})$. We show that the models $\overline{\mathcal{M}}$ and \mathcal{N} are $L_{\infty, \omega}$ -equivalent. To do this, it suffices to establish that between $\overline{\mathcal{M}}$ and \mathcal{N} there are finite partial isomorphisms $f : A \simeq A'$ which are mutually extendable for any self-sufficient sets $A \subseteq \overline{\mathcal{M}}$ and $A' \subseteq \mathcal{N}$ realizing self-sufficient types $\Phi(A)$ and $\Phi'(A')$, where $\Phi'(A') = \Phi(A')$.

Let $f : A \simeq A'$ be a finite partial isomorphism satisfying the conditions above. Consider a self-sufficient type $\Psi'(B') \in (\mathbf{T}_0; \leq)$ with $\Phi'(A') \leq \Psi'(B')$ and $\mathcal{N} \models \Psi'(B')$. Since $\overline{\mathcal{M}}$ is a $(\mathbf{T}_0; \leq)$ -generic model, $\overline{\mathcal{M}}$ has an isomorphic copy B of B' over A' which is realized over A and is such that $\overline{\mathcal{M}} \models \Psi(B)$, $\Psi(B) = \Psi'(B)$, and $\Phi(A) \leq \Psi(B)$, for a self-sufficient type $\Psi(B)$. This means that the required extension $g : B \simeq B'$ of the partial isomorphism f exists.

Consider a self-sufficient type $\Psi(B) \in (\mathbf{T}_0; \leq)$ with $\Phi(A) \leq \Psi(B)$ and $\overline{\mathcal{M}} \models \Psi(B)$. Since the formulas

$$\forall \overline{x} ((\chi_{\overline{\Phi}}(X) \wedge \varphi(X)) \rightarrow \exists Y (\chi_{\overline{\Phi}}(X, Y) \wedge \psi(X, Y)))$$

are true in $\overline{\mathcal{M}}$, they are also true in \mathcal{N} . The set $\{(\chi_{\Psi}(A', Y)) \cup \{\psi(A', Y) \mid \psi(A, B \setminus A) \in \Psi(B)\}\}$ is locally satisfiable and the model \mathcal{N} is ω -saturated; so the set at hand is satisfiable in \mathcal{N} , that is, there exists a set $B' \subseteq \mathcal{N}$ satisfying $\mathcal{N} \models \Psi'(B')$, $\Psi'(B') = \Psi(B')$, and $\Phi'(A') \leq \Psi'(B')$, for a self-sufficient type $\Psi'(B')$. Hence again we are faced up to the required extension $g : B \simeq B'$ of f .

In view of the possibility for extending isomorphisms $f : A \simeq A'$, so as to preserve formulas of corresponding self-sufficient types $\Phi(A)$ and $\Phi'(A')$, on the basis of the back-and-forth argument, we con-

clude that the model $\overline{\mathcal{M}}$ with a constantly distinguished set A is isomorphic to a countable elementary submodel of the model \mathcal{N} with a constantly distinguished set A' . Since the sets A and A' are chosen arbitrarily and the model \mathcal{N} is saturated, we see that any type over a finite set in $\overline{\mathcal{M}}$ is realized in, that is, $\overline{\mathcal{M}}$ is a saturated model.

A possibility for extending partial isomorphisms $f : B \simeq B'$ so as to preserve formulas of corresponding self-sufficient types $\Psi(B)$ and $\Psi'(B')$, implies that if $\Psi(B) = \Psi'(B)$ then there exists an automorphism of $\overline{\mathcal{M}}$ extending a given partial isomorphism between B and B' . Hence, $\text{tp}_{\overline{\mathcal{M}}}(B) = \text{tp}_{\overline{\mathcal{M}}}(B')$. Since any type $\Phi(A)$ can be extended to its self-sufficient closure $\overline{\Phi(A)}$ (see Theorem 2.3.5 above), any finite set $A \subseteq M$ is extendable to its self-sufficient closure $\overline{A} \subseteq M$ so that the type $\text{tp}(\overline{A})$ contains the type $\overline{\Phi(Y)}$ for a self-sufficient type $\overline{\Phi(A)}$, and $\overline{\Phi(Y)} \vdash \text{tp}(\overline{A})$. \square

Theorem 2.5.1, combined with Compactness Theorem and Lemma 1.2.3, implies that, for any self-sufficient class $(\mathbf{T}_0; \leq)$ with uniform t - t -amalgamation, if the class \mathbf{K} has finite closures, then the $(\mathbf{T}_0; \leq)$ -generic theory $T = \text{Th}(\overline{\mathcal{M}})$ is $\Delta(\mathbf{T}_0)$ -based, where $\Delta(\mathbf{T}_0)$ is a set consisting of all possible formulas obtained by existentially quantifying over conjunctions $\bigwedge_{i=1}^n \varphi_i(X)$ of formulas $\varphi_i(X) \in \Phi(X)$, $i = 1, \dots, n$, where $\Phi(A) \in \mathbf{T}_0$ for some set A . Thus Theorem 2.5.1 gives rise to the following:

Corollary 2.5.2. *If P is some property of formulas preserved under Boolean combinations of the formulas, $(\mathbf{T}_0; \leq)$ is a self-sufficient class possessing the uniform t -amalgamation property, and the class \mathbf{K} has finite closures, then any formula of $(\mathbf{T}_0; \leq)$ -generic theory possesses P iff any formula in $\Delta(\mathbf{T}_0)$ P has the property P .*

In work by V. Harnik and L. Harrington [72] was stated that any Boolean combination of stable formulas is itself a stable formula. Based on Corollary 2.5.2 we derive

Corollary 2.5.3. *If $(\mathbf{T}_0; \leq)$ is a self-sufficient class possessing the uniform t -amalgamation property, and the class \mathbf{K} has finite closures, then the $(\mathbf{T}_0; \leq)$ -generic theory is stable iff any formula in the set $\Delta(\mathbf{T}_0)$ is stable.*

§ 2.6. On the finite closure property in fusions of generic classes

E. Hrushovski [91] defined a mechanism of fusion of two generic theories for obtaining a strongly minimal theory, having a structure with fields of two different characteristics. His technique has developed essentially last time by questions of fusions of fields and of fusions of vector spaces having various required properties (see A. Baudisch, A. Martin-Pizarro, M. Ziegler [53], [51], [54], [52]; A. Hasson, M. Hils [77]; K. Holland [85], [86]; M. Ziegler [208]). Considered in Section 1.2 combinations and colorings of models, when these models are countable and homogeneous, are interpretable as partial cases of fusions for correspondent generic classes.

As shown in Section 2.3, any self-sufficient generic class generates operations of self-sufficient closures on its generic models. After fusion of generic closures, these operations, via a transitive closure, are extended to the operation of self-sufficient closure on a generic model of this fusion. Thus, the system of finite closures arises, generating new and more general operation of finite closures.

Since a saturation of generic model is conditioned by formula-definability of operation of self-sufficient closure \bar{A} of each finite set A , a natural question arises on possibility of creating for fusion of generic models, having formula-definable operations of self-sufficient closures, with formula-definability of resulted operation of self-sufficient closure.

In this Section, we shall formulate exactly this problem on fusion of generic classes and shall propose sufficient conditions for existence of such fusions with that models possess finite closures.

Let $(\mathbf{T}; \leq)$ be a generic class. We say that $(\mathbf{T}; \leq)$ has the *finite closure property* if any model of $(\mathbf{T}; \leq)$ -generic theory has finite closures.

The following theorem proposes a characterization of the finite closure property for generic classes in terms of domination relation.

Theorem 2.6.1. *A generic class $(\mathbf{T}; \leq)$ of language Σ has the finite closure property iff $(\mathbf{T}; \leq)$ is dominated by a generic class $(\mathbf{T}'; \leq')$ of language Σ , satisfying the following conditions:*

(1) *each type $\Phi(A)$ of \mathbf{T}' contains a description of some its minimal self-sufficient extension and has a restriction to a type, in \mathbf{T} , over A ;*

(2) each type $p(\bar{x}) \in S(\emptyset)$ of $(\mathbf{T}'; \leq')$ -generic theory has an extension $q(\bar{y}) \in S(\emptyset)$, containing a type $[\Phi(A)]_Y^A$, where Y is the set of coordinates of tuple \bar{y} and $\Phi(A) \in \mathbf{T}'$.

P r o o f. Suppose, that the class $(\mathbf{T}; \leq)$ has the finite closure property. Then each finite set in a model of $(\mathbf{T}; \leq)$ -generic theory T is extensible to a self-sufficient set. Since the set of all types $[\Phi(A)]_X^A$, correspondent to types $\Phi(A) \in \mathbf{T}$, is countable, all possible pairwise inconsistent extensions of types $\Phi(A)$ to types $\Psi(A)$, containing descriptions of their self-sufficient extensions, form required generic class $(\mathbf{T}'; \leq')$, where \leq' inherits the relation \leq .

Conversely, suppose that the generic class $(\mathbf{T}; \leq)$ is dominated by a generic class $(\mathbf{T}'; \leq')$ of the same language and such that any type $\Phi(A)$ of \mathbf{T}' contains a description of some its minimal self-sufficient extension and has a restriction to a type, in \mathbf{T} , over A , and also each type $p(\bar{x}) \in S(\emptyset)$ of $(\mathbf{T}'; \leq')$ -generic theory T has an extension $q(\bar{y}) \in S(\emptyset)$, containing a type $[\Phi(A)]_Y^A$, where Y is the set of coordinates of tuple \bar{y} and $\Phi(A) \in \mathbf{T}'$. Then any $(\mathbf{T}'; \leq')$ -generic model is $(\mathbf{T}; \leq)$ -generic. Since any type $p \in S(T)$ contains an information on existence of self-sufficient extensions of its realizations, we have the finite closure property for the generic class $(\mathbf{T}; \leq)$. \square

In view of the condition (2) in Theorem 2.6.1, if the class $(\mathbf{T}; \leq)$ has the finite closure property, then there are countably many restrictions of types $q(\bar{y}) \in S(\emptyset)$ to types $[\Phi(A)]_Y^A$.

The generic class $(\mathbf{T}'; \leq')$, described in Theorem 2.6.1, is said to be a *generic class, witnessing on the finite closure property* for the class $(\mathbf{T}; \leq)$. Similar addition of external information on properties of types of given generic class $(\mathbf{T}; \leq)$, forming an *expansion* of generic class, is said to be a *witness* on correspondent property.

Let $(\mathbf{T}_0; \leq_0)$, $(\mathbf{T}_1; \leq_1)$, and $(\mathbf{T}_2; \leq_2)$ be generic classes of languages Σ_0 , Σ_1 , and Σ_2 respectively, $\Sigma_0 = \Sigma_1 \cap \Sigma_2$, $\leq_0 = \leq_1 \cap \leq_2$. A generic class $(\mathbf{T}_3; \leq_3)$ of language $\Sigma_1 \cup \Sigma_2$, such that $(\mathbf{T}_3; \leq_3) \upharpoonright \Sigma_i = (\mathbf{T}_i; \leq_i)$, $i = 1, 2$, is said to be a *fusion* of classes $(\mathbf{T}_1; \leq_1)$ and $(\mathbf{T}_2; \leq_2)$ over $(\mathbf{T}_0; \leq_0)$. Here, a $(\mathbf{T}_3; \leq_3)$ -generic model (theory) is a *fusion* of $(\mathbf{T}_1; \leq_1)$ - and $(\mathbf{T}_2; \leq_2)$ -generic models (theories).

Fusions of $(\mathbf{T}_1; \leq_1)$ and $(\mathbf{T}_2; \leq_2)$ over $(\mathbf{T}_0; \leq_0)$ will be denoted by

$$(\mathbf{T}_1; \leq_1) \mathcal{F}_{(\mathbf{T}_0; \leq_0)} (\mathbf{T}_2; \leq_2).$$

A fusion of $(\mathbf{T}_1; \leq_1)$ -generic model \mathcal{M}_1 (theory T_1) and $(\mathbf{T}_2; \leq_2)$ -generic model \mathcal{M}_2 (theory T_2) over $(\mathbf{T}_0; \leq_0)$ -generic model \mathcal{M}_0 (theory T_0) is denoted by $\mathcal{M}_1 \mathcal{F}_{\mathcal{M}_0} \mathcal{M}_2$ (respectively $T_1 \mathcal{F}_{T_0} T_2$).

Clearly, fusions of classes $(\mathbf{T}_1; \leq_1)$ and $(\mathbf{T}_2; \leq_2)$ is allowed to not exist (if, for instance, a chain of self-sufficient closures for a set relative to \leq_1 and to \leq_2 is not stabilized), and if it exists, then it can be not unique. Moreover, a presence of the finite closure property or the uniform t -amalgamation for the classes $(\mathbf{T}_1; \leq_1)$ and $(\mathbf{T}_2; \leq_2)$ doesn't imply correspondent properties hold for fusions.

Besides notice, that the finite closure property may be satisfied both with uniform estimates for powers of closures, depending on cardinalities of initial finite sets, and in a case of absence of such estimates, provided that powers and structures of closures are described in types of given finite sets. Generic classes with power estimates are called *PE-classes* and generic classes without such estimates are *NPE-classes*.

All examples of generic classes are PE-classes, that similar to Hrushovski example generated non-negative predimension functions δ and have saturated generic models (see surveys by J. T. Baldwin [38], [40] and B. P. Poizat [150]). Generic classes of free acyclic (it will be considered in Chapter 3) and of cubic theories, being NPE-classes, are described in the works by the author [176] and [184].

In view of Theorem 2.6.1, the finite closure property for fusions of generic classes is obviously characterized in terms of expansions of generic classes.

Let \mathcal{M}_i be a $(\mathbf{T}_i; \leq_i)$ -generic model, $i = 0, 1, 2$, \mathcal{M}_3 be a $(\mathbf{T}_1; \leq_1) \mathcal{F}_{(\mathbf{T}_0; \leq_0)} (\mathbf{T}_2; \leq_2)$ -generic model, where $(\mathbf{T}_i; \leq_i)$ and $(\mathbf{T}_1; \leq_1) \mathcal{F}_{(\mathbf{T}_0; \leq_0)} (\mathbf{T}_2; \leq_2)$ are self-sufficient generic classes.

Clearly, \mathcal{M}_0 is elementary embeddable in $\mathcal{M}_1 \upharpoonright \Sigma_0$ and in $\mathcal{M}_2 \upharpoonright \Sigma_0$; the models \mathcal{M}_1 and \mathcal{M}_2 are elementary embeddable in $\mathcal{M}_3 \upharpoonright \Sigma_1$ and in $\mathcal{M}_3 \upharpoonright \Sigma_2$ respectively. So we shall assume, that \mathcal{M}_0 is an elementary submodel of $\mathcal{M}_1 \upharpoonright \Sigma_0$ and of $\mathcal{M}_2 \upharpoonright \Sigma_0$; \mathcal{M}_1 and \mathcal{M}_2 are elementary submodels of $\mathcal{M}_3 \upharpoonright \Sigma_1$ and of $\mathcal{M}_3 \upharpoonright \Sigma_2$ respectively. Moreover, $\mathcal{M}_3 = \mathcal{M}_1 \mathcal{F}_{\mathcal{M}_0} \mathcal{M}_2$.

Denote the operations of self-sufficient closures in \mathcal{M}_i by Cl_i , $i = 1, 2, 3$.

Clearly, for any finite set $A \subseteq \mathcal{M}_3$, the inclusion $\text{Cl}_3(A) \supseteq \bigcup_{n \in \omega} A_n$ holds, where $A_0 = A$, $A_{n+1} = \text{Cl}_1(\text{Cl}_2(A_n))$. Moreover, since the set

$\text{Cl}_3(A)$ is finite, the chain of sets A_n , $n \in \omega$, is stabilized, starting with some n . That number is said to be an *iterative number* and it is denoted by $n_A(\mathcal{M}_3)$ or by n_A .

If $\text{Cl}_3(A) = \bigcup_{n \in \omega} A_n$ for any finite set $A \subseteq M_3$, we say that the operation Cl_3 is *generated* by Cl_1 and Cl_2 and write $\text{Cl}_3 = \langle \text{Cl}_1, \text{Cl}_2 \rangle$.

Notice, that conditions of coincidence or non-coincidence of operators Cl_3 and $\langle \text{Cl}_1, \text{Cl}_2 \rangle$ are witnessed by expansions of fusions of generic classes.

Hrushovski style fusions of generic classes (*Hrushovski fusions*) (see E. Hrushovski [91]; A. Baudisch, A. Martin-Pizarro, M. Ziegler [53]–[52]; A. Hasson, M. Hils [77]; K. Holland [85], [86]; M. Ziegler [208]) are defined by non-negative linear predimension functions δ_i for classes $(\mathbf{T}_i; \leq_i)$, $i = 0, 1, 2$, with non-negative linear predimension functions

$$\delta(A) = \delta_1(A) + \delta_2(A) - \delta_0(A),$$

of fusions. In general, these fusions don't have closures of form $\langle \text{Cl}_1, \text{Cl}_2 \rangle$, since closed, relative to Cl_1 and Cl_2 , sets (with nondecreasing values $\delta_1(A)$ and $\delta_2(A)$) can be not closed relative Cl_3 (summarized number of weights of connections w.r.t. $\delta_1(A)$ and $\delta_2(A)$ can exceed the number of elements that used in calculation of $\delta(A)$). Moreover, iterative numbers n_A can be unbounded: $\sup\{n_A\} = \infty$.

Theories of *Herwig graphs* [81] and theories of digraphs, that will be described in Chapter 4, can be also considered as Hrushovski fusions. Moreover, a countable graph structure, supplied by weights of edges or arcs, allows to interpret resulted theories T as fusions of countable set of theories T_k of languages $\{I_k^{(2)}\}$, $k \in \omega$, with $\text{Cl}_T \neq \langle \text{Cl}_k \rangle_{k \in \omega}$, where Cl_T is a self-sufficient closure in a generic model of T , and Cl_k are self-sufficient closures in generic models of T_k , $k \in \omega$.

Inessential combinations \mathcal{M}_3 of models \mathcal{M}_1 and \mathcal{M}_2 with identical closures Cl_1 and Cl_2 generate identical closures Cl_3 . Another examples of fusions for generic classes with $\text{Cl}_3 = \langle \text{Cl}_1, \text{Cl}_2 \rangle$ are represented in third Chapter.

Below, in this Section, we shall consider closure operations Cl_3 , generated by Cl_1 and Cl_2 . Fix a fusion of generic classes

$$(\mathbf{T}_3; \leq_3) \equiv (\mathbf{T}_1; \leq_1) \mathcal{F}_{(\mathbf{T}_0; \leq_0)} (\mathbf{T}_2; \leq_2).$$

The following statement represents an obvious (by Compactness Theorem) characterization for the preservation of the finite closure property under transition from the classes $(\mathbf{T}_1; \leq_1)$ and $(\mathbf{T}_2; \leq_2)$ to the class $(\mathbf{T}_3; \leq_3)$.

Proposition 2.6.2. *A generic class $(\mathbf{T}_3; \leq_3)$ doesn't have the finite closure property iff, in $(\mathbf{T}_3; \leq_3)$ -generic model, there exists a sequence $A_n, n \in \omega$, of finite sets having the same cardinality, such that closures $\text{Cl}_3(A_n)$ can be obtained by at least n iterations w.r.t. Cl_1 and Cl_2 , and a description of unbounded number of iterations for these sets is consistent with $(\mathbf{T}_3; \leq_3)$ -generic theory.*

For the illustration, we represent an example of fusion $(\mathbf{T}_3; \leq_3)$ of generic classes, for which $\text{Cl}_3 = \langle \text{Cl}_1, \text{Cl}_2 \rangle$ and the finite closure property is not satisfied.

Example 2.6.1. Let $(\mathbf{T}_i; \leq_i)$ be generic classes of graph languages $\{Q_i^{(2)}\}, i = 1, 2$, with types describing pairwise disjoint edges such that any vertex is either isolated or belongs to unique edge, being not a loop. Moreover, we add the following requirements:

- 1) the number of edges and the number of isolated vertices are unbounded;
- 2) each finite graph with given numbers of edges and of isolated vertices is represented by a type in \mathbf{T}_i ;
- 3) if a vertex a belongs to a set A , where $\Phi(A) \in \mathbf{T}_i$ and the description $\Phi(A)$ contains an information that a belongs to an edge $[a, b]$, then $b \in A$;
- 4) relations \leq_1 and \leq_2 coincides with the inclusion relations.

Notice, that each self-sufficient closure of finite set A in a $(\mathbf{T}_i; \leq_i)$ -generic model is obtained by adding to each endpoint of edge, belonging to A , an opposite endpoint of the edge.

Now we define a fusion of the generic classes $(\mathbf{T}_1; \leq_1)$ and $(\mathbf{T}_2; \leq_2)$, allowing for each vertex to be either isolated, or belong to unique edge, or belong to two edges of different colors (Q_1 and Q_2), such that descriptions of types contain only information about *finite* chains, such that every finite length is represented.

Self-sufficient sets of $(\mathbf{T}_3; \leq_3)$ -generic model are finite sets, closed under adding of opposite endpoints of edges. At the same time, a presence of unbounded chains means that there exists a countable model of $(\mathbf{T}_3; \leq_3)$ -generic theory, having an infinite chain. There are no elements in such chain belonging to self-sufficient sets, since such sets should be finite. \square

For practical creation of operation Cl_3 with non-identical closures Cl_1 and Cl_2 and with preservation of the finite closure property, it is appropriate to use the *minimization iterative principal*, or *MI-principal*, by which iterative numbers n_A are minimal. This minimization can be *majorized* by estimates f of numbers n_A depending on cardinalities of $|A|$: $n_A \leq f(|A|)$. If there exists a *majorizing estimate for number of iterations f* for all sets A , included in self-sufficient types $\Phi(A) \in \mathbf{T}_3$, and this estimate is preserved for self-sufficient amalgams in the class \mathbf{T}_3 , then this estimate will be valid for all models of $(\mathbf{T}_3; \leq_3)$ -generic theory. Having a majorizing estimate for generic class $(\mathbf{T}_3; \leq_3)$, the finite closure property for this class will be satisfied. Thus we have the following statement:

Proposition 2.6.3. *Let a class $(\mathbf{T}_3; \leq_3)$ coincide with a generic class $(\mathbf{T}_1; \leq_1) \mathcal{F}_{(\mathbf{T}_0; \leq_0)}(\mathbf{T}_2; \leq_2)$, $\text{Cl}_3 = \langle \text{Cl}_1, \text{Cl}_2 \rangle$. If classes $(\mathbf{T}_i; \leq_i)$ have the finite closure property, $i = 1, 2$, and there exists a majorizing estimate for iterative numbers for the class $(\mathbf{T}_3; \leq_3)$, then the class $(\mathbf{T}_3; \leq_3)$ has the finite closure property too.*

Now we consider a sufficient condition for existence of minimal majorizing estimate ($n_A \equiv 1$) of fusion

$$(\mathbf{T}_3; \leq_3) \rightleftharpoons (\mathbf{T}_1; \leq_1) \mathcal{F}_{(\mathbf{T}_0; \leq_0)}(\mathbf{T}_2; \leq_2),$$

where closures Cl_1 and Cl_2 are, possibly both, non-identical.

Suppose, that on the universe of $(\mathbf{T}_3; \leq_3)$ -generic model \mathcal{M}_3 , there is (not necessary formula-definable) an equivalent relation E , satisfying the following conditions for any finite set $A \subseteq M_3$:

- (1) $\text{Cl}_1(A) = \bigcup_{a \in A} \text{Cl}_1(A \cap E(a))$;
- (2) $\text{Cl}_2(C) = C$ for any set C with $\text{Cl}_2(A) \subseteq C \subseteq \bigcup_{a \in \text{Cl}_2(A)} E(a)$.

Having these conditions, we say that $(\text{Cl}_1, \text{Cl}_2)$ is an *E-stepped special closure system with minimality condition*, or an *ESSM-system*.

If $(\text{Cl}_1, \text{Cl}_2)$ is an ESSM-system, then there exists a minimal majorizing estimate for iterative numbers of self-sufficient class $(\mathbf{T}_3; \leq_3)$. Indeed, let A be a finite set in a model of $(\mathbf{T}_3; \leq_3)$ -generic theory. Then the set $B \rightleftharpoons \text{Cl}_1(\text{Cl}_2(A))$ is Cl_1 -closed, since the operation Cl_1 is transitive, and the inclusion $B \subseteq \bigcup_{a \in \text{Cl}_2(A)} E(a)$ implies

that B is Cl_2 -closed.

The following generalization of concept of ESSM-system guarantees an existence of majorizing estimate for iterative numbers in the fusion $(\mathbf{T}_3; \leq_3)$.

Suppose, that on the universe of $(\mathbf{T}_3; \leq_3)$ -generic model \mathcal{M}_3 there is (not necessary formula-definable) an equivalent relation E , satisfying the following conditions for any finite set $A \subseteq M_3$, where $\mathcal{M}_3 \models \Phi(A)$ for some type $\Phi(A) \in \mathbf{T}_3$:

$$(1) \text{Cl}_1(A) = \bigcup_{a \in A} \text{Cl}_1(A \cap E(a));$$

$$(2) \text{ if } C \subseteq \bigcup_{a \in \text{Cl}_2(A)} E(a) \text{ and } \text{Cl}_2(C) \subseteq \bigcup_{a \in \text{Cl}_2(A)} E(a), \text{ then } \text{Cl}_2(C) = C;$$

$$(3) \text{ there exists a finite number } m_A \text{ of } E\text{-classes } E_1, \dots, E_{m_A}, \text{ defined by some formula in } \Phi(A) \text{ such that } \text{Cl}_3(A) \subseteq \bigcup_{i=1}^{m_A} E_i.$$

Having these conditions, we say that $(\text{Cl}_1, \text{Cl}_2)$ is an *E-stepped special closure system*, or *ESS-system*.

If $(\text{Cl}_1, \text{Cl}_2)$ is an ESS-system, then there exists a minimal majorizing estimate for iterative numbers of self-sufficient class $(\mathbf{T}_3; \leq_3)$. Indeed, let A be a finite set in a model of $(\mathbf{T}_3; \leq_3)$ -generic theory. Then the iterative number is bounded by $m_A + 1$, since each iteration defines a subset of $\bigcup_{i=1}^{m_A} E_i$, and since stabilization of number of E -classes, containing the result of two sequential iterations, the conditions (1) and (2) imply that the resulted set is simultaneously Cl_1 - and Cl_2 -closed.

Thus we have the following

Theorem 2.6.4. *Let $(\mathbf{T}_3; \leq_3)$ be a class*

$$(\mathbf{T}_1; \leq_1) \mathcal{F}_{(\mathbf{T}_0; \leq_0)} (\mathbf{T}_2; \leq_2),$$

$(\text{Cl}_1, \text{Cl}_2)$ be an ESS-system, and the classes $(\mathbf{T}_i; \leq_i)$, $i = 1, 2$, have the finite closure property. Then the class $(\mathbf{T}_3; \leq_3)$ has the finite closure property too.

We denote the class $(\mathbf{T}_3; \leq_3)$, being in Theorem 2.6.4, by $(\mathbf{T}_1; \leq_1) \mathcal{F}_{(\mathbf{T}_0; \leq_0)}^{\text{ESS}} (\mathbf{T}_2; \leq_2)$.

Let $(\mathbf{T}_i; \leq_i)$, $(\mathbf{T}'_i; \leq'_i)$, $i = 1, \dots, n$, be generic classes satisfying the following conditions:

- 1) $(\mathbf{T}'_1; \leq'_1) = (\mathbf{T}_1; \leq_1)$;
- 2) $(\mathbf{T}'_{i+1}; \leq'_{i+1}) = (\mathbf{T}'_i; \leq'_i) \mathcal{F}_{(\mathbf{T}'_i; \leq'_i) \cap (\mathbf{T}_i; \leq_i)}^{\text{ESS}}(\mathbf{T}_i; \leq_i)$, $i = 1, \dots, n - 1$.

The generic class $(\mathbf{T}'_n; \leq'_n)$ is denoted by $(\mathcal{F}_{i=1}^{\text{ESS}})^n(\mathbf{T}_i; \leq_i)$.

Theorem 2.6.4 implies, that the finite closure property is preserved under finite iterations of creation of generic classes on a base of ESS-systems, i.e., by transition from classes $(\mathbf{T}_i; \leq_i)$, $i = 1, \dots, n$, to the class $(\mathcal{F}_{i=1}^{\text{ESS}})^n(\mathbf{T}_i; \leq_i)$.

Corollary 2.6.5. *Any class of form $(\mathcal{F}_{i=1}^{\text{ESS}})^n(\mathbf{T}_i; \leq_i)$ has the finite closure property.*

Chapter 3

GENERIC EHRENFEUCHT THEORIES

§ 3.1. Generic theory with a non-symmetric semi-isolation relation

The construction, outlined in this Section (and in the next), establishes the existence of a powerful digraph $\Gamma_{\text{gen}} = \langle X, Q \rangle$ with shortest routes of unbounded lengths, which — via some inessential Q -ordered coloring — is expanded to a countable saturated model \mathcal{M} with a nonprincipal powerful type $p_\infty(x) \in S^1(\emptyset)$ such that the digraph

$$\langle p_\infty(\mathcal{M}); R_Q^{p_\infty}(\mathcal{M}) \rangle$$

is isomorphic to Γ_{gen} , where

$$R_Q^{p_\infty}(\mathcal{M}) = \{(a, b) \in (p_\infty(\mathcal{M}))^2 \mid \mathcal{M} \models Q(a, b)\}.$$

We construct Γ_{gen} while simultaneously coloring it, using the syntactic method, described in previous Chapter for constructing generic models.

Let $\Gamma_1 = \langle X_1; Q_1 \rangle$ be a colored subgraph of an acyclic colored digraph $\Gamma_2 = \langle X_2; Q_2 \rangle$ with coloring $\text{Col} : X_2 \rightarrow \omega \cup \{\infty\}$, a and b be vertices in X_1 , S be an (a, b) -route, not entirely in Γ_1 . The route S is *external* (over Γ_1) if only endpoints in S belong to X_1 . Denote by $W(\Gamma_1, \Gamma_2)$ the set of triples (a, b, n) , $a, b \in X_1$, $n \in \omega \setminus \{0, 1\}$, such that a and b are connected in Γ_2 by a shortest (a, b) -route of length n , and moreover, every shortest (a, b) -route is external over Γ_1 . A triple $\langle X_1, Q_1, W_1 \rangle$, where $W_1 = W(\Gamma_1, \Gamma_2)$, is called a *c_0 -subgraph* of the digraph Γ_2 if the vertex set X_1 is finite.

The relation “is a c_0 -subgraph” is denoted by \subseteq_{c_0} , that is, if the set W_1 exists, we write $\langle \Gamma_1, W_1 \rangle \subseteq_{c_0} \Gamma_2$. A structure $\langle \Gamma_1, W_1 \rangle$ will often be treated independently; we call $\langle \Gamma_1, W_1 \rangle$ a *c-graph* and denote it also by $\langle X_1, Q_1, W_1 \rangle$, where $\Gamma_1 = \langle X_1; Q_1 \rangle$. Here, X_1 is called the *universe* of the *c-graph* $\langle \Gamma_1, W_1 \rangle$.

For a *c-graph* $\Gamma_1 = \langle X_1, Q_1, W_1 \rangle$, $cc(\Gamma_1)$ denotes the minimal digraph $\Gamma \supseteq \Gamma_1$, which, for any triple $(a, b, n) \in W_1$ contains a shortest (a, b) -route of length n , with every vertex in $\Gamma \setminus \Gamma_1$ being of degree 2.

We define the relation \subseteq_c on the class of *c-graphs*. Thus, the *c-graph* $\tilde{\Gamma}_1 = \langle X_1, Q_1, W_1 \rangle$ is called a *c-subgraph* of the *c-graph* $\tilde{\Gamma}_2 = \langle X_2, Q_2, W_2 \rangle$, written $\tilde{\Gamma}_1 \subseteq_c \tilde{\Gamma}_2$, if $X_1 \subseteq X_2$, $Q_1 = Q_2 \cap (X_1)^2$, and $W_1 = W(\tilde{\Gamma}_1, cc(\tilde{\Gamma}_2))$.

Clearly, the relation \subseteq_c induces partial ordering on any set of *c-graphs*.

Below in this Section, the *c-graphs* are denoted by $\mathcal{A}, \mathcal{B}, \dots$ (possibly with indices), and their corresponding universes — A, B, \dots . The empty set \emptyset in this instance, is assumed to be the universe of a *c-graph* having the form $\langle \emptyset, \emptyset, \emptyset \rangle$. Also, we often write $\mathcal{A} \subseteq_c \mathcal{N}$ in place of $\mathcal{A} \subseteq_{c_0} \mathcal{N}$.

Denote by \mathbf{K}^* the class of all *c-graphs* $\mathcal{A} = \langle A, Q_{\mathcal{A}}, W_{\mathcal{A}} \rangle$ such that, for any vertices $a, b \in A$, the existence of an (a, b) -route (in the graph $\langle A; Q_{\mathcal{A}} \rangle$, or else as the condition that $(a, b, n) \in W_{\mathcal{A}}$) implies $\text{Col}(a) \leq \text{Col}(b)$.

Clearly, any *c-subgraph* of a *c-graph* in \mathbf{K}^* is also a *c-graph* in \mathbf{K}^* .

Denote by \mathbf{K} the class of all colored acyclic digraphs whose every *c-subgraph* belongs to \mathbf{K}^* .

If \mathcal{A} and \mathcal{B} are c_0 -subgraphs of a digraph \mathcal{N} in \mathbf{K} , then sets $A \cap B$ and $A \cup B$ are universes of the c_0 -subgraphs of \mathcal{N} , which we denote by $\mathcal{A} \cap_{\mathcal{N}} \mathcal{B}$ and $\mathcal{A} \cup_{\mathcal{N}} \mathcal{B}$, respectively.

Obviously, the value of $\mathcal{A} \cap_{\mathcal{N}} \mathcal{B}$ does not depend on the choice of \mathcal{N} , whereas the value of $\mathcal{A} \cup_{\mathcal{N}} \mathcal{B}$ may vary with \mathcal{N} . In what follows, we drop the index \mathcal{N} from the above-mentioned denotations if there is clarity as to which digraph is being spoken of.

If $\mathcal{A}, \mathcal{B} = \langle B, Q_{\mathcal{B}}, W_{\mathcal{B}} \rangle$, and $\mathcal{C} = \langle C, Q_{\mathcal{C}}, W_{\mathcal{C}} \rangle$ are *c-graphs*, and $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$, then we call the *c-graph* $\langle B \cup C, Q_{\mathcal{B}} \cup Q_{\mathcal{C}}, W_{\mathcal{B}} \cup W_{\mathcal{C}} \rangle$ a *free c-amalgam* of the *c-graphs* \mathcal{B} and \mathcal{C} over \mathcal{A} and denote it by $\mathcal{B} *_A \mathcal{C}$.

Clearly, the free c -amalgam $\mathcal{B} *_A \mathcal{C}$ exists for any c -graphs \mathcal{A}, \mathcal{B} , and \mathcal{C} with $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$. In this case c -graphs \mathcal{A}, \mathcal{B} , and \mathcal{C} are c -subgraphs of the c -graph $\mathcal{B} *_A \mathcal{C}$.

A one-to-one map $f : A \rightarrow B$ is called a c -embedding of the c -graph $\mathcal{A} = \langle A, Q_A, W_A \rangle$ into the c -graph $\mathcal{B} = \langle B, Q_B, W_B \rangle$ (written $f : \mathcal{A} \rightarrow_c \mathcal{B}$) if f is an embedding of the colored graph $\mathcal{A} = \langle A, Q_A \rangle$ into the colored graph $\mathcal{B} = \langle B, Q_B \rangle$ such that

$$W_B \cap (f(A) \times f(A) \times \omega) = \{(f(a_1), f(a_2), n) \mid (a_1, a_2, n) \in W_A\}.$$

We say that c -graphs \mathcal{A} and \mathcal{B} are c -isomorphic if there is a c -embedding $f : \mathcal{A} \rightarrow_c \mathcal{B}$ with $f(A) = B$. In this event f is called a c -isomorphism between \mathcal{A} and \mathcal{B} , and c -graphs \mathcal{A} and \mathcal{B} are called c -isomorphic copies.

A one-to-one map $f : A \rightarrow N$ is a c -embedding of the c -graph \mathcal{A} into the digraph \mathcal{N} (written $f : \mathcal{A} \rightarrow_c \mathcal{N}$) if f is a c -embedding of the c -graph \mathcal{A} into a c_0 -subgraph $f(\mathcal{A})$ of \mathcal{N} with universe $f(A)$.

Lemma 3.1.1. (amalgamation lemma). *The class \mathbf{K}^* has the c -amalgamation property (c -AP), that is, for any c -embeddings $f_0 : \mathcal{A} \rightarrow_c \mathcal{B}$ and $g_0 : \mathcal{A} \rightarrow_c \mathcal{C}$, where $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}^*$, there exist a c -graph $\mathcal{D} \in \mathbf{K}^*$ and c -embeddings $f_1 : \mathcal{B} \rightarrow_c \mathcal{D}$ and $g_1 : \mathcal{C} \rightarrow_c \mathcal{D}$ such that $f_0 \circ f_1 = g_0 \circ g_1$.*

P r o o f. There is no loss of generality in assuming that $\mathcal{A} \subseteq_c \mathcal{B}$ and $\mathcal{A} \subseteq_c \mathcal{C}$. Obviously, as \mathcal{D} we can take the c -graph $\mathcal{B} *_A \mathcal{C}$. \square

Denote by \mathbf{K}_0^* the subclass of \mathbf{K}^* , generated from the set of colored digraphs $\Gamma_{\alpha, \beta, \gamma} = \langle \{0, 1, 2\}, \{(0, 1), (0, 2), (1, 2)\} \rangle$, where $\text{Col}(0) = \alpha$, $\text{Col}(1) = \beta$, $\text{Col}(2) = \gamma$, $\alpha \leq \beta \leq \gamma$, $\gamma \in \omega \cup \{\infty\}$, by operations of taking c -subgraphs, c -isomorphic copies, free c -amalgams, by an operation which, for any c -graph \mathcal{A} and any pair (a, b) of its vertices, with $\text{Col}(a) \leq \text{Col}(b)$, not connected by routes in the graph $cc(\mathcal{A})$, allows the set $W_{\mathcal{A}}$ to be added one arbitrarily chosen triple (a, b, m) , where m is a natural number greater than the maximal of lengths of shortest routes in $cc(\mathcal{A})$, and by the inverse operation allowing an arbitrary triple (a, b, m) to be removed from the set $W_{\mathcal{A}}$ of that c -graph \mathcal{A} .

The operation allowing the set W to be added an information on aforesaid routes is called a *tracing*, and the inverse operation of removing of information about that routes is a *Detracing*.

By definition, every c -graph \mathcal{A} is endowed with some coloring $\text{Col} : A \rightarrow \omega \cup \{\infty\}$. The function $\text{Col}' : A \rightarrow \omega \cup \{\infty\}$ is called

an *admissible recoloring* of \mathcal{A} if, after replacing the function Col by Col' , we obtain a c -graph in the class \mathbf{K}_0 . Denote the c -graph produced by the recoloring by $\mathcal{A}(\text{Col}')$.

Lemma 3.1.2. *If \mathcal{A} is a c -graph in the class \mathbf{K}_0^* and Col' is its admissible recoloring then the c -graph $\mathcal{A}(\text{Col}')$ belongs to \mathbf{K}_0^* .*

Proof is by induction on the number of steps to be taken in constructing a c -graph \mathcal{A} from the graphs $\Gamma_{\alpha,\beta,\gamma}$. \square

Denote by \mathbf{K}_0 the class of all colored acyclic digraphs whose every finite subgraph forms a c -graph in \mathbf{K}_0^* .

Theorem 3.1.3. *There exists a countable, colored, saturated digraph $\mathcal{M} \in \mathbf{K}_0$ satisfying the following:*

- (1) *if $f : \mathcal{A} \rightarrow_c \mathcal{M}$ and $g : \mathcal{A} \rightarrow_c \mathcal{B}$ are c -embeddings, and $\mathcal{B} \in \mathbf{K}_0^*$, then there exists a c -embedding $h : \mathcal{B} \rightarrow_c \mathcal{M}$ such that $f = g \circ h$;*
- (2) *if \mathcal{A} and \mathcal{B} are c -isomorphic c -subgraphs of the digraph \mathcal{M} , then $\text{tp}_{\mathcal{M}}(\mathcal{A}) = \text{tp}_{\mathcal{M}}(\mathcal{B})$;*
- (3) *the coloring of the restriction $\mathcal{M} \upharpoonright Q$ of the model \mathcal{M} to the graph language $\Sigma = \{Q\}$ is inessential and Q -ordered;*
- (4) *$Q(x, y)$ is a principal formula in $\text{Th}(\mathcal{M} \upharpoonright Q)$.*

Proof. Using the amalgamation lemma, we construct \mathcal{M} as the union of c -graphs $(\mathcal{A}_n)_{n \in \omega}$, $\mathcal{A}_n \subseteq_c \mathcal{A}_{n+1}$, in \mathbf{K}_0^* . In so doing, we require that the following condition is met: for any c -graphs $\mathcal{A} \subseteq_c \mathcal{A}_n$ and $\mathcal{B} \in \mathbf{K}_0^*$ with $\mathcal{A} \subseteq_c \mathcal{B}$, there exists a copy $\mathcal{C} \subseteq_c \mathcal{M}$ of \mathcal{B} over \mathcal{A} such that \mathcal{C} is a c - c -subgraph of the digraph \mathcal{A}_m , for some $m > n$. Moreover, the colors of the elements of $\mathcal{C} \setminus \mathcal{A}$, under taking graph copies over \mathcal{A} , are distributed over \mathcal{A} in any admissible way, that is, so that for any vertices $a, b \in \mathcal{C}$, the fact that an (a, b) -route exists would imply $\text{Col}(a) \leq \text{Col}(b)$. The possibility for such distributions of colors to be realized follows from Lemma 3.1.2.

The number of the requirements being countable entails the existence of a countable model \mathcal{M} satisfying all the conditions specified.

We claim that \mathcal{M} is saturated. Let \mathcal{M}' be an ω -saturated model of $\text{Th}(\mathcal{M})$, $\mathcal{A} \subseteq_c \mathcal{M}$, $\mathcal{A}' \subseteq_c \mathcal{M}'$, and $f : \mathcal{A} \rightarrow_c \mathcal{A}'$ be a c -isomorphism. If $\mathcal{A}' \subseteq_c \mathcal{B}' \subseteq_c \mathcal{M}'$ then the construction of \mathcal{M} implies that there exists a c -isomorphic copy \mathcal{B} of the c -graph \mathcal{B}' over \mathcal{A}' , which is realized over \mathcal{A} in \mathcal{M} . Hence, there is a c -isomorphism $g : \mathcal{B} \rightarrow_c \mathcal{B}'$, extending the c -isomorphism f .

Now, let $\mathcal{A} \subseteq_c \mathcal{B} \subseteq_c \mathcal{M}$ and X and Y be disjoint sets of variables, which are in 1-1 correspondence with sets A and $B \setminus A$. Assume that the formula $\varphi_n(X)$ ($\psi_n(X, Y)$, respectively), $n \in \omega$, describes the following:

- (a) finite colors of elements of \mathcal{A} (of \mathcal{B});
- (b) negations of colors not exceeding n for elements of \mathcal{A} (of \mathcal{B}) that are infinite in color;
- (c) the existence and lengths of shortest routes connecting elements of \mathcal{A} (of \mathcal{B});
- (d) the non-existence of routes of length at most n connecting elements of \mathcal{A} (of \mathcal{B}), if the elements are not linked via routes.

By the construction of \mathcal{M} ,

$$\mathcal{M} \models \forall X (\varphi_n(X) \rightarrow \exists Y \psi_n(X, Y)),$$

and hence

$$\mathcal{M}' \models \forall X (\varphi_n(X) \rightarrow \exists Y \psi_n(X, Y)).$$

This implies that the set $\{\psi_n(A', Y) \mid n \in \omega\}$ of formulas is locally realizable in \mathcal{M}' ; hence, it is realizable in \mathcal{M}' since \mathcal{M}' is ω -saturated. Therefore there exist a c -graph $\mathcal{B}' \subseteq_c \mathcal{M}'$, where $\mathcal{A}' \subseteq_c \mathcal{B}'$, and c -isomorphism $g : \mathcal{B} \rightarrow \mathcal{B}'$ extending the c -isomorphism f .

The possibility for extending any c -isomorphisms $f : \mathcal{A} \rightarrow_c \mathcal{A}'$ and the known back-and-forth method show that a model \mathcal{M} with distinguished constants forming universe of a c -graph \mathcal{A} is isomorphic to a countable elementary submodel of the model \mathcal{M}' with distinguished constants forming universe of the c -graph \mathcal{A}' . Since the c -isomorphic c -graphs \mathcal{A} and \mathcal{A}' are chosen arbitrarily, and \mathcal{M}' is saturated, we conclude that \mathcal{M} realizes any type over a finite set, \mathcal{M} is saturated, and $\text{Th}(\mathcal{M})$ is small.

The possibility for extending c -isomorphisms of c -graphs lying in saturated models also implies that for c -isomorphic c -subgraphs \mathcal{A} and \mathcal{B} of the colored digraph \mathcal{M} , there exists an automorphism of \mathcal{M} , extending the initial c -isomorphism between \mathcal{A} and \mathcal{B} . Consequently, $\text{tp}_{\mathcal{M}}(A) = \text{tp}_{\mathcal{M}}(B)$.

Since the type of any c -graph in \mathcal{M} is defined by formulas containing at most two free variables and describing colors of elements, and since there exist sequences linking the elements, we conclude that the coloring of a digraph $\mathcal{M} \upharpoonright Q$ is inessential. Since the numbers of the colors, in moving over the route, remain non-decreasing, the coloring is Q -ordered.

If elements a and b in $\mathcal{M} \upharpoonright Q$ are connected by an arc, then the type $\text{tp}(a \hat{\ } b)$ is defined by a formula $Q(x, y)$, and hence $Q(x, y)$ is principal in $\text{Th}(\mathcal{M} \upharpoonright Q)$. \square

Notice, that the proof of Theorem 3.1.3 in fact repeats the proof of Theorem 2.5.1 for the generic class \mathbf{T}_0 of types, correspondent to c -graphs. Moreover, we have shown, that the class \mathbf{T}_0 has the uniform t -amalgamation property.

The theory $T_0 \equiv \text{Th}(\mathcal{M})$ for the color digraph constructed in the proof of Theorem 3.1.3, is said to be \mathbf{K}_0^* -generic, and its countable saturated model \mathcal{M} is \mathbf{K}_0^* -generic.

By the construction of T_0 , for any model \mathcal{M}' of T_0 , $\mathcal{M}' \models Q(a, b)$ implies $(a, b) \in \text{IECT}_{\mathcal{M}'}$, that is, $\text{tp}_{x \hat{\ } y}(a \hat{\ } b)$ is defined by $Q(x, y)$, and also by the colors of the elements a and b . Thus, from Proposition 1.2.13 and Theorem 3.1.3, we infer

Corollary 3.1.4. *The semi-isolation relation $\text{SI}_{p_\infty(x)}$ is non-symmetric.*

Below, we state that the digraph $\Gamma_{\text{gen}} \equiv \mathcal{M} \upharpoonright Q$ possesses the properties specified at the beginning of the present Section.

Let \mathcal{A} be a c -subgraph of \mathcal{M} , and a, b be elements in A . Consider a c -graph \mathcal{B} obtained by adding to \mathcal{A} an element c such that $\text{Col}(c) \leq \min\{\text{Col}(a), \text{Col}(b)\}$, and also arcs (c, a) and (c, b) . Obviously, \mathcal{B} belongs to the class \mathbf{K}_0^* and its copy extends \mathcal{A} in \mathcal{M} . Consequently

$$T_0 \vdash \forall x, y (\text{Col}_k(x) \wedge \text{Col}_m(y) \rightarrow \exists \geq^\omega z (\text{Col}_n(z) \wedge Q(z, x) \wedge Q(z, y)))$$

for any k, m, n with $n \leq \min\{k, m\}$. In particular, if a and b are realizations of type $p_\infty(x)$, then there exists a realization $c \models p_\infty$ such that $\models Q(c, a) \wedge Q(c, b)$. Hence the digraph

$$\Gamma_\infty = \langle p_\infty(M); R_Q^{p_\infty}(\mathcal{M}) \rangle$$

enjoys the pairwise intersection property. That the group $\text{Aut}(\Gamma_\infty)$ is transitive is obvious. The formula $R_Q^{p_\infty}(x, y)$ is principal in $\text{Th}(\Gamma_\infty)$ by Theorem 3.1.3. By the construction of T_0 , $\text{acl}_{\Gamma_\infty}(\{a\}) = \{a\}$ for any $a \in p_\infty(\mathcal{M})$. Consequently Γ_∞ is a powerful digraph.

By construction, the digraph Γ_∞ is isomorphic to the digraph Γ_{gen} , and the latter likewise is a powerful digraph. In view of being constructed, note, the digraph $\Gamma_{\text{gen}}^{-1} = \langle M; Q^{-1} \rangle$ is isomorphic to Γ_{gen} . Thus, we have

Corollary 3.1.5. *The digraphs Γ_{gen} and Γ_{gen}^{-1} are powerful.*

By the construction of \mathcal{M} , for any elements a_1, \dots, a_n , there are elements $b_i \in Q(a_i, \mathcal{M})$, $i = 1, \dots, n$, which are pairwise incomparable under $\bigcup_{n \in \omega} Q^n$. By the definition of \mathbf{K}_0^* , there exists an

element $c \in \bigcap_{i=1}^n Q(b_i, \mathcal{M})$, and hence $\bigcap_{i=1}^n Q^2(a_i, \mathcal{M}) \neq \emptyset$. In view of

Compactness Theorem and the property of a digraph with Q^2 being acyclic, there is a sequence $(a_n)_{n \in \omega}$ satisfying $\models Q^2(a_i, a_j) \Leftrightarrow i < j$. This, combined with the fact that Γ_{gen} is acyclic, implies that the formula $\varphi(x, y_1 \hat{\ } y_2) \equiv Q^2(y_1, x) \wedge Q^2(x, y_2)$ has the tree property (see [3]). Now, appealing to the simplicity criterion in the book by F. O. Wagner [27] (Corollary 2.8.9), we arrive at

Corollary 3.1.6. *Theory T_0 is not simple.*

A question whether T_0 lack in the strict-order property remains open. In favor of the positive answer is the fact that there are no infinite partial orders over relations Q^n on the universe, and in constructing the generic model, use is made of free amalgams only.

Theorem 3.1.7. (1) *A type q of T_0 is principal iff every two distinct elements, a_i and a_j , in any realization \bar{a} of q are connected by some (a_i, a_j) - or (a_j, a_i) -route, and all elements of the realizations of q are finite in color.*

(2) *A type q of T_0 is realized in the model \mathcal{M}_{p_∞} iff for any realization \bar{a} of q , every two distinct elements, a_i and a_j , of \bar{a} are connected by some (a_i, a_j) - or (a_j, a_i) -route, and the following condition holds:*

Let elements of a tuple \bar{a} contain members in finite colors, assume that a_f is an element of finite color, which is the common point at which terminate the routes, connecting all elements of finite colors with a_f , and suppose that among the elements of \bar{a} are members of infinite colors and that a_∞ is an element of infinite color, which is the common point at which commence the routes, connecting all elements of infinite colors with a_f . Then there exists an (a_f, a_∞) -route.

P r o o f. Let q be any type of the theory T_0 and \bar{a} be a realization of q .

(1) Suppose q is a principal type. Since the type p_∞ is non-isolated, all elements $a_i \in \bar{a}$ are finite in color. Assume that there exist distinct elements $a_i, a_j \in \bar{a}$ which are not connected both by

(a_i, a_j) - and (a_j, a_i) -routes. By the definition of \mathbf{K}_0^* , starting with some n , there exist tuples \bar{a}^n whose corresponding elements a_i^n and a_j^n are connected by an (a_i^n, a_j^n) -route of length n but not by shorter routes, and all other lengths of the shortest routes between them are the same as for elements in \bar{a} ; moreover, the elements of \bar{a}^n coincide in color with the respective elements of \bar{a} . Hence q is not isolated by any formula, that is, it cannot be a principal type.

Assume, now, that every two distinct elements $a_i, a_j \in \bar{a}$ are connected by some (a_i, a_j) - or (a_j, a_i) -route, and that all elements of \bar{a} are finite in color. By Theorem 3.1.3(2), then, a formula, describing colors of elements as well as lengths of shortest routes, will isolate q , that is, q is a principal type.

(2) Let q be a type realized in \mathcal{M}_{p_∞} , a be a realization of type p_∞ , and $\varphi(a, \bar{y})$ — be a consistent formula for which $\varphi(a, \bar{y}) \vdash q(\bar{y})$ and $\models \varphi(a, \bar{a})$. We claim that all distinct elements of a tuple $a \hat{\ } \bar{a}$ are coupled by routes. Indeed, if not, that is, some distinct elements $a_i, a_j \in a \hat{\ } \bar{a}$ are not linked by (a_i, a_j) - or (a_j, a_i) -routes, by the definition of \mathbf{K}_0^* , then, starting with some n , there exist tuples $a^n \hat{\ } \bar{a}^n$, whose respective elements a_i^n, a_j^n are connected by an (a_i^n, a_j^n) -route of length n but not by shorter routes, and all other lengths of the shortest sequences couplings are the same as for elements in $a \hat{\ } \bar{a}$; moreover, the elements of $a^n \hat{\ } \bar{a}^n$ coincide in color with the respective elements of $a \hat{\ } \bar{a}$. By the quantifier elimination in T_0 , in view of Theorem 3.1.3(2), the formula $\varphi(x, \bar{y})$ satisfies $\models \varphi(a^n, \bar{a}^n)$, starting with some n . Since $\models p_\infty(a^n)$, the condition $\models \varphi(a^n, \bar{a}^n)$ conflicts with $\varphi(a, \bar{y}) \vdash q(\bar{y})$. Thus all distinct elements in $a \hat{\ } \bar{a}$ are coupled by some routes.

Now, the property of Γ_{gen} being acyclic and the fact that the coloring is ordered imply that there exists an element $a_f \in \bar{a}$ of finite color which is maximal among all the elements of finite color (if any) and is such that all the routes, connecting a_f with the elements of finite colors in \bar{a} , terminate at a_f .

At the same time, among all members of infinite colors in \bar{a} (if any) is an element a_∞ in infinite color such that all routes connecting a_∞ with the other elements of infinite colors commence at a_∞ .

It remains to observe that the coloring being ordered implies the existence of an (a_f, a) -route, and that the condition of a_∞ being semi-isolated over a (since $\mathcal{M}_{p_\infty} = \mathcal{M}_a$) entails the existence of an (a, a_∞) -route. We have thus shown that the (a_f, a_∞) -route exists, and that the condition specified in the statement that we are proving is necessary for q to be realized in \mathcal{M}_{p_∞} .

Suppose, now, that all distinct elements in \bar{a} are coupled via routes. If \bar{a} has no elements of infinite colors, then q is principal by item (1); hence, it is realized in \mathcal{M}_{p_∞} . If \bar{a} contains elements of infinite colors then the property of being ordered for colorings implies that q is isolated by the set $p_\infty(y_\infty)$ of formulas (where a_∞ is an element of infinite color in \bar{a} at which originate all routes, connecting a_∞ with all elements \bar{a} , of infinite colors), and also by a formula describing finite colors of elements in \bar{a} and lengths of shortest routes, connecting elements in \bar{a} . Hence q is realized in \mathcal{M}_{a_∞} . \square

§ 3.2. Generic theories with nonprincipal powerful types

In this section we come up with a description of the construction which allows us to build theories with non-principal powerful types shaped as an expansion of the theory T_0 for the color digraph \mathcal{M} from previous Section.

Obviously, since the lengths of the shortest sequences are not 185 bounded, there exists a non- p_∞ -principal p_∞ -type in T_0 . In Example 1.3.1, we described a mechanism for a non- p -principal p -type to be realized in model \mathcal{M}_p . We use the trick from that Example for finding an expansion of T_0 , having a powerful 1-type defined by the set $p_\infty(x)$.

A type $r(y_1, \dots, y_k)$ in $S(T_0)$ is said to be (p_∞, y_1) -principal if $p_\infty(y_1) \subseteq r(y_1, \dots, y_k)$ and $\{\varphi(y_1, \dots, y_k)\} \cup p_\infty(y_1) \vdash r(y_1, \dots, y_k)$ for some formula $\varphi(y_1, \dots, y_k) \in r$.

Clearly, \mathcal{M}_{p_∞} realizes exactly those types $q(y_2, \dots, y_k)$ in $S(T_0)$ that are contained in (p_∞, y_1) -principal types $r(y_1, y_2, \dots, y_k) \in S(T_0)$.

Below, in order to turn p_∞ into a powerful type, for every type $q(y_2, \dots, y_k)$ not contained in any (p_∞, y_1) -principal type $r(y_1, y_2, \dots, y_k)$, we introduce a new k -ary predicate R_q so that the set $\{R_q(y_1, \dots, y_k)\} \cup p_\infty(y_1)$ is consistent and $\{R_q(y_1, \dots, y_k)\} \cup p_\infty(y_1) \vdash q(y_2, \dots, y_k)$.

With this goal in mind, we renumber the set \mathbf{q} of all types $q(y_1, \dots, y_k)$ for tuples \bar{a}_q with mutually distinct coordinates for which the sets of elements of \bar{a}_q contain elements of finite colors whose number is not equal to 1, and $\mathcal{M}_{\bar{a}_q}$ is not isomorphic to a prime model or to \mathcal{M}_{p_∞} : $\mathbf{q} = \{q_m(y_1, \dots, y_{k_m}) \mid m \in \omega\}$.

In this event, by Theorem 3.1.3, $q_m(y_1, \dots, y_{k_m})$ are defined by c -isomorphism types \mathbf{A}_m of their realizations \bar{a}_m . In what follows, therefore, the types $q_m(y_1, \dots, y_{k_m})$ will be identified with the c -isomorphism types \mathbf{A}_m .

For every type $q_m(\bar{y}) \in \mathbf{q}$ and for a corresponding c -isomorphism type \mathbf{A}_m , we fix the set $\Phi_{\mathbf{A}_m}(\bar{y})$ of formulas $\varphi_n(\bar{y})$, $n \in \omega$, which isolates $q_m(\bar{y})$. The formulas of this set describe the following:

- (a) finite colors of elements in \bar{a}_m ;
- (b) negations of colors lesser than n for elements of infinite colors in \bar{a}_m ;
- (c) the existence and lengths of shortest routes connecting elements of \bar{a}_m ;
- (d) the non-existence of routes of length lesser than n connecting elements of \bar{a}_m , if the elements are not linked by routes.

Now, consider a c -isomorphism type \mathbf{A} of an arbitrary tuple $\bar{a} = (a_1, \dots, a_k)$ on a set $A = \{a_1, \dots, a_k\}$ of cardinality k which is not in \mathcal{M}_{p_∞} and contains $s \neq 1$ elements of finite color.¹ Denote by $\max_{\mathbf{A}}$ the value $\max\{\text{Col}(a_i) \mid \text{Col}(a_i) \in \omega, a_i \in A\}$ if the set A contains elements of finite colors. Otherwise, that is, if all elements of A are infinite in color, we put $\max_{\mathbf{A}} \doteq 0$.

We define $(k+1)$ -ary relations $R_{\mathbf{A}}$, as follows:

$$(1) \vdash \exists \bar{y} R_{\mathbf{A}}(x, \bar{y}) \leftrightarrow \bigwedge_{n < \max_{\mathbf{A}}} \neg \text{Col}_n(x);$$

(2) for any $n \geq \max_{\mathbf{A}}$, the formula $R_{\mathbf{A}}(x, y_1, \dots, y_k) \wedge \text{Col}_n(x)$ is equivalent to a conjunction of $\varphi_n(\bar{y}) \in \Phi_{\mathbf{A}}(\bar{y})$,² and the formula describing the following properties:

- (a) if $\langle a_{i_1}, \dots, a_{i_r} \rangle$ (where $i_1 < \dots < i_r$) is a tuple of all elements a_i of infinite color in \bar{a} , and $\langle a_{j_1}, \dots, a_{j_s} \rangle$ (where $j_1 < \dots < j_s$) is a tuple of all elements a_j of finite colors in \bar{a} , then there exist elements z_0, \dots, z_{r-1} and u_0, \dots, u_{s-1} such that $z_{r-1} = y_{i_r}$, $Q(z_{m-1}, z_m) \wedge Q(z_{m-1}, y_{i_m})$, $\text{Col}(z_{m-1}) = n$ for $m = 1, \dots, r-1$,

¹The restriction $s \neq 1$ is introduced only for the sake of convenience of our further reasoning. It does not diminish generality in treating c -isomorphism types for subsequent realizations of appropriate types in \mathcal{M}_{p_∞} , since any set having one element of finite color can be added yet another element of finite color, and for any types q_1 and q_2 , the conditions $\mathcal{M}_{p_\infty} \models q_2$ and $q_1 \subseteq q_2$ imply $\mathcal{M}_{p_\infty} \models q_1$.

²Isomorphism types of tuples realizing the formula $R_{\mathbf{A}}(a, \bar{y})$, $\text{Col}(a) = n \geq \max_{\mathbf{A}}$, approximate the description of a c -isomorphism type of a tuple \bar{a} , and the limit (as $n \rightarrow \infty$) of these approximations corresponds to a description of the c -isomorphism type of \bar{a} .

$u_{s-1} = y_{j_s}$, $Q(u_m, u_{m-1}) \wedge Q(y_{j_m}, u_{m-1})$, $m = 1, \dots, s-1$, $\text{Col}(u_{m-1}) = \max\{\text{Col}(u_m), \text{Col}(y_{j_m})\}$ for $1 < m < s$, and $\text{Col}(u_0) = n$ for $s > 1$; if $s = 0$ then $x = z_0$; if $r = 0$ then $x = u_0$; if $r \geq 1$ and $s \geq 2$ then $\vdash Q(x, z_0) \wedge Q(x, u_0)$;

(b) a c -graph consisting of elements $x, y_1, \dots, y_k, z_0, \dots, z_{r-1}, u_0, \dots, u_{s-1}$ contains no arcs other than the arcs specified in (a) and in the description of \mathbf{A} for the elements \bar{y} ; nor does it contain external shortest routes of length at most n but for the external shortest routes connecting elements of \bar{y} and the elements described in \mathbf{A} .³

Item (2) implies that if A and A' are sets that are non-coincident or are coincident but are not c -isomorphic while the numberings are kept fixed, then the corresponding relations $R_{\mathbf{A}}$ and $R_{\mathbf{A}'}$ will be disjoint, starting with some color w.r.t. the first coordinate.

Notice, that predicates $R_{\mathbf{A}}$, where \mathbf{A} are isomorphism types of elements of infinite colors, refine the graph structure while not increasing sets of binary relations defined by projections such as

$$\exists y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_k (R_{\mathbf{A}}(x, \bar{y}) \wedge \varphi(x)) \quad (3.1)$$

and

$$\exists x, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{j-1}, y_{j+1}, \dots, y_k (R_{\mathbf{A}}(x, \bar{y}) \wedge \varphi(x)), \quad (3.2)$$

where $\varphi(x)$ is a formula distinguishing some set of elements having a finite or cofinite set of colors. In fact, by the definition of $R_{\mathbf{A}}$, any relation corresponding to a projection of form (3.1), is defined by a formula describing transitions from x to y_i via some Q -routes. The relations corresponding to projections of form (3.2) are defined by formulas describing the presence or absence of links between y_i and y_j via Q -routes of some bounded length.

The same effect is observed in taking projections of $R_{\mathbf{A}}$, where \mathbf{A} are isomorphism types of elements of finite colors.

If a tuple \bar{a} contains elements of both infinite and finite colors, then the relation $R_{\mathbf{A}}$, for a corresponding isomorphism type \mathbf{A} , leads to new binary relations arising via formulas $\psi(x, y_i)$ in form (3.1), where y_i corresponds to an element of some finite color l . Moreover,

³This means that c -isomorphism types of sets A have a unique extension to c -isomorphism types including elements $x, z_0, \dots, z_{r-1}, u_0, \dots, u_{s-1}$, where x satisfies type p_∞ .

satisfiability or non-satisfiability of the formula $\psi(a^n, a_i^l)$ at realizations a^n and a_i^l of natural colors n and l , respectively, is characterized by a relation between the color n and the length of a shortest (a_i^l, a_n) - Q -route. Formulas of form (3.2) preserve, as noted, the former graph structure.

We claim that with the above-mentioned refinement of the graph structure (via relations $R_{\mathbf{A}}$) in hand, the required expansion may be realized by newly constructing the generic model \mathcal{M} from the c -graphs expanded by finite records saying of positive links between elements via intermediate elements through projections of the relations $R_{\mathbf{A}}$.

We start our construction by describing the class \mathbf{K}_1^* of finite structures endowed with finite records holding of interrelations of the elements satisfying conditions (1) and (2). Since the desired generic model expands a \mathbf{K}_0^* -generic model, we assume that every finite set A , which enters \mathbf{K}_1^* and is restricted to the graph signature $\{Q\}$ with coloring Col , forms a c -graph $\langle A, Q, W \rangle$ in \mathbf{K}_0^* . Moreover, introduction of the relations $R_{\mathbf{A}_m}$ requires that the record W is added positive information on interrelations of the elements w.r.t. projections $\exists y_{l_1}, \dots, y_{l_t} R_{\mathbf{A}_m}(x, \bar{y})$ in accordance with item (2).

Before we end to define structures in the class \mathbf{K}_1^* , we observe the following. As shown in Theorem 3.1.3, a c -isomorphism type of every c -graph \mathcal{A} (i.e., information on colors of elements and on couplings of the elements via shortest routes) determines type of the set A in generic model. In defining every relation $R_{\mathbf{A}}$ the belonging of every tuple $a \hat{=} \bar{a}$ to that relation is specified either by a principal formula describing relations between the elements in prime model or by a sequence of formulas (see item (2)) which locally describe the absence of links between some elements of \bar{a} via routes, while keeping the links between the element a and the elements of \bar{a} fixed in length.

The last description — as noted above for binary relations — depends directly on the interrelations between colors of approximations a^n (in prime model) of an element a (these approximations will be called *sources*) and lengths of shortest routes between appropriate elements of approximations \bar{a}^n (in prime model) of a tuple \bar{a} (these approximations will be called *successors*). Namely, if the color number of a source a^n does not exceed (unbounded as $n \rightarrow \infty$) lengths of shortest routes between elements of successors \bar{a}^n , then the relation $R_{\mathbf{A}}$ holds, provided that the elements $z_0, \dots, z_{r-1}, u_0, \dots, u_{s-1}$,

described in item (2), are in hand. But if the color number of a^n is greater than is some (unbounded as $n \rightarrow \infty$) length of shortest routes between elements of \bar{a}^n , then $R_{\mathbf{A}}$ — under the same conditions — will fail. Below, the relation which the color number of a source a^n associates with a collection of pairwise non-bounded (as $n \rightarrow \infty$) lengths of shortest routes between elements of a successor $\bar{a}^{n''}$, as well as relations between the sources and the elements of finite colors in the successors, are for brevity referred to as *CL-correlation*.

In generic theory, all n -types are defined by 2-types describing colors of elements and lengths of shortest routes; therefore, the CL-correlation may be characterized by formulas $\rho(x, y_i, y_j)$, which express the following:

(i) the possibility for transiting via a Q -route from an element x , corresponding to a^n , to an intermediate element z , which has the same color as x and is separated from x , by a distance equal to the maximal of lengths of shortest (a^n, a_i^n) - and (a^n, a_j^n) -routes, if a_i and a_j are infinite in color;

(ii) the possibility for transiting via a Q^{-1} -route from an element x , corresponding to a^n , to an intermediate element z , which has the same color as x and is separated from x , by a distance equal to the maximal of lengths of shortest (a^n, a_i^n) - and (a^n, a_j^n) -routes, if a_i and a_j are finite in color;

(iii) the possibility for transiting from an intermediate element z to elements y_i and y_j by some ternary relation $R_{\mathbf{A}^*}(z, y_i, y_j)$, if a_i and a_j are both infinite or finite in color together;

(iv) the possibility for transiting via a Q -route of fixed length from an element x , corresponding to a^n , to an element y_i , corresponding to a_j^n , and the impossibility of transiting via a Q^{-1} -route of length at most n from an element x to an element y_j , where the color of a_j is less than n .⁴

In fact, if the tuple $a^n \wedge \bar{a}^n$ belongs to the relation $R_{\mathbf{A}}$ then the condition for the CL-correlation holds. By the construction of a generic model, therefore, there are intermediate elements z which are involved in the description of formulas $\rho(x, y_i, y_j)$; these z are colored as is a^n and are connected with y_i and y_j by the above-specified relations.

⁴The relation for (x, y_i, y_j) described in this item corresponds to a projection of some relation $R_{\mathbf{A}}$ leaving the coordinates x, y_i and y_j free.

Conversely, if, for the tuple $a^n \wedge \bar{a}^n$, we have descriptions of fixed colors of elements and lengths of shortest routes corresponding to $R_{\mathbf{A}}$, and also of the formulas $\rho(x, y_i, y_j)$, then the coincidence of the color of intermediate elements z with that of x gives rise to the CL-correlation. Consequently, $a^n \wedge \bar{a}^n$ belongs to $R_{\mathbf{A}}$ by definition.

In what follows, we assume that projections of form (3.1) with elements y_i , of finite colors likewise are defined by formulas $\rho(x, y_i, y_j)$, with $y_i = y_j$.

Notice also, that using distinct relations $R_{\mathbf{A}}$ in the formulas ρ is equivalent to employing fixed relations $R_{\mathbf{A}^*}$, whose taking depends only on the configuration of colors and lengths of shortest sequences for pairs of elements. Thus the formulas $\rho(x, y_i, y_j)$ with $R_{\mathbf{A}^*}$ are ternary indicators for positive or negative entries of the formulas $\exists y_{l_1} \dots y_{l_t} R_{\mathbf{A}_m}(x, \bar{y})$ in the types of the theory for the generic model in question. We add these indicators to the descriptions of c -isomorphism types \mathbf{A}_m in defining the relations $R_{\mathbf{A}_m}$ themselves (the formulas $\rho(y_i, y_j, y_k)$, or add their negations conjunctively to $\varphi_n(\bar{y})$) and to the general descriptions of the types of tuples. The adding of ρ , in this instance, extends the very language determined by the c -isomorphism types \mathbf{A}_m with the formulas $\rho^\delta(y_i, y_j, y_k)$, $\delta \in \{0, 1\}$, added. This extended language remains countable due to there being finitely many versions of adding positive formulas ρ to every c -isomorphism type.

Ultimately we specify that the class \mathbf{K}_1^* consists of all finite structures of the language $\text{Col} \cup \{Q\} \cup \{R_{\mathbf{A}_m} \mid m \in \omega\}$, which are obtained from c -graphs belonging to \mathbf{K}_0^* by adding the relations $R_{\mathbf{A}_m}$ consistent with item (2) (with the c -isomorphism types \mathbf{A}_m extended by ρ), and all possible admissible formulas $\rho(a_i, a_j, a_k)$.

Finite structures \mathcal{A} with finite records $W_{\mathcal{A}}$, forming the class \mathbf{K}_1^* are called c_1 -structures. Denote by \mathbf{K}_1 the class of all models of the language $\text{Col} \cup \{Q\} \cup \{R_{\mathbf{A}_m} \mid m \in \omega\}$, whose every finite subset forms a c_1 -structure in \mathbf{K}_1^* .

The concept of a c_1 -embedding $f : \mathcal{A} \rightarrow_{c_1} \mathcal{B}$ for c_1 -structures \mathcal{A} and \mathcal{B} , under which an appropriate record $W_{\mathcal{A}}$ ($W_{f(\mathcal{A})} = W_{\mathcal{B}} \upharpoonright f(\mathcal{A})$) is preserved, is a natural generalization of the concept of a c -embedding. Thereby we also define the concept of a c_1 -embedding $f : \mathcal{A} \rightarrow_{c_1} \mathcal{N}$ of a c_1 -structure \mathcal{A} into a model \mathcal{N} in \mathbf{K}_1 .

Two c_1 -structures, \mathcal{A} and \mathcal{B} , are said to be c_1 -isomorphic if there exists a c_1 -embedding $f : \mathcal{A} \rightarrow_{c_1} \mathcal{B}$ with $f(\mathcal{A}) = \mathcal{B}$.

Theorem 3.2.1. *There exists a countable saturated model $\mathcal{M} \in \mathbf{K}_1$ satisfying the following conditions:*

(a) *if $f : \mathcal{A} \rightarrow_{c_1} \mathcal{M}$ and $g : \mathcal{A} \rightarrow_{c_1} \mathcal{B}$ are c_1 -embeddings, and $\mathcal{B} \in \mathbf{K}_1^*$, then there exists a c_1 -embedding $h : \mathcal{B} \rightarrow_{c_1} \mathcal{M}$ such that $f = g \circ h$;*

(b) *if \mathcal{A} and \mathcal{B} are c_1 -isomorphic c_1 -structures in \mathcal{M} , then $\text{tp}_{\mathcal{M}}(A) = \text{tp}_{\mathcal{M}}(B)$;*

(c) *the restriction of \mathcal{M} to the language $\text{Col} \cup \{Q\}$ is a \mathbf{K}_0^* -generic model;*

(d) *every formula $R_{\mathbf{A}}(a, \bar{y})$, where $\models p_{\infty}(a)$, is principal, and a c_1 -isomorphism type of every realization of $R_{\mathbf{A}}(a, \bar{y})$ coincides with a c_1 -isomorphism type \mathbf{A} .*

P r o o f. The existence of a countable saturated model \mathcal{M} in \mathbf{K}_1 satisfying (a)-(c) is proved by following essentially the same line of argument as we used in proving Theorem 3.1.3. In so doing, to the descriptions of the formulas $\varphi_n(X)$ and $\psi_n(X, Y)$, for every pair (a, b) of vertices, we add the following:

(a) positive information on links of triples of elements via formulas ρ , if the links in question in c_1 -structures exist;

(b) negations of the links of triples of elements via formulas ρ in which lengths of shortest routes to intermediate elements z do not exceed n , if the links in question in c_1 -structures do not exist.

We claim that for any realization a of type p_{∞} and for any predicate symbol $R_{\mathbf{A}}$, the formula $R_{\mathbf{A}}(a, \bar{y})$ is principal. Let \bar{b} and \bar{c} be arbitrary tuples for which $\models R_{\mathbf{A}}(a, \bar{b}) \wedge R_{\mathbf{A}}(a, \bar{c})$. By the definition of a relation $R_{\mathbf{A}}$, we then see that the c_1 -structures \mathcal{B} and \mathcal{C} , consisting of the elements $a \hat{\ } \bar{b}$ and $a \hat{\ } \bar{c}$, are c_1 -isomorphic. Since the types $\text{tp}_{\mathcal{M}}(B)$ and $\text{tp}_{\mathcal{M}}(C)$ coincide, there exists an automorphism fixing an element a and mapping \bar{b} to \bar{c} . Consequently $\text{tp}(\bar{b}/a) = \text{tp}(\bar{c}/a)$, and hence $R_{\mathbf{A}}(a, \bar{y})$ is a principal formula. That the c_1 -isomorphism types of the tuples \bar{b} and \bar{c} coincide with the c_1 -isomorphism type \mathbf{A} follows from the definition of $R_{\mathbf{A}}(x, \bar{y})$. \square

The theory $T_1 \equiv \text{Th}(\mathcal{M})$ for \mathcal{M} , which is constructed for proving Theorem 3.2.1, and the countable saturated model \mathcal{M} are said to be \mathbf{K}_1^* -generic.

Since every type over the empty set in T_1 is determined by the type of a corresponding c_1 -structure, and for every type q of a c_1 -

structure not in the prime model, there exists a principal formula $\exists y_{l_1} \dots y_{l_t} R_{\mathbf{A}}(a, \bar{y})$ (where $\models p_\infty(a)$) for which

$$\exists y_{l_1} \dots y_{l_t} R_{\mathbf{A}}(x, \bar{y})(a, \bar{y}) \vdash q,$$

it follows that the model \mathcal{M}_{p_∞} realizes all types of T_1 . Thus $p_\infty(x)$ is a powerful type, and we have the following:

Theorem 3.2.2. *There exists a complete theory with a non-principal powerful type expanding the theory T_0 .*

We start with the requirements to be followed in order to construct the theory $T_2 \supset T_0$ in which all non-principal types are powerful. With this goal in mind, we redefine relations $R_{\mathbf{A}}$ by replacing condition (1) by the following:

$$(1') \vdash \left(\exists \bar{y} R_{\mathbf{A}}(x, \bar{y}) \leftrightarrow \bigwedge_{n < \max_{\mathbf{A}}} \neg \text{Col}_n(x) \right) \wedge$$

$$\left(\exists \bar{y} (R_{\mathbf{A}}(x, \bar{y}) \wedge \varphi_n(\bar{y})) \leftrightarrow \bigwedge_{t < \max\{\max_{\mathbf{A}}, n\}} \neg \text{Col}_t(x) \right), \quad n \in \omega,$$

and replacing (2) by (2'), which is obtained from the former by replacing the formulas $\varphi_n(\bar{y})$ by formulas $\varphi_n(\bar{y}) \wedge \neg \varphi_{n+1}(\bar{y})$ with the formulas ρ entered into the latter.

Repeating the description of \mathbf{K}_1^* subject to conditions (1') and (2'), again we obtain the class \mathbf{K}_2^* of a countable language, which (up to renaming of symbols) may be conceived of as coincident with the language of \mathbf{K}_1^* . Finite structures \mathcal{A} with finite records forming the class \mathbf{K}_2^* are called *c₂-structures*. Denote by \mathbf{K}_2 the class of all models of the language $\text{Col} \cup \{Q\} \cup \{R_{\mathbf{A}_m} \mid m \in \omega\}$, every finite subset of which forms a *c₂-structure* in \mathbf{K}_2^* . Similarly to the concept of a *c₁-embedding*, the concept of a *c₂-embedding* $f : \mathcal{A} \rightarrow_{c_2} \mathcal{B}$ for *c₂-structures* \mathcal{A} and \mathcal{B} , under which an appropriate record $W_{\mathcal{A}}$ ($W_{f(\mathcal{A})} = W_{\mathcal{B}} \upharpoonright f(\mathcal{A})$) is preserved, is a generalization of the concept of a *c-embedding*. Thereby we also define the concept of a *c₂-embedding* $f : \mathcal{A} \rightarrow_{c_2} \mathcal{N}$ of a *c₂-structure* \mathcal{A} into a model \mathcal{N} in \mathbf{K}_2 .

Theorem 3.2.3. *There exists a countable saturated model $\mathcal{M} \in \mathbf{K}_2$ satisfying the following:*

(a) *if $f : \mathcal{A} \rightarrow_{c_2} \mathcal{M}$ and $g : \mathcal{A} \rightarrow_{c_2} \mathcal{B}$ are c_2 -embeddings, and $\mathcal{B} \in \mathbf{K}_2^*$, then there exists a c_2 -embedding $h : \mathcal{B} \rightarrow_{c_2} \mathcal{M}$ such that $f = g \circ h$;*

(b) *if \mathcal{A} and \mathcal{B} are c_2 -isomorphic c_2 -structures in \mathcal{M} , then $\text{tp}_{\mathcal{M}}(A) = \text{tp}_{\mathcal{M}}(B)$;*

(c) *the restriction of \mathcal{M} to the language $\text{Col} \cup \{Q\}$ is a \mathbf{K}_0^* -generic model;*

(d) *every formula $R_{\mathbf{A}}(a, \bar{y})$, where $\models p_{\infty}(a)$, is principal, and a c_2 - c_2 -isomorphism type of every realization of $R_{\mathbf{A}}(a, \bar{y})$ coincides with the c_2 -isomorphism type \mathbf{A} ;*

(e) *every formula $R_{\mathbf{A}}(x, \bar{a})$, where \mathbf{A} is the c_2 -isomorphism type of a tuple \bar{a} , is principal, and every realization of the formula $R_{\mathbf{A}}(x, \bar{a})$ is a realization of the type p_{∞} .*

P r o o f of (a)-(d) repeats the argument for the respective items in Theorem 3.2.1.

(e) Consider an arbitrary formula $R_{\mathbf{A}}(x, \bar{a})$, where \mathbf{A} is the c_2 -isomorphism type of a tuple \bar{a} . We claim that $R_{\mathbf{A}}(x, \bar{a})$ is principal. Let b and c be any elements for which $\models R_{\mathbf{A}}(b, \bar{a}) \wedge R_{\mathbf{A}}(c, \bar{a})$. By the definition of $R_{\mathbf{A}}$, then, we see that the c_2 -structures \mathcal{B} and \mathcal{C} , consisting of the elements $b \hat{\ } \bar{a}$ and $c \hat{\ } \bar{a}$ are c_2 -isomorphic. Since $\text{tp}_{\mathcal{M}}(B)$ and $\text{tp}_{\mathcal{M}}(C)$ coincide, there exists an automorphism fixing \bar{a} and mapping b to c . Consequently $\text{tp}(b/\bar{a}) = \text{tp}(c/\bar{a})$, and hence $R_{\mathbf{A}}(x, \bar{a})$ is a principal formula. The condition that $R_{\mathbf{A}}(x, \bar{a}) \vdash p_{\infty}(x)$ follows from the definition of $R_{\mathbf{A}}(x, \bar{y})$. \square

The theory $T_2 \equiv \text{Th}(\mathcal{M})$ for \mathcal{M} , which is constructed for proving Theorem 3.2.3, and the countable saturated model \mathcal{M} are said to be \mathbf{K}_2^* -generic.

From Theorem 3.2.3(e), it follows that for every non-principal type $q(\bar{y})$ of T_2 , the model \mathcal{M}_q realizes p_{∞} , and hence every non-principal type is powerful. Moreover, in view of Proposition 1.1.3, introduction of the predicates $R_{\mathbf{A}}$ allows us, for every tuple \bar{a} , having some c_2 -isomorphism type \mathbf{A}_m , to find a realization a of $p_{\infty}(x)$ such that $\models R_{\mathbf{A}}(a, \bar{a})$, from which, in view of Proposition 1.1.3, it follows that the model $\mathcal{M}_{\bar{a}}$ coincides with \mathcal{M}_a . Thus all prime models over tuples realizing non-principal types are isomorphic to $\mathcal{M}_{p_{\infty}}$, and we have the following:

Theorem 3.2.4. *There exists a small theory T_2 expanding the theory T_0 and satisfying the condition $|\text{RK}(T_2)| = 2$.*

§ 3.3. A theory with three countable models

First, we point out general principles for expanding T_2 , which lead to the construction of a theory T with $|\text{RK}(T)| = 2$ and the property (CEP).

Let $a_0, a'_0, a_1, a'_1, \dots, a_n, a'_n, a''_n, b_0, b_1$ (where $b_0 \neq b_1$) be realizations of a powerful type $p_\infty(x)$ of a theory T_2 such that $\mathcal{M}_{a_i} = \mathcal{M}_{a'_i}$, $i = 0, \dots, n$, $\mathcal{M}_{a_n} = \mathcal{M}_{a''_n}$, $\mathcal{M}_{a_i} \prec \mathcal{M}_{a_{i+1}}$, $\models Q(a_{i+1}, a'_i)$, $i = 0, \dots, n-1$, $\mathcal{M}_{a_n} \prec \mathcal{M}_{b_0}$, $\mathcal{M}_{a_n} \prec \mathcal{M}_{b_1}$, $\models Q(b_0, a'_n) \wedge Q(b_1, a''_n)$, and the elements b_0 and b_1 are not linked via (b_0, b_1) - or (b_1, b_0) -routes. A *corealization amalgam* of models \mathcal{M}_{b_0} and \mathcal{M}_{b_1} over a type q of a tuple $\langle a_0, a'_0, a_1, a'_1, \dots, a_n, a'_n, a''_n, b_0, b_1 \rangle$ is a model \mathcal{M} (denoted by $\mathcal{M}_{b_0} *_{q} \mathcal{M}_{b_1}$), which is an expansion of $\mathcal{M}_{b_0} \cup \mathcal{M}_{b_1}$ by binary predicates $R_q = \{(b_0, b_1)\}$ and $R'_q = \{(b_1, b_0)\}$.⁵

We expand T_2 to T using all possible binary predicate symbols R_q and R'_q so that for any of the above-mentioned realizations $a_0, a'_0, a_1, a'_1, \dots, a_n, a'_n, a''_n, b_0, b_1$ of p_∞ , the model \mathcal{M}_{b_0} extends to a prime model over b_0 containing the corealization amalgam $\mathcal{M}_{b_0} *_{q} \mathcal{M}_{b_1}$.

Moreover, we require that a saturated model for the expanded theory satisfies the conditions (1') and (2') specified in Section 3.2, and the following:

- (3) $R_q(b_0, y)$ and $R'_q(b_1, y)$ are principal formulas for any type q ;
- (4) the relation $R^* = \bigcup_q (R_q \cup R'_q)$ forms an acyclic undigraph

with shortest routes of unbounded lengths, which is composed of mutually disjoint relations $R_q \cup R'_q$, where $R'_q = (R_q)^{-1}$, $R_q \cap R'_q = \emptyset$ or $R_q = R'_q$, with infinitely many images or preimages w.r.t. each one of R_q, R'_q ;

(5) every connected component w.r.t. R^* consists of one-color elements, and for each color $\alpha \in \omega \cup \{\infty\}$, there are infinitely many connected components consisting of the elements in color α ;

(6) the transitive closure of the relation R^* is disjoint from a transitive closure of the relation Q ;

⁵Notice, that for c_2 -structures, in treating corealization amalgams, the elements a_i and a'_i coincide since every model \mathcal{M}_{p_∞} has just one element over which all types realized in that model are principal. After introducing relations R_q and R'_q , we obtain infinitely many such elements, and in dealing with elementary chains, we need to take into account a possible difference between a_i and a'_i .

(7) the connected components w.r.t. R^* form equivalence classes which are partially ordered under the transitive closure of $Q \cup \text{id}$.

Subject to the above-specified conditions, models \mathcal{M}_{b_0} will elementary include \mathcal{M}_{b_1} and vice versa. Moreover, by construction, for any models \mathcal{M}_a and \mathcal{M}_b , where $a, b \models p_\infty$, the fact that $\mathcal{M}_a \prec \mathcal{M}_b$ and \mathcal{M}_b realizes some non-principal type over a implies that there exists a sequence $a_0, a'_0, \dots, a_n, a'_n$ of realizations of type p_∞ such that $a_0 = a$, $\mathcal{M}_{a_i} = \mathcal{M}_{a'_i}$, $i = 0, \dots, n$, $\mathcal{M}_{a_i} \prec \mathcal{M}_{a_{i+1}}$, $\models Q(a_{i+1}, a'_i)$, $i = 0, \dots, n-1$, and $\mathcal{M}_{a_n} = \mathcal{M}_b$. Consequently, any limit model \mathcal{M} over p_∞ is representable as the union of an elementary chain $(\mathcal{M}_{a_n})_{n \in \omega}$ over p_∞ , equal to an elementary chain $(\mathcal{M}_{a'_n})_{n \in \omega}$, where $\models Q(a_{n+1}, a'_n)$, $\models p_\infty(a'_n)$. So by construction, any two models limit over p_∞ will be isomorphic. Thus the condition (CEP) will hold, which, by Corollary 1.1.15 and in view of $|\text{RK}(T)| = 2$, will imply $I(T, \omega) = 3$.

Prior to constructing T , we look at the theory T_a , described under items (4) and (5), of the language $\Sigma = \text{Col} \cup \{R_q \mid q \in \mathcal{Q}\} \cup \{R'_q \mid q \in \mathcal{Q}\}$, where \mathcal{Q} is some countable index set. This theory is a subtheory of T , and we call it a *free acyclic theory with fixed-color connected components*, or, briefly, an *facc-theory*.

Let \mathcal{M} be a model of T_a and A be a finite set in \mathcal{M} . A c_a -graph is a structure \mathcal{A} of the language Σ consisting of the universe A colored by the function Col , relations R_q and R'_q on A , and a finite record W saying of the existence, lengths and tuples of names for arcs of shortest routes, pairwise connecting the elements of A . By analogy with c -graphs, the c_a -graphs \mathcal{A} with corresponding universes A are denoted by $\langle A, R_q, R'_q, W \rangle_{q \in \mathcal{Q}}$.

For a c_a -graph $\mathcal{A} = \langle A, R_q, R'_q, W \rangle_{q \in \mathcal{Q}}$, $\text{cc}(\mathcal{A})$ denotes the minimal graph $\Gamma \supseteq \langle A, R_q, R'_q \rangle_{q \in \mathcal{Q}}$ with marked arcs containing, for every pair $(a, b) \in A^2$, coupled via a route in accordance with the entry in W , a shortest (a, b) -route with the tuple of names of arcs specified in W .

Define a relation \subseteq_{c_a} on the class of c_a -graphs. We call the c_a -graph $\mathcal{A} = \langle A, R_{q,\mathcal{A}}, R'_{q,\mathcal{A}}, W_{\mathcal{A}} \rangle_{q \in \mathcal{Q}}$ a c_a -subgraph of the c_a -graph $\mathcal{B} = \langle B, R_{q,\mathcal{B}}, R'_{q,\mathcal{B}}, W_{\mathcal{B}} \rangle_{q \in \mathcal{Q}}$, and write $\mathcal{A} \subseteq_{c_a} \mathcal{B}$ if $A \subseteq B$, $R_{q,\mathcal{A}} = R_{q,\mathcal{B}} \cap A^2$, and $W_{\mathcal{A}}$ is the record saying of shortest routes linking the elements of A in the graph $\text{cc}(\mathcal{B})$.

We say that the c_a -graph $\mathcal{A} = \langle A, R_q, R'_q, W \rangle_{q \in \mathcal{Q}}$ is *closed* if \mathcal{A} contains all routes specified in the record W , that is, $\mathcal{A} = \text{cc}(\mathcal{A})$.

If \mathcal{A} , $\mathcal{B} = \langle B, R_{q,\mathcal{B}}, R'_{q,\mathcal{B}}, W_{\mathcal{B}} \rangle_{q \in \mathcal{Q}}$, and $\mathcal{C} = \langle C, R_{q,\mathcal{C}}, R'_{q,\mathcal{C}}, W_{\mathcal{C}} \rangle_{q \in \mathcal{Q}}$ are closed c_a -graphs, and $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$, then a *free c_a -amalgam* of the c_a -graphs \mathcal{B} and \mathcal{C} over \mathcal{A} (denoted by $\mathcal{B} *_{\mathcal{A}} \mathcal{C}$) is the c_a -graph $\langle B \cup C, R_{q,\mathcal{B}} \cup R_{q,\mathcal{C}}, R'_{q,\mathcal{B}} \cup R'_{q,\mathcal{C}}, W_{\mathcal{B}} \cup W_{\mathcal{C}} \rangle_{q \in \mathcal{Q}}$.

A one-to-one map $f : A \rightarrow B$ is called *c_a -embedding* of the c_a -graph $\mathcal{A} = \langle A, R_{q,\mathcal{A}}, R'_{q,\mathcal{A}}, W_{\mathcal{A}} \rangle_{q \in \mathcal{Q}}$ into the c_a -graph $\mathcal{B} = \langle B, R_{q,\mathcal{B}}, R'_{q,\mathcal{B}}, W_{\mathcal{B}} \rangle_{q \in \mathcal{Q}}$ (written $f : \mathcal{A} \rightarrow_{c_a} \mathcal{B}$) if f is an embedding of the colored graph $\mathcal{A} = \langle A, R_{q,\mathcal{A}}, R'_{q,\mathcal{A}} \rangle_{q \in \mathcal{Q}}$ into the colored graph $\mathcal{B} = \langle B, R_{q,\mathcal{B}}, R'_{q,\mathcal{B}} \rangle_{q \in \mathcal{Q}}$ such that the record $W_{f(A)}$ of a c_a -subgraph of the c_a -graph \mathcal{B} with universe $f(A)$ coincides with a record obtained from $W_{\mathcal{A}}$ by replacing all elements $a \in A$ by $f(a)$.

Two c_a -graphs, \mathcal{A} and \mathcal{B} , are said to be *c_a -isomorphic* if there exists a c_a -embedding $f : \mathcal{A} \rightarrow_{c_a} \mathcal{B}$ with $f(A) = B$. The map f , in this instance, is called a *c_a -isomorphism* between \mathcal{A} and \mathcal{B} , and the c_a -graphs \mathcal{A} and \mathcal{B} are called *c_a -isomorphic copies*.

Lemma 3.3.1. (*c_a -amalgamation lemma*). *The class of all closed c_a -graphs has the c_a -amalgamation property (c_a -AP), that is, for any c_a -embeddings $f_0 : \mathcal{A} \rightarrow_{c_a} \mathcal{B}$ and $g_0 : \mathcal{A} \rightarrow_{c_a} \mathcal{C}$, where \mathcal{A} , \mathcal{B} , and \mathcal{C} are closed c_a -graphs, there are a closed c_a -graph \mathcal{D} and c_a -embeddings $f_1 : \mathcal{B} \rightarrow_{c_a} \mathcal{D}$ and $g_1 : \mathcal{C} \rightarrow_{c_a} \mathcal{D}$ such that $f_0 \circ f_1 = g_0 \circ g_1$.*

P r o o f. There is no loss of generality in assuming that $\mathcal{A} \subseteq_{c_a} \mathcal{B}$ and $\mathcal{A} \subseteq_{c_a} \mathcal{C}$. As \mathcal{D} we can take a closed c_a -graph $\mathcal{B} *_{\mathcal{A}} \mathcal{C}$. \square

Clearly, a saturated model of the facc-theory is representable as a generic model which is constructed from all possible closed c_a -graphs using the c_a -amalgamation lemma.

The next proposition gives a list of basic properties for facc-theories.

Proposition 3.3.2. *For any facc-theory T_a , the following statements hold:*

- (1) *a countable saturated model \mathcal{M} of T_a consists of countably many connected components for every color $\alpha \in \omega \cup \{\infty\}$;*
- (2) *if \mathcal{A} and \mathcal{B} are c_a -isomorphic c_a -subgraphs of the countable saturated model \mathcal{M} of T_a , then $\text{tp}_{\mathcal{M}}(\mathcal{A}) = \text{tp}_{\mathcal{M}}(\mathcal{B})$;*
- (3) *a type $\text{tp}_{\mathcal{M}}(\mathcal{A})$ of T_a is principal iff all elements of \mathcal{A} are finite in color, and all one-color elements belong to one connected component; moreover, the prime model \mathcal{M}_0 consists of elements of finite colors each forming one connected component for every finite color;*

(4) a type $\text{tp}_{\mathcal{M}}(A)$ of T_a is realized in \mathcal{M}_{p_∞} iff the subtype of all elements of finite colors in A is principal and the elements of infinite colors in A belong to one connected component.

P r o o f is obvious. \square

The construction of a theory T expanding both T_2 and some facc-theory proceeds by steps, similarly to how was T_2 .

We define the notion of a c_3 -graph subject to conditions given below, assuming that the language of every c_3 -graph consists of unary predicate symbols for the coloring Col , the binary predicate symbol Q , and a countable well-ordered set of binary predicate symbols R_q and R'_q , for some c_3 -isomorphism types corresponding to types q .

I. If $\mathcal{A} = \langle A, Q, W \rangle$ is a c -graph in the class \mathbf{K}_0^* , then the structure $\langle A, Q, R_q, R'_q, W' \rangle_{q \in \mathcal{Q}}$, obtained from \mathcal{A} by adding the empty relations R_q and R'_q and the entries recording lengths of all shortest Q -routes, linking the elements of A , is a c_3 -graph.

II. Every c_a -graph, to which the empty relation Q is added, is a c_3 -graph.

III. Let $\Gamma_\delta = \langle A_\delta, Q, R_{q,\delta}, R'_{q,\delta}, W_\delta \rangle_{q \in \mathcal{Q}}$, $\delta \in \{0, 1\}$, be c_3 -graphs with universes $A_\delta = \{a_0, a'_0, a_1, a'_1, \dots, a_n, a'_n, a''_n, b_\delta\}$, a_i and a'_i be one-color elements linked via a (sole) shortest route in an undigraph with R^* , $i = 0, 1, \dots, n$, and a_n and a''_n also be one-color elements linked via a (sole) shortest route in an undigraph with R^* , $\models Q(a_{i+1}, a'_i)$, $i = 0, \dots, n-1$, $\models Q(b_0, a'_n) \wedge Q(b_1, a''_n)$. Assume that b_0 and b_1 are one-color elements which are not connected by (b_0, b_1) - or (b_1, b_0) -routes in a graph with the relation represented as the union of Q and the (finitely many) relations $R_q \cup R'_q$, involved in the shortest routes, connecting the elements of $A_0 \cup A_1$. Then the structure

$$\Gamma = \langle A_0 \cup A_1, Q, R_q, R'_q, W_0 \cup W_1 \cup W_r(b_0, b_1) \rangle_{q \in \mathcal{Q}},$$

is a c_3 -graph, which is called a *corealization amalgam* of Γ_0 and Γ_1 , where the relations Q and R_q , defined on the c_3 -graphs Γ_0 and Γ_1 , are joined, and to the empty relations R_r and R'_r (where r is a type describing quantifier-free links between elements of $A_0 \cup A_1$, and also the interrelations between the elements of $A_0 \cup A_1$ according to the record $W_0 \cup W_1$) we add the pairs (b_0, b_1) and (b_1, b_0) respectively; $W_r(b_0, b_1)$ is a record saying of the existence of a set $A_0 \cup A_1$ extending to Γ the graph $\langle \{b_0, b_1\}, Q, R_q, R'_q \rangle_{q \in \mathcal{Q}}$ with the empty relations Q , R_q , and R'_q except $R_r = \{(b_0, b_1)\}$ and $R'_r = \{(b_1, b_0)\}$.

IV. We define an irreflexive relation $<$ on the set \mathcal{Q} as follows: $q_1 < q_2 \Leftrightarrow$ iff $W_{q_2}(b_0, b_1)$ mentions R_{q_1} . Obviously, in defining R_q and R'_q by steps, the transitive closure of $<$, in view of item III, can be extended to a well-order relation \leq on \mathcal{Q} , consisting of all types r described in III. Below, we assume that the set $\Sigma_2 = \{Q\} \cup \{R_q \mid q \in \mathcal{Q}\} \cup \{R'_q \mid q \in \mathcal{Q}\}$ of binary signature symbols is well ordered, where Q is the least element, R_{q_1} does not exceed R_{q_2} if $q_1 \leq q_2$, and R'_{q_1} is sandwiched between R_{q_1} and its immediate successor R_{q_2} .⁶

V. If $\Gamma = \langle A, Q, R_q, R'_q, W \rangle_{q \in \mathcal{Q}}$ is a c_3 -graph then, for any set $A_0 \subseteq A$, a c_3 -graph (which is a c_3 -subgraph of the c_3 -graph Γ) is a structure like

$$\langle A_0, Q \cap A_0^2, R_q \cap A_0^2, R'_q \cap A_0^2, W_0 \rangle_{q \in \mathcal{Q}},$$

where W_0 consists of the following (induced by the c_3 -graph):

(a) records $W_q(b_0, b_1)$ for all pairs (b_0, b_1) belonging to relations $R_q \cap A_0^2$;

(b) entries describing lengths of shortest routes in the c -subgraph of a c -graph $\langle A, Q, W_c \rangle$ having universe A_0 , where W_c is an entry recording shortest routes w.r.t. Q in the c_3 -graph Γ ;

(c) entries describing lengths and tuples of names for arcs of shortest routes in the c_a -subgraph of a c_a -graph $\langle A, R_q, R'_q, W_{c_a} \rangle_{q \in \mathcal{Q}}$ having universe A_0 , where W_{c_a} is an entry recording shortest lengths and tuples of names for arcs of shortest routes w.r.t. relations $R_q, R'_q, q \in \mathcal{Q}$, in the c_3 -graph Γ ;

(d) entries recording the existence and lengths n of shortest (a, b) -routes (if any) via intermediate elements c , for which there are (a, c) -routes w.r.t. Q , and $(c, b) \in R_0$, where R_0 is the least signature symbol in \mathcal{Q} ; moreover, the formula $\theta_n(x, y)$, describing the shortest (a, b) -routes via intermediate elements c , is equivalent to every formula describing the shortest (a, b) -routes via intermediate elements c' , where the (a, c') -routes w.r.t. Q are of the same length n , while c' and b are linked in the c_a -graph by a given set of names of arcs.

We require that the entries specified in (a)–(d) cancel out for any fixed vertex pair (a, b) , and that the record W itself, too, consists of

⁶Since the symbols of Σ_2 are well ordered, every record $W_r(b_0, b_1)$ in the construction of a generic model allows the c_3 -graph having universe $\{b_0, b_1\}$ and the same record to be extended to a finite c_3 -graph in which all records like $W_q(a, b)$ are realized.

mutually exclusive entries for the pairs $(a, b) \in A^2$ described in (a)–(d). Furthermore, we insist on some formula $\theta_n(a, b)$ being involved in the record W for any pair (a, b) coupled by some route in a graph with $Q \cup R^*$ but not by any routes in a graph with Q , or with R^* .

VI. Let $\mathcal{A}, \mathcal{B} = \langle B, Q_{\mathcal{B}}, R_{q,\mathcal{B}}, R'_{q,\mathcal{B}}, W_{\mathcal{B}} \rangle_{q \in \mathcal{Q}}$ and $\mathcal{C} = \langle C, Q_{\mathcal{C}}, R_{q,\mathcal{C}}, R'_{q,\mathcal{C}}, W_{\mathcal{C}} \rangle_{q \in \mathcal{Q}}$ be *closed* c_3 -graphs, that is, those which, together with any two vertices belonging to one connected component w.r.t. R^* , contain a shortest R^* -route, and $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$. Then the *free* c_3 -*amalgam* $\langle B \cup C, Q_{\mathcal{B}} \cup Q_{\mathcal{C}}, R_{q,\mathcal{B}} \cup R_{q,\mathcal{C}}, W_{\mathcal{B}} \cup W_{\mathcal{C}} \cup W \rangle$ of the c_3 -graphs \mathcal{B} and \mathcal{C} over \mathcal{A} (denoted by $\mathcal{B} *_{\mathcal{A}} \mathcal{C}$) is a c_3 -graph, where W is a record including all formulas $\theta_n(a, b)$ for vertices $a, b \in B \cup C$ which both do not belong to B and C together and are connected by a shortest route in which the least number $n \geq 1$ of Q -arcs and not less than one R^* -arc are involved.

VII. Let $\mathcal{A} = \langle A, Q, R_q, R'_q, W \rangle_{q \in \mathcal{Q}}$ be a c_3 -graph, (a, b) be a pair of vertices for which some formula $\theta_n(a, b)$ belongs to the record W , and $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$ be a tuple of signature symbols in Σ_2 in which Q occurs at least n times, and the symbols of \mathcal{Q} occur at least once. Then the structure, which is obtained by adding to \mathcal{A} an external (a, b) -route whose tuple of arcs has the same set of names as has $\bar{\alpha}$ and the degrees of the new vertices are equal to two, is a c_3 -graph. The operation of adding the above-mentioned external routes is called a c_3 -*tracing*.

Denote by $\mathbf{\Delta}_0$ the class consisting of all c_3 -graphs corresponding to some c - or c_a -graph in accordance with items I and II, and including all possible c_3 -graphs of the forms:

$\Gamma_{0,q} = \langle \{0, 1, 2\}, \{(0, 1), (0, 2)\}, R_q, R'_q, W \rangle_{q \in \mathcal{Q}}$, $\text{Col}(0) \leq \text{Col}(1)$, $\text{Col}(1) = \text{Col}(2)$, $\Gamma_{1,q} = \langle \{0, 1, 2\}, \{(1, 0), (2, 0)\}, R_q, R'_q, W \rangle_{q \in \mathcal{Q}}$, $\text{Col}(0) \geq \text{Col}(1)$, $\text{Col}(1) = \text{Col}(2)$, where $R_q = \{(1, 2)\}$ if $R_q \cap R'_q = \emptyset$, and $R_q = \{(1, 2), (2, 1)\}$ if $R_q = R'_q$, $W = W_q(1, 2)$;

$\Gamma_{0,q_1,\delta_1,\dots,q_n,\delta_n} = \langle \{0, 1, 2\}, \{(0, 1), (0, 2)\}, R_q, R'_q, W \rangle_{q \in \mathcal{Q}}$, $\text{Col}(0) \leq \text{Col}(1)$, $\text{Col}(1) = \text{Col}(2)$, $\Gamma_{1,q_1,\delta_1,\dots,q_n,\delta_n} = \langle \{0, 1, 2\}, \{(1, 0), (2, 0)\}, R_q, R'_q, W \rangle_{q \in \mathcal{Q}}$, $n \geq 2$, where the relations R_q and R'_q are empty and W contains information on there being a shortest R^* -route, connecting 1 and 2 via a sequence of arcs, having a tuple of names $((R_{q_1})^{\delta_1}, \dots, (R_{q_n})^{\delta_n})$, $\delta_1, \dots, \delta_n \in \{-1, 1\}$;

$\Gamma_{0,n} = \langle \{0, 1, 2\}, \{(0, 1), (0, 2)\}, R_q, R'_q, W \rangle_{q \in \mathcal{Q}}$, $\text{Col}(0) \leq \text{Col}(1) \leq \text{Col}(2)$, $\Gamma_{1,n} = \langle \{0, 1, 2\}, \{(1, 0), (2, 0)\}, R_q, R'_q, W \rangle_{q \in \mathcal{Q}}$, $\text{Col}(1) \leq \text{Col}(2) \leq \text{Col}(0)$, $n \geq 2$, where R_q and R'_q are empty and W contains a formula $\theta_n(1, 2)$.

Similarly to the concepts of a c_1 -embedding and of a c_2 -embedding, the notion of a c_3 -embedding $f : \mathcal{A} \rightarrow_{c_3} \mathcal{B}$ for c_3 -graphs \mathcal{A} and \mathcal{B} under which an appropriate record $W_{\mathcal{A}}$ ($W_{f(\mathcal{A})} = W_{\mathcal{B}} \upharpoonright f(\mathcal{A})$) is preserved generalizes the notions of a c -embedding and of a c_a -embedding.

Two c_3 -graphs, \mathcal{A} and \mathcal{B} , are said to be c_3 -isomorphic if there exists a c_3 -embedding $f : \mathcal{A} \rightarrow_{c_3} \mathcal{B}$ with $f(\mathcal{A}) = \mathcal{B}$. The map f , in this event, is called a c_3 -isomorphism between \mathcal{A} and \mathcal{B} , and the c_3 -graphs \mathcal{A} and \mathcal{B} are called c_3 -isomorphic copies.

VIII. Any c_3 -graph is a finite structure and is obtained from the c_3 -graphs belonging to the class $\mathbf{\Delta}_0$ by applying some finitely many operations of taking admissible recolorings, corealization amalgams, c_3 -subgraphs, free c_3 -amalgams, c_3 -tracings, and c_3 -isomorphic copies in accordance with items III–VII, an operation which, for any c_3 -graph $\mathcal{A} = \langle A, Q, R_q, R'_q, W \rangle_{q \in Q}$ and any pair of its vertices $(a, b) \in A^2$, $\text{Col}(a) \leq \text{Col}(b)$, not connected via Q - or R^* -routes in the minimal graph including all the routes described in W , allows W to be added one of arbitrarily chosen records saying of the existence of a shortest (a, b) -route of length greater than the maximal of lengths of shortest sequences w.r.t. Q , and the inverse operation which, for any pair of elements, allows the record to be cleared of positive information on external shortest Q -routes and information on the elements belonging to one $R^* \cup \text{id}$ -equivalence class, or on their connectivity by formulas θ_n .

Denote by \mathbf{K}_3^* the class of all c_3 -graphs, and by \mathbf{K}_3 the class of all colored graphs whose every finite subgraph forms a c_3 -graph in \mathbf{K}_3^* .

A one-to-one map $f : A \rightarrow N$ is a c_3 -embedding of the c_3 -graph \mathcal{A} into the graph $\mathcal{N} \in \mathbf{K}_3$ (written $f : \mathcal{A} \rightarrow_{c_3} \mathcal{N}$) if f is a c_3 -embedding of the c_3 -graph \mathcal{A} into a c_3 -subgraph $f(\mathcal{A})$ of \mathcal{N} with universe $f(A)$.

Combining the proofs of Lemmas 3.1.1 and 3.3.1 we arrive at the following:

Lemma 3.3.3. (c_3 -amalgamation lemma). *The class \mathbf{K}_3^* has the c_3 -amalgamation property (c_3 -AP), that is, for any c_3 -embeddings $f_0 : \mathcal{A} \rightarrow_{c_3} \mathcal{B}$ and $g_0 : \mathcal{A} \rightarrow_{c_3} \mathcal{C}$, where \mathcal{A} , \mathcal{B} , and \mathcal{C} are closed c_3 -graphs in \mathbf{K}_3^* , there exist a closed c_3 -graph $\mathcal{D} \in \mathbf{K}_3^*$ and c_3 -embeddings $f_1 : \mathcal{B} \rightarrow_{c_3} \mathcal{D}$ and $g_1 : \mathcal{C} \rightarrow_{c_3} \mathcal{D}$ such that $f_0 \circ f_1 = g_0 \circ g_1$.*

Theorem 3.3.4. *There exists a countable, colored, saturated graph $\mathcal{M} \in \mathbf{K}_3$, satisfying the following:*

- (1) *if $f : \mathcal{A} \rightarrow_{c_3} \mathcal{M}$ and $g : \mathcal{A} \rightarrow_{c_3} \mathcal{B}$ are c_3 -embeddings, and $\mathcal{B} \in \mathbf{K}_3^*$, then there exists a c_3 -embedding $h : \mathcal{B} \rightarrow_{c_3} \mathcal{M}$ such that $f = g \circ h$;*
- (2) *if \mathcal{A} and \mathcal{B} are c_3 -isomorphic closed c_3 -subgraphs of the graph \mathcal{M} , then $\text{tp}_{\mathcal{M}}(\mathcal{A}) = \text{tp}_{\mathcal{M}}(\mathcal{B})$;*
- (3) *the coloring of the restriction $\mathcal{M} \upharpoonright Q$ of the model \mathcal{M} to the graph language $\Sigma = \{Q\}$ is inessential and Q -ordered;*
- (4) *$Q(x, y)$ is a principal formula in $\text{Th}(\mathcal{M} \upharpoonright \Sigma_2)$, and the formulas $R_q(a, y)$ and $R'_q(a, y)$ are principal for any element $a \in M$.*

P r o o f is similar to the proof of Theorem 3.1.3, with the c -graphs replaced by closed c_3 -graphs and Lemma 3.3.3 applied to these. To the descriptions of the formulas $\varphi_n(X)$ and $\psi_n(X, Y)$, in this instance, for every vertex pair (a, b) , we add the following:

- (a) positive information on tuples of names of shortest R^* -routes, if such (a, b) -routes exist;
- (b) negations of R^* -routes of length at most n connecting a and b and consisting of arcs whose names belong to the initial segment of length n in the well-ordered set \mathcal{Q} , if a and b are not linked by the R^* -routes;
- (c) formulas θ_m in appropriate variables for all elements satisfying these formulas;
- (d) formulas $\neg\theta_1 \wedge \dots \wedge \neg\theta_n$ in appropriate variables for all elements not satisfying any one of θ_m .

The description of c_3 -graphs implies that the coloring Col is inessential and Q - Q -ordered, and the property of being closed for c_3 -graphs consisting of one of Q - or R^* -arc implies being isolated for a formula $Q(x, y)$ in $\text{Th}(\mathcal{M} \upharpoonright \Sigma_2)$, and for formulas $R_q(a, y)$ and $R'_q(a, y)$ for any element $a \in M$. \square

The theory $T_3 \equiv \text{Th}(\mathcal{M})$ of \mathcal{M} , constructed for proving Theorem 3.3.4, and the countable saturated model \mathcal{M} are said to be \mathbf{K}_3^* -generic.

Similarly to Theorem 3.1.7, we prove the following:

Theorem 3.3.5. (1) *A type q of T_3 is principal iff q is extended to a type r of T_3 realized by closed c_3 -graphs every two distinct elements of which are connected by some Q - or R^* -route, and all elements of realizations of r are finite in color.*

(2) A type q of T_3 is realized in the model \mathcal{M}_{p_∞} iff q is extended to a type r of T_3 realized by closed c_3 -graphs \mathcal{A} in which any two distinct elements a_i and a_j are connected by some Q - or R^* -route, and the following condition is met: if among elements of the set A are elements of finite colors and elements of infinite colors then:

(a) there exist elements $a_{f,1}, \dots, a_{f,k}$ of finite colors which are connected by R^* -routes and are such that for every element $a \in A \setminus \{a_{f,1}, \dots, a_{f,k}\}$ of finite color and every $i \in \{1, \dots, k\}$, there is a Q -route leading from a to $a_{f,i}$;

(b) there exist elements $a_{\infty,1}, \dots, a_{\infty,l}$ of infinite color which are connected by R^* -routes and are such that for every element $a \in A \setminus \{a_{\infty,1}, \dots, a_{\infty,l}\}$ of infinite color and every $j \in \{1, \dots, l\}$, there is a Q -route leading from $a_{\infty,j}$ to a ;

(c) for any $i \in \{1, \dots, k\}$ and any $j \in \{1, \dots, l\}$, there exist Q -routes leading from $a_{f,i}$ to $a_{\infty,j}$.

We define the class \mathbf{K}_4^* of finite structures equipped with finite records saying of interrelations of the elements satisfying the conditions (1') and (2') specified in the previous Section, where instead of c_1 -isomorphism types \mathbf{A} , we treat all possible c_3 -isomorphism types which are endowed with formulas ρ and are such that all prime models over these types are not isomorphic either to a prime model or to \mathcal{M}_{p_∞} . We assume that every finite set A , which belongs to \mathbf{K}_4^* and is restricted to the graph language Σ_2 with coloring Col , forms a c_3 -graph $\langle A, Q, R_q, R'_q, W \rangle_{q \in \mathcal{Q}}$ in \mathbf{K}_3^* , and W is added positive information on the interrelations of the elements w.r.t. projections $\exists y_1 \dots y_l R_{\mathbf{A}_m}(x, \bar{y})$, and w.r.t. formulas ρ in accordance with item (2').

Finite structures \mathcal{A} with finite records $W_{\mathcal{A}}$ forming \mathbf{K}_4^* , are called c_4 -structures, and the c_4 - c_4 -structures which are closed c_3 -graphs are also said to be *closed*.

Denote by \mathbf{K}_4 the class of all models in the language $\text{Col} \cup \Sigma_2 \cup \{R_{\mathbf{A}_m} \mid m \in \omega\}$ whose every finite subset forms a c_4 -structure in \mathbf{K}_4^* .

The notion of a c_4 -embedding $f : \mathcal{A} \rightarrow_{c_4} \mathcal{B}$ for c_4 -structures \mathcal{A} and \mathcal{B} , under which an appropriate record $W_{\mathcal{A}}$ ($W_{f(\mathcal{A})} = W_{\mathcal{B}} \upharpoonright f(\mathcal{A})$) is preserved, is a natural generalization of the concepts of c_i -embeddings introduced above. Thereby we also define the concept of a c_4 -embedding $f : \mathcal{A} \rightarrow_{c_4} \mathcal{N}$ of a c_4 -structure \mathcal{A} into a model \mathcal{N} in \mathbf{K}_4 .

Two c_4 -structures, \mathcal{A} and \mathcal{B} , are c_4 -isomorphic if there is a c_4 -embedding $f : \mathcal{A} \rightarrow_{c_4} \mathcal{B}$ with $f(\mathcal{A}) = \mathcal{B}$.

Theorem 3.3.6. *There exists a countable saturated model $\mathcal{M} \in \mathbf{K}_4$ satisfying the following:*

(a) *if $f : \mathcal{A} \rightarrow_{c_4} \mathcal{M}$ and $g : \mathcal{A} \rightarrow_{c_4} \mathcal{B}$ are c_4 -embeddings, and $\mathcal{B} \in \mathbf{K}_4^*$, then there is a c_4 -embedding $h : \mathcal{B} \rightarrow_{c_4} \mathcal{M}$ such that $f = g \circ h$;*

(b) *if \mathcal{A} and \mathcal{B} are c_4 -isomorphic closed c_4 -structures in \mathcal{M} , then $\text{tp}_{\mathcal{M}}(A) = \text{tp}_{\mathcal{M}}(B)$;*

(c) *the restriction of \mathcal{M} to the language $\text{Col} \cup \Sigma_2$ is a \mathbf{K}_3^* -generic model;*

(d) *every formula $R_{\mathbf{A}}(a, \bar{y})$, where $\models p_{\infty}(a)$, is principal, and the c_4 -isomorphism type of any realization of the formula $R_{\mathbf{A}}(a, \bar{y})$ coincides with a c_4 -isomorphism type \mathbf{A} ;*

(e) *every formula $R_{\mathbf{A}}(x, \bar{a})$, where \mathbf{A} is the c_4 -isomorphism type of a tuple \bar{a} , is principal, and any realization of the formula $R_{\mathbf{A}}(x, \bar{a})$ is a realization of the type p_{∞} .*

P r o o f is an obvious combination of the proofs of Theorems 3.2.3 and 3.3.4. \square

The theory $T_4 \equiv \text{Th}(\mathcal{M})$ of \mathcal{M} , which is constructed for proving Theorem 3.3.7, and the countable saturated model \mathcal{M} are said to be \mathbf{K}_4^* -generic.

Following the argument of Theorem 3.2.4 and using Theorem 3.3.6, we state the following:

Theorem 3.3.7. *The theory T_4 satisfies the condition $|\text{RK}(T_4)| = 2$.*

Theorem 3.3.8. *There is a limit model of the theory T_4 over the type p_{∞} which is unique up to isomorphism.*

P r o o f. The existence of a limit model follows from Proposition 1.1.8 and Corollary 1.1.9, in view of the semi-isolation relation $\text{SI}_{p_{\infty}}$ being nonsymmetric w.r.t. $Q(x, y)$. In order to prove that our limit model is unique, it suffices to show that any limit model \mathcal{M} over p_{∞} is saturated.

Let $(\mathcal{M}_{a_n})_{n \in \omega}$ be any elementary chain over p_{∞} whose union coincides with the limit model \mathcal{M} . By Theorem 3.3.5, for every elementary extension of the model \mathcal{M}_{a_n} to the model $\mathcal{M}_{a_{n+1}}$, there exists an R^* - or a Q -route leading from a_{n+1} to a_n . Since the limit model is not isomorphic to $\mathcal{M}_{p_{\infty}}$, and the model \mathcal{M}_{a_n} is elementarily extended to a prime model over a_n which coincides with $\mathcal{M}_{a_{n+1}}$ provided the R^* -route from a_{n+1} to a_n exists, from $(\mathcal{M}_{a_n})_{n \in \omega}$ we can remove all models \mathcal{M}_{a_n} for which R^* -routes from a_n to a_{n-1} are available.

With all the above-specified models removed, appealing to the construction of a generic model, we can extend the resulting elementary chain $(\mathcal{M}_{a'_n})_{n \in \omega}$ by models \mathcal{M}_a , where a are elements forming, for any $n \in \omega$, one of the shortest Q -routes from a'_{n+1} to a'_n , if such Q -routes exist. Furthermore, the resulting elementary chain is extended to an elementary chain such that every model on one side is neighbored by its coincident model, and on the other side — by a proper elementary substructure or superstructure.

Now, we consider an arbitrary 1-type $q(x, \bar{b}) \in S(\bar{b})$, where \bar{b} is a tuple in M , and argue to show that $q(x, \bar{b})$ is realized in \mathcal{M} . Indeed, the tuple \bar{b} belongs to some model \mathcal{M}_{a_n} , and, by definition of $R_{\mathbf{A}}$, for some c_4 -isomorphism type \mathbf{A} , containing realizations of the type $q(x, \bar{y})$, and for some $n' \geq n$, there exists an element $c \in M_{a_{n'}}$ such that some projection $\exists z_{i_1}, \dots, z_{i_m} R_{\mathbf{A}}(c, \bar{z})$ is realized by \bar{b} . Since the formula $R_{\mathbf{A}}(c, \bar{z})$ is principal, there exists a tuple $\bar{d} \in M$, realizing that formula and extending \bar{b} . By the choice of the c_4 -isomorphism type \mathbf{A} , some coordinate of \bar{d} realizes the type $q(x, \bar{b})$. Since the chosen type q is arbitrary, \mathcal{M} is saturated. \square

Appealing to Corollary 1.1.15 and Theorems 3.3.7 and 3.3.8, we conclude that the following theorem holds.

Theorem 3.3.9. *There exists a complete generic theory T expanding the generic theory T_3 and with $I(T, \omega) = 3$.*

§ 3.4. Realizations of basic characteristics of complete theories with finitely many countable models

Recall that Theorem 1.1.13 contains a characterization of the class of complete theories with finitely many countable models. We claim that all situations described in that theorem are realized out.

Theorem 3.4.1. *For any finite preordered set $\langle X; \leq \rangle$ with the least element x_0 and the greatest class \widetilde{x}_1 in the ordered factor set $\langle X; \leq \rangle / \sim$ w.r.t. \sim (where $x \sim y \Leftrightarrow x \leq y$ and $y \leq x$), and for any function $f : X / \sim \rightarrow \omega$ satisfying the conditions $f(\widetilde{x}_0) = 0$, $f(\widetilde{x}_1) > 0$ for $|X| > 1$, and $f(\widetilde{y}) > 0$ for $|\widetilde{y}| > 1$, there exist a complete theory T and an isomorphism $g : \langle X; \leq \rangle \xrightarrow{\sim} \text{RK}(T)$ such that $\text{IL}(g(\widetilde{y})) = f(\widetilde{y})$ for any $\widetilde{y} \in X / \sim$.*

P r o o f. Let $\langle X; \leq \rangle$ and f be as above. Without loss of generality, we may assume that $|X| > 1$. We fix a numbering $\nu : |X| \rightarrow X$ such that $\nu(m) < \nu(n)$ and $\nu(m) \not\sim \nu(n)$ imply $m < n$, and in correspondence with every \sim -class is an interval in $|X|$. Consider a theory T_0 of unary predicates $P_1, \dots, P_{|X|-1}$ forming a partition on $|X| - 1$ infinite classes with inessential coloring $\text{Col} : M \rightarrow \omega \cup \{\infty\}$ such that $\vdash \exists^{\geq \omega} (P_i(x) \wedge \text{Col}_n(x))$, $i = 1, \dots, |X| - 1$, $n \in \omega$. We claim that there exists an expansion T of T_0 with an isomorphism $g : \langle X; \leq \rangle \cong \text{RK}(T)$ such that:

- (a) $g(\nu(i)) = \mathbf{M}_{p_i}$, where \mathbf{M}_{p_i} is an isomorphism type of the prime model \mathcal{M}_{p_i} over a realization of the type $p_i(x)$ in $S^1(\emptyset)$ for $p_i(x)$ being isolated by the set $\{P_i(x) \wedge \neg \text{Col}_n(x) \mid n \in \omega\}$ of formulas, $i = 1, \dots, |X| - 1$, and $p_1(x), \dots, p_{|X|-1}(x)$ are all non-principal 1-types over \emptyset in the variable x ;
- (b) $\text{IL}(g(\tilde{y})) = f(\tilde{y})$ for any $\tilde{y} \in X/\sim$.

We construct $T = \bigcup_{i < |X|} T_i$ by induction so as to respect the numbering ν . Assume T_0, \dots, T_{k-1} are already constructed and $\nu(k), \nu(k+1), \dots, \nu(k+l)$ form a \sim -class.

If $f(\nu(k)) = 0$, then $l = 0$, and we define T_k by expanding the language of T_{k-1} by new binary predicate symbols R_{ki} (where the class $\nu(k)$ covers $\nu(i)$, $i \neq 0$) so as to satisfy the following conditions:

- (1) $R_{ki}(a, y)$ is a principal formula and $R_{ki}(a, y) \vdash p_i(y)$ for any $a \models p_k$;
- (2) for any $a, b \models p_i$, there are infinitely many elements $c \models p_k$ and infinitely many d not realizing types $p_1(x), \dots, p_{|X|-1}$ such that

$$\models R_{ki}(c, a) \wedge R_{ki}(c, b) \wedge R_{ki}(d, a) \wedge R_{ki}(d, b);$$

moreover, $c \models p_k$ and $\models R_{ki}(c, a)$ imply that a does not semi-isolate c .

Clearly, (1) and (2) can be realized so that the model \mathcal{M}_{p_k} of T_k has a unique realization of the type p_k and hence $\text{IL}(g(\nu(k))) = 0 = f(\nu(k))$. In addition, \mathcal{M}_{p_k} , by induction, will realize all types p_i dominated by p_k , that is, satisfying the relation $\nu(i) \leq \nu(k)$.

Suppose $f(\nu(k)) = r > 0$. We define T_k^0 by expanding the language of T_{k-1} by new binary predicate symbols R_{ki} (where $\nu(k)$ covers $\nu(i)$, $i \neq 0$) satisfying (1) and (2), and by binary predicate symbols R'_{ij} (where $\nu(i), \nu(j) \in \nu(k)$) with the following conditions:

(3) $R'_{ij}(a, y)$ is a principal formula and $R'_{ij}(a, y) \vdash p_j(y)$ for any $a \models p_i$;

(4) for any $a, b \models p_j$, there are infinitely many elements $c \models p_i$ and infinitely many d not realizing types $p_1(x), \dots, p_{|X|-1}$, with

$$\models R'_{ij}(c, a) \wedge R'_{ij}(c, b) \wedge R'_{ij}(d, a) \wedge R'_{ij}(d, b);$$

moreover, $c \models p_i$ and $\models R'_{ij}(c, a)$ imply that a does not semi-isolate c ;

(5) for any elements a and b not realizing types $p_{k+l+1}, \dots, p_{|X|-1}$, there exist infinitely many $c \models p_j$ such that $\models R'_{ij}(a, c) \wedge R'_{ij}(b, c)$;

(6) the relation $R'_k = \bigcup_{\nu(i), \nu(j) \in \widetilde{\nu(k)}} R'_{ij}$ forms a digraph isomorphic

to the digraph Γ_{gen} .

As above, predicates R_{ki} guarantee that types p_i are dominated by p_k for $\nu(i) < \nu(k)$ and $\nu(i) \not\sim \nu(k)$, while relations R'_{ij} ensure that p_i and p_j are domination equivalent and that models \mathcal{M}_{p_i} and \mathcal{M}_{p_j} are non-isomorphic, for $i \neq j$.

Now, similarly to the conditions (1') and (2') specified in Section 3.2 with Q replaced by R'_k , we extend the language by predicate symbols $R_{\mathbf{A}}$ so that types p_i , $i = k, \dots, k+l$, dominate all types $q(\bar{x}) \in S(T_k)$ with $p_j(x_i) \not\subseteq q(\bar{x})$ for $j > k+l$, the types q in question that are not dominated by p_i , $i < k$, dominate all types p_k, \dots, p_{k+l} , and the models \mathcal{M}_q are isomorphic to \mathcal{M}_{p_k} .

For the condition $\text{IL}(g(\widetilde{\nu(k)})) = r$ to be satisfied, on the structure of realizations of p_k , we define a graph structure with binary relations R''_1, \dots, R''_r such that:

(7) $R''_i(a, y)$ is a principal formula and $R''_i(a, y) \vdash p_k(y)$ for any $a \models p_k$, $i \leq r$;

(8) for any $a, b \models p_k$, there are infinitely many elements $c \models p_k$ and infinitely many d not realizing types $p_1(x), \dots, p_{|X|-1}$ for which

$$\models R''_i(c, a) \wedge R''_i(c, b) \wedge R''_i(d, a) \wedge R''_i(d, b);$$

moreover, $\models R''_i(c, a)$ and $c \models p_j$ imply that a does not semi-isolate c ;

(9) the relation $R'_k \cup \bigcup_{i=1}^r R''_i$ forms a digraph isomorphic to the digraph Γ_{gen} .

If \mathcal{M}_a and \mathcal{M}_b are prime models over realizations a and b of p_k , respectively, such that $\models R''_i(a, b)$ and $\mathcal{M}_b \prec \mathcal{M}_a$, then we call \mathcal{M}_a an R''_i -extension of the model \mathcal{M}_b . An elementary chain $(\mathcal{M}_s)_{s \in \omega}$ over p_k is called an R''_i -chain if \mathcal{M}_{s+1} is an R''_i -extension of \mathcal{M}_s for any s .

If \mathcal{M}_a and \mathcal{M}_b are prime models over realizations a and b of p_i , respectively, with $\nu(i) \sim \nu(k)$, such that $\models R''_{ii}(a, b)$ and $\mathcal{M}_b \prec \mathcal{M}_a$, then the model \mathcal{M}_a is called an R''_{ii} -extension of \mathcal{M}_b .

Similarly to the conditions (3)-(7) given in Section 3.3, we extend the language by symbols R_q so as to satisfy the following:

(10) for any limit model \mathcal{M} over a type p_{k+i} , $0 \leq i \leq l$, there is a relation R''_j such that \mathcal{M} is the union of an R''_j -chain $(\mathcal{M}_s)_{s \in \omega}$ over the type p_k ;

(11) limit models \mathcal{M}_1 and \mathcal{M}_2 over p_k are equivalent iff there is a predicate R''_i such that \mathcal{M}_1 and \mathcal{M}_2 are unions of R''_i -chains but are not unions of R''_j -chains, for $j > i$.

Notice, that (10) and (11) are realized via predicates R_q "saying" the following:

(a) every R''_{kk} -extension \mathcal{M}_a of \mathcal{M}_b contains an R''_1 -extension and vice versa;

(b) for any i , $\nu(i) \sim \nu(k)$, $i \neq k$, and for any finite elementary chain $\mathcal{M}_{a_1}, \dots, \mathcal{M}_{a_s}$, $a_1, \dots, a_s \models p_i$, there exist realizations b_1, \dots, b_{s-1} of p_k such that the sequence $\mathcal{M}_{a_1}, \mathcal{M}_{b_1}, \dots, \mathcal{M}_{b_{s-1}}, \mathcal{M}_{a_s}$ is also an elementary chain;

(c) if \mathcal{M}_{b_0} and \mathcal{M}_{b_1} are R''_s -extensions of \mathcal{M}_{a_n} , q is the type of a tuple $\langle a_0, a'_0, \dots, a_n, a'_n, a''_n, b_0, b_1 \rangle$ of elements which realize type p_k and are such that $\mathcal{M}_{a_{i+1}}$ is an R''_{ti} -extension of a model $\mathcal{M}_{a'_i}$, equal to \mathcal{M}_{a_i} , $\models R''_{ti}(a_{i+1}, a'_i)$, \mathcal{M}_{b_0} and \mathcal{M}_{b_1} are R''_{tn} -extensions of the model \mathcal{M}_{a_n} , equal to $\mathcal{M}_{a'_n}$ and to $\mathcal{M}_{a''_n}$, $\models R''_{tn}(b_0, a'_n) \wedge R''_{ti}(b_1, a''_n)$, and elements b_0 and b_1 are not connected by (b_0, b_1) - or (b_1, b_0) -routes, then \mathcal{M}_{b_0} contains a corealization amalgam $\mathcal{M}_{b_0} *_q \mathcal{M}_{b_1}$;

(d) if $\mathcal{M}_{a_1}, \dots, \mathcal{M}_{a_s}$ is a finite elementary chain such that the model $\mathcal{M}_{a_{j+1}}$ is an R''_{ij} -extension of \mathcal{M}_{a_j} , $j = 1, \dots, s-1$, and $\max\{i_1, \dots, i_{s-2}\} < i_{s-1}$, then \mathcal{M}_{a_s} contains an $R''_{i_{s-1}}$ -extension of \mathcal{M}_{a_1} and vice versa.

With the expansions exercised above we obtain a theory $T_k = T_{k+1} = \dots = T_{k+l}$ such that types p_k, \dots, p_{k+l} are domination equivalent, $\mathcal{M}_{p_k}, \dots, \mathcal{M}_{p_{k+l}}$ are pairwise non-isomorphic, and the number of limit models over p_k, \dots, p_{k+l} is equal to $f(\widetilde{\nu(k)})$.

Proceeding further with this process, at step $|X| - 1$, we obtain a theory $T = T_{|X|-1}$ and an isomorphism $g : \langle X; \leq \rangle \xrightarrow{\sim} \text{RK}(T)$ such that $g(\nu(0))$ is an isomorphism type of the prime model for T , $g(\nu(m))$ is an isomorphism type of \mathcal{M}_{p_m} , $1 \leq m \leq |X| - 1$, and $\text{IL}(g(\tilde{y})) = f(\tilde{y})$ for any $\tilde{y} \in X/\sim$.

The possibility for the above-specified properties to be realized is verified in essentially the same way as was sketched in constructing the theory with three countable models under Section 3.3 using Theorem 2.5.1. \square

§ 3.5. Theories with finite Rudin — Keisler preorders

The previous Sections have revealed a technique for constructing all possible theories with finitely many pairwise nonisomorphic countable models w.r.t. Rudin — Keisler preorders and the distribution functions of the number of limit models. The aim of this Section is to generalize the main result of Section 3.4 from the classification of theories with finitely many countable models to the case of an arbitrary theory with a finite Rudin — Keisler preorder that has at most countably many or continuum many limit models for each equivalence class.

Theorem 3.5.1. *For any finite preordered set $\langle X; \leq \rangle$ with the least element x_0 and the greatest class \tilde{x}_1 in the ordered factor set $\langle X; \leq \rangle/\sim$ w.r.t. \sim (where $x \sim y \Leftrightarrow x \leq y$ and $y \leq x$), and for any function $f : X/\sim \rightarrow \omega \cup \{\omega, 2^\omega\}$, satisfying the conditions $f(\tilde{x}_0) = 0$, $f(\tilde{x}_1) > 0$ for $|X| > 1$, and $f(\tilde{y}) > 0$ for $|\tilde{y}| > 1$, there exist a complete theory T and an isomorphism $g : \langle X; \leq \rangle \xrightarrow{\sim} \text{RK}(T)$ such that $\text{IL}(g(\tilde{y})) = f(\tilde{y})$ for any $\tilde{y} \in X/\sim$.*

Henceforth in this Section, we assume that the structure $\text{RK}(T)$ is finite.

It was shown in Section 1.1 that each preordered set $\text{RK}(T)$ contains the least element, which corresponds to the prime model, and the finiteness of $\text{RK}(T)$ implies the existence of the largest \sim_{RK} -class corresponding to the prime models over the realizations of powerful types.

Repeating the proof of Proposition 1.1.7 and using the condition $|\text{RK}(T)| < \omega$ instead of the condition $I(T, \omega) < \omega$, we obtain

Proposition 3.5.2. *If $|\text{RK}(T)| < \omega$ then each countable model of T is prime over a realization of some type in $S(T)$ or limit over some type in $S(T)$.*

Notice, that by Morley Theorem (see M. Morley [127]), for each class $\widetilde{\mathbf{M}} \in \text{RK}(T)/\sim_{RK}$, the number of limit models $\text{IL}(\widetilde{\mathbf{M}})$ belongs to $\omega \cup \{\omega, \omega_1, 2^\omega\}$.

Using this remark, by analogy with Theorem 1.1.13, we come to the following theorem, which may be regarded as a syntactic characterization of the class of complete theories with finite Rudin — Keisler preorders.

Theorem 3.5.3. *Any small theory T with a finite Rudin — Keisler preorder satisfies the following conditions:*

(a) $\text{RK}(T)$ contains the least element \mathbf{M}_0 (the isomorphism type of a prime model), and $\text{IL}(\widetilde{\mathbf{M}}_0) = 0$;

(b) $\text{RK}(T)$ contains the greatest \sim_{RK} -class $\widetilde{\mathbf{M}}_1$ (the class of isomorphism types of all prime models over realizations of powerful types), and $|\text{RK}(T)| > 1$ implies $\text{IL}(\widetilde{\mathbf{M}}_1) \geq 1$;

(c) if $|\widetilde{\mathbf{M}}| > 1$, then $\text{IL}(\widetilde{\mathbf{M}}) \geq 1$.

Moreover, we have the following decomposition formula:

$$I(T, \omega) = |\text{RK}(T)| + \sum_{i=0}^{|\text{RK}(T)/\sim_{RK}|-1} \text{IL}(\widetilde{\mathbf{M}}_i),$$

where $\widetilde{\mathbf{M}}_0, \dots, \widetilde{\mathbf{M}}_{|\text{RK}(T)/\sim_{RK}|-1}$ are all elements of the partially ordered set $\text{RK}(T)/\sim_{RK}$ and $\text{IL}(\widetilde{\mathbf{M}}_i) \in \omega \cup \{\omega, \omega_1, 2^\omega\}$ for each i .

Therefore, by analogy with the theories with finitely many countable models, the number of, and the relationship between, the countable models of a theory with a finite Rudin — Keisler preorder is determined by the following two characteristics: the Rudin — Keisler preorder itself and the distribution function $\text{IL}(\cdot)$ of the number of limit models that for each \sim_{RK} -class may take a value in $\omega \cup \{\omega, \omega_1, 2^\omega\}$.

Show that all possibilities, described in Theorem 3.5.3 and satisfying the condition

$$\text{rang}(\text{IL}(\cdot)) \subset \omega \cup \{\omega, 2^\omega\}$$

can be realized. This will prove Theorem 3.5.1 conversing Theorem 3.5.3 in assumption of Continuum Hypothesis.

We use the proof of Theorem 3.4.1. The construction of the theory T^0 with a given domination preorder $\langle X; \leq \rangle$ repeats the construction of an analogous theory with finitely many countable models step by step. The required expansion T of T^0 , satisfying the conditions $\text{IL}(g(\tilde{y})) = f(\tilde{y})$ for all $\tilde{y} \in X/\sim$, will be constructed by a scheme similar to that of the proof of Theorem 3.4.1.

By the construction of T^0 , it suffices to consider the step k of induction for the case $\text{IL}(g(\nu(k))) = \lambda$, $\lambda \in (\omega \cup \{\omega, 2^\omega\}) \setminus \{0\}$. Define on the structure of realizations of the type p_k a graph structure with arcs, colored by pairwise disjoint binary relations R''_n , for $n \in \omega$, satisfying the following conditions:

- (1) $R''_n(a, y)$ is a principal formula and $R''_n(a, y) \vdash p_k(y)$ for any $a \models p_k$, $n \in \omega$;
- (2) for any $a, b \models p_k$, there exists infinitely many elements $c \models p_k$ and infinitely many elements d that do not realize the types $p_1(x)$, \dots , $p_{|X|-1}(x)$ for which

$$\models R''_n(c, a) \wedge R''_n(c, b) \wedge R''_n(d, a) \wedge R''_n(d, b),$$

$n \in \omega$; moreover, $\models R''_n(c, a)$ and $c \models p_i$ imply that a does not semi-isolate c ;

- 3) the relation $R'_k \cup \bigcup_{i=1}^r R''_i$ forms a digraph isomorphic to the digraph Γ_{gen} .

Let \mathcal{M}_a and \mathcal{M}_b be prime models over realizations a and b of p_k , respectively, such that $\models R''_n(a, b)$ and $\mathcal{M}_b \prec \mathcal{M}_a$. Then \mathcal{M}_a is called an n -extension of \mathcal{M}_b for $n \in \omega$. Let $w = \langle n_1, \dots, n_m \rangle \in \omega^m$ be an m -tuple, \mathcal{M}_{a_i} , for $i = 1, \dots, m+1$, be models such that $\mathcal{M}_{a_{i+1}}$ is an n_i -extension of \mathcal{M}_{a_i} , $i = 1, \dots, m$. Then the model $\mathcal{M}_{a_{m+1}}$ is called a w -extension of \mathcal{M}_{a_1} . An elementary chain $(\mathcal{M}_s)_{s \in \omega}$ is said to be an f -chain (where $f \in \omega^\omega$) if \mathcal{M}_{s+1} is an $f(s)$ -extension of \mathcal{M}_s for any $s \in \omega$.

Likewise in the proof of Theorem 3.4.1, expanding the theory by the predicates R_q and R'_q , we obtain a theory in which each limit model with the isomorphism type in $g(\nu(k))$ can be represented as the union of an f -chain for some sequence $f \in \omega^\omega$. Consequently, the number $\text{IL}(g(\nu(k)))$ is equal to the number of pairwise non-isomorphic unions of f -chains. The introduction of the additional predicates R_q and R'_q reduces the existence of the isomorphism

of the unions of f_1 - and f_2 -chains to the existence of the words $w_1^m, w_2^m \in \omega^{<\omega}$, for $m \in \omega$, satisfying the following conditions:

- (a) the sequence f_i is “similar” to the countable concatenation of the words $w_i^0, w_i^1, \dots, w_i^m, \dots, i = 0, 1$;
- (b) any w_0^m -extension is a w_1^m -extension, and vice versa, for $m \in \omega$.

Thus, the problem of constructing some extension of the theory with the condition $\text{IL}(g(\widehat{\nu(k)})) = \lambda$ reduces to the problem of finding a quotient of the set ω^ω by identifying words of the form w_1^m and w_2^m that would contain exactly λ classes. We now formally define and investigate the possibilities of taking the quotients of ω^ω .

Consider the semigroup $S_0 = \langle W; \widehat{\ } \rangle$ of all *nonempty* words in the alphabet ω under the operation $\widehat{\ }$ of concatenation. If w_1 and w_2 are words in W , we call the formula $w_1 \approx w_2$ an *identity*, as usual. Given some set I of identities $w_1^j \approx w_2^j$ with $j \in J$ including the set I_0 of all identities of view $w \approx w$, we define the set of identities deducible from I . An identity $w_1 \approx w_2$ is called *deducible* from I if, there exists some finite sequence $w_1^1 \approx w_2^1, \dots, w_1^t \approx w_2^t$ of identities such that $w_1^1 = w_1, w_2^t = w_2$, and each identity in the sequence either belongs to I or follows from the previous identities by applying one of the following deduction rules:

- (1) $\frac{w_1 \approx w_2}{w_2 \approx w_1}$, where $w_1, w_2 \in W$;
- (2) $\frac{w_1 \approx w_2; w_2 \approx w_3}{w_1 \approx w_3}$, where $w_1, w_2, w_3 \in W$;
- (3) $\frac{w_1 \approx w_2; w'_1 \approx w'_2}{w_1 w'_1 \approx w_2 w'_2}$, where $w_1, w'_1, w_2, w'_2 \in W$;
- (4) $\frac{w_1 w_2 w_3 \approx w_1 w'_2 w_3}{w_2 \approx w'_2}$, where $w_1, w_2, w'_2, w_3 \in W$.

Henceforth we consider sets of identities $I \supseteq I_0$ closed with respect to deducibility. Any set I of identities has a bijective correspondence with the semigroup $S_I = \langle W; \widehat{\ } \rangle / I$ that is the quotient of the semigroup S_0 by the congruence \sim_I :

$$w_1 \sim_I w_2 \Leftrightarrow (w_1 \approx w_2) \in I.$$

Let $\langle \mathcal{L}, \leq \rangle$ be the pair in which \mathcal{L} is the set of all semigroups S_I , and $S_{I_1} \leq S_{I_2}$ is equivalent to $I_1 \subseteq I_2$. Clearly, the structure $\langle \mathcal{L}; \leq \rangle$

is a complete lattice, where $\sup \left\{ S_{I_j} \mid j \in J \right\} = S_I$ for the set I of all identities deducible from $\bigcup_{j \in J} I_j$, and $\inf \left\{ S_{I_j} \mid j \in J \right\} = S_{\bigcap_{j \in J} I_j}$. The semigroup S_0 is the bottom element in the lattice $\langle \mathcal{L}; \leq \rangle$, and the one-element semigroup, in which all words are identified, is its top element.

Define the quotients of ω^ω corresponding to sets I of identities. Two sequences, f_0 and f_1 , in ω^ω are called *almost similar* if there exist $l_0, l_1 \in \omega$ such that $f_0(n + l_0) = f_1(n + l_1)$ for all $n \in \omega$. Two sequences, f_0 and f_1 , are called *strongly I -equivalent* if f_i is almost similar to a finite concatenation of words $w_i^m \in W$, $m \in \omega$, $i = 0, 1$, where the identity $w_0^m \approx w_1^m$ belongs to I for all $m \in \omega$. Two sequences, f and f' , are called *I -equivalent* if there exists a sequence $f_0, f_1, \dots, f_n \in \omega^\omega$ with $f_0 = f$, $f_n = f'$, and f_i and f_{i+1} are strongly I -equivalent for all $i = 0, \dots, n - 1$.

Clearly, that after all pairwise identifications of w_1 - and w_2 -chains for all identities $w_1 \approx w_2$ in I , the existence of an isomorphism between the unions of f - and f' -chains becomes equivalent to the I -equivalence of f and f' .

For any set I of identities, we denote by ω^ω/I the quotient of ω^ω by the relation of I -equivalence. Denote by T^0/I the theory obtained from T^0 by pairwise identifications of w_1 - and w_2 -chains for all identities $w_1 \approx w_2$ in I . Clearly, we may assume that the sets ω^ω/I are bijective with the theories T^0/I .

Denote by $\text{IL}_k(T^0/I)$ the number $\text{IL}(\widetilde{\mathbf{M}})$ for the class $\widetilde{\mathbf{M}}$ of isomorphism types of models of T^0/I , that includes the isomorphism type of \mathcal{M}_{p_k} .

- Lemma 3.5.4.** 1. If $I_1 \subseteq I_2$, then $\text{IL}_k(T^0/I_1) \geq \text{IL}_k(T^0/I_2)$.
 2. $\text{IL}_k(T^0/I_0) = 2^\omega$.
 3. $\text{IL}_k(T^0/\{w_1 \approx w_2 \mid w_1, w_2 \in W\}) = 1$.

P r o o f is obvious.

The next lemma was implicitly proved in Section 3.4.

Lemma 3.5.5. For any $n \in \omega \setminus \{0\}$, there exists a set I_n of identities such that $\text{IL}_k(T^0/I_n) = n$.

P r o o f. Fix some integer $n \geq 1$ and consider the set I_n of identities deducible from the set of identities

$$n - 1 \approx m,$$

where $m \geq n$, and

$$n_1 n_2 \dots n_s \approx n_s,$$

where $\max\{n_1, n_2, \dots, n_{s-1}\} < n_s$. It is easy to see that each sequence in ω^ω is I_n -In-equivalent to some constant sequence f_r , that is, $f_r(j) \equiv r$, for all $j \in \omega$, $0 \leq r < n$. Indeed, the identities written imply that each sequence f is I_n -equivalent to the sequence f_r , where r is equal to the largest value below $n - 1$ of infinite constant subsequences of f , if existent, and $r = n - 1$ if such subsequences fail to exist. In addition, it is obvious that the sequences f_0, \dots, f_{n-1} are pairwise I_n -nonequivalent. Therefore, $\text{IL}_k(T^0/I_n) = n$. \square

Lemma 3.5.6. *There exists a set I_ω of identities such that $\text{IL}_k(T^0/I_\omega) = \omega$.*

P r o o f. Denote by I_ω the set of identities deducible from the identities

$$n_1 n_2 \dots n_s \approx n_s,$$

where $\max\{n_1, n_2, \dots, n_{s-1}\} < n_s$, and

$$n_1 n_2 \approx n_1(n_1 + 1)(n_1 + 2) \dots (n_2 - 1)n_2,$$

where $n_1 < n_2$. We claim that $\text{IL}_k(T^0/I_\omega) = \omega$. Indeed, by the first system of identities each bounded subsequence $f \in \omega^\omega$ is I_ω -equivalent to the constant sequence $f_r \in \omega^\omega$, $f_r(j) \equiv r$, $j \in \omega$, where r is the largest value of an infinite constant subsequence of f . Moreover, the two systems of identities imply that each unbounded sequence is I_ω -equivalent to the sequence $f_\omega \in \omega^\omega$, where $f_\omega(j) = j$, $j \in \omega$. Therefore, each sequence $f \in \omega^\omega$ is I_ω -equivalent to some sequence f_μ with $\mu \leq \omega$. It is obvious that the sequences f_μ are pairwise I_ω -nonequivalent. Consequently, $\text{IL}_k(T^0/I_\omega) = \omega$. \square

Theorem 3.5.1 is immediate from Lemmas 3.5.4 — 3.5.6.

The question of the existence of a set I_{ω_1} such that $\text{IL}_k(T^0/I_{\omega_1}) = \omega_1$ remains open. Apparently, a positive answer could yield an alternative for Knight's approach (see R. W. Knight [107]) to constructing a counterexample to the Vaught conjecture on the nonexistence of a theory T for which $I(T, \omega) = \omega_1$.

§ 3.6. Rudin — Keisler preorders in small theories

The following modification of Theorem 3.4.1 represents a description of preordered sets $\text{RK}(T)$ in small theories T .

Theorem 3.6.1. (1) *For any small theory T , the preordered set $\text{RK}(T)$ is not more than countable, upward directed, and has a least element.*

(2) *For any finite or countable, preordered, upward directed set $\langle X; \leq \rangle$ having a least element, there exists a small theory T for which $\text{RK}(T) \simeq \langle X; \leq \rangle$.*

P r o o f. (1) That $|\text{RK}(T)| \leq \omega$ follows from the property of T being small. The property for the preordered set $\text{RK}(T) = \langle \mathbf{PM}; \leq_{RK} \rangle$ to be upward directed is implied by the following: if \mathbf{M}_1 and \mathbf{M}_2 are isomorphism types of \mathbf{PM} corresponding to models $\mathcal{M}_{\bar{a}_1}$ and $\mathcal{M}_{\bar{a}_2}$, then types $\text{tp}(\bar{a}_1)$ and $\text{tp}(\bar{a}_2)$ are dominated by $q = \text{tp}(\bar{a}_1 \hat{\ } \bar{a}_2)$; hence, $\mathbf{M}_1 \leq_{RK} \mathbf{M}$ and $\mathbf{M}_2 \leq_{RK} \mathbf{M}$, where \mathbf{M} is the isomorphism type of \mathcal{M}_q . The least element in $\text{RK}(T)$ is an isomorphism type of the prime model.

(2) In view of Theorem 3.4.1, there is no loss of generality in assuming that the set X is countable. A small theory T with $\text{RK}(T) \simeq \langle X; \leq \rangle$ is constructed similarly to how were the theories constructed in proving Theorem 3.4.1, with the theory of unary predicates $P_1, \dots, P_{|X|-1}$ replaced by a theory of pairwise disjoint unary predicates P_i , $i \in \omega$, each containing infinitely many elements. \square

§ 3.7. Theories with non-dense structures of powerful digraphs and theories with powerful types and without powerful digraphs

In this Section, we define a modification of construction in Section 3.2, producing a powerful digraph $\hat{\Gamma} = \langle \hat{X}; \hat{Q} \rangle$ having transitive closure being a partial order with infinitely many covering elements for any element in \hat{X} .

The relation \hat{Q} is a disjoint union of relations Q_0 and Q_1 such that $Q_0(x, y)$ and $Q_1(x, y)$ are principal formulas satisfying the following:

$$\vdash Q_i(x, y) \leftrightarrow \hat{Q}(x, y) \wedge \left(\exists z (\hat{Q}(x, z) \wedge \hat{Q}(y, z)) \right)^{1-i} \wedge \left(\exists u (\hat{Q}(x, u) \wedge \hat{Q}(u, y)) \right)^i,$$

$i = 0, 1$. Moreover, $Q_0(a, \hat{\Gamma})$ is the set of successors for a in $\text{TC}(\hat{\Gamma})$.

Denote by $\hat{\mathbf{K}}^*$ the class of all c -graphs $\mathcal{A} = \langle A, \hat{Q}, W \rangle$, $\hat{Q} = Q_0 \dot{\cup} Q_1$ such that $\models Q_0(a, b)$ implies an absence of (c, b) -routes for any $c \in \hat{Q}(a, \mathcal{A}) \setminus \{b\}$, and $\models Q_1(a, b)$ implies an absence of (b, c) -routes for any $c \in \hat{Q}(a, \mathcal{A}) \setminus \{b\}$. Here, if $(a, b) \in Q_i$ then the index i is called an *arc color* (a, b) .

Below, considering c -graphs in $\hat{\mathbf{K}}_0$, we distinguish colors of arcs and use the language $\langle Q_0, Q_1 \rangle$ instead of (or as an addition to) \hat{Q} in digraphs and c -graphs.

The concepts of c -embedding and c -isomorphism are obviously transformed to the class of c -graphs of the language $\langle Q_0, Q_1 \rangle$ preserving arc colors via mappings.

Denote by $\hat{\mathbf{K}}_0^*$ the subclass of $\hat{\mathbf{K}}^*$ generated by stated below c -graphs $\langle A, Q_0, Q_1, W \rangle$ by taking of c -subgraphs, c -isomorphic copies, free c -amalgams, tracings (allowing for any chosen pair of vertices (a, b) , that are not linked by routes, where $\text{Col}(a) \leq \text{Col}(b)$, to extend records W by an information on shortest (a, b) -routes of arbitrary length m , exceeding lengthes of all shortest routes being in given c -graph), and also detracings (allowing to remove aforesaid information):

(a) $\Gamma_{\alpha, \beta, \gamma, 0} = \langle \{0, 1, 2\}, \{(0, 1), (1, 2)\}, \{(0, 2)\}, W \rangle$, where $W = \emptyset$;

(b) $\Gamma_{\alpha, \beta, \gamma, 1} = \langle \{0, 1, 2\}, \{(0, 1)\}, \{(0, 2), (1, 2)\}, W \rangle$, where $W = \emptyset$;

(c) $\Gamma_{\alpha, \beta, \gamma, s} = \langle \{0, 1, 2\}, \{(0, 1)\}, \{(0, 2)\}, W \rangle$, where $W = \{(1, 2, s)\}$, $2 \leq s < \omega$, $\text{Col}(0) = \alpha$, $\text{Col}(1) = \beta$, $\text{Col}(2) = \gamma$, $\alpha \leq \beta \leq \gamma$, $\gamma \in \omega \cup \{\infty\}$.

Lemma 3.7.1. ($\hat{\cdot}$ -amalgamation lemma). *The class $\hat{\mathbf{K}}_0^*$ has the $\hat{\cdot}$ -amalgamation property ($\hat{\cdot}$ -AP), that is, for any c -embeddings $f_0 : \mathcal{A} \rightarrow_c \mathcal{B}$ and $g_0 : \mathcal{A} \rightarrow_c \mathcal{C}$, where $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \hat{\mathbf{K}}_0^*$, there exist a c -graph $\mathcal{D} \in \hat{\mathbf{K}}_0^*$ and c -embeddings $f_1 : \mathcal{B} \rightarrow_c \mathcal{D}$ and $g_1 : \mathcal{C} \rightarrow_c \mathcal{D}$ such that $f_0 \circ f_1 = g_0 \circ g_1$.*

P r o o f is obvious. \square

Denote by $\hat{\mathbf{K}}_0$ the class of colored acyclic digraphs in which any finite subgraph forms a c -graph in $\hat{\mathbf{K}}_0^*$.

Theorem 3.7.2. *There exists countable, colored, saturated digraph $\hat{\mathcal{M}} \in \hat{\mathbf{K}}_0$ satisfying the following:*

- (1) *if $f : \mathcal{A} \rightarrow_c \hat{\mathcal{M}}$ and $g : \mathcal{A} \rightarrow_c \mathcal{B}$ are c -embeddings and $\mathcal{B} \in \hat{\mathbf{K}}_0^*$, then there exists a c -embedding $h : \mathcal{B} \rightarrow_c \hat{\mathcal{M}}$ such that $f = g \circ h$;*
- (2) *if \mathcal{A} and \mathcal{B} are c -isomorphic c -subgraphs of $\hat{\mathcal{M}}$, then $\text{tp}_{\hat{\mathcal{M}}}(A) = \text{tp}_{\hat{\mathcal{M}}}(B)$;*
- (3) *the coloring of restriction $\hat{\mathcal{M}} \upharpoonright \hat{Q}$ of $\hat{\mathcal{M}}$ to the graph language $\Sigma = \{\hat{Q}\}$ is inessential and \hat{Q} -ordered;*
- (4) *the formula $\hat{Q}(x, y)$ is equivalent in $\text{Th}(\hat{\mathcal{M}} \upharpoonright \hat{Q})$ to the disjunction of principal formulas $Q_0(x, y)$ and $Q_1(x, y)$.*

P r o o f is similar to the proof of Theorem 3.1.3. The construction of the saturated model $\hat{\mathcal{M}}$ repeats steps of construction for the model \mathcal{M} in the proof of Theorem 3.1.3 with replacing of \mathbf{K}_0^* to the class $\hat{\mathbf{K}}_0^*$ and of \mathbf{K}_0 to $\hat{\mathbf{K}}_0$. \square

Elements in $Q_0(a, \hat{\mathcal{M}})$ are successors of a in the digraph $\text{TC}(\hat{\Gamma})$, where $\hat{\Gamma} = \hat{\mathcal{M}}|_{\hat{Q}}$, since, by construction, there are no elements $b \in \hat{Q}(a, \hat{\mathcal{M}})$ such that there exists a (b, c) -route with $c \in Q_0(a, \hat{\mathcal{M}}) \setminus \{b\}$. Thus, in $\text{TC}(\hat{\Gamma})$, the partial order is not dense.

Similar Corollaries 3.1.4–3.1.6, we state that the relation $\text{SI}_{p_\infty(x)}$ is non-symmetric, the digraph $\langle p_\infty(\hat{\mathcal{M}}); R_{\hat{Q}}^{p_\infty}(\hat{\mathcal{M}}) \rangle$ is powerful, and the theory $\text{Th}(\hat{\mathcal{M}})$ is not simple.

Notice, that constructions, described in Section 3.2, can be applied to $\hat{\mathcal{M}}$ and also lead to creation of not ω -categorical, not simple theory, for which any nonprincipal type is powerful. Thus, we obtain the following theorem.

Theorem 3.7.3. *There exists a generic theory T satisfying the following conditions:*

- (1) $|\text{RK}(T)| = 2$;
- (2) *the structure of realizations of some powerful type $p \in S(T)$ contains a powerful digraph Γ with unbounded lengths of shortest routes and with infinitely many covering elements for each element in the transitive closure $\text{TC}(\Gamma)$.*

Now, we describe another modification of aforesaid construction, allowing to create a small generic theory with a type having the local pairwise intersection property, but without the global pairwise intersection property.

Let $\mathcal{A} = \langle A, Q, W \rangle$ be a c -graph. We replace the graph language introducing a sequence of pairwise disjoint binary predicates Q_n , $n \in \omega$, such that $Q = \bigcup_{n \in \omega} Q_n$, and substitute in W , instead of records on existence of external routes connecting elements in A , the records on existence of these routes with an information of colors m_i for sequential arcs $(b_i, b_{i+1}) \in Q_{m_i}$ forming these routes. Moreover, we allow, in obtained record W' , for any two elements $a, b \in A$, to write a formula-definable information of form

$$\begin{aligned} \varphi_{k,m}(a, b) \iff & \exists x \left(\bigwedge_{i=0}^k \neg \text{Col}_i(x) \wedge \left(\bigvee_{i=0}^m Q_i(x, a) \right) \wedge \left(\bigvee_{i=0}^m Q_i(x, b) \right) \right) \wedge \\ & \neg \exists y \left(\bigwedge_{i=0}^k \neg \text{Col}_i(y) \wedge \left(\bigvee_{i=0}^{m-1} \bigwedge_{j=0}^k Q_i^j(y, a) \right) \wedge \left(\bigvee_{i=0}^{m-1} \bigwedge_{j=0}^k Q_i^j(y, b) \right) \right) \end{aligned}$$

such that the set of formulas over A , correspondent to W' , is consistent. The obtained structure $\mathcal{A} = \langle A, Q_n, W' \rangle_{n \in \omega}$ is called a *cv-graph*.

The concepts of c -embedding and c -isomorphism, as above, are transformed to the class of *cv-graphs* of the language $\langle Q_0, Q_1, \dots, Q_n, \dots \rangle$ with the preservation, via mappings, of arc colors and of satisfiability for $\varphi_{k,m}(a, b)$ in the records.

A *cv-graph* $\mathcal{A} = \langle A, Q_{n,\mathcal{A}}, W_{\mathcal{A}} \rangle_{n \in \omega}$ is called a *cv-subgraph* of *cv-graph* $\mathcal{B} = \langle B, Q_{n,\mathcal{B}}, W_{\mathcal{B}} \rangle_{n \in \omega}$ (written $\mathcal{A} \subseteq_{cv} \mathcal{B}$) if $A \subseteq B$, $Q_{n,\mathcal{A}} = Q_{n,\mathcal{B}} \cap A^2$, $n \in \omega$, and $W_{\mathcal{A}}$ consists of all records in $W_{\mathcal{B}}$, in which pairs of elements in A are used, and also of records, induced by the *cv-graph* \mathcal{B} , on existence or on absence of elements x and y in accordance of $\varphi_{k,m}(a, b)$.

Let $\mathcal{A} = \langle A, Q_{n,\mathcal{A}}, W_{\mathcal{A}} \rangle_{n \in \omega}$, $\mathcal{B} = \langle B, Q_{n,\mathcal{B}}, W_{\mathcal{B}} \rangle_{n \in \omega}$, and $\mathcal{C} = \langle C, Q_{n,\mathcal{C}}, W_{\mathcal{C}} \rangle_{n \in \omega}$ be *cv-graphs* such that $A = B \cap C$, $Q_{n,\mathcal{A}} = Q_{n,\mathcal{B}} \cap Q_{n,\mathcal{C}}$, $n \in \omega$, and $W_{\mathcal{A}}$ consists of all records, included both in $W_{\mathcal{B}}$ and in $W_{\mathcal{C}}$, in which pairs of elements in A take place, that induced by *cv-graphs* \mathcal{B} and \mathcal{C} . We call the *cv-graph* $\langle B \cup C, Q_{\mathcal{B}} \cup Q_{\mathcal{C}}, W_{\mathcal{B}} \cup W_{\mathcal{C}} \cup W \rangle$, a *cv-amalgam* of \mathcal{B} and \mathcal{C} over \mathcal{A} if W consists of all possible formulas, describing, for any elements $b \in B \setminus A$ and $c \in C \setminus A$, an existence and colors of elements d , for which there are

(d, b) - and (d, c) -routes (if their existence is implied by relations $Q_B \cup Q_C$ and $W_B \cup W_C$), and also an existence of elements e of all finite colors m , not exceeding $\min\{\text{Col}(b), \text{Col}(c)\}$ and such that $\models \bigvee_{i=0}^m Q_i(e, b) \wedge \bigvee_{i=0}^m Q_i(e, c)$. The record W also contains a finite set of formulas of form $\varphi_{m-1, m}(b, c)$, $m \leq \min\{\text{Col}(b), \text{Col}(c)\}$, $m \in \omega$, for the rest pairs (b, c) , where $b \in B \setminus A$, $c \in C \setminus A$.

We denote by $\check{\mathbf{K}}_0$ the class of all cv-graphs that can be obtained by all possible aforesaid transformations of c -graphs in \mathbf{K}_0 . Denote by $\check{\mathbf{K}}_0^*$ the subclass of $\check{\mathbf{K}}_0$, generated by taking of cv-subgraphs, cv-isomorphic copies and free cv-amalgams, and also by tracings and detracings starting by the following cv-graphs $\langle A, Q_n, W \rangle_{n \in \omega}$, where the index $\cdot n$ in $(a, b)_n$ points out that $(a, b) \in Q_n$:

$$\Gamma_{\alpha, \beta, \gamma, n_1, n_2, n_3, W} = \langle \{0, 1, 2\}, \{(0, 1)_{n_1}, (1, 2)_{n_2}, (0, 2)_{n_3}\}, W \rangle,$$

where $\text{Col}(0) = \alpha$, $\text{Col}(1) = \beta$, $\text{Col}(2) = \gamma$, $\alpha \leq \beta \leq \gamma$, $\gamma \in \omega \cup \{\infty\}$, W consists of nonempty finite set of formulas $\varphi_{m-1, m}(0, 1)$, $m \leq \min\{\text{Col}(0), \text{Col}(1)\}$, $m \in \omega$, and also of nonempty finite set of formulas $\varphi_{m-1, m}(0, 2)$, $m \leq \min\{\text{Col}(0), \text{Col}(2)\}$, $m \in \omega$.

Lemma 3.7.4. ($\check{\sim}$ -amalgamation lemma). *The class $\check{\mathbf{K}}_0^*$ has the $\check{\sim}$ -amalgamation property ($\check{\sim}$ -AP), that is, for any cv-embeddings $f_0 : \mathcal{A} \rightarrow_{\text{cv}} \mathcal{B}$ and $g_0 : \mathcal{A} \rightarrow_{\text{cv}} \mathcal{C}$, where $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \check{\mathbf{K}}_0^*$, there exist a cv-graph $\mathcal{D} \in \check{\mathbf{K}}_0^*$ and cv-embeddings $f_1 : \mathcal{B} \rightarrow_{\text{cv}} \mathcal{D}$ and $g_1 : \mathcal{C} \rightarrow_{\text{cv}} \mathcal{D}$ such that $f_0 \circ f_1 = g_0 \circ g_1$.*

P r o o f is obvious. \square

Denote by $\check{\mathbf{K}}$ the class of colored acyclic digraphs in which each finite subgraph forms a cv-graph in $\check{\mathbf{K}}_0^*$.

Theorem 3.7.5. *There exists countable, colored, saturated digraph $\check{\mathcal{M}} \in \check{\mathbf{K}}$ satisfying the following:*

(1) *if $f : \mathcal{A} \rightarrow_{\text{cv}} \mathcal{M}$ and $g : \mathcal{A} \rightarrow_{\text{cv}} \mathcal{B}$ are cv-embeddings and $\mathcal{B} \in \check{\mathbf{K}}_0^*$, then there exists a cv-embedding $h : \mathcal{B} \rightarrow_{\text{cv}} \mathcal{M}$ such that $f = g \circ h$;*

(2) *if \mathcal{A} and \mathcal{B} are cv-isomorphic cv-subgraphs in $\check{\mathcal{M}}$, then $\text{tp}_{\check{\mathcal{M}}}(A) = \text{tp}_{\check{\mathcal{M}}}(B)$;*

(3) *the coloring of restriction $\check{\mathcal{M}} \upharpoonright \hat{Q}$ of $\check{\mathcal{M}}$ to the graph language $\Sigma = \langle Q_0, Q_1, \dots, Q_n, \dots \rangle$ is inessential and Q_n -ordered for each $n \in \omega$;*

(4) *$p_\infty(x)$ has the (LPIP) but doesn't have the (GPIP).*

P r o o f is similar to the proof of Theorem 3.1.3. The construction of the saturated model $\hat{\mathcal{M}}$ repeats steps of construction for the model \mathcal{M} in the proof of Theorem 3.1.3 with replacing of \mathbf{K}_0^* to $\check{\mathbf{K}}_0^*$ and of \mathbf{K} to $\check{\mathbf{K}}$. The property (LPIP) for $p_\infty(x)$ holds, since, by the generic construction, for any two elements, a and b , having the same color at least $n + 1$, there exists an element c of color n such that $\models \bigvee_{i=0}^n Q_i(c, a) \wedge \bigvee_{i=0}^n Q_i(c, b)$. An absence of (GPIP) is followed, since, by Compactness Theorem, for any $n \in \omega$ there exist realizations a and b of p_∞ for which there are no realizations c of p_∞ such that $(c, a), (c, b) \in \bigcup_{i=0}^n \left\{ \bigcup_{j \in \omega} Q_i^j \mid j \in \omega \right\}$. \square

Using the construction of $\check{\mathcal{M}}$, we have that for any two realizations, a and b , of $p_\infty(x)$, there exist numbers $m, n \in \omega$ and a realization c of p_∞ such that $\models Q_m(c, a) \wedge Q_n(c, b)$. Moreover, for any elements d_1 and d_2 and for any color $\alpha \geq \max\{\text{Col}(d_1), \text{Col}(d_2)\}$, there exists an element e of color α , for which $\models Q_0(d_1, e) \wedge Q_0(d_2, e)$. Therefore, an obvious modification of construction in Section 3.2, applied for the generic model \mathcal{M} , states the following theorem.

Theorem 3.7.6. *There exists a generic theory T satisfying the following conditions:*

- (1) $|\text{RK}(T)| = 2$;
- (2) *some powerful type $p \in S(T)$ has the (LPIP), but doesn't have the (GPIP).*

Constructions, similar to aforesaid in Sections 3.3–3.6, show that the generic models $\hat{\mathcal{M}}$ and $\check{\mathcal{M}}$ can be extended to models with theories having given domination preorder and given distribution function of numbers of limit models.

Chapter 4

STABLE GENERIC EHRENFUCHT THEORIES (A SOLUTION OF THE LACHLAN PROBLEM)

§ 4.1. Small stable generic graphs with infinite weight. Bipartite digraphs

B. Herwig [81] described a generic construction that was a modification of the *Hrushovski construction* [89]. Herwig's invention made it possible to construct a small stable theory of a graph with colored edges which possesses a unique 1-type such that its own weight is infinite. Thus, the realization was given for one of the essential properties shared by all stable Ehrenfeucht theories.

In the present Section, grounding on *Herwig construction* [81], we describe a generic construction that yields a family of small stable bipartite edgeless digraphs with colored arcs such that all 1-types have infinite weight. We thus solve the existence problem for a small stable acyclic digraph with colored arcs which has types with infinite weight.

1. Definitions and properties. The language Σ_{bp} of a generic bipartite graph Γ_{bp} consists of binary symbols I_p , $p \in \omega$, and unary symbols J_0 and J_1 . The relations corresponding to J_0 and J_1 are disjoint and divide the universe in two parts. The relations corresponding to I_p are assumed irreflexive, antisymmetric, and pairwise disjoint, i.e., each pair of vertices (a, b) belongs to at most one relation I_p , so that if $(a, b) \in I_p$ then $a \in J_0$, $b \in J_1$. In what follows in this Section, a *graph* is a bipartite digraph of language Σ_{bp} sat-

isfying the above conditions. The structure consisting of the empty set and empty relations is also a graph, and we denote it by \emptyset . The class of all graphs under consideration is denoted by Γ^{bp} .

We fix a positive real β . Let $\frac{l_n}{m_n}$ stand for the *best rational approximation* of β such that $\frac{l_n}{m_n} \leq \beta$ and $m_n \leq n$. The *index* of β is $\text{Ind}(\beta) = \sum_{i>0} (\beta - \frac{l_i}{m_i})$. In [81], B. Herwig observed that the set $\{\beta \in \mathbb{R}^+ \mid \text{Ind}(\beta) > N\}$ is dense and open for each $N \in \omega$. Thus, the set $\{\beta \mid \text{Ind}(\beta) = \infty\}$ is dense in \mathbb{R}^+ . Furthermore, if $\text{Ind}(\beta) = \infty$ and $q \in \mathbb{Q}$ then $\text{Ind}(\beta q) = \infty$.

Choose a positive real $\alpha_1 < \frac{1}{2}$ with $\text{Ind}(\frac{1}{\alpha_1}) = \infty$. Below we will define a sequence $(\alpha_k)_{k \in \omega \setminus \{0\}}$ of *weights* of I_k -arcs, where $\alpha_{k+1} = \frac{\alpha_k}{N_k}$ with N_k sufficiently large naturals.

Define a *prerank function* y that assigns a real number to every finite graph \mathcal{A} by the rule

$$y(\mathcal{A}) = |A| - \sum_{k=1}^{\infty} \alpha_k \cdot e_k(\mathcal{A}),$$

where $e_k(\mathcal{A})$ is the number of I_k -arcs in \mathcal{A} .

Notice that, in the definition of a prerank function, the summation is always finite due to the finiteness of \mathcal{A} .

A *p-approximation* of a prerank function y is a function y_p assigning a real to every finite graph \mathcal{A} by the rule

$$y_p(\mathcal{A}) = |A| - \sum_{k=1}^p \alpha_k \cdot e_k(\mathcal{A}).$$

Given a graph \mathcal{A} , we consider its representation as the point $s_{\mathcal{A}}^1 = (|A|; y_1(\mathcal{A}))$ in the *lattice*¹

$$L_1 = \{(n; n - \alpha_1 \cdot m) \mid n, m \in \omega\}.$$

Define a monotonically increasing unbounded sequence $(b_n^1)_{n \geq 1}$: $b_1^1 = 1$, $b_{n+1}^1 = b_n^1 + (1 - \alpha_1 \cdot \frac{l_n}{m_n})$, where $\frac{l_n}{m_n}$ is the best rational approximation of $\frac{1}{\alpha_1}$ satisfying $m_n \leq n$. The unboundedness of the sequence follows from the condition $\text{Ind}(\frac{1}{\alpha_1}) = \infty$.

¹As in [81], here and below, a lattice is some discrete set of points on the co-ordinate plane.

Given a graph \mathcal{A} , we write $\mathcal{A} \in \mathbf{K}_1^{\text{bp}}$ iff \mathcal{A} is finite and $y_1(\mathcal{A}') \geq b_n^1$ for every nonempty graph $\mathcal{A}' \subseteq \mathcal{A}$, where $n = |\mathcal{A}'|$. We put $k_1 \doteq 1$ and thus conclude the definition of step 1.

Suppose that at step p , we have defined the number α_p , the lattice L_p of the possible y_p -values, and the monotonically increasing sequence $(b_n^p)_{n \geq k_p}$. Choose a natural $k_{p+1} > k_p$ subject to $b_{k_{p+1}}^p > p+2$ and a sufficiently small real $\varepsilon_{p+1} > 0$ ($\varepsilon_{p+1} < \varepsilon_p$ for $p > 1$) such that, for all $(p_1, y_1), (p_2, y_2) \in L_p$ with $p_1, p_2 \leq k_{p+1}$ and $y_1 \neq y_2$, we have $|y_1 - y_2| > \varepsilon_{p+1}$. Choose a number $\alpha_{p+1} = \frac{\alpha_p}{N_p}$ with $N_p \in \omega$ and $N_p > 2$ so that $\alpha_{p+1} \cdot \frac{k_{p+1}(k_{p+1}-1)}{2} < \varepsilon_{p+1}$. The lattice

$$L_{p+1} \doteq \{(n; n - \alpha_{p+1} \cdot m) \mid n, m \in \omega\}$$

refines L_p .

The sequence $(b_n^{p+1})_{n \geq k_{p+1}}$ is constructed according to the following relations: $b_{k_{p+1}}^{p+1} = b_{k_{p+1}}^p - 2\varepsilon_{p+1}$, $b_{n+1}^{p+1} = b_n^{p+1} + (1 - \alpha_{p+1}) \cdot \frac{l_n}{m_n}$, where $\frac{l_n}{m_n}$ is the best rational approximation of $\frac{1}{\alpha_{p+1}}$ satisfying $m_n \leq n$. The condition $\text{Ind}(\frac{1}{\alpha_{p+1}}) = \infty$ implies the unboundedness of the monotonically increasing sequence $(b_n^{p+1})_{n \geq k_{p+1}}$. Moreover, the least element $b_{k_{p+1}}^{p+1}$ of this sequence satisfies the inequality

$$b_{k_{p+1}}^{p+1} > p + 1 + \varepsilon_2, \quad (4.1)$$

in view of $\varepsilon_2 < \frac{1}{3}$.

Lemma 4.1.1. *For every positive $p \in \omega$ and arbitrary finite graphs \mathcal{A} and \mathcal{B} , where \mathcal{A} is a proper subgraph of \mathcal{B} , $|\mathcal{B}| = n < k_{p+1}$, the following assertions hold:*

- (1) *We have $y_p(\mathcal{B}) - y(\mathcal{B}) < \varepsilon_{p+1}$.*
- (2) *The inequality $y_p(\mathcal{A}) < y_p(\mathcal{B})$ holds iff $y(\mathcal{A}) < y(\mathcal{B})$. The inequality $y_p(\mathcal{A}) < y_p(\mathcal{B})$ holds iff $y_q(\mathcal{A}) < y_q(\mathcal{B})$ for every $q \geq p$.*
- (3) *The least possible positive slope² $\text{sl}((p_1, y_1), (p_2, y_2))$ between the points (p_1, y_1) and (p_2, y_2) ($p_1 < p_2 \leq n$, $n \geq k_p$) in the lattice L_p coincides with the slope $\text{sl}((n, b_n^p), (n+1, b_{n+1}^p))$ between (n, b_n^p) and $(n+1, b_{n+1}^p)$.*
- (4) *If $q < p$ and $m \geq k_p$ then $b_m^p \leq b_m^q - 2\varepsilon_{q+1}$.*

²that is, the tangent of slope angle for the line segment, connecting points.

P r o o f. (1) Only the unordered pairs of distinct elements of B are used in counting the values $e_k(\mathcal{B})$. Each pair contributes at most one value $e_k(\mathcal{B})$ to calculation. The number of such pairs does not exceed $\frac{k_{p+1}(k_{p+1}-1)}{2}$. Therefore, $y_p(\mathcal{B})$ can differ from $y(\mathcal{B})$ by at most $\alpha_{p+1} \cdot \frac{k_{p+1}(k_{p+1}-1)}{2} < \varepsilon_{p+1}$.

(2) Since $y(\mathcal{A}) = y_q(\mathcal{A})$ and $y(\mathcal{B}) = y_q(\mathcal{B})$ starting from some q , it is sufficient to show the equivalence $y_p(\mathcal{A}) < y_p(\mathcal{B}) \Leftrightarrow y_q(\mathcal{A}) < y_q(\mathcal{B})$ for every $q \geq p$. Choose $q \geq p$ arbitrarily and suppose that $y_p(\mathcal{A}) < y_p(\mathcal{B})$. Then $y_p(\mathcal{A}) < y_p(\mathcal{B}) - \varepsilon_{p+1}$ by the definition of ε_{p+1} . In view of assertion (1), we have

$$y_q(\mathcal{A}) \leq y_p(\mathcal{A}) < y_p(\mathcal{B}) - \varepsilon_{p+1} < y(\mathcal{B}) \leq y_q(\mathcal{B}).$$

Therefore, $y_q(\mathcal{A}) < y_q(\mathcal{B})$.

To show the converse, suppose that $y_p(\mathcal{A}) \geq y_p(\mathcal{B})$ and, without loss of generality, assume that $A \neq B$. We claim that $y_p(\mathcal{A}) > y_p(\mathcal{B})$. Indeed, by hypothesis, the positive part $|A|$ of $y_p(\mathcal{A})$ is less than the positive part $|B|$ of $y_p(\mathcal{B})$. Since $y_p(\mathcal{A})$ and $y_p(\mathcal{B})$ have the forms $|A| - M_{\mathcal{A}} \cdot \alpha_p$ and $|B| - M_{\mathcal{B}} \cdot \alpha_p$ respectively, from the irrationality of α_p , it follows that $y_p(\mathcal{A}) \neq y_p(\mathcal{B})$.

We now obtain the inequality $y_q(\mathcal{A}) > y_q(\mathcal{B})$ from the following relations:

$$y_q(\mathcal{A}) \geq y(\mathcal{A}) > y_p(\mathcal{A}) - \varepsilon_{p+1} > y_p(\mathcal{B}) \geq y_q(\mathcal{B}).$$

(3) The minimal possible positive slope equals

$$\min \left\{ \frac{p_1 - \alpha_p \cdot m_1}{p_1} \mid \frac{p_1}{m_1} > \alpha_p, p_1 \leq n \right\}$$

which, in turn, equals $1 - \alpha_p \cdot \frac{l_n}{m_n}$, where $\frac{l_n}{m_n}$ is the best rational approximation of $\frac{1}{\alpha_p}$ satisfying $m_n \leq n$.

(4) It is sufficient to show that $b_m^{q+1} \leq b_m^q - 2\varepsilon_{q+1}$. This can be established by induction on m on using assertion (3) and the fact that L_{q+1} is a refinement of L_q . \square

2. Generic class and generic theory. Given a finite graph \mathcal{A} , we write $\mathcal{A} \in \mathbf{K}_{p+1}^{\text{bp}}$ iff $\mathcal{A} \in \mathbf{K}_p^{\text{bp}}$ and $y_p(\mathcal{A}') \geq b_n^p$ for any graph $\mathcal{A}' \subseteq \mathcal{A}$, where $n = |\mathcal{A}'|$, $k_p \leq n < k_{p+1}$.

Put $\mathbf{K}_0^{\text{bp}} \equiv \bigcap_{p=1}^{\infty} \mathbf{K}_p^{\text{bp}}$ and denote by \mathbf{K}^{bp} the class of all graphs

whose every finite subgraph belongs to \mathbf{K}_0^{bp} .

Let \mathcal{A} be a subgraph of \mathcal{M} , where \mathcal{M} belongs to \mathbf{K}^{bp} . We say that \mathcal{A} is a *self-sufficient subgraph* of \mathcal{M} and write $\mathcal{A} \leq \mathcal{M}$ if $y(\mathcal{A}) \leq y(\mathcal{B})$ for every finite graph \mathcal{B} , where $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M}$. If $\mathcal{A} \leq \mathcal{M}$ and \mathcal{M} is a finite graph then \mathcal{A} is called a *strong subgraph* of \mathcal{M} .

We have to show that the class \mathbf{T}_0^{bp} of all quantifier-free types corresponding to the graphs in \mathbf{K}_0^{bp} with the relation \leq' (where $\Phi(\mathcal{A}) \leq' \Psi(\mathcal{B}) \Leftrightarrow \mathcal{A} \leq \mathcal{B}$) is a self-sufficient generic class that (upon adding the necessary formulas to the types describing the self-sufficient closures) satisfies the property of uniform t -amalgamation. According to Theorem 2.5.1, this fact will yield the ω -saturation of the $(\mathbf{T}_0^{\text{bp}}; \leq')$ -generic model.

We start with the following:

Remark 4.1.2. (1) *The conditions $\mathcal{A} \in \mathbf{K}_0^{\text{bp}}$ and $k_p \leq |A| = n < k_{p+1}$ do not imply that $y(\mathcal{A}) \geq b_n^p$. Nevertheless, $y(\mathcal{A})$ cannot be much less than b_n^p : since $y_p(\mathcal{A}) - y(\mathcal{A}) < \varepsilon_{p+1}$, we have $y(\mathcal{A}) > b_n^p - \varepsilon_{p+1}$.*

Moreover, in view of (4.1), given a graph $\mathcal{A} \in \mathbf{K}_0^{\text{bp}}$ of cardinality $|A| \geq \max\{2, k_p\}$, the inequality $y(\mathcal{A}) > p$ holds.

(2) *Let \mathcal{A} be a graph in \mathbf{K}_0^{bp} , $|A| = p$, \mathcal{M} be a graph in \mathbf{K}^{bp} , and $\mathcal{A} \subseteq \mathcal{M}$. Then $\mathcal{A} \leq \mathcal{M}$ iff $y_p(\mathcal{A}) \leq y_p(\mathcal{B})$ for all finite graphs \mathcal{B} , where $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M}$.*

Indeed, if $|B| < k_{p+1}$ then $y_p(\mathcal{A}) \leq y_p(\mathcal{B})$ is equivalent to $y(\mathcal{A}) \leq y(\mathcal{B})$ by Lemma 4.1.1(2). If $|B| \geq k_p$ then

$$y_p(\mathcal{B}) \geq y(\mathcal{B}) > p \geq y_p(\mathcal{A}).$$

Moreover, in order to verify that \mathcal{A} is self-sufficient in \mathcal{M} , it suffices to choose $n_{\mathcal{A}} = k_p$ and check the validity of the relation $y_p(\mathcal{A}) \leq y_p(\mathcal{B})$ only for the graphs \mathcal{B} , $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M}$, satisfying $|B| < n_{\mathcal{A}}$.

Therefore, the condition $\mathcal{A} \leq \mathcal{M}$ is definable by a formula that describes the absence of $n < n_{\mathcal{A}}$ new elements in $M \setminus A$ such that $n < \alpha_1 \cdot e_1 + \dots + \alpha_p \cdot e_p$, where $p = |A|$, e_s is the number of new I_s -arcs.

The following lemma is a direct consequence of the definition:

Lemma 4.1.3. (1) If $\mathcal{A} \leq \mathcal{B}$ then $\mathcal{A} \subseteq \mathcal{B}$.

(2) If $\mathcal{A} \leq \mathcal{C}$, $\mathcal{B} \in \mathbf{K}_0^{\text{bp}}$, and $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{A} \leq \mathcal{B}$.

(3) The empty graph \emptyset is the least element of the structure $(\mathbf{K}_0^{\text{bp}}; \leq)$.

Lemma 4.1.4. If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}_0^{\text{bp}}$, $\mathcal{A} \leq \mathcal{B}$, and $\mathcal{C} \subseteq \mathcal{B}$ then $\mathcal{A} \cap \mathcal{C} \leq \mathcal{C}$.

P r o o f. Assume the contrary. Then there exist n new elements in $\mathcal{C} \setminus \mathcal{A}$ such that $n < \alpha_1 \cdot e_1 + \dots + \alpha_p \cdot e_p$, where e_s is the number of new I_s -arcs. Since all new elements lie in $\mathcal{B} \setminus \mathcal{A}$, they violate the condition $\mathcal{A} \leq \mathcal{B}$. \square

Lemma 4.1.5. The relation \leq is a partial order on \mathbf{K}_0^{bp} .

P r o o f. The reflexivity and antisymmetry of \leq are evident.

We show that \leq is transitive. Assume the contrary and consider finite graphs $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}_0^{\text{bp}}$ such that $\mathcal{A} \leq \mathcal{B}$, $\mathcal{B} \leq \mathcal{C}$, and $\mathcal{A} \not\leq \mathcal{C}$. Denote by p the cardinality of \mathcal{A} . Choose a graph \mathcal{A}' , $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{C}$, with the least possible value $y_p(\mathcal{A}')$. This value is attainable by a graph of cardinality less than $n_{\mathcal{A}}$, since the set $\{y_p(\mathcal{D}) \mid \mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{C}, y_p(\mathcal{D}) \leq y_p(\mathcal{A})\}$ is a subset of the finite set $\{n - \alpha_p \cdot m \mid n < n_{\mathcal{A}}, m < \frac{n_{\mathcal{A}}}{\alpha_p}\}$.

We claim that \mathcal{A}' is a minimal strong subgraph of \mathcal{C} containing \mathcal{A} . Indeed, we have $\mathcal{A}' \leq \mathcal{C}$, since, for any graph \mathcal{D} with $\mathcal{A}' \subseteq \mathcal{D} \subseteq \mathcal{C}$, if $|\mathcal{D}| < k_{p+1}$ then $y(\mathcal{D}) \geq y(\mathcal{A}')$ in view of $y_p(\mathcal{D}) \geq y_p(\mathcal{A}')$ by Lemma 4.1.1(2); and if $|\mathcal{D}| \geq k_{p+1}$ then $y(\mathcal{D}) > y(\mathcal{A}) \geq y(\mathcal{A}')$. Moreover, every subgraph \mathcal{D} with $\mathcal{A} \subseteq \mathcal{D} \subsetneq \mathcal{A}'$ is not a strong subgraph of \mathcal{C} , since the minimality of $y_p(\mathcal{A}')$ implies $y_p(\mathcal{D}) \geq y_p(\mathcal{A}')$, and the relation $|\mathcal{A}'| \neq |\mathcal{D}|$ yields $y(\mathcal{D}) > y(\mathcal{A}')$ by Lemma 4.1.1(2).

The minimality of \mathcal{A}' and the existence of a strong subgraph \mathcal{B} of \mathcal{C} imply that $\mathcal{A}' \subsetneq \mathcal{C}$. Now, from the inequality $|\mathcal{A}'| \neq |\mathcal{C}|$ and from the irrationality of α_s , we conclude that $y_p(\mathcal{A}') \neq y_p(\mathcal{C})$, and Lemma 4.1.1(2) yields $y(\mathcal{A}') < y(\mathcal{C})$. By Lemma 4.1.4, we have $\mathcal{B} \cap \mathcal{A}' \leq \mathcal{A}'$. Therefore, $\mathcal{B} \cap \mathcal{A}' = \mathcal{A}'$ and $\mathcal{A}' \subseteq \mathcal{B}$. The last relation contradicts $\mathcal{A} \leq \mathcal{B}$. \square

Let $\mathcal{A}, \mathcal{B} = \langle B; I_{p,\mathcal{B}}, J_{0,\mathcal{B}}, J_{1,\mathcal{B}} \rangle$, and $\mathcal{C} = \langle C; I_{p,\mathcal{C}}, J_{0,\mathcal{C}}, J_{1,\mathcal{C}} \rangle$ be graphs, with $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$. The *free amalgam* of \mathcal{B} and \mathcal{C} over \mathcal{A} (written $\mathcal{B} *_{\mathcal{A}} \mathcal{C}$) is the structure $\langle B \cup C; I_{p,\mathcal{B}} \cup I_{p,\mathcal{C}}, J_{0,\mathcal{B}} \cup J_{0,\mathcal{C}}, J_{1,\mathcal{B}} \cup J_{1,\mathcal{C}} \rangle$.

Observe that, in constructing a free amalgam, the initials (endpoints) of arcs can be joined only with the initials (endpoints) of

arcs. Therefore, the structure $\mathcal{B} *_{\mathcal{A}} \mathcal{C}$ is a graph containing \mathcal{A} , \mathcal{B} , and \mathcal{C} as subgraphs. This graph is bipartite with irreflexive and antisymmetric pairwise disjoint relations I_p and unary relations J_0 and J_1 that have no common elements and form a partition of the universe in two parts so that if $(a, b) \in I_p$ then $a \in J_0$ and $b \in J_1$.

An embedding f of a graph \mathcal{A} in a graph \mathcal{B} ($f : \mathcal{A} \rightarrow \mathcal{B}$) is said to be *strong* if $f(\mathcal{A}) \leq \mathcal{B}$.

Lemma 4.1.6. (*amalgamation lemma*). *The class \mathbf{K}_0^{bp} has the amalgamation property (AP), that is, for any strong embeddings $f_0 : \mathcal{A} \rightarrow \mathcal{B}$ and $g_0 : \mathcal{A} \rightarrow \mathcal{C}$, where $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}_0^{\text{bp}}$, there exist a graph $\mathcal{D} \in \mathbf{K}_0^{\text{bp}}$ and strong embeddings $f_1 : \mathcal{B} \rightarrow \mathcal{D}$ and $g_1 : \mathcal{C} \rightarrow \mathcal{D}$ such that $f_0 \circ f_1 = g_0 \circ g_1$.*

P r o o f. Without loss of generality we may assume that $\mathcal{A} \leq \mathcal{B}$, $\mathcal{A} \leq \mathcal{C}$, and $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$. We claim that the graph $\mathcal{D} \doteq \mathcal{B} *_{\mathcal{A}} \mathcal{C}$ is as required. To this end, by the symmetry of the definition of a free amalgam, it is sufficient to show that $\mathcal{B} \leq \mathcal{D}$ and $\mathcal{D} \in \mathbf{K}_0^{\text{bp}}$.

Assume that $\mathcal{B} \not\leq \mathcal{D}$. Then there exist n new elements in $D \setminus B$ such that $n < \alpha_1 \cdot e_1 + \dots + \alpha_p \cdot e_p$, where $p = |B|$ and e_s is the number of new I_s -arcs. Since all new elements lie in $C \setminus A$, they violate the condition $\mathcal{A} \leq \mathcal{C}$.

Since each subgraph of \mathcal{D} has the form $\mathcal{B}_0 *_{\mathcal{A}_0} \mathcal{C}_0$, where $\mathcal{A}_0 \leq \mathcal{B}_0$ and $\mathcal{A}_0 \leq \mathcal{C}_0$, in order to show that $\mathcal{D} \in \mathbf{K}_0^{\text{bp}}$, it suffices to verify that $y_p(\mathcal{D}) \geq b_n^p$, where $n = |D|$, $k_p \leq n < k_{p+1}$. Suppose that $|B| \leq |C| = m$, $k_q \leq m < k_{q+1}$, and $\mathcal{A} \subsetneq \mathcal{B}$. By Lemma 4.1.1(3) and by condition $\mathcal{A} \leq \mathcal{B}$, it follows that

$$\text{sl}((|A|, y_q(\mathcal{A})), (|B|, y_q(\mathcal{B}))) \geq \text{sl}((l, b_l^q), (l+1, b_{l+1}^q))$$

for any $l \geq m$. Then the equality

$$\text{sl}((|C|, y_q(\mathcal{C})), (|D|, y_q(\mathcal{D}))) = \text{sl}((|A|, y_q(\mathcal{A})), (|B|, y_q(\mathcal{B})))$$

yields

$$\text{sl}((|C|, y_q(\mathcal{C})), (|D|, y_q(\mathcal{D}))) \geq \text{sl}((m, b_m^q), (|D|, b_{|D|}^q)). \quad (4.2)$$

The fact that \mathcal{C} belongs to \mathbf{K}_0^{bp} implies that the point $(|C|, y_q(\mathcal{C}))$ is above the point (m, b_m^q) . Then, by (4.2), we conclude that $(|D|, y_q(\mathcal{D}))$ is above $(|D|, b_{|D|}^q)$, i.e., $y_q(\mathcal{D}) \geq b_{|D|}^q$.

If $p = q$ then the required inequality $y_p(\mathcal{D}) \geq b_n^p$ holds. Otherwise, we have $q < p$ and then $y_p(\mathcal{D})$ differs from $y_q(\mathcal{D})$ by less than $\alpha_{q+1} \cdot k_{q+1} \cdot (k_{q+1} - 1) < 2\varepsilon_{q+1}$, since \mathcal{D} contains less than $k_{q+1} \cdot (k_{q+1} - 1)$ arcs. By Lemma 4.1.1(4), we conclude that $y_p(\mathcal{D}) \geq b_n^p$. \square

Lemmas 4.1.3–4.1.6 imply

Corollary 4.1.7. *The class $(\mathbf{T}_0^{\text{bp}}; \leq')$ is self-sufficient.*

Denote the $(\mathbf{T}_0^{\text{bp}}; \leq')$ -generic theory by T^{bp} .

We claim that, upon adding a formula $\chi_{\overline{\Phi}}(\overline{A})$ such that $(T^{\text{bp}}, \overline{A}) \vdash \chi_{\overline{\Phi}}(\overline{A})$ to every quantifier-free type $\overline{\Phi}(\overline{A}) \in \mathbf{T}_0^{\text{bp}}$, we obtain a self-sufficient class $(\mathbf{T}_0; \leq'')$ satisfying the uniform t -amalgamation property.

Indeed, in view of Remark 4.1.2 for every graph $\mathcal{A} \in \mathbf{K}_0^{\text{bp}}$ of cardinality p and every graph $\mathcal{M} \models 1T^{\text{bp}}$, where $\mathcal{A} \subseteq \mathcal{M}$, we have:

$$\mathcal{A} \leq \mathcal{M} \Leftrightarrow y_p(\mathcal{A}) \leq y_p(\mathcal{B}) \text{ for any graph } \mathcal{B}$$

$$\text{with } \mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M} \text{ and } |\mathcal{B}| \leq k_p.$$

Since the cardinalities of the graphs \mathcal{B} are bounded only in terms of the cardinality of \mathcal{A} and since checking the condition $y_p(\mathcal{A}) \leq y_p(\mathcal{B})$ involves only counting the links over the relations I_1, \dots, I_p , the self-sufficiency condition $\mathcal{A} \leq \mathcal{M}$ is expressible by a universal formula $\chi_{\mathcal{A}}(X)$ of the language $\{I_1, \dots, I_p\}$, where the set of variables X is bijective with A .

Let \mathcal{A} and \mathcal{B} be graphs in \mathbf{K}_0^{bp} , \mathcal{M} be a generic model of the theory T^{bp} with $\mathcal{A} \leq \mathcal{B} \leq \mathcal{M}$. Denote by $\psi_{\mathcal{A},s}(X)$ (by $\psi_{\mathcal{B},s}(X, Y)$, respectively) a quantifier-free formula that describes the quantifier-free $\{I_1, \dots, I_s, J_0, J_1\}$ -type of \mathcal{A} (of \mathcal{B}), where X and Y are disjoint sets of variables bijective with A and $B \setminus A$, respectively. Then, for every $s \geq |B|$, the following formula is true in \mathcal{M} :

$$\forall X ((\chi_{\mathcal{A}}(X) \wedge \psi_{\mathcal{A},s}(X)) \rightarrow \exists Y (\chi_{\mathcal{B}}(X, Y) \wedge \psi_{\mathcal{B},s}(X, Y))).$$

The last relation implies the uniform t -amalgamation property for the class $(\mathbf{T}_0; \leq'')$ that we obtain from $(\mathbf{T}_0^{\text{bp}}; \leq')$ by adding the formulas to the types which establish the cardinality bounds and the $\{I_1, \dots, I_p, J_0, J_1\}$ -structures of the self-sufficient closures, as well as the formulas $\chi_{\mathcal{A}}(A)$ to the types of self-sufficient sets A .

The addition of the above-mentioned formulas ensures the existence of finite closures for arbitrary finite sets of models of T^{bp} .

In view of Theorem 2.5.1, the following holds:

Theorem 4.1.8. *A $(\mathbf{T}_0^{\text{bp}}; \leq')$ -generic model \mathcal{M} is saturated. In addition, each finite set $A \subseteq M$ can be extended to its self-sufficient closure $\bar{A} \subseteq M$, and the type $\text{tp}_X(\bar{A})$ is deducible from the set $\{\chi_{\bar{A}}(X)\} \cup \{\psi_{\bar{A},s}(X) \mid s \in \omega \setminus \{0\}\}$.*

Let \mathcal{N} be an ω -saturated model of T^{bp} .

Proposition 4.1.9. *For each finite set A in \mathcal{N} , we have $\text{acl}(A) = \bar{A}$.*

P r o o f. The inclusion $\bar{A} \subseteq \text{acl}(A)$ follows from the uniqueness of \bar{A} in \mathcal{N} . We now take an arbitrary element $b \in N \setminus \bar{A}$ and put $B = \overline{A \cup \{b\}}$. By the construction of a generic model, there exist infinitely many isomorphic pairwise disjoint copies of the set $B \setminus A$ over A , that is, $b \notin \text{acl}(A)$. \square

3. Stability of the generic theory. We claim that the generic theory T^{bp} is stable. Let \mathcal{N} be a sufficiently saturated model of T^{bp} . The *rank function* in \mathcal{N} is the function

$$r_{\mathcal{N}} : \{\mathcal{A} \mid \mathcal{A} \text{ is a finite subgraph of } \mathcal{N}\} \rightarrow \mathbb{R}^+,$$

that is defined by $r_{\mathcal{N}}(\mathcal{A}) = \inf\{y(\mathcal{B}) \mid \mathcal{A} \subseteq \mathcal{B} \subseteq_{\text{fin}} \mathcal{N}\}$ (here and elsewhere the notation $\mathcal{B} \subseteq_{\text{fin}} \mathcal{N}$ means that \mathcal{B} is a finite substructure of \mathcal{N} , and $B \subseteq_{\text{fin}} N$ means that B is a finite subset of N).

Observe that, in view of the self-sufficiency of the class $(\mathbf{T}_0^{\text{bp}}; \leq')$, the infimum in the above equality is always attainable and coincides with $y(\bar{\mathcal{A}})$, i.e., $r_{\mathcal{N}}(\mathcal{A}) = y(\bar{\mathcal{A}})$.

In what follows, we fix a model \mathcal{N} , denote the function $r_{\mathcal{N}}$ by r for brevity, and replace the graphs \mathcal{A} by their universes in \mathcal{N} . We also assume that all sets under consideration are subsets of N .

The *relative prerank function* $y(A/B)$ of a graph A over a graph B is given by the relation

$$y(A/B) = y(A \cup B) - y(B)$$

and the *relative rank function* $r(A/B)$ of A over B is given by

$$r(A/B) = r(A \cup B) - r(B).$$

We establish some intermediate properties of the relative rank function.

Clearly, $r(A/B) \geq 0$.

Lemma 4.1.10. *For any finite sets B and C , we have*

$$y(B) + y(C) \geq y(B \cup C) + y(B \cap C).$$

*The inequality becomes an equality iff $\mathcal{B} \cup \mathcal{C} = \mathcal{B} *_{\mathcal{B} \cap \mathcal{C}} \mathcal{C}$.*

P r o o f. By definition, the value

$$y(B) + y(C) - y(B \cup C) - y(B \cap C)$$

equals $\sum_{k=1}^{\infty} \alpha_k \cdot e_k$, where e_k is the number of I_k -arcs that connect the elements of $B \setminus C$ with the elements of $C \setminus B$. Therefore, the required inequality holds and becomes an equality precisely in the absence of the specified I_k -arcs, i.e., when the graph with universe $B \cup C$ is the free amalgam $\mathcal{B} *_{\mathcal{B} \cap \mathcal{C}} \mathcal{C}$. \square

Lemma 4.1.11. *If $B_1 \subseteq B_2$ then $r(A/B_1) \geq r(A/B_2)$.*

P r o o f. Put $C \doteq \overline{A \cup B_1}$, $D \doteq \overline{B_1}$, $E \doteq \overline{B_2}$. By Lemma 4.1.10 and in view of $D \subseteq C$, $D \subseteq E$, and $D \leq C \cap E$, it follows that

$$y(C) + y(E) \geq y(C \cup E) + y(C \cap E) \geq y(C \cup E) + y(D).$$

Then $y(C) - y(D) \geq y(C \cup E) - y(E)$. This implies

$$\begin{aligned} r(A/B_1) &= y(C) - y(D) \geq y(C \cup E) - y(E) \geq y(\overline{C \cup E}) - y(E) = \\ &= r(A \cup B_2) - r(B_2) = r(A/B_2). \quad \square \end{aligned}$$

If A is a finite set and X is some (not necessarily finite) set then

$$r(A/X) \doteq \inf\{r(A/B) \mid B \subseteq_{\text{fm}} X\}.$$

Lemma 4.1.12. *Let A_1 and A_2 be finite sets and let X be a set such that $A_1 \setminus X = A_2 \setminus X$. Then $r(A_1/X) = r(A_2/X)$.*

P r o o f. Assume the contrary. Without loss of generality we may assume that $r(A_1/X) = r(A_2/X) + \varepsilon$ for some $\varepsilon > 0$. Choose a finite set $B_2 \subseteq X$ such that $r(A_2/B_2) < r(A_2/X) + \varepsilon$. Denote the set $(A_1 \cap X) \cup (A_2 \cap X) \cup B_2$ by C . Then

$$r(A_1/X) \leq r(A_1/C) = r(A_2/C) \leq r(A_2/B_2) < r(A_2/X) + \varepsilon.$$

We have $r(A_1/X) < r(A_2/X) + \varepsilon$, which contradicts the choice of ε . \square

A set $X \subseteq N$ is called *closed* (in \mathcal{N}) (written $X \leq N$) if $\overline{A} \subseteq X$ for any finite set $A \subseteq X$. The closeness is equivalent to the absence in $N \setminus X$ of any new n elements such that $n < \sum_{k=1}^{\infty} \alpha_k \cdot e_k$, where e_k is the number of new I_k -arcs.

Lemma 4.1.13. *Let X and Y be sets in \mathcal{N} . Then the following hold:*

- (1) *if $X \leq N$ and $Y \leq N$ then $X \cap Y \leq N$;*
- (2) *there exists a least closed set $\overline{X} \supseteq X$; moreover, $\overline{X} = \bigcup\{\overline{A} \mid A \text{ is a finite subset of } X\}$, and $\overline{X} = \text{acl}(X)$;*
- (3) *if $X \subset Y$ then $\overline{X} \subseteq \overline{Y}$.*

The set \overline{X} is called the *intrinsic closure* of X (in \mathcal{N}).

P r o o f. (1) Suppose that $X \leq N$ and $Y \leq N$. Let A be a finite subset of $X \cap Y$. Since $A \subset X$, $A \subset Y$, and the sets \overline{X} and \overline{Y} are closed, we have $\overline{A} \subseteq \overline{X}$ and $\overline{A} \subseteq \overline{Y}$. Consequently, $\overline{A} \subseteq X \cap Y$. Therefore, $X \cap Y$ is closed.

(2) Denote the set $\bigcup\{\overline{A} \mid A \text{ is a finite subset of } X\}$ by Z . Clearly, each closed superset of X includes Z and $X \subseteq Z$. On the other hand, if A is a finite subset of Z then there exists a finite set $B \subseteq X$ such that $A \subseteq \overline{B}$. So $\overline{A} \subseteq \overline{B} \subseteq Z$. Therefore, Z is the least closed set containing X . The equality $\overline{X} = \text{acl}(X)$ now follows from Proposition 4.1.9.

(3) It is sufficient to observe that, for arbitrary finite graphs \mathcal{A} and \mathcal{B} , if $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$. Indeed, $A \subseteq \overline{A \cap B} \leq N$. Consequently, $\overline{A} \subseteq \overline{A \cap B}$ and $\overline{A} \subseteq \overline{B}$. \square

By analogy with Definition 3.30 in the work by J. T. Baldwin and N. Shi [37], we say that finite sets A and B are *independent* over Z and write $A \downarrow_Z^r B$ if $r(A/Z) = r(A/(Z \cup B))$ and $\overline{A \cup Z} \cap \overline{B \cup Z} \subseteq \overline{Z}$.

We say that sets X and Y are *independent* over Z and write $X \downarrow_Z^r Y$ if $A \downarrow_Z^r B$ for any finite sets $A \subseteq X$ and $B \subseteq Y$.

The following assertion is an analog of Lemma 3.31 in the work by J. T. Baldwin and N. Shi [37]:

Lemma 4.1.14. *If X and Y are closed sets, $Z = X \cap Y$, and $X \downarrow_Z^r Y$, then $X \cup Y$ is closed.*

P r o o f. First, observe that, by Lemma 4.1.13(1), the set Z is closed.

Assume now, that $X \cup Y$ is not closed. Then there exist finite sets $A \subseteq X$, $B \subseteq Y$, and C such that $(A \cup B, C)$ is a minimal pair and C is not contained in $A \cup B$. Put $\varepsilon = y(A \cup B) - y(C)$. By the choice of A , B , and C , for any finite sets A' and B' with $A \subseteq A' \subseteq X$ and $B \subseteq B' \subseteq Y$, we have $y(C/(A' \cup B')) \leq -\varepsilon$. Then $r(A' \cup B' \cup C) \leq y(A' \cup B' \cup C) \leq y(A' \cup B') - \varepsilon$. In order to arrive at a contradiction, using the independence of X and Y over Z , we choose a pair (A_1, B_1) such that $A \subseteq A_1 \subseteq X$, $B \subseteq B_1 \subseteq Y$, and $r(A_1 \cup B_1 \cup C) > y(A_1 \cup B_1) - \varepsilon$.

Since $X \downarrow_Z^r Y$, we have $r(A/Z) = r(A/(Z \cup B))$. Choose a finite set $Z_0 \subseteq Z$ such that $r(A/Z_0) < r(A/Z) + \frac{\varepsilon}{2}$. Put $A_0 = A \cup Z_0$, $B_0 = B \cup Z_0$ and $D_0 = A_0 \cap B_0$. By Lemma 4.1.12, we have $r(A/Z) = r(A_0/Z)$. Moreover, from $A \cup Z_0 = A_0$ it follows that $r(A/Z_0) = r(A_0/Z_0) \geq r(A_0/D_0)$. Therefore, $r(A_0/D_0) < r(A_0/Z) + \frac{\varepsilon}{2}$.

Put $D_1 = \overline{D_0}$, $A_1 = \overline{A_0}$, and $B_1 = \overline{B_0}$. Clearly, $y(A_1/D_1) = r(A_1/D_1)$ and $|y(A_1/D_1) - r(A_0/Z_0)| < \frac{\varepsilon}{2}$. In addition, $D_1 \subseteq A_1 \cap B_1$.

We claim that A_1 and B_1 are the required sets. Since $r(A_1 \cup B_1 \cup C) \geq r(A_0 \cup B_0)$, it is sufficient to show that $r(A_0 \cup B_0) > y(A_1 \cup B_1) - \varepsilon$. We have

$$\begin{aligned} r(A_0 \cup B_0) &= r(A_0/B_0) + r(B_0) \geq r(A_0/B_0 \cup Z) + r(B_0) = \\ &r(A_0/Z) + r(B_0) > r(A_0/Z_0) - \frac{\varepsilon}{2} + r(B_0) > \\ &\left(y(A_1/D_1) - \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{2} + y(B_1) = y(A_1/D_1) + y(B_1) - \varepsilon. \end{aligned}$$

In view of the self-sufficiency of D_1 , we have $y((A_1 \cap B_1)/D_1) \geq 0$. Then

$$\begin{aligned} &y(A_1/D_1) + y(B_1) - \varepsilon = \\ &y(A_1/(A_1 \cap B_1)) + y((A_1 \cap B_1)/D_1) + y(B_1) - \varepsilon \geq \end{aligned}$$

$$y(A_1/(A_1 \cap B_1)) + y(B_1) - \varepsilon \geq y(A_1 \cup B_1) - \varepsilon.$$

Comparing the first and last expressions, we obtain

$$r(A_0 \cup B_0) > y(A_1 \cup B_1) - \varepsilon. \quad \square$$

The following definition naturally generalizes the above notion of a free amalgam of graphs. A set U (in \mathcal{N}) is called a *free amalgam* of X and Y over Z and denoted by $X *_Z Y$ if $X \cup Y = U$, $X \cap Y = Z$, and there are no arcs that connect elements in $X \setminus Y$ with elements in $Y \setminus X$.

Lemma 4.1.15. *If X is a self-sufficient set, Y is a closed set, and $Z = \overline{X \cup Z} \cap Y$, then $X \downarrow_Z^r Y$ iff $\overline{X \cup Z} \cup Y$ is a closed set that coincides with the free amalgam $\overline{X \cup Z} *_Z Y$.*

P r o o f. Denote $\overline{X \cup Z}$ by X' .

Suppose that $X \downarrow_Z^r Y$. The fact that $X' \cup Y$ is closed was shown in Lemma 4.1.14. We prove that $X' \cup Y = X' *_Z Y$. Assume to the contrary that $X' \cup Y \neq X' *_Z Y$, i.e., there exists an I_p -arc that connects $x \in X' \setminus Z$ with $y \in Y \setminus Z$. Choose a finite set $Z_0 \subseteq Z$ such that $x \in \overline{X \cup Z_0}$. Then, for every finite self-sufficient set $Y_0 \supseteq Z_0 \cup \{y\}$, we have $r(X/Z \cap Y_0) \geq r(X/Y_0) + \alpha_p$ and, therefore, $r(X/Z) \geq r(X/Y) + \alpha_p$. The latter contradicts the equality $r(X/Z) = r(X/Y)$ which is a consequence $X \downarrow_Z^r Y$.

Suppose now, that $X' \cup Y$ is a closed set that coincides with the free amalgam $\overline{X \cup Z} *_Z Y$. We show that $X \downarrow_Z^r Y$. It is sufficient to establish that

$$\inf\{r(X/A) \mid A \subseteq_{\text{fin}} Z\} \leq \inf\{r(X/B) \mid B \subseteq_{\text{fin}} Y\}.$$

Choose an arbitrary finite set $B \subseteq Y$. Denote by C the self-sufficient set $\overline{B \cup X}$ which is contained in $X' \cup Y$ by hypothesis. Since $C = (C \cap X') *_Z (C \cap Y)$ and the set $C = (C \cap X') \cup (C \cap Y)$ is self-sufficient, it follows that $\overline{X \cup (C \cap Z)} \cup (C \cap Y)$ is also self-sufficient and coincides with $\overline{X \cup (C \cap Z)} *_Z C \cap Y$. Since $B \subseteq C \cap Y$, we have $r(X/B) \geq r(X/C \cap Y)$. It remains to observe that $r(X/C \cap Z) = r(X/C \cap Y)$. Indeed, denoting by X'' the set $\overline{X \cup (C \cap Z)}$, we obtain

$$\begin{aligned} r(X/C \cap Z) &= y(X'') - y(C \cap Z) \geq y(X'') - y(X'' \cap (C \cap Y)) \geq \\ &y(X'' \cup (C \cap Y)) - y(C \cap Y) \geq r(X''/C \cap Y) = r(X/C \cap Y). \end{aligned}$$

The first inequality is an equality due to $C \cap Z = X'' \cap (C \cap Y)$. The second inequality is an equality by Lemma 4.1.10 in view of $X'' \cup (C \cap Y) = X'' *_{C \cap Z} (C \cap Y)$. The third inequality is also an equality, since the set $X'' \cup (C \cap Y)$ is self-sufficient. \square

Lemma 4.1.15 readily implies

Corollary 4.1.16. *If X is a self-sufficient set, Y is a closed set, $Z = \overline{X \cup Z} \cap Y$, and $X \downarrow_Z^r Y$ then the type $\text{tp}(X/Y)$ is uniquely defined by the type $\text{tp}(X/Z)$, a description of $\overline{X \cup Z \cup Y}$ to be closed, and a quantifier-free type that describes the coincidence of $\overline{X \cup Z \cup Y}$ with the free amalgam $X *_Z Y$.*

Lemma 4.1.17. *If X is a closed set and a is an element in \mathcal{N} not belonging to X , then there exists at most countable, closed subset $X' \subseteq X$ such that $\{a\} \downarrow_{X'}^r X$.*

P r o o f is similar to that of Lemma 3.32 in the work by J. T. Baldwin and N. Shi [37]. We choose a sequence $X_n, n \in \omega$, of finite subsets of X such that $X_{n-1} \subseteq X_n$ and $r(\{a\}/X_n) - r(\{a\}/X) < \frac{1}{n}, n \in \omega \setminus \{0\}$. Put $U \equiv \bigcup_{n \in \omega} X_n$ and $X' \equiv \overline{X \cap \{a\} \cup U}$. The set X' is closed and at most countable in view of Lemma 4.1.13. Applying Lemma 4.1.11, we infer $r(\{a\}/X') = r(\{a\}/X) = r(\{a\}/(X' \cup X))$. In addition, $(\overline{\{a\} \cup X'}) \cap X = X'$. Consequently, $\{a\} \downarrow_{X'}^r X$. \square

Recall the notion of a weight (see S. Shelah [22]). Let p be a type of a theory T , λ be a cardinal. The *weight* $w(p)$ of type p is greater than or equal to λ if there exists a nonforking extension $\text{tp}(a/A)$ of p and a sequence $(a_i)_{i \in \lambda}$ independent over A such that every a_i is dependent on a over A . We assign $w(p) \equiv \lambda$ if $w(p) \geq \lambda$ and $w(p) \not\equiv \lambda^+$.

Theorem 4.1.18. (1) *The theory T^{bp} is stable, small, and has precisely two 1-types (of elements of color J_0 and those of color J_1).*

(2) *Each 1-type of T^{bp} has infinite weight; namely, for every element a of color $J_i, i = 0, 1$, there are infinitely many independent elements $b_n, n \in \omega$, of color J_{1-i} that depend on a .*

P r o o f. (1) Theorem 4.1.8 implies that T^{bp} is small. In order to prove the stability of T^{bp} , we estimate the number of 1-types in $S(N)$, where \mathcal{N} is a model of T^{bp} . Consider an arbitrary element a . By Lemma 4.1.17, there exists at most countable, closed set $X \subseteq N$ such that $\overline{\{a\} \cup X} \cap N = X$ and $\{a\} \downarrow_X^r N$. By Corollary

4.1.16, the type $\text{tp}(a/N)$ is defined by $\text{tp}(a/X)$, by a description of the closeness of $\overline{\{a\} \cup X} \cup N$, and by a quantifier-free type that describes the coincidence of $\overline{\{a\} \cup X} \cup N$ with the free amalgam $\overline{\{a\} \cup X} *_X N$. Therefore, in order to count the number of types in $S(N)$, it is sufficient to count the number of types in $S(X)$ and the number of ways of choosing the countable set X in N . Then

$$|S(N)| \leq 2^\omega \cdot |N|^\omega = |N|^\omega.$$

Consequently, T^{bp} is stable.

By construction, every singleton $\{a\}$ is self-sufficient. In view of Theorem 4.1.8, this means that there exist precisely two 1-types of T^{bp} ; moreover, one of them contains the formula $J_0(x)$ and the other contains $J_1(x)$.

(2) Let \mathcal{A} be a graph with universe $\{a\}$, and let $A_p = \{a, b_p\}$ be the universe of a two-element graph \mathcal{A}_p that contains the I_p -arc between a and b_p . Put $\mathcal{B}_1 \equiv \mathcal{A}_1$, $\mathcal{B}_{p+1} \equiv \mathcal{B}_p *_A \mathcal{A}_{p+1}$, $\mathcal{B} \equiv \bigcup_{p \in \omega \setminus \{0\}} \mathcal{B}_p$. By construction, \mathcal{B} is a closed subgraph of some generic \mathcal{M} of T^{bp} . Since $\alpha_1 + \dots + \alpha_{p+1} < 1$ and $\{b_1, \dots, b_{p+1}\} \leq \{a, b_1, \dots, b_{p+1}\} \leq B \leq M$, we have $\{b_{p+1}\} \downarrow_{\emptyset}^r \{b_1, \dots, b_p\}$. Therefore, $(b_p)_{p \in \omega \setminus \{0\}}$ is an infinite independent sequence of elements of the same type, where each b_p is dependent on a . \square

Let \mathcal{M} be a model, $\varphi(x)$, $\psi(x, y)$ be formulas of $\text{Th}(\mathcal{M})$, and X be a set in \mathcal{M} , which is defined by $\varphi(x)$: $X = \varphi(\mathcal{M})$. We say that X has the *pairwise ψ -intersection property* if

$$\mathcal{M} \models \forall x, y (\varphi(x) \wedge \varphi(y) \rightarrow \exists z (\psi(z, x) \wedge \psi(z, y))).$$

Since $\alpha_p < \frac{1}{2}$ for every natural $p \geq 1$, it follows that arbitrary elements a and \bar{b} of color J_0 are connected via an element c of color J_1 so that $aI_p c$ and $bI_p c$. In the same way, arbitrary elements a and b of color J_1 are connected via an element c of color J_1 so that $cI_p a$ and $cI_p b$. Furthermore, in either case, there exist infinitely many such elements c . Thus, we have

Proposition 4.1.19. *For every natural $p \geq 1$, the set of elements of color J_i has the pairwise $I_p^{(-1)^{1-i}}$ -intersection property, $i = 0, 1$.*

The following two assertions clarify the structure of the prime models \mathcal{M}_A over finite sets A .

Lemma 4.1.20. *If A and B are self-sufficient sets in \mathcal{N} and $A \leq B$ then the type $\text{tp}(B/A)$ is isolated iff \mathcal{B} is a complete bipartite graph over \mathcal{A} , i.e., arbitrary two elements $a \in B$ and $b \in B \setminus A$, different in J -color, are connected by an I_p -arc.*

P r o o f. Let Y be a set of variables bijective with $B \setminus A$. If \mathcal{B} is a complete bipartite graph over \mathcal{A} then Theorem 4.1.8 implies that the type $\text{tp}((B \setminus A)/A)$ is isolated by the formula $\chi_{\overline{\mathcal{B}}}(A \cup Y) \wedge \psi_{\overline{\mathcal{B}},s}(A \cup Y)$, where s is a number that exceeds all color numbers for the arcs in \mathcal{B} . If \mathcal{B} is not a complete bipartite graph over \mathcal{A} , then Theorem 4.1.8 implies that $\text{tp}((B \setminus A)/A)$ is isolated by the set of formulas $\{\chi_{\overline{\mathcal{B}}}(A \cup Y)\} \cup \{\psi_{\overline{\mathcal{B}},s}(A \cup Y) \mid s \in \omega\}$, but is not isolated by finite parts of this set. \square

Lemma 4.1.20 implies

Corollary 4.1.21. *Let A be a self-sufficient set in \mathcal{N} . The model \mathcal{M}_A is a complete bipartite graph over A . The set of isomorphism types of prime models over finite sets coincides with the set of isomorphism types of the models of form \mathcal{M}_A , where A is a self-sufficient set such that \mathcal{M}_A is a complete bipartite graph over A .*

§ 4.2. Small stable generic graphs with infinite weight. Digraphs without furcations

In the present Section, we introduce the concept of a digraph without furcations and, on the basis of Section 4.1, describe a generic construction that yields a family of small *stable* digraphs without furcations such that all 1-types of non-intermediate elements have infinite weight. We thus solve the existence problem for small stable theories of finite (graph) language that possess types with infinite weight.

1. Definitions and properties. Let $\mathcal{A} = \langle A, Q, W \rangle$ be a c -graph. A vertex $a \in A$ is an *upper (lower) furcation* in \mathcal{A} if there are vertices $b, c, d \in A$, $c \neq d$, such that the following conditions hold:

- (a) $(b, a) \in Q$ or $(b, a, n) \in W$ (respectively, $(a, b) \in Q$ or $(a, b, n) \in W$) for some n ;
- (b) $(a, c) \in Q$ or $(a, c, n) \in W$ (respectively, $(c, a) \in Q$ or $(c, a, n) \in W$) for some n ;
- (c) $(a, d) \in Q$ or $(a, d, n) \in W$ (respectively, $(d, a) \in Q$ or $(d, a, n) \in W$) for some n .

Upper (lower) furcations are often called just *furcations*.

A c -graph or acyclic digraph is called a graph *without furcations*. The class of all graphs without furcations is denoted by $\mathbf{\Gamma}^{\text{nf}}$.

We denote by $\mathbf{\Gamma}_c^{\text{bp}}$ the class of all graphs with language $\{Q\}$ that can be obtained from graphs of the class $\mathbf{\Gamma}^{\text{bp}}$ by replacing all I_p -arcs (a, b) by some number of (a, b) - Q -routes of length at least p , including the shortest route of length p , such that all intermediate vertices in these routes are of degree 2. Clearly, any graph in $\mathbf{\Gamma}_c^{\text{bp}}$ has no furcations. Furthermore, the acyclic digraphs in which degrees of vertices are at most 2 are graphs without furcations. We denote the class of all such graphs by $\mathbf{\Gamma}^{\leq 2}$.

Obviously, every graph in $\mathbf{\Gamma}^{\leq 2}$ consists of a number of connected components \mathcal{K} , and each component is a finite or infinite route with or without initial element, as well as with or without final element. As above, we mark initial elements by color J_0 and final elements by color J_1 . The class $\mathbf{\Gamma}^{\leq 2}$ is closed under adding isolated vertices, i.e., vertices of degree 0. We assume that the set of isolated vertices is disjoint with $J_0 \cup J_1$.

Let $\Gamma_i = \langle X_i, Q_i \rangle$, $i \in I$, be graphs with the unique common element a of color J_j , $j \in \{0, 1\}$. The graph $\langle \bigcup_{i \in I} (X_i), \bigcup_{i \in I} (Q_i) \rangle$ is said to be the *free amalgam* of Γ_i , $i \in I$, over a and denoted by $*_{i \in I} \Gamma_i$.

We denote by $\mathbf{\Gamma}_0^{\text{nf}}$ the closure of $\mathbf{\Gamma}_c^{\text{bp}} \cup \mathbf{\Gamma}^{\leq 2}$ under disjoint unions and free amalgams over vertices.

It is easy to observe that any graph in $\mathbf{\Gamma}^{\text{nf}}$ has no furcations. Furthermore, each graph without furcations can be represented as a disjoint union of amalgams over vertices of graphs in $\mathbf{\Gamma}_c^{\text{bp}} \cup \mathbf{\Gamma}^{\leq 2}$. Therefore, $\mathbf{\Gamma}^{\text{nf}} = \mathbf{\Gamma}_0^{\text{nf}}$.

Denoting the operation of taking amalgams over vertices by \mathbf{A} and the operation of taking disjoint unions by \mathbf{D} , we obtain the following formula:

$$\mathbf{\Gamma}^{\text{nf}} = \mathbf{D}(\mathbf{A}(\mathbf{\Gamma}_c^{\text{bp}} \cup \mathbf{\Gamma}^{\leq 2})).$$

In what follows, we consider graphs without furcations and their c -subgraphs $\mathcal{A} = \langle A, Q, W \rangle$ in which the functions Col are one-color: $\rho_{\text{Col}} = \{0\}$, and each vertex is either *initial* (of color J_0), or *final* (of color J_1), or *intermediate* (of degree 2 in some extension of \mathcal{A} , and of color distinct from both J_0 and J_1). Moreover, for each

vertex $a \in A$ not belonging to $J_0 \cup J_1$, the record W is assumed to contain information on the length of the shortest (b, a) - Q -route, where $b \in J_0$ (if there exists such a vertex b in a c -graph or some its extension), as well as information on the length of the shortest (a, c) - Q -route, where $c \in J_1$ (if there exists such a vertex c in c -graph or some its extension). c -Subgraphs considered independently are called c -graphs as usual.

Let $\mathcal{A} = \langle A, Q, W \rangle$ be a c -graph. We denote by $e_1(\mathcal{A})$ the number $|Q|$; and by $e_k(\mathcal{A})$, $k \in \omega \setminus \{0, 1\}$, the number of pairs (a, b) of vertices in \mathcal{A} such that $(a, b, k) \in W$.

We define a *prerank function* y , that assigns a real to each c -graph \mathcal{A} by the rule

$$y(\mathcal{A}) = |A| - \sum_{k=1}^{\infty} \alpha_k \cdot e_k(\mathcal{A}),$$

where α_k are defined in Section 4.1. Notice that, in the definition of a prerank function, the summation is always finite due to the finiteness of c -graphs \mathcal{A} .

A *p-approximation* of a prerank function y is a function y_p that assigns a real to each c -graph \mathcal{A} by the rule

$$y_p(\mathcal{A}) = |A| - \sum_{k=1}^p \alpha_k \cdot e_k(\mathcal{A}).$$

Given a c -graph \mathcal{A} , we write $\mathcal{A} \in \mathbf{K}_1^{\text{nf}}$ iff $y_1(\mathcal{A}') \geq b_n^1$ for every nonempty c -graph $\mathcal{A}' \subseteq_c \mathcal{A}$, where $n = |\mathcal{A}'|$.

Lemma 4.2.1. *For every positive $p \in \omega$ and arbitrary c -graphs \mathcal{A} and \mathcal{B} , where \mathcal{A} is a proper c -subgraph of \mathcal{B} , $|\mathcal{B}| = n < k_{p+1}$, the following assertions hold:*

- (1) *We have $y_p(\mathcal{B}) - y(\mathcal{B}) < \varepsilon_{p+1}$.*
- (2) *The inequality $y_p(\mathcal{A}) < y_p(\mathcal{B})$ holds iff $y(\mathcal{A}) < y(\mathcal{B})$. The inequality $y_p(\mathcal{A}) < y_p(\mathcal{B})$ holds iff $y_q(\mathcal{A}) < y_q(\mathcal{B})$ for every $q \geq p$.*

P r o o f is similar to that of Lemma 4.1.1. \square

2. Generic class and generic theory. Given a c -graph \mathcal{A} , we write $\mathcal{A} \in \mathbf{K}_{p+1}^{\text{nf}}$ iff \mathcal{A} has no vertices that are forbidden for adding arcs to, $\mathcal{A} \in \mathbf{K}_p^{\text{nf}}$, and $y_p(\mathcal{A}') \geq b_n^p$ for every c -graph $\mathcal{A}' \subseteq_c \mathcal{A}$, where $n = |\mathcal{A}'|$, $k_p \leq n < k_{p+1}$.

Put $\mathbf{K}_0^{\text{nf}} \equiv \bigcap_{p=1}^{\infty} \mathbf{K}_p^{\text{nf}}$ and denote by \mathbf{K}^{nf} the class of all graphs

(without furcations) whose all c -subgraphs belong to \mathbf{K}_0^{nf} .

Let \mathcal{A} be a c -subgraph of a graph (of c -graph) \mathcal{M} (of a graph) in \mathbf{K}^{nf} . We say that \mathcal{A} is a *self-sufficient c -subgraph* of the graph (respectively, c -graph) \mathcal{M} and write $\mathcal{A} \leq_c \mathcal{M}$ if $y(\mathcal{A}) \leq y(\mathcal{B})$ for every c -graph \mathcal{B} , where $\mathcal{A} \subseteq_c \mathcal{B} \subseteq_c \mathcal{M}$. If $\mathcal{A} \leq_c \mathcal{M}$ and \mathcal{M} is a c -graph then \mathcal{A} is called a *strong c -subgraph* of \mathcal{M} .

Notice that, for each c -graph $\mathcal{A} = \langle A, Q, W \rangle \in \mathbf{K}_0^{\text{nf}}$ and each intermediate vertex $a \in A$, the set of all elements, that form a Q -route S_a including the vertex a , is contained in the definable closure of the set $\{a\}$: $S_a \subseteq \text{dcl}\{a\}$. This means that every extension of $\{a\}$ to a maximal Q -route S_a is uniquely defined. The set

$$A \cup \{S_a \mid a \text{ is an intermediate vertex in } A\}$$

is called the *route closure* of A (inside a given graph that includes \mathcal{A} as a c -subgraph) and denoted by $\text{ccl}(A)$.

Clearly, the set $\text{ccl}(A) \cap (J_0 \cup J_1)$ is finite for every c -graph \mathcal{A} .

Below, we assume that the record W contains information on interconnection between the elements of $\text{ccl}(A)$ by only external shortest Q -routes. Thus, the record W of the *expanded c -graph* $\mathcal{A} = \langle A, Q, W \rangle$ contains the structure description of the set $\text{ccl}(A)$ (possibly, including infinite information on the routes of infinite length whose intermediate vertices are of degree 2), together with finite information on lengths of the external (w.r.t. $\text{ccl}(A)$) shortest Q -routes that connect the elements of $\text{ccl}(A) \cap J_0$ with the elements of $\text{ccl}(A) \cap J_1$ only by external shortest Q -routes.

We are to prove that the class \mathbf{T} of all types corresponding to the expanded c -graphs in \mathbf{K}_0^{nf} with the relation \leq'_c (where $\Phi(A) \leq'_c \Psi(B) \Leftrightarrow \mathcal{A} \leq_c \mathcal{B}$) includes a self-sufficient generic subclass \mathbf{T}_0^{nf} that dominates \mathbf{T} and satisfies (after adding the necessary formulas describing the self-sufficient closures to the types) the uniform t -amalgamation property. This fact yields the ω -saturation of the $(\mathbf{T}_0^{\text{nf}}; \leq'_c)$ -generic model that realizes all types $\Phi(X)$ corresponding to the types $\Phi(A)$ in \mathbf{T} .

Remark 4.2.2. (1) *The conditions $\mathcal{A} \in \mathbf{K}_0^{\text{nf}}$ and $k_p \leq |A| = n < k_{p+1}$ do not imply that $y(\mathcal{A}) \geq b_n^p$. Nevertheless, $y(\mathcal{A})$ cannot be much less than b_n^p : since $y_p(\mathcal{A}) - y(\mathcal{A}) < \varepsilon_{p+1}$, we have $y(\mathcal{A}) > b_n^p - \varepsilon_{p+1}$.*

Moreover, in view of the inequality (4.1), for a graph $\mathcal{A} \in \mathbf{K}_0^{\text{nf}}$ of cardinality $|A| \geq \max\{2, k_p\}$, the inequality $y(\mathcal{A}) > p$ holds.

(2) Let \mathcal{A} be a c -graph in \mathbf{K}_0^{nf} , $|A| = p$, let \mathcal{M} be a graph in \mathbf{K}^{nf} , and $\mathcal{A} \subseteq_c \mathcal{M}$. Then $\mathcal{A} \leq_c \mathcal{M}$ iff $y_p(\mathcal{A}) \leq y_p(\mathcal{B})$ for each c -graph \mathcal{B} , where $\mathcal{A} \subseteq_c \mathcal{B} \subseteq_c \mathcal{M}$.

Indeed, if $|B| < k_{p+1}$, then $y_p(\mathcal{A}) \leq y_p(\mathcal{B})$ is equivalent to $y(\mathcal{A}) \leq y(\mathcal{B})$ by Lemma 4.2.1(2), and if $|B| \geq k_p$ then

$$y_p(\mathcal{B}) \geq y(\mathcal{B}) > p \geq y_p(\mathcal{A}).$$

Furthermore, in order to verify that \mathcal{A} is self-sufficient in \mathcal{M} , it suffices to choose $n_{\mathcal{A}} = k_p$ and check the validity of the relation $y_p(\mathcal{A}) \leq y_p(\mathcal{B})$ only for c -graphs \mathcal{B} , $\mathcal{A} \subseteq_c \mathcal{B} \subseteq_c \mathcal{M}$, satisfying $|B| < n_{\mathcal{A}}$.

Thus, the condition $\mathcal{A} \leq_c \mathcal{M}$ is definable by a formula that describes the absence of $n < n_{\mathcal{A}}$ new elements of $M \setminus A$ such that

$$n < \alpha_1 \cdot e_1 + \alpha_2 \cdot (e_2 - e'_2) + \dots + \alpha_p \cdot (e_p - e'_p),$$

where $p = |A|$, e_1 is the number of new arcs, e_s is the number of new pairs of vertices (a, b) that are connected only by external shortest routes of length s , and e'_s is the number of pairs of vertices (a, b) that are no longer connected only by external shortest routes of length s in the extended c -graph.

Moreover, each triple (a, b, s) that enters the calculation of e'_s turns into a sequence of triples $(a_0, a_1, s_1), \dots, (a_{k-1}, a_k, s_k)$ such that $a_0 = a$, $a_k = b$, the elements a_1, \dots, a_{k-1} are of degree 2, the elements a_{i-1} and a_i are connected by an arc (for $s_i = 1$) or the unique external shortest route of length s_i (for $s_i > 1$) in the extended c -graph, and $s_1 + \dots + s_k = s$. Therefore, when we extend a c -graph and remove a triple (a, b, s) that enters the calculation of $y(\cdot)$, the value $-\alpha_s$ is replaced by the positive (in view of $\alpha_i < \frac{1}{2}$, $i \geq 1$) value $(k-1) - \alpha_{s_1} - \dots - \alpha_{s_k}$ in the new calculation of $y(\cdot)$. Thus, the value $y(\cdot)$ can be diminished only by adding vertices of color J_0 or J_1 to a c -graph; and the condition $\mathcal{A} \leq_c \mathcal{M}$ is definable by a formula that describes the absence of $n < n_{\mathcal{A}}$ new elements of $M \setminus A$ such that $n < \alpha_1 \cdot e_1 + \dots + \alpha_p \cdot e_p$, where $p = |A|$ and e_s is the number of new pairs of vertices (a, b) connected only by external shortest routes of length s .

The following lemma is a direct consequence of the definition:

Lemma 4.2.3. (1) If $\mathcal{A} \leq_c \mathcal{B}$ then $\mathcal{A} \subseteq_c \mathcal{B}$.

(2) If $\mathcal{A} \leq_c \mathcal{C}$, $\mathcal{B} \in \mathbf{K}_0^{\text{nf}}$, and $\mathcal{A} \subseteq_c \mathcal{B} \subseteq_c \mathcal{C}$ then $\mathcal{A} \leq_c \mathcal{B}$.

(3) The empty graph \emptyset is the least element of the structure $(\mathbf{K}_0^{\text{nf}}; \leq_c)$.

Lemma 4.2.4. If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}_0^{\text{nf}}$, $\mathcal{A} \leq_c \mathcal{B}$, and $\mathcal{C} \subseteq_c \mathcal{B}$ then $\mathcal{A} \cap \mathcal{C} \leq_c \mathcal{C}$.

P r o o f. Assume the contrary. Then there exist n new elements in $\mathcal{C} \setminus \mathcal{A}$ such that $n < \alpha_1 \cdot e_1 + \dots + \alpha_p \cdot e_p$, where $p = |\mathcal{A}|$, e_s is the number of new pairs of vertices (a, b) connected only by external shortest routes of length s . Since all new elements lie in $\mathcal{B} \setminus \mathcal{A}$, they violate the condition $\mathcal{A} \leq_c \mathcal{B}$. \square

Lemma 4.2.5. The relation \leq_c is a partial order on \mathbf{K}_0^{nf} .

P r o o f is similar to that of Lemma 4.1.5 upon replacing the graphs of \mathbf{K}_0^{bp} by the c -graphs in \mathbf{K}_0^{nf} , and the relations \subseteq and \leq , by the relations \subseteq_c and \leq_c , respectively. \square

A c -graph \mathcal{A} is *J-closed* if \mathcal{A} contains, together with each intermediate vertex a , the initial vertex (if it exists) and the final vertex (if it exists) of the route S_a .

Let $\mathcal{A}, \mathcal{B} = \langle B, Q_B, W_B \rangle$, and $\mathcal{C} = \langle C, Q_C, W_C \rangle$ be *expanded J-closed c-graphs*, $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$ and $\text{ccl}(\mathcal{A}) = \text{ccl}(\mathcal{B}) \cap \text{ccl}(\mathcal{C})$ (the last equality indicates that, together with each common Q -route S of length ≥ 2 which is described at once in \mathcal{B} and \mathcal{C} , the set \mathcal{A} contains an intermediate vertex common for \mathcal{B} and \mathcal{C} which belongs to S). The *free c-amalgam* of c -graphs \mathcal{B} and \mathcal{C} over \mathcal{A} (in symbols, $\mathcal{B} *_\mathcal{A} \mathcal{C}$) is the c -graph $\langle B \cup C, Q_B \cup Q_C \cup Q, W_B \cup W_C \cup W \rangle$, where Q (respectively W) is the set of the arcs (of the records on external shortest routes over $B \cup C$) with endpoints in $B \cup C$ that are described in the graph with universe $\text{ccl}(\mathcal{A})$.

By definition, every free c -amalgam is *J-closed*. Moreover, the records on route closures ensure that the c -graph $\mathcal{B} *_\mathcal{A} \mathcal{C}$ is a graph without furcations.

A c -embedding f of a c -graph \mathcal{A} into a c -graph \mathcal{B} (written $f : \mathcal{A} \rightarrow_c \mathcal{B}$) is said to be *strong* if $f(\mathcal{A}) \leq_c \mathcal{B}$.

Lemma 4.2.6. (*amalgamation lemma*). The class \mathbf{K}_0^{nf} has the *c-amalgamation property c-(AP)*, i.e., for arbitrary strong c -embeddings $f_0 : \mathcal{A} \rightarrow_c \mathcal{B}$ and $g_0 : \mathcal{A} \rightarrow_c \mathcal{C}$ such that $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}_0^{\text{nf}}$ are *J-closed c-graphs*, there exist a *J-closed c-graph* $\mathcal{D} \in \mathbf{K}_0^{\text{nf}}$ and strong c -embeddings $f_1 : \mathcal{B} \rightarrow_c \mathcal{D}$ and $g_1 : \mathcal{C} \rightarrow_c \mathcal{D}$ such that $f_0 \circ f_1 = g_0 \circ g_1$.

P r o o f. Without loss of generality, we may assume that $\mathcal{A} \leq_c \mathcal{B}$, $\mathcal{A} \leq_c \mathcal{C}$, $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$, and $\text{ccl}(\mathcal{A}) = \text{ccl}(\mathcal{B}) \cap \text{ccl}(\mathcal{C})$. We claim that c -graph $\mathcal{D} \doteq \mathcal{B} *_{\mathcal{A}} \mathcal{C}$ is the one sought. To this end, in view of the symmetry of the definition of free c -amalgam, it suffices to show that $\mathcal{B} \leq_c \mathcal{D}$ and $\mathcal{D} \in \mathbf{K}_0^{\text{nf}}$.

Suppose that $\mathcal{B} \not\leq_c \mathcal{D}$. Then there exist n new elements in $D \setminus B$ such that $n < \alpha_1 \cdot e_1 + \dots + \alpha_p \cdot e_p$, where $p = |B|$ and e_s is the number of new pairs of vertices (a, b) that are connected only by external shortest routes of length s . Since all new elements lie in $C \setminus A$, they violate the condition $\mathcal{A} \leq_c \mathcal{C}$.

Since each intermediate vertex is of degree 2 and all numbers α_p are less than $\frac{1}{2}$, the property of c -graph to belong to the class \mathbf{K}_0^{nf} is preserved under adding finitely many intermediate vertices. Therefore, it suffices to establish that the c -graph \mathcal{D} belongs to \mathbf{K}_0^{nf} in the case when \mathcal{D} has no intermediate vertices. In this case, each J -closed c -subgraph of \mathcal{D} has the form $\mathcal{B}_0 *_{\mathcal{A}_0} \mathcal{C}_0$, where $\mathcal{A}_0 \leq_c \mathcal{B}_0$ and $\mathcal{A}_0 \leq_c \mathcal{C}_0$. Therefore, to verify that $\mathcal{D} \in \mathbf{K}_0^{\text{nf}}$, it is sufficient to ascertain that $y_p(\mathcal{D}) \geq b_n^p$, where $n = |D|$, $k_p \leq n < k_{p+1}$.

Suppose that $|B| \leq |C| = m$, $k_q \leq m < k_{q+1}$, and $A \subsetneq B$. By Lemma 4.1.1(3) and the condition $\mathcal{A} \leq_c \mathcal{B}$, we have

$$\text{sl}((|A|, y_q(\mathcal{A})), (|B|, y_q(\mathcal{B}))) \geq \text{sl}((l, b_l^q), (l+1, b_{l+1}^q))$$

for any $l \geq m$. Then from

$$\text{sl}((|C|, y_q(\mathcal{C})), (|D|, y_q(\mathcal{D}))) = \text{sl}((|A|, y_q(\mathcal{A})), (|B|, y_q(\mathcal{B})))$$

we derive

$$\text{sl}((|C|, y_q(\mathcal{C})), (|D|, y_q(\mathcal{D}))) \geq \text{sl}((m, b_m^q), (|D|, b_{|D|}^q)). \quad (4.3)$$

Since the c -graph \mathcal{C} belongs to \mathbf{K}_0^{nf} , the point $(|C|, y_q(\mathcal{C}))$ is above the point (m, b_m^q) . Hence, the inequality (4.3) implies that the point $(|D|, y_q(\mathcal{D}))$ is above the point $(|D|, b_{|D|}^q)$, i.e., $y_q(\mathcal{D}) \geq b_{|D|}^q$.

If $p = q$, then we establish the required inequality $y_p(\mathcal{D}) \geq b_n^p$. Otherwise, i.e., if $q < p$ then the value $y_p(\mathcal{D})$ differs from $y_q(\mathcal{D})$ by less than $\alpha_{q+1} \cdot k_{q+1} \cdot (k_{q+1} - 1) < 2\varepsilon_{q+1}$, since \mathcal{D} contains less than $k_{q+1} \cdot (k_{q+1} - 1)$ arcs together with records on Q -routes. In view of Lemma 4.1.1(4), we conclude that $y_p(\mathcal{D}) \geq b_n^p$. \square

Denote by \mathbf{T}_0^{nf} the class of all types of \mathbf{T} that correspond to J -closed c -graphs; and by \leq_c'' , the self-sufficiency relation on \mathbf{T}_0^{nf} induced by \leq_c' . Lemmas 4.2.3–4.2.6 imply

Corollary 4.2.7. *The class $(\mathbf{T}_0^{\text{nf}}; \leq_c'')$ is self-sufficient.*

Denote the $(\mathbf{T}_0^{\text{nf}}; \leq_c'')$ -generic theory by T^{nf} .

We claim that the self-sufficient class $(\mathbf{T}_0; \leq_c''')$ possessing the uniform t -amalgamation property arises after adding a formula $\chi_{\overline{\Phi}}(\overline{A})$ such that $(T^{\text{nf}}, \overline{A}) \vdash \chi_{\overline{\Phi}}(\overline{A})$ to each self-sufficient type $\overline{\Phi}(\overline{A}) \in \mathbf{T}_0^{\text{nf}}$.

Indeed, each intermediate vertex generates at most two vertices in $J_0 \cup J_1$; therefore, the J -closure of p -element c -graph contains at most $3p$ elements. Hence, by Remark 4.2.2, for an arbitrary c -graph $\mathcal{A} \in \mathbf{K}_0^{\text{nf}}$ of cardinality p and an arbitrary graph $\mathcal{M} \models T^{\text{nf}}$, where $\mathcal{A} \subseteq_c \mathcal{M}$, the following relation holds:

$$\mathcal{A} \leq_c \mathcal{M} \Leftrightarrow y_{3p}(\mathcal{A}) \leq y_{3p}(\mathcal{B}) \text{ for any } c\text{-graph } \mathcal{B}$$

$$\text{such that } \mathcal{A} \subseteq_c \mathcal{B} \subseteq_c \mathcal{M} \text{ and } |B| \leq k_{3p}.$$

Since the cardinalities of c -graphs \mathcal{B} are bounded in terms of the cardinality of \mathcal{A} , while the verification of $y_{3p}(\mathcal{A}) \leq y_{3p}(\mathcal{B})$ involves only calculation of connections by the relations Q^1, \dots, Q^{3p} , the self-sufficiency condition $\mathcal{A} \leq_c \mathcal{M}$ is expressed by a formula $\chi_{\mathcal{A}}(X)$ of graph language $\{Q\}$, where the set X of variables is bijective with the set A .

Let \mathcal{A} and \mathcal{B} be c -graphs in \mathbf{K}_0^{nf} , \mathcal{M} be a generic model of T^{nf} , and $\mathcal{A} \leq_c \mathcal{B} \leq_c \mathcal{M}$. Denote by $\psi_{\mathcal{A},s}(X)$ (respectively, by $\psi_{\mathcal{B},s}(X, Y)$) the formula that describes the $\{Q^1, \dots, Q^s, J_0, J_1\}$ -type of \mathcal{A} (respectively, of \mathcal{B}), where X and Y are disjoint sets of variables which are bijective with A and $B \setminus A$. Then for every $s \geq |B|$, the following formula is true in \mathcal{M} :

$$\forall X ((\chi_{\mathcal{A}}(X) \wedge \psi_{\mathcal{A},s}(X)) \rightarrow \exists Y (\chi_{\mathcal{B}}(X, Y) \wedge \psi_{\mathcal{B},s}(X, Y))).$$

The last relation implies the uniform t -amalgamation property for the class $(\mathbf{T}_0; \leq)$ obtained from $(\mathbf{T}_0^{\text{nf}}; \leq_c''')$ by adding the formulas that define cardinality bounds and $\{Q^1, \dots, Q^{3p}, J_0, J_1\}$ -structures of self-sufficient closures to the types as well as the formulas $\chi_{\mathcal{A}}(A)$ to the types of self-sufficient sets A .

By adding the above-stated formulas, we ensure that every finite set of models of the theory T^{nf} possesses a finite closure.

For a finite set A in a model \mathcal{N} of T^{nf} , we denote the self-sufficient closure of the J -closure of A by \overline{A} and say that \overline{A} is the J -self-sufficient closure of A . The sets \overline{A} considered independently are called J -self-sufficient sets.

Theorem 2.5.1 implies

Theorem 4.2.8. *The $(\mathbf{T}_0^{\text{nf}}; \leq_c''')$ -generic model \mathcal{M} is saturated. Moreover, every finite set $A \subseteq M$ can be extended to its J -self-sufficient closure $\bar{A} \subseteq M$, and the type $\text{tp}_X(\bar{A})$ is deduced from the set $\{\chi_{\bar{A}}(X)\} \cup \{\psi_{\bar{A},s}(X) \mid s \in \omega \setminus \{0\}\}$.*

Let \mathcal{N} be an ω -saturated model of T^{nf} .

Proposition 4.2.9. *For each finite set A in \mathcal{N} , the equality $\text{acl}(A) = \bigcup\{\bar{B} \mid B \subseteq_{\text{fin}} \text{ccl}(A)\}$ is valid. Moreover, the sets $\bar{A} \setminus A$ and $\text{acl}(A) \setminus \text{ccl}(A)$ do not contain intermediate elements.*

P r o o f. The set $\bar{A} \setminus A$ does not contain intermediate elements, since, after removing an intermediate element a from a finite set B , the value $y(\cdot)$ diminishes at least by a positive value $1 - \alpha_{s_1} - \alpha_{s_2} + \alpha_s$, where s_1 and s_2 — are the lengths of the external shortest routes over B that connect a with the remaining elements b and c , and $s = s_1 + s_2$ is the length of the shortest route, external over $B \setminus \{a\}$, that connects b and c .

Denote the set $\bigcup\{\bar{B} \mid B \subseteq_{\text{fin}} \text{ccl}(A)\}$ by X . The inclusion $X \subseteq \text{acl}(A)$ follows from the fact that the route S_a is uniquely determined for every intermediate vertex $a \in A$, and the sets \bar{B} are unique in \mathcal{N} , where $B \subseteq_{\text{fin}} \text{ccl}(A)$. Given an element $b \in N \setminus X$, put $C \equiv \bar{A} \cup \{b\}$. Since $C \setminus (A \cup \{b\})$ does not contain intermediate elements, by the construction of generic model, there are infinitely many c -isomorphic pairwise disjoint copies of the set $C \setminus X$ over X , i.e., $b \notin \text{acl}(A)$. Thus, $\text{acl}(A) \subseteq X$.

Now, since $\bar{B} \setminus B$ does not contain intermediate elements, the equality $\text{acl}(A) = X$ implies that $\text{acl}(A) \setminus \text{ccl}(A)$ does not contain intermediate elements either. \square

3. Stability of the generic theory. We claim that the generic theory T^{nf} is stable. Let \mathcal{N} be a sufficiently saturated model of T^{nf} . The *rank function* in \mathcal{N} is the function

$$r_{\mathcal{N}} : \{\mathcal{A} \mid \mathcal{A} \text{ is a } c\text{-subgraph of } \mathcal{N}\} \rightarrow \mathbb{R}^+,$$

that is defined by the equality $r_{\mathcal{N}}(\mathcal{A}) = \inf\{y(\mathcal{B}) \mid \mathcal{A} \subseteq_c \mathcal{B} \subseteq_c \mathcal{N}\}$.

Observe that, since the class $(\mathbf{T}_0^{\text{nf}}; \leq_c'')$ is self-sufficient, the infimum in the above equality is always attainable and coincides with $y(\bar{\mathcal{A}})$: $r_{\mathcal{N}}(\mathcal{A}) = y(\bar{\mathcal{A}})$.

In what follows, we fix a model \mathcal{N} , denote the function $r_{\mathcal{N}}$ by r for brevity, and replace c -graphs \mathcal{A} by their universes in \mathcal{N} . We also assume that all sets under consideration are subsets of N .

The *relative prerank function* $y(A/B)$ of a c -graph A over a c -graph B is given by the relation

$$y(A/B) = y(A \cup B) - y(B),$$

and the *relative rank function* $r(A/B)$ of a c -graph A over a c -graph B is given by the relation

$$r(A/B) = r(A \cup B) - r(B).$$

We now establish some necessary properties of the relative rank function.

Clearly, $r(A/B) \geq 0$.

Lemma 4.2.10. *For any finite sets B and C , we have*

$$y(B) + y(C) \geq y(B \cup C) + y(B \cap C).$$

The inequality becomes an equality iff there are no elements $b \in B \setminus C$ and $c \in C \setminus B$ connected by external shortest routes over $B \cup C$ and none of external shortest routes over $B \cap C$ contains intermediate vertices of $B \cup C$.

P r o o f. By hypothesis, the value

$$V \doteq y(B) + y(C) - y(B \cup C) - y(B \cap C)$$

is equal to

$$\sum_{k=1}^{\infty} \alpha_k \cdot (e_k(\mathcal{B} \cap \mathcal{C}) + e_k(\mathcal{B} \cup \mathcal{C}) - e_k(\mathcal{B}) - e_k(\mathcal{C})).$$

All summands in the last expression, and therefore the expression itself, are equal to zero if there are no elements $b \in B \setminus C$ and $c \in C \setminus B$ connected by external shortest routes over $B \cup C$ and none of external shortest routes over $B \cap C$ contains intermediate vertices of $B \cup C$.

If there are elements $b \in B \setminus C$ and $c \in C \setminus B$ connected by external shortest routes over $B \cup C$ and none of external shortest routes over either $B \cap C$ or B or C contains intermediate vertices

of $B \cup C$ then the positive value V equals $\sum_{k=1}^{\infty} \alpha_k \cdot e_k$, where e_1 is the number of Q -arcs, e_k , $k > 1$, is the number of pairs $(b, c) \in (B \setminus C) \times (C \setminus B)$ of elements that are connected only by external shortest Q -routes over $B \cup C$ of length k .

Suppose that some of external shortest route over either $B \cap C$ or B or C contains intermediate vertices of $B \cup C$. Consider all external shortest routes over $B \cap C$, as well as over B and C , such that their lengths do not change if we consider $B \cup C$. Observe that, in calculation of V , the quantity for each of these routes equals zero while, for the pairs $(b, c) \in (B \setminus C) \times (C \setminus B)$ of elements connected by arcs or only by the external shortest routes over $B \cup C$ of length k , that are not proper subroutes of external shortest routes over $B \cap C$ or B or C , the value V includes the nonnegative value $\sum_{k=1}^{\infty} \alpha_k \cdot e_k$, where e_1 is the number of Q -arcs, e_k , $k > 1$, is the number of pairs $(b, c) \in (B \setminus C) \times (C \setminus B)$ of elements connected only by external shortest Q -routes over $B \cup C$ of length k . Therefore, it suffices to establish that the positive values in V form the elements corresponding to each external shortest route over $B \cap C$ or B or C that contains intermediate vertices of $B \cup C$.

Let $(a_1, a_2) \in B^2 \cup C^2$ be a pair of vertices that are connected over $B \cap C$ or B or C only by external shortest (a_1, a_2) -routes of length s , and let a'_0, \dots, a'_n , $n \geq 2$, be all elements of $B \cup C$ that are intermediate vertices or endpoints of (a_1, a_2) -routes. For an arbitrary pair (a'_i, a'_j) of elements connected by a (unique) external shortest (a'_i, a'_j) -route S_i over $B \cup C$, denote its length by s_i . By the choice of vertices, we have $s_i < s$.

Consider the case when there exist pairs $(b_1, b_2) \in B^2$ and $(c_1, c_2) \in C^2$ of vertices that are connected only by external shortest (b_1, b_2) - and (c_1, c_2) -routes S_B and S_C over B and C , respectively, including the route S_i . Denote the lengths of these routes by s_B and s_C . By the choice of vertices, we have $s_i < s_B$ or $s_i < s_C$. The following cases are possible:

- (a) $s_i = s_B$ and $s_i < s_C$, i.e., $(a'_i, a'_j) = (b_1, b_2)$ and $(a'_i, a'_j) \neq (c_1, c_2)$;
- (b) $s_i < s_B$ and $s_i = s_C$, i.e., $(a'_i, a'_j) \neq (b_1, b_2)$ and $(a'_i, a'_j) = (c_1, c_2)$;
- (c) $s_i < s_B$ and $s_i < s_C$, i.e., $(a'_i, a'_j) \neq (b_1, b_2)$ and $(a'_i, a'_j) \neq (c_1, c_2)$.

In the first case, the information on S_i and S_B is presented in V by the difference $\alpha_{s_i} - \alpha_{s_B}$ equal 0. In the second case, the information on S_i and S_C is presented in V by the difference $\alpha_{s_i} - \alpha_{s_C}$ equal 0. Furthermore, the route S_C (in the first case), or the route S_B (in the second case), includes an external shortest $(a'_{i'}, a'_{j'})$ -route over $B \cup C$ such that $(a'_{i'}, a'_{j'}) \notin B^2 \cup C^2$. Thus, we reduce the calculation of V to examining the third case, when the information on S_i , S_B , and S_C is presented in V by the expression

$$\alpha_{s_i} - \alpha_{s_B} - \alpha_{s_C}. \quad (4.4)$$

which is positive in view of $\alpha_{s_B} < \frac{\alpha_{s_i}}{2}$ and $\alpha_{s_C} < \frac{\alpha_{s_i}}{2}$. The sum of values of the positive expressions (4.4) gives a lower bound for V .

Suppose that the above pair $(b_1, b_2) \in B^2$ exists while the pair $(c_1, c_2) \in C^2$ does not exist. Then, by examining the cases $s_i = s_B$ and $s_i < s_B$ in a similar way, we come to the expression $\alpha_{s_i} - \alpha_{s_B}$ which is zero in the first case and positive in the second case; and again the positive expressions give a lower bound for V .

Arguing similarly in the case when the pair $(c_1, c_2) \in C^2$ exists while the pair $(b_1, b_2) \in B^2$ does not exist, we obtain a positive lower bound for V as well. \square

Lemma 4.2.11. *If $B_1 \subseteq B_2$ then $r(A/B_1) \geq r(A/B_2)$.*

P r o o f is a word-for-word repetition of the proof of Lemma 4.1.11. \square

If A is a finite set and X is some set (not necessarily finite) then

$$r(A/X) = \inf\{r(A/B) \mid B \subseteq_{\text{fn}} X\}.$$

Lemma 4.2.12. *Let A_1 and A_2 be finite sets and let X be a set such that $A_1 \setminus X = A_2 \setminus X$. Then $r(A_1/X) = r(A_2/X)$.*

P r o o f is similar to that of Lemma 4.1.12. \square

A set $X \subseteq N$ is called *closed* (in \mathcal{N}) (written $X \leq N$) if $\text{ccl}(X) = X$ and $\overline{A} \subseteq X$ for every finite set $A \subseteq X$. The last condition is equivalent to the fact that $N \setminus X$ does not contain n new elements such that $n < \sum_{k=1}^{\infty} \alpha_k \cdot e_k$, where e_1 is the number of new Q -arcs, e_k , $k > 1$, is the number of new pairs of elements connected only by external shortest Q -routes of length k .

Lemma 4.2.13. *Let X and Y be sets in \mathcal{N} . Then the following hold:*

- (1) *if $X \leq N$ and $Y \leq N$ then $X \cap Y \leq N$;*
- (2) *there exists a smallest closed set $\overline{X}^c \supseteq X$; moreover, $\overline{X}^c = \bigcup \{\overline{A} \mid A \subseteq_{\text{fin}} \text{ccl}(X)\}$ and $\overline{X}^c = \text{acl}(X)$;*
- (3) *if $X \subset Y$ then $\overline{X}^c \subseteq \overline{Y}^c$.*

The set \overline{X}^c is called an *intrinsic closure* of X (in \mathcal{N}).

P r o o f. (1) Suppose that $X \leq N$ and $Y \leq N$. Clearly, $\text{ccl}(X \cap Y) = \text{ccl}(X) \cap \text{ccl}(Y) = X \cap Y$, i.e., the set $X \cap Y$ is route-closed. Let A be a finite subset of $X \cap Y$. Since $A \subseteq X$, $A \subseteq Y$, and both X and Y are closed, we have $\overline{A} \subseteq X$ and $\overline{A} \subseteq Y$. Therefore, $\overline{A} \subseteq X \cap Y$. Thus, $X \cap Y$ is closed.

(2) Denote the set $\bigcup \{\overline{A} \mid A \subseteq_{\text{fin}} \text{ccl}(X)\}$ by Z . Obviously, each closed superset of X includes Z , and $X \subseteq Z$. On the other hand, in view of Proposition 4.2.9, we obtain $\text{ccl}(Z) = Z$, and if A is a finite subset of Z then there exists a finite set $B \subseteq \text{ccl}(X)$ such that $A \subseteq \overline{B}$. We claim that $\overline{A} \subseteq \overline{B}$. Indeed, $A \subseteq \overline{A} \cap \overline{B} \leq_c N$. Therefore, $\overline{A} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \subseteq \overline{B}$. The inclusions $\overline{A} \subseteq \overline{B}$ and $\overline{B} \subseteq Z$ imply $\overline{A} \subseteq Z$. Thus, Z is the least closed set that includes X .

(3) Clearly, $X \subset Y$ implies $\text{ccl}(X) \subseteq \text{ccl}(Y)$. It remains to observe that, for arbitrary c -graphs \mathcal{A} and \mathcal{B} , if $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$. Indeed, $A \subseteq \overline{A} \cap \overline{B} \leq_c N$, as above. Therefore, $\overline{A} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \subseteq \overline{B}$. \square

By analogy with Section 4.1, we say that finite sets A and B are *independent* over Z and write $A \downarrow_Z^r B$ if $r(A/Z) = r(A/(Z \cup B))$ and $\overline{A \cup Z}^c \cap \overline{B \cup Z}^c \subseteq \overline{Z}$.

We say that sets X and Y are *independent* over Z and write $X \downarrow_Z^r Y$ if $A \downarrow_Z^r B$ for all finite sets $A \subseteq X$ and $B \subseteq Y$.

Lemma 4.2.14. *If X and Y are closed sets, $Z = X \cap Y$, and $X \downarrow_Z^r Y$ then $X \cup Y$ is closed.*

P r o o f. Since $\text{ccl}(X \cup Y) = \text{ccl}(X) \cup \text{ccl}(Y)$ and both X and Y are route-closed, the set $X \cup Y$ is route-closed. The implication

$$A \subseteq_{\text{fin}} (X \cup Y) \Rightarrow \overline{A} \subseteq_{\text{fin}} (X \cup Y)$$

is verified similarly to the proof of Lemma 4.1.14 on using Lemma 4.2.11. \square

Lemma 4.2.15. *If X is a J -self-sufficient set, Y is a closed set, and $Z = \overline{X \cup Z^c} \cap Y$ then $X \downarrow_Z^r Y$ iff $\overline{X \cup Z^c} \cup Y$ is a closed set such that there are no elements $x \in \overline{X \cup Z^c} \setminus Y$ and $y \in Y \setminus \overline{X \cup Z^c}$ connected by external shortest routes over $\overline{X \cup Z^c} \cup Y$, and none of external shortest routes over Z contains intermediate vertices of $\overline{X \cup Z^c} \cup Y$.*

P r o o f is similar to that of Lemma 4.1.15 on using Lemmas 4.2.11 and 4.2.13. \square

Lemma 4.2.15 readily implies

Corollary 4.2.16. *If X is a J -self-sufficient set, Y is a closed set, $Z = \overline{X \cup Z^c} \cap Y$ and $X \downarrow_Z^r Y$ then the type $\text{tp}(X/Y)$ is uniquely defined by the type $\text{tp}(X/Z)$, a description of the property for $\overline{X \cup Z^c} \cup Y$ to be closed, and the type that describes the absence of elements $x \in \overline{X \cup Z^c} \setminus Y$ and $y \in Y \setminus \overline{X \cup Z^c}$, connected by external shortest routes over $\overline{X \cup Z^c} \cup Y$ and the absence of intermediate vertices of $\overline{X \cup Z^c} \cup Y$ for external shortest routes over Z .*

Lemma 4.2.17. *If X is a closed set and a is an element of the model \mathcal{N} that does not belong to X then there exists at most countable closed subset $X' \subseteq X$ such that $\{a\} \downarrow_{X'}^r X$.*

P r o o f is a word-for-word repetition of that of Lemma 4.1.17 on using Lemmas 4.2.11 and 4.2.13. \square

Theorem 4.2.18. (1) *The theory T^{nf} is stable, small and has countably many 1-types: of elements of color J_0 , of elements of color J_1 , of intermediate elements that are at some finite distance from elements of color J_0 and (or) elements of color J_1 , and of intermediate elements that are not connected by routes with elements of color $J_0 \cup J_1$.*

(2) *Any 1-type of non-intermediate elements of T^{nf} has infinite weight; namely, for every element a of color J_i , $i = 0, 1$, there are infinitely many independent elements b_n , $n \in \omega$, of color J_{1-i} that depend on a .*

P r o o f. (1) Theorem 4.2.8 implies that T^{nf} is small. The proof of stability for T^{nf} is similar to that for T^{bp} in view of Corollary 4.2.16 and Lemma 4.2.17.

By construction, every singleton $\{a\}$ is self-sufficient. In view of Theorem 4.2.8, this means that there exist countably many 1-types of T^{nf} , and each of these types is defined by one of the following sets:

(a) $\{J_0(x)\}$;

(b) $\{J_1(x)\}$;

(c) the set of formulas describing the fact that an intermediate element x is at some fixed finite distance from an element of J_0 and at some fixed finite distance from an element of J_1 ;

(d) the set of formulas describing the fact that an intermediate element x is at some fixed finite distance from an element of J_i and that there are no connections between x and elements of J_{1-i} by a Q -route, $i = 0, 1$;

(e) the set of formulas describing the absence of connections between an element x and elements of $J_0 \cup J_1$ by Q -routes.

(2) Let \mathcal{A} be a c -graph with universe $\{a\}$, $A_p = \{a, b_p\}$ be the universe of two-element c -graph \mathcal{A}_p that contains the Q^p -arc between a and b_p , where $a \in J_i$ and $b_p \in J_{1-i}$. Put $\mathcal{B}_1 \equiv \mathcal{A}_1$, $\mathcal{B}_{p+1} \equiv \mathcal{B}_p *_A \mathcal{A}_{p+1}$, $\mathcal{B} \equiv \bigcup_{p \in \omega \setminus \{0\}} \mathcal{B}_p$. By construction, \mathcal{B} is a self-

sufficient subgraph of some generic model \mathcal{M} of T^{nf} . Since $\alpha_1 + \dots + \alpha_{p+1} < 1$ and $\{b_1, \dots, b_{p+1}\} \leq_c \{a, b_1, \dots, b_{p+1}\} \leq_c B \leq_c M$, we have $\{b_{p+1}\} \downarrow_{\emptyset}^r \{b_1, \dots, b_p\}$. Thus, $(b_p)_{p \in \omega \setminus \{0\}}$ is an infinite independent sequence of elements of the type $\{J_{1-i}(x)\}$ such that each element b_p depends on a . \square

Since $\alpha_p < \frac{1}{2}$ for every natural $p \geq 1$, it follows that every two elements a and b of color J_0 are connected via an element c of color J_1 so that aQc and bQc . In the same way, every two elements a and b of color J_1 are connected via an element c of color J_0 so that cQa and cQb . Furthermore, in either case, there are infinitely many such elements c . Thus, we obtain

Proposition 4.2.19. *The set of elements of color J_i has the pairwise $Q^{(-1)^{1-i}}$ -intersection property, $i = 0, 1$.*

The following two assertions clarify the structure of the prime models \mathcal{M}_A over finite sets A .

Lemma 4.2.20. *If A and B are J -self-sufficient sets in \mathcal{N} and $A \leq_c B$ then the type $\text{tp}(B/A)$ is isolated iff the following conditions hold:*

(1) *each intermediate element of $B \setminus A$ that belongs to an infinite chain is connected with an intermediate element of A by a Q -route;*

(2) *every two differently J -colored elements $a \in B \cap (J_0 \cup J_1)$ and $b \in (B \setminus A) \cap (J_0 \cup J_1)$ are connected by Q -routes.*

P r o o f. Let Y be the set of variables bijective in $B \setminus A$. If the conditions (1) and (2) are satisfied then, by Theorem 4.2.8, the type $\text{tp}((B \setminus A)/A)$ is isolated by the formula $\chi_{\overline{\mathcal{B}}}(A \cup Y) \wedge \psi_{\overline{\mathcal{B}},s}(A \cup Y)$, where s is a positive number that exceeds all lengths of external shortest routes connecting elements of \mathcal{B} . If any of the conditions (1) and (2) is violated then, by Theorem 4.2.8, the type $\text{tp}((B \setminus A)/A)$ is isolated by the set of formulas $\{\chi_{\overline{\mathcal{B}}}(A \cup Y)\} \cup \{\psi_{\overline{\mathcal{B}},s}(A \cup Y) \mid s \in \omega\}$, but is not isolated by any finite part of this set. \square

Lemma 4.2.20 implies

Corollary 4.2.21. *Let A be a J -self-sufficient set in \mathcal{N} . In the model \mathcal{M}_A , each intermediate element of $M_A \setminus A$ that belongs to an infinite chain is connected with an intermediate element of A by a Q -route. Furthermore, every two differently J -colored elements $a \in M \cap (J_0 \cup J_1)$ and $b \in (M \setminus A) \cap (J_0 \cup J_1)$ are connected by Q -routes. The set of isomorphism types of prime models over finite sets coincides with the set of isomorphism types of the models \mathcal{M}_A , where A are J -self-sufficient sets.*

§ 4.3. Small stable generic graphs with infinite weight. Powerful digraphs

In this Section, on the base of generic constructions in Sections 4.1 and 4.2, we describe a generic construction producing a family of *stable* powerful digraphs with almost inessential ordered colorings.

1. Tandem c -graphs without furcations. A *furcation-tandem* in a c -graph \mathcal{A} (in a graph \mathcal{M} , respectively) is a pair (a_b, a_t) of vertices such that the following conditions hold in $\text{cc}(\mathcal{A})$ (in \mathcal{M}):

- (a) a_b has zero outcoming semi-degree $\text{deg}^+ a_b$ and non-zero input semi-degree $\text{deg}^- a_b$;
- (b) a_t has zero input semi-degree $\text{deg}^- a_t$ and non-zero outcoming semi-degree $\text{deg}^+ a_t$;
- (c) $\text{deg}^- a_b + \text{deg}^+ a_t \geq 3$.

Clearly that, for any c -graph (respectively, for acyclic digraph), an identification of vertices of furcation-tandem (a_b, a_t) forms a furcation. Conversely, any furcation-tandem is a result of replacement of a furcation a by a pair (a_b, a_t) of different vertices, where endpoints of routes, that input in a , are replaced by a_b , and initials of routes, outcoming from a , are replaced by a_t . Furthermore, a c -graph (a digraph) without furcations is formed as result

of all possible replacements of furcations by furcation-tandems in given c -graph (acyclic digraph). If resulted c -graph (digraph) is equipped by an equivalence E with singleton classes for old vertices and with two-element classes, that contain furcation-tandems w.r.t. old vertices, then it is called a *tandem c -graph without furcations* $T(\mathcal{A}) = \langle T(\mathcal{A}), Q, W, E \rangle$ (*tandem graph without furcations* $T(\mathcal{M}) = \langle T(\mathcal{M}), Q, E \rangle$), correspondent to c -graph \mathcal{A} (digraph \mathcal{M}). Moreover, we consider, for the definiteness, that any vertex a of degree 0, lying in $T(\mathcal{A})$, belongs to J_1 .

Clearly, for any c -graph $\mathcal{A} = \langle A, Q, W \rangle$, the number of arcs in $T(\mathcal{A})$ is equal to the number of arcs in Q , and the number of records on its external shortest routes is the number of triples $(a, b, n) \in W$.

For any c -graph \mathcal{A} (respectively, for acyclic digraph \mathcal{M}), we denote by \mathcal{A}_f (respectively, by \mathcal{M}_f) the set of all furcations in \mathcal{A} (in \mathcal{M}), and we denote by \mathcal{A}_{nf} (\mathcal{M}_{nf}) the set of all vertices in \mathcal{A} (in \mathcal{M}) that are not furcations in \mathcal{A} (in \mathcal{M}): $\mathcal{A}_{\text{nf}} = A \setminus \mathcal{A}_f$ ($\mathcal{M}_{\text{nf}} = M \setminus \mathcal{M}_f$). We shall use notations A_f , A_{nf} , M_f , and M_{nf} instead of \mathcal{A}_f , \mathcal{A}_{nf} , \mathcal{M}_f , and \mathcal{M}_{nf} , respectively, if a c -graph (a digraph) is fixed.

Clearly, if $\mathcal{A} \subseteq_c \mathcal{B}$ then $A_f \subseteq B_f$.

Constructing c -graph $T(\mathcal{B})$ w.r.t. c -graph \mathcal{A} , where $\mathcal{A} \subseteq_c \mathcal{B}$, we shall assume that any vertex $a \in A \cap (B_f \setminus A_f)$ coincides with a_b if $\deg_A^+ a = 0$, and is equal to a_t , otherwise. Each vertex a in $T(\mathcal{B})$ w.r.t. \mathcal{A} will be denoted by $a_{T(\mathcal{A})}$.

Let $f_0 : \mathcal{A} \rightarrow_c \mathcal{B}$ be a c -embedding. A c -embedding $g : T(\mathcal{A}) \rightarrow_c T(\mathcal{B})$ is called a *canonical c -embedding* of the c -graph $T(\mathcal{A})$ into the c -graph $T(\mathcal{B})$ w.r.t. f_0 (written $g : T(\mathcal{A}) \rightarrow_{c, f_0} T(\mathcal{B})$) if $g(a) = f_0(a)_{T(f_0(\mathcal{A}))}$ for any vertex $a \in A_{\text{nf}}$, and $g(a_b) = f_0(a)_b$ and $g(a_t) = f_0(a)_t$ for any vertex $a \in A_f$.

Clearly, for any c -embedding $f_0 : \mathcal{A} \rightarrow_c \mathcal{B}$, a canonical c -embedding $g : T(\mathcal{A}) \rightarrow_{c, f_0} T(\mathcal{B})$ exists.

Two c -graphs, $T(\mathcal{A})$ and $T(\mathcal{B})$, are called *canonically c -isomorphic* if there exist c -isomorphisms $f_0 : \mathcal{A} \rightarrow_c \mathcal{B}$ and $g : T(\mathcal{A}) \rightarrow_{c, f_0} T(\mathcal{B})$. Furthermore, the map g is said to be a *canonical c -isomorphism* between $T(\mathcal{A})$ and $T(\mathcal{B})$ w.r.t. c -isomorphism f_0 , and c -graphs $T(\mathcal{A})$ and $T(\mathcal{B})$ are called *canonically c -isomorphic copies* (w.r.t. c -isomorphism f_0).

For any c -graphs \mathcal{A} , \mathcal{B} , and \mathcal{C} , the following assertions hold, connecting, by canonical c -isomorphisms, set-theoretic operations over c -graphs with correspondent operations over their tandem images.

Proposition 4.3.1. *Two c -graphs, $T(\mathcal{A} \cap \mathcal{B})$ and $T(\mathcal{A}) \cap T(\mathcal{B})$, are canonically c -isomorphic iff $(A \cap B)_f = A_f \cap B_f$.*

P r o o f. Suppose, that c -graphs $T(\mathcal{A} \cap \mathcal{B})$ and $T(\mathcal{A}) \cap T(\mathcal{B})$ are canonically c -isomorphic. Then each furcation-tandem (a_b, a_t) in $T(\mathcal{A} \cap \mathcal{B})$ is a furcation-tandem both in $T(\mathcal{A})$ and in $T(\mathcal{B})$, and conversely, each common furcation-tandem of $T(\mathcal{A})$ and of $T(\mathcal{B})$ is a furcation-tandem in $T(\mathcal{A} \cap \mathcal{B})$. The first conclusion means that $(A \cap B)_f \subseteq A_f \cap B_f$, and the second that $A_f \cap B_f \subseteq (A \cap B)_f$. Thus, $(A \cap B)_f = A_f \cap B_f$.

Now, suppose that $(A \cap B)_f = A_f \cap B_f$. Then constructing c -graph $T(\mathcal{A} \cap \mathcal{B})$, we observe that vertices in $A \cap B$ become furcation-tandems iff these vertices become furcation-tandems both in $T(\mathcal{A})$ and $T(\mathcal{B})$. Thus, having simultaneous transformations of furcations to furcation-tandems, c -graphs $T(\mathcal{A} \cap \mathcal{B})$ and $T(\mathcal{A}) \cap T(\mathcal{B})$ equally preserve route connections between elements in $\mathcal{A} \cap \mathcal{B}$, i.e., the c -graphs $T(\mathcal{A} \cap \mathcal{B})$ and $T(\mathcal{A}) \cap T(\mathcal{B})$ are canonically c -isomorphic. \square

Proposition 4.3.2. *Let \mathcal{M} be an acyclic digraph containing c -subgraphs \mathcal{A} and \mathcal{B} so that $M_f \cap A = A_f$ and $M_f \cap B = B_f$. Then c -graphs $T(\mathcal{A} \cup_{\mathcal{M}} \mathcal{B})$ and $T(\mathcal{A}) \cup_{T(\mathcal{M})} T(\mathcal{B})$ are canonically c -isomorphic iff $(A \cup B)_f = A_f \cup B_f$.*

P r o o f is similar the proof of Proposition 4.3.1 with replacements of symbols \cap by \cup . \square

Proposition 4.3.3. *If $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$ then the c -graphs $T(\mathcal{B} *_A \mathcal{C})$ and $T(\mathcal{B}) *_T(\mathcal{A}) T(\mathcal{C})$ are canonically c -isomorphic iff $A_f = B_f \cap C_f$.*

P r o o f is analogous to the proof of previous Propositions. It should only be noticed that, by definition of free amalgam, the equality $A_f = B_f \cap C_f$ implies $(B \cup C)_f = B_f \cup C_f$. \square

If $\mathcal{A} \subseteq_c \mathcal{B}$ then the c -graph \mathcal{A} will be denoted by $\mathcal{B} \upharpoonright \mathcal{A}$.

For a c -graph \mathcal{A} , we denote the set $A \cup \{\{a_b, a_t\} \mid a \in A\}$ by $\Upsilon(\mathcal{A})$. Then for all possible c -graphs \mathcal{B} such that $\mathcal{A} \subseteq_c \mathcal{B}$ and the set B , together with each vertex $a \in A$, doesn't contain vertices a_b and a_t distinguished from a , and together with each vertex $a_j \in A$, doesn't contain the vertex a distinguished from a_b and a_t ,³ restrictions $T(\mathcal{B}) \upharpoonright \Upsilon(\mathcal{A})$ correspond all possibilities of replacements of vertices in A_{nf} by furcation-tandems w.r.t. c -extensions of \mathcal{A} . A

³For c -graphs, the condition of inconsistency for presence of different elements a and a_j is stated to avoid collisions, and it will be always supposed for finite sets of c -graphs.

c -graph $T(\mathcal{B}) \upharpoonright \Upsilon(A)$, in which all vertices of \mathcal{A} are replaced by furcation-tandems, is denoted by $T^*(\mathcal{A})$.

A c -graph $T(\mathcal{A})$ is called a *canonical c -subgraph* of c -graph $T(\mathcal{B})$ if $\mathcal{A} \subseteq_c \mathcal{B}$ and $T(\mathcal{A}) = T(\mathcal{B}) \upharpoonright \Upsilon(A)$.

Let \mathcal{A} be a c -subgraph of c -graph \mathcal{B} . The c -graph \mathcal{B} is called a *hereditary extension in furcations* of \mathcal{A} (written $\mathcal{A} \subseteq_f \mathcal{B}$) if $A_f = B_f \cap A^{\text{in}}$, where A^{in} is the set of vertices in \mathcal{A} having nonzero input and outgoing semi-degrees.

Clearly, the relation \subseteq_f is reflexive, antisymmetric and transitive.

Proposition 4.3.4. *Let \mathcal{A} be a c -subgraph of c -graph \mathcal{B} , and the set B , together with each vertex $a \in A$, doesn't contain vertices a_b and a_t distinguished from a , and together with each vertex $a_j \in A$, doesn't contain the vertex a distinguished from a_b and a_t . Then $T(\mathcal{A})$ is a canonical c -subgraph of $T(\mathcal{B})$ iff $\mathcal{A} \subseteq_f \mathcal{B}$.*

P r o o f. Suppose, that $T(\mathcal{A})$ is a canonical c -subgraph of $T(\mathcal{B})$. Then the furcation-tandem (a_b, a_t) of each vertex a in B_f coincides with the furcation-tandem of a in \mathcal{A} or contains the vertex a in \mathcal{A} . For the first case, we have $a \in A_f$, and for the second case, a can not be intermediate in \mathcal{A} , i.e., have non-zero input and outgoing semi-degrees, since $\deg^+ a_b = 0$ and $\deg^- a_t = 0$ for the vertex a being in \mathcal{B} . Thus, $A_f = B_f \cap A^{\text{in}}$, that is, $\mathcal{A} \subseteq_f \mathcal{B}$.

Now suppose, that $\mathcal{A} \subseteq_f \mathcal{B}$. Then constructing $T(\mathcal{A})$ and $T(\mathcal{B})$, each intermediate vertex of \mathcal{A} is (not) replaced by furcation-tandem both for \mathcal{A} and for \mathcal{B} , and each vertex $a \in A$, transforming to the furcation-tandem for \mathcal{B} , is equal to a_b if $\deg^+_{\mathcal{A}} a = 0$, and is equal to a_t otherwise. Since arcs and an information on external shortest routes is preserved by constructions of $T(\mathcal{A})$ and $T(\mathcal{B})$, we have that $T(\mathcal{A})$ is a canonical c -subgraph of $T(\mathcal{B})$. \square

Let \mathcal{A} be a c -subgraph of c -graph \mathcal{B} , and let \mathcal{A}' be a c -subgraph of c -graph $T^*(\mathcal{A})$, in which all furcations a in $B_f \cap A$ are represented by furcation-tandems (a_b, a_t) , and the rest vertices in A by themselves or by furcation-tandems. Then the tandem c -graph without furcations, correspondent to c -graph that obtained from \mathcal{A}' by adding of all vertices in $B \setminus A$, all arcs, and the information on external shortest routes, connecting (in \mathcal{B}) elements of $B \setminus A$ with elements of B , is said to be a *tandem c -graph without furcations* correspondent to \mathcal{B} w.r.t. \mathcal{A}' , and denoted by $T(\mathcal{B}_{\mathcal{A}'})$.

Clearly, $T(\mathcal{B}) = T(\mathcal{B}_{\emptyset})$ for any c -graph \mathcal{B} , and if $\mathcal{A}' \neq \emptyset$ then $T(\mathcal{B})$ is naturally embeddable in $T(\mathcal{B}_{\mathcal{A}'})$.

2. Prerank functions. We define a *preprerank function* y^0 that assigns a real to every c -graph \mathcal{A} by the rule

$$y^0(\mathcal{A}) = 2 \cdot |A_f| + |A_{nf}| - \sum_{k=1}^{\infty} \alpha_k \cdot e_k(\mathcal{A}).$$

Notice that, in the definition of a preprerank function, the summation is always finite due to the finiteness of c -graphs. Moreover, for any c -graph \mathcal{A} , the value $y^0(\mathcal{A})$ is equal to $y(T(\mathcal{A}))$, where $2 \cdot |A_f| + |A_{nf}| = |T(\mathcal{A})|$. Thus,

$$y^0(\mathcal{A}) = |T(\mathcal{A})| - \sum_{k=1}^{\infty} \alpha_k \cdot e_k(\mathcal{A}).$$

A p -*approximation* of preprerank function y^0 is a function y_p^0 assigning a real to every c -graph \mathcal{A} by the rule

$$y_p^0(\mathcal{A}) = 2 \cdot |A_f| + |A_{nf}| - \sum_{k=1}^p \alpha_k \cdot e_k(\mathcal{A}).$$

Clearly, $y_p^0(\mathcal{A}) = y_p(T(\mathcal{A}))$.

Below we describe constructions simultaneously for c -graphs \mathcal{A} and correspondent c -graphs $T(\mathcal{A})$ without furcations.

The function $y^0(\cdot)$, defined as prerank functions above, has the following essential lack. If one take c -graphs with non-negative values $y^0(\cdot)$, then c -subgraphs, obtained by removing of furcations, can have values less than any given negative number.

Indeed, if c -graph \mathcal{A}_n consists of furcation a , vertices $b_1, \dots, b_n, c_1, \dots, c_n$, set $Q = \{(b_1, a), \dots, (b_n, a), (a, c_1), \dots, (a, c_n)\}$ of arcs, and empty set W , then the c -subgraph \mathcal{B}_n of \mathcal{A}_n , that formed from \mathcal{A}_n by removing of a , has the following value of preprerank function: $y^0(\mathcal{B}_n) = 2n - \alpha_2 \cdot n^2$. Therefore, $\lim_{n \rightarrow \infty} y(\mathcal{B}_n) = -\infty$.

For the control of real balance, in c -graphs \mathcal{A} , between the number of elements and the number of links, by prerank function, we have to take into consideration a virtual presence (as an addition to records W) of finitely many so-called “compulsory” furcations, i.e., furcations a such that there are no superfluous shortest routes connecting different pairs $(a', a'') \in \bigcup_{n \in \omega \setminus \{0\}} Q^n(a, \mathcal{A}) \times \bigcup_{n \in \omega \setminus \{0\}} Q^n(\mathcal{A}, a)$

and avoiding these furcations. Moreover, before adding of all compulsory furcations, belonging to shortest (a', a'') -routes, the pairs (a', a'') don't take into consideration for the calculation of value of prerank function, and after adding of all compulsory furcations, we take into consideration all pairs connected by external shortest routes only. Thus, the prerank function will have only non-negative values and it will allow to include all compulsory furcations in self-sufficient closures of c -graphs.

Thus, a furcation a of c -graph \mathcal{A} is called \mathcal{A} -*compulsory* if

$$y(T^*(\mathcal{B})) > y(T((\mathcal{A} \uparrow (B \cup \{a\})))_{T^*(\mathcal{B})})$$

for some c -subgraph $\mathcal{B} \subseteq_c \mathcal{A}$.⁴ Furthermore, the furcation a is also called \mathcal{B} -*compulsory*.⁵

An \mathcal{A} -compulsory furcation is called \mathcal{B} -*external* (where $\mathcal{B} \subseteq_c \mathcal{A}$) if it doesn't belong to \mathcal{B} . An \mathcal{A} -compulsory furcation is called *compulsory* if \mathcal{A} is fixed. A \mathcal{B} -external compulsory furcation is said to be *external* if c -graph \mathcal{B} is fixed.

By definition of \mathbf{K}_0^{nf} , for any fixed number $n \in \omega$ and any c -graph $T^*(\mathcal{A}) \in \mathbf{K}_0^{\text{nf}}$, there are cardinality bounds, depending on cardinality $|T^*(\mathcal{A})|$, for number of new external shortest routes of lengths at most k , connecting vertices in $T^*(\mathcal{A})$ with some new vertex a so that obtained c -graph \mathcal{B} with universe $T^*(\mathcal{A}) \cup \{a\}$ or $T^*(\mathcal{A}) \cup \{a_b, a_t\}$ belongs to \mathbf{K}_0^{nf} . Therefore, a sequential choice of numbers α_s , satisfying all aforesaid condition, allows to produce the property that some vertex, a_b or a_t , for each compulsory furcation a , belongs to a self-sufficient closure for the set of its predecessors or for the set of its successors in $T^*(\mathcal{A})$.

Indeed, removing any furcation a connected with $b_1, \dots, b_l, c_1, \dots, c_m$ only by external shortest (b_1, a) -, \dots , (b_l, a) -, (a, c_1) - and (a, c_m) -routes of lengths $s_1, \dots, s_l, s'_1, \dots, s'_m$, respectively, such that for any pair (b_i, c_j) the length s_{ij} of shortest (b_i, c_j) -route equals $s_i + s'_j$, we obtain only external shortest routes between b_i and c_j .

⁴By definition of $y(\cdot)$, for checking that the furcation a is compulsory, it suffices to consider c -subgraphs $\mathcal{B} \subseteq_c \mathcal{A} \uparrow \left(\bigcup_{n \in \omega \setminus \{0\}} Q^n(a, \mathcal{A}) \cup \bigcup_{n \in \omega \setminus \{0\}} Q^n(\mathcal{A}, a) \right)$.

⁵Here, the difference of \mathcal{A} -compulsion and \mathcal{B} -compulsion is that the furcation a belongs to \mathcal{A} and doesn't belong to \mathcal{B} . Moreover, belonging to a self-sufficient closure \bar{B} , the vertex a is an element integrated in \mathcal{B} .

Moreover, in prerank function $y(\cdot)$, sums $\sum_{i,j}(\alpha_{s_i} + \alpha_{s'_j})$ are replaced by $\sum_{i,j} \alpha_{s_{ij}}$, where $\alpha_{s_{ij}} < \min\{\alpha_{s_i}, \alpha_{s_j}\}$. Considering all possible configurations of c -graphs with universes $\{a, b_1, \dots, b_l, c_1, \dots, c_m\}$ and having only external shortest routes of lengths at most $k - 1$, we can, on each sequential step of definition for the number α_k , choose this number so small that the following conditions (*) are satisfied:

(1) if for c -graphs \mathcal{B} and \mathcal{C} with universes $\{a, b_1, \dots, b_l\}$ and $\{a, c_1, \dots, c_m\}$ respectively, where $\deg_{\mathcal{B}}^+(a) = \deg_{\mathcal{C}}^-(a) = 0$, the relations

$$y(T(\mathcal{B}_{T^*}(\mathcal{B} \uparrow \{b_1, \dots, b_l\}))) \geq y(T^*(\mathcal{B} \uparrow \{b_1, \dots, b_l\})) \quad (4.5)$$

and

$$y(T(\mathcal{C}_{T^*}(\mathcal{C} \uparrow \{c_1, \dots, c_m\}))) \geq y(T^*(\mathcal{C} \uparrow \{c_1, \dots, c_m\})) \quad (4.6)$$

hold then, for c -graph \mathcal{D} with the universe $\{a, b_1, \dots, b_l, c_1, \dots, c_m\}$, the relation $Q_{\mathcal{B}} \cup Q_{\mathcal{C}}$, the record $W_{\mathcal{B}} \cup W_{\mathcal{C}}$, and for the c -graph $\mathcal{E} = \mathcal{D} \uparrow \{b_1, \dots, b_l, c_1, \dots, c_m\}$, the inequality

$$y(T(\mathcal{D}_{T^*}(\mathcal{E}))) \geq y(T^*(\mathcal{E})) \quad (4.7)$$

is satisfied;

(2) if a vertex a is not an \mathcal{E} -compulsory furcation then, for the c -graphs \mathcal{B}' , \mathcal{C}' , and \mathcal{E}' that obtained from \mathcal{B} , \mathcal{C} , and \mathcal{E} , respectively, by removing of all (b_i, b_j) - and (c_i, c_j) -routes, we have

$$\sum_{k=1}^{\infty} \alpha_k \cdot e_k(\mathcal{E}') < \frac{1}{2} \cdot \min \left\{ \sum_{k=1}^{\infty} \alpha_k \cdot e_k(\mathcal{B}'), \sum_{k=1}^{\infty} \alpha_k \cdot e_k(\mathcal{C}') \right\}. \quad (4.8)$$

Considering two-element c -graph, \mathcal{B} and \mathcal{C} , and using the condition (2), we conclude that

$$\sum_{k=1}^{\infty} \alpha_k \cdot e_k(\mathcal{E}') < \frac{1}{2} \cdot \alpha_l,$$

where l is the greatest index, till which, inclusive, all values $e_k(\mathcal{E}')$ equal zero. Moreover, taking all possible c -graphs satisfying (*),

we may rewrite the inequality (4.8) by the following sequence of inequalities for all natural $p \geq 1$:

$$\sum_{k=1}^p \alpha_k \cdot e_k(\mathcal{E}') < \frac{1}{2} \cdot \min \left\{ \sum_{k=1}^p \alpha_k \cdot e_k(\mathcal{B}'), \sum_{k=1}^p \alpha_k \cdot e_k(\mathcal{C}') \right\}.$$

Below, we suppose that (*) is satisfied and this property is taken into consideration in the definition of α_k 's. It means, in particular, that if (4.7) is violated then (4.5) or (4.6) is not true, i.e., if a \mathcal{D} -compulsory furcation a belongs to the self-sufficient closure of \mathcal{E} then a belongs to the self-sufficient closure of $\mathcal{B} \upharpoonright (B \setminus \{a\})$ or of $\mathcal{C} \upharpoonright (C \setminus \{a\})$.

The conditions (*) also imply that, for any non-compulsory furcation a of c -graph \mathcal{A} , all connections between elements in $A \setminus \{a\}$ shortest routes by external over $A \setminus \{a\}$, containing a , can be replaced by links being external shortest routes over $A \setminus \{a\}$ such that sets of intermediate elements are pairwise disjoint.

The following modification of the preprerank function allows to add all \mathcal{A} -compulsory furcations to the self-sufficient closure of \mathcal{A} , that will be guaranteed by extension of the record $W_{\mathcal{A}}$.

We allow for each c -graph $\mathcal{B} \subseteq_c \mathcal{A}$ to have less than $k_{|T(\mathcal{B})|}$ compulsory furcations (these furcations may be not only \mathcal{B} -compulsory, but also compulsory w.r.t. c -graphs that obtained by extensions of \mathcal{B} by other, of less than $k_{|T(\mathcal{B})|}$, compulsory furcations).⁶ Now, we add to each record $W_{\mathcal{B}}$ a type describing the structure of c -graph \mathcal{B} together with all its compulsory furcations, and also describing an absence of any another compulsory furcations. Obtained expanded c -graphs are called *cc-graphs*.

We denote by A_{cf} the set of all \mathcal{A} -compulsory furcations of the cc -graph \mathcal{A} belonging \mathcal{A} . The cc -graph \mathcal{A} is called *cf-closed* if A_{cf} contains all \mathcal{A} -compulsory furcations.

The operation of adding to a cc -graph \mathcal{A} of all its compulsory furcations with the structure, described in $W(\mathcal{A})$, is called the operation of *cf-closure* of cc -graph \mathcal{A} . The cc -graph, being the result of *cf-closure* of cc -graph \mathcal{A} , is denoted by $\text{cfc}(\mathcal{A})$.

⁶In view of constructions in Section 4.2, the stated condition is satisfied for any c -graph \mathcal{A} such that $T(\mathcal{A}) \upharpoonright B \in \mathbf{K}_0^{\text{nf}}$ for any c -subgraph $\mathcal{B} \subseteq_c \mathcal{A}$.

A cc-graph \mathcal{A} is a *cc-subgraph* of cc-graph \mathcal{B} (written $\mathcal{A} \subseteq_{cc} \mathcal{B}$) if the following conditions hold:

(1) the c -graph $\langle A, Q_{\mathcal{A}}, W_1 \rangle$ with the record W_1 of the structure of c -graph in \mathcal{A} is a c -subgraph of the c -graph $\langle B, Q_{\mathcal{B}}, W_2 \rangle$ with the record W_2 of the structure of c -graph in \mathcal{B} ;

(2) the record $W_{\mathcal{A}}$ coincides with the restriction of the record $W_{\mathcal{B}}$ to the set A .

A cc-graph \mathcal{A} is called a *cc-subgraph* of graph \mathcal{M} if the triple $\langle A, Q_{\mathcal{A}}, W \rangle$, with the record W of structure of c -graph in \mathcal{A} , is a c -subgraph of \mathcal{M} and the record $W_{\mathcal{A}}$ of the cc-graph \mathcal{A} is co-ordinated with $\text{tp}_{\mathcal{M}}(A)$. We write $\mathcal{A} \subseteq_{cc} \mathcal{M}$ if \mathcal{A} is a cc-subgraph of \mathcal{M} .

Clearly, if $\mathcal{A} \subseteq_{cc} \mathcal{B}$ then any \mathcal{A} -compulsory furcation is \mathcal{B} -compulsory and, in particular, $A_{cf} \subseteq B_{cf}$.

We define a *prerank function* y^1 that assigns a real to every cc-graph \mathcal{A} by the rule

$$y^1(\mathcal{A}) = 2 \cdot |A_f| + |A_{nf}| - \sum_{k=1}^{\infty} \alpha_k \cdot e_k^1(\mathcal{A}),$$

where $e_1^1(\mathcal{A})$ is the number of arcs in \mathcal{A} , $e_k^1(\mathcal{A})$, $k \geq 2$, is the number of pairs $(a, a') \in A^2$, connected only by external shortest (a, a') -routes of length k and such that there are no (a, a') -routes of length k containing \mathcal{A} -external \mathcal{A} -compulsory furcations.

The construction of tandem cc-graph $T_c(\mathcal{A})$ ($T_c^*(\mathcal{A})$) without furcations, correspondent to cc-graph \mathcal{A} , consists of removing from c -graph $T(\mathcal{A})$ ($T^*(\mathcal{A})$) of all tuples (a_1, a_2, n) , for which there exist external shortest (a_1, a_2) -routes of length n containing external compulsory furcations.

By definition, we have $|T_c^*(\mathcal{A})| = 2 \cdot |A|$, $|T_c(\mathcal{A})| = |T(\mathcal{A})|$ and $y^1(\mathcal{A}) = y(T_c(\mathcal{A})) = y(T_c^*(\mathcal{A})) - |A_{nf}|$.

A *p-approximation* of prerank function y^1 is a function y_p^1 assigning a real to every cc-graph \mathcal{A} by the rule

$$y_p^1(\mathcal{A}) = 2 \cdot |A_f| + |A_{nf}| - \sum_{k=1}^p \alpha_k \cdot e_k^1(\mathcal{A}).$$

Clearly, $y_p^1(\mathcal{A}) = y_p(T_c(\mathcal{A}))$.

3. Generic class and generic theory. We are going to define a class of cc-graphs, in which like the classes of c -graphs from Sections 4.1 and 4.2, on the one hand, there is a balance between numbers of elements and number of connections by lower estimates b_n^p for values of prerank function, and, on the other hand, a decreasing of values of prerank function with extensions of universes of cc-graphs determines that elements of extensions belong to self-sufficient closures of given cc-graphs. At the same time, doubling weights of vertices for elements $a \in A_{\text{nf}}$ being in B_f , where $\mathcal{A} \subseteq_{\text{cc}} \mathcal{B}$, we can not observe the self-sufficiency of cc-graphs \mathcal{A} (meaning an exceeding of number of new element, not of virtual pairs of elements, over the numbers of new links) by inequalities $y^1(\mathcal{A}) \leq y^1(\mathcal{B})$. The same cause, even for the following definition of self-sufficiency, doesn't allow to obtain the finite closure property for models of generic theory and, as corollary, to have saturated generic model taking, as above, the class of cc-graphs \mathcal{A} for which all cc-subgraphs \mathcal{A}' satisfy $y_p^1(\mathcal{A}') \geq b_n^p$, where $n = |T(\mathcal{A}')|$ and $k_p \leq n < k_{p+1}$.

For the realization of required properties, we restrict the class of considered cc-graphs to a subclass, allowing to bound the number of iterations for self-sufficient closures of cc-subgraphs dependent on cardinalities of given cc-graphs. The self-sufficiency condition will be defined as nondecreasing of values of prerank balance between number of new vertices and number of new links in tandem c -graphs without furcations w.r.t. tandem c -graph without furcations of given self-sufficient cc-graph for which each vertex is represented by a furcation-tandem.

Let \mathcal{A} be a cc-subgraph of cc-graph \mathcal{B} , and \mathcal{A}' be a cc-subgraph of $T_c^*(\mathcal{A})$, in which all furcation vertices a in $\mathcal{B}_f \cap \mathcal{A}$ are represented by furcation-tandems (a_b, a_t) , and the rest vertices of \mathcal{A} by themselves or by furcation-tandems. Then the tandem cc-graph without furcations, correspondent to cc-graph that obtained from \mathcal{A}' by adding all vertices of $B \setminus A$ and of correspondent arcs and the information on compulsory furcations and on external shortest routes, connecting (in \mathcal{B}) elements of $B \setminus A$ with elements of B , is said to be a *tandem cc-graph without furcations* correspondent to \mathcal{B} w.r.t. \mathcal{A}' , and is denoted by $T_c(\mathcal{B}_{\mathcal{A}'})$.

Recall [25], that a graph \mathcal{A} is a *part* of graph \mathcal{B} if $A \subseteq B$ and \mathcal{B} contains all arcs of \mathcal{A} . Similarly, a cc-graph \mathcal{A} is called a *part* of cc-graph \mathcal{B} (written $\mathcal{A} \sqsubseteq \mathcal{B}$) if $A \subseteq B$, \mathcal{B} contains all arcs of \mathcal{A} , and the record $W_{\mathcal{B}}$ includes all records in $W_{\mathcal{A}}$. We also write $\mathcal{A} \sqsubseteq \mathcal{M}$ if \mathcal{A} is a part of some cc-graph \mathcal{B} being a cc-subgraph of graph \mathcal{M} .

Notice, that replacing a cc-graph \mathcal{A} by its parts with the universe A , we can consider all subsets of A , consisting of furcations, as non-furcations and, by observation of parts, to test cc-subgraphs \mathcal{A}' of \mathcal{A} being self-sufficient in \mathcal{A} . Moreover, by (*), the non-self-sufficiency in \mathcal{A} satisfies the following *hereditary property*: if \mathcal{A} contains a subset $B \subseteq A \setminus \mathcal{A}'$ that by adding to \mathcal{A}' produces, w.r.t. \mathcal{A}' , a negative difference between number of new vertices and weighted number of new links in a part of \mathcal{A} with the universe $\mathcal{A}' \cup B$, then for any cc-graph \mathcal{A}'' , where $\mathcal{A}' \subseteq_{\text{cc}} \mathcal{A}'' \subseteq_{\text{cc}} \mathcal{A}$ and $B \subseteq A \setminus \mathcal{A}''$, we also have a negative difference between number of new vertices and weighted number of new links, w.r.t. \mathcal{A}'' , in some part of \mathcal{A} with the universe $\mathcal{A}'' \cup B$.

Lemma 4.3.5. *For every positive $p \in \omega$ and arbitrary cc-graphs \mathcal{A} and \mathcal{B} , where \mathcal{A} is a proper part of \mathcal{B} , $|T(\mathcal{B})| = n < k_{p+1}$, the following assertions hold:*

- (1) *We have $y_p^1(\mathcal{B}) - y^1(\mathcal{B}) < \varepsilon_{p+1}$.*
- (2) *The inequality $y_p^1(\mathcal{A}) < y_p^1(\mathcal{B})$ holds iff $y^1(\mathcal{A}) < y^1(\mathcal{B})$. The inequality $y_p^1(\mathcal{A}) < y_p^1(\mathcal{B})$ holds iff $y_q^1(\mathcal{A}) < y_q^1(\mathcal{B})$ for every $q \geq p$.*

P r o o f is similar to that of Lemma 4.1.1 with replacements of y by y^1 , and graphs \mathcal{A} and \mathcal{B} by cc-graphs $T_c(\mathcal{A})$ and $T_c(\mathcal{B})$, respectively. \square

Given a cc-graph \mathcal{A} , we write $\mathcal{A} \in \mathbf{K}_1^f$ iff $y_1^1(\mathcal{A}') \geq b_n^1$ for any nonempty cc-graph $\mathcal{A}' \sqsubseteq \text{cfc}(\mathcal{A})$, where $n = |T(\mathcal{A}')|$.

For a cc-graph \mathcal{A} , we write $\mathcal{A} \in \mathbf{K}_{p+1}^f$ iff $\mathcal{A} \in \mathbf{K}_p^f$ and $y_p^\Delta(\mathcal{A}') \geq b_n^p$ for any cc-graph $\mathcal{A}' \sqsubseteq \text{cfc}(\mathcal{A})$, where $y_p^\Delta(\mathcal{A}')$ is a minimal value $y_p(T_c(\mathcal{A}_0)) + \Delta_1 + \dots + \Delta_m$, $\Delta_i = y_p(T_c((\mathcal{A}_{i+1})_{T_c^*(\mathcal{A}_i)})) - y_p(T_c^*(\mathcal{A}_i))$, $\mathcal{A}_0 \subseteq_{\text{cc}} \mathcal{A}_1 \subseteq_{\text{cc}} \dots \subseteq_{\text{cc}} \mathcal{A}_m = \mathcal{A}'$, n is a natural number, for which $y_p^\Delta(\mathcal{A}') = n - s \cdot \alpha_1$ with some $s \in \omega$, $k_p \leq n < k_{p+1}$. Notice, that the definition is correct, since the value n in $n - s \cdot \alpha_1$ doesn't depend on p .

Put $\mathbf{K}_0^f \equiv \bigcap_{p=1}^{\infty} \mathbf{K}_p^f$ and denote by \mathbf{K}^f the class of all acyclic digraphs whose cc-subgraphs belong to \mathbf{K}_0^f .

Let \mathcal{A} be a cc-subgraph of a graph (respectively, of a cf-closed cc-subgraph) \mathcal{M} (of a graph) in \mathbf{K}^f . We say that \mathcal{A} is a *self-sufficient cc-subgraph* of the graph (respectively, cc-graph) \mathcal{M} and write $\mathcal{A} \leq_{\text{cc}} \mathcal{M}$ if for any cc-graphs $\mathcal{A}' \sqsubseteq \mathcal{A}$, $\mathcal{B}' \sqsubseteq \mathcal{M}$, $\mathcal{A}' = \mathcal{A}_0 \subseteq_{\text{cc}} \mathcal{A}_1 \subseteq_{\text{cc}} \dots \subseteq_{\text{cc}} \mathcal{A}_m = \mathcal{B}'$, the inequality $\Delta_1 + \dots + \Delta_m < 0$, where $\Delta_i =$

$y(T_c((\mathcal{A}_{i+1})_{T_c^*(\mathcal{A}_i)})) - y(T_c^*(\mathcal{A}_i))$, implies $B' \subseteq A$. If $\mathcal{A} \leq_{\text{cc}} \mathcal{M}$ and \mathcal{M} is a cc-graph then \mathcal{A} is called a *strong cc-subgraph* of \mathcal{M} .

In particular, if a cc-graph \mathcal{A} is self-sufficient then \mathcal{A} is cf-closed.

We have to show that the class \mathbf{T}_0^f of all types corresponding to the cc-graphs of \mathbf{K}_0^f with the relation \leq'_{cc} (where $\Phi(A) \leq'_{\text{cc}} \Psi(B) \Leftrightarrow \mathcal{A} \leq_{\text{cc}} \mathcal{B}$) is self-sufficient and satisfies (after adding the necessary formulas describing the self-sufficient closures to the types) the uniform t -amalgamation property. This fact yields the ω -saturation of $(\mathbf{T}_0^f; \leq'_{\text{cc}})$ -generic model that realizes all types $\Phi(X)$ corresponding to the types $\Phi(A)$ in \mathbf{T}_0^f .

Remark 4.3.6. (1) *The conditions $\mathcal{A} \in \mathbf{K}_0^f$ and $k_p \leq |T(A)| = n < k_{p+1}$ do not imply that $y^\Delta(\mathcal{A}) = \inf_p y_p^\Delta(\mathcal{A}) \geq b_n^p$. Nevertheless, $y^\Delta(\mathcal{A})$ cannot be much less than b_n^p : since $y_p^1(\mathcal{A}) - y^1(\mathcal{A}) < \varepsilon_{p+1}$ then $y^\Delta(\mathcal{A}) > b_n^p - \varepsilon_{p+1}$.*

Moreover, in view of the inequality (4.1), for a cc-graph $\mathcal{A} \in \mathbf{K}_0^f$ with $|T(A)| \geq \max\{2, k_p\}$, we have $y^\Delta(\mathcal{A}) > p$.

(2) *Suppose that \mathcal{A} is a cc-graph of \mathbf{K}_0^f , $|T^*(A)| = p$, \mathcal{M} is a graph in \mathbf{K}^f , $\mathcal{A} \subseteq_{\text{cc}} \mathcal{M}$. Then $\mathcal{A} \leq_{\text{cc}} \mathcal{M}$ iff, for any cc-graphs $\mathcal{A}' \sqsubseteq \mathcal{A}$, $\mathcal{B}' \sqsubseteq \mathcal{M}$, $\mathcal{A}' = \mathcal{A}_0 \subseteq_{\text{cc}} \mathcal{A}_1 \subseteq_{\text{cc}} \dots \subseteq_{\text{cc}} \mathcal{A}_m = \mathcal{B}'$, the inequality $\Delta_1^p + \dots + \Delta_m^p < 0$, where $\Delta_i^p = y_p(T_c((\mathcal{A}_{i+1})_{T_c^*(\mathcal{A}_i)})) - y_p(T_c^*(\mathcal{A}_i))$, implies $B' \subseteq A$.*

Indeed, if $|T_c((\mathcal{B}')_{T_c^*(\mathcal{A}')})| < k_{p+1}$ then $\Delta_1^p + \dots + \Delta_m^p < 0$ is equivalent to $\Delta_1 + \dots + \Delta_m < 0$ by Lemma 4.3.5(2), and if $|T_c((\mathcal{B}')_{T_c^*(\mathcal{A}')})| \geq k_p$ then $\Delta_1^p + \dots + \Delta_m^p \geq \Delta_1 + \dots + \Delta_m > p - y_p(T_c^*(\mathcal{A}')) \geq 0$.

Moreover, in order to verify that cc-graph \mathcal{A} is self-sufficient in \mathcal{M} , it suffices to choose $n_{\mathcal{A}} = k_p$ and check the validity of the relations $B' \subseteq A$ only for cc-graphs $\mathcal{A}' \sqsubseteq \mathcal{A}$ and \mathcal{B}' with $\mathcal{A}' \subseteq_{\text{cc}} \mathcal{B}'$, $\mathcal{B}' \sqsubseteq \mathcal{B} \subseteq_{\text{cc}} \mathcal{M}$, $\Delta_1^p + \dots + \Delta_m^p < 0$, $|T^*(\mathcal{B}')| < n_{\mathcal{A}}$.

Thus, the condition $\mathcal{A} \leq_{\text{cc}} \mathcal{M}$ is definable by a formula that describes the absence of n_i new, w.r.t. $T^*(A_{i-1})$ (where $\mathcal{A}_0 \subseteq_{\text{cc}} \mathcal{A}_1 \subseteq_{\text{cc}} \dots \subseteq_{\text{cc}} \mathcal{A}_m \sqsubseteq \mathcal{M}$, $\mathcal{A}_0 \sqsubseteq \mathcal{A}$), vertices of $A_m \cup \Upsilon(A_m)$, where $n_1 + \dots + n_m < n_{\mathcal{A}}$, such that the set of these vertices is not contained in $A \cup \Upsilon(A)$ and

$$n < \alpha_1 \cdot e_1 + \alpha_2 \cdot (e_2 - e'_2) + \dots + \alpha_p \cdot (e_p - e'_p), \quad (4.9)$$

where $p = |T^*(A)|$; e_1 is the number of new arcs; e_s , $s > 1$, is

the number of new pairs of vertices (a, b) that are connected only by external shortest routes of length s , and such that there are no shortest (a, b) -routes containing \mathcal{A}_m -external \mathcal{A}_m -compulsory furcations; e'_s is the number of pairs of vertices (a, b) used in calculation of Δ_i and that stop to be linked only by external shortest routes of length s in extended, w.r.t. \mathcal{A}_i , cc-subgraph of the cc-graph \mathcal{A}_m .

Below, we suppose that the record $W_{\mathcal{A}}$ of each cc-graph \mathcal{A} is extended by an information on (im)possibility of an extension for \mathcal{A} , by less than $n_{\mathcal{A}}$ new vertices, to obtain the value $y(\cdot)$ that is non-decreasing for extensions. Thus, any type, describing a place of \mathcal{A} w.r.t. external elements, is expanded by an information on minimal self-sufficient extension of \mathcal{A} , including an information on compulsory furcations.

The following lemma is a direct consequence of the definition:

Lemma 4.3.7. (1) *If $\mathcal{A} \leq_{\text{cc}} \mathcal{B}$ then $\mathcal{A} \subseteq_{\text{cc}} \mathcal{B}$.*

(2) *If $\mathcal{A} \leq_{\text{cc}} \mathcal{C}$, $\mathcal{B} \in \mathbf{K}_0^f$, and $\mathcal{A} \subseteq_{\text{cc}} \mathcal{B} \subseteq_{\text{cc}} \mathcal{C}$, then $\mathcal{A} \leq_{\text{cc}} \mathcal{B}$.*

(3) *The empty graph \emptyset is the least element of the structure $(\mathbf{K}_0^f; \leq_{\text{cc}})$.*

Lemma 4.3.8. *If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}_0^f$, $\mathcal{A} \leq_{\text{cc}} \mathcal{B}$, and $\mathcal{C} \subseteq_{\text{cc}} \mathcal{B}$, then $\mathcal{A} \cap \mathcal{C} \leq_{\text{cc}} \mathcal{C}$.*

P r o o f. Assume the contrary. Then there exist n_1, \dots, n_m new elements in $\Upsilon(\mathcal{C})$ such that

$$n_1 + \dots + n_m < \alpha_1 \cdot e_1 + \alpha_2 \cdot (e_2 - e'_2) + \dots + \alpha_p \cdot (e_p - e'_p),$$

where $p = |T^*(A)|$; e_s and e'_s are values as in (4.9). Since all new elements belong to $\Upsilon(\mathcal{B})$, they violate the condition $\mathcal{A} \leq_{\text{cc}} \mathcal{B}$. \square

Lemma 4.3.9. *The relation \leq_{cc} is a partial order on \mathbf{K}_0^f .*

P r o o f. The reflexivity and antisymmetry of \leq_{cc} are obvious.

We claim that \leq_{cc} is transitive. Assume the contrary and consider cc-graphs $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}_0^f$ such that $\mathcal{A} \leq_{\text{cc}} \mathcal{B}$, $\mathcal{B} \leq_{\text{cc}} \mathcal{C}$ and $\mathcal{A} \not\leq_{\text{cc}} \mathcal{C}$. By assumption, there exists a part \mathcal{A}' of \mathcal{A} and its extension \mathcal{C}' , being a part of \mathcal{C} and not being a part of \mathcal{B} , such that $\mathcal{A}' = \mathcal{A}_0 \subseteq_{\text{cc}} \mathcal{A}_1 \subseteq_{\text{cc}} \dots \subseteq_{\text{cc}} \mathcal{A}_m = \mathcal{C}'$ and $\Delta_1 + \dots + \Delta_m < 0$. Since \mathcal{A}' is a part of \mathcal{B} , the last inequality contradicts to the self-sufficiency of \mathcal{B} in \mathcal{C} . \square

Lemma 4.3.10. *For any cc-subgraph \mathcal{A} of graph \mathcal{M} in \mathbf{K}^f , there exists the least cc-graph $\overline{\mathcal{A}}$ that contains \mathcal{A} is a self-sufficient cc-subgraph of \mathcal{M} . Moreover, $|\overline{\mathcal{A}}| < k_{|T^*(A)|}$, each vertex in $T(\overline{\mathcal{A}}) \setminus \Upsilon(\mathcal{A})$ has degree not less than two and belongs to $J_0 \cup J_1$.*

P r o o f. An existence of self-sufficient extensions of the cc-graph \mathcal{A} in the graph \mathcal{M} is implied by definition of \mathbf{K}^f , namely, from inequalities $y_p^\Delta(\mathcal{A}') \geq b_n^p$, where \mathcal{A}' are parts of \mathcal{A} with universes A . The same inequalities implies that there exists an upper cardinality estimate $k_{|T^*(A)|}$ for minimal self-sufficient extensions of \mathcal{A} . By Lemma 4.3.8, an intersection of any two such self-sufficient extensions is also a self-sufficient cc-subgraph of \mathcal{M} containing \mathcal{A} . This means, that there exists the least self-sufficient extension of \mathcal{A} , and it is equal to the self-sufficient closure $\overline{\mathcal{A}}$.

Each vertex in $T(\overline{\mathcal{A}}) \setminus \Upsilon(A)$ has degree not less than two, since the self-sufficient extension $\overline{\mathcal{A}}$ is minimal. The condition $T(\overline{\mathcal{A}}) \setminus \Upsilon(A) \subseteq J_0 \cup J_1$ is implied by the fact that, removing an intermediate vertex a of degree 2, the value of the prerank function becomes less on positive quantity $1 - \alpha_s + \alpha_{s_1} + \alpha_{s_2}$, where $s = s_1 + s_2$ is the length of external (when a is removed) shortest route containing a . Thus, there are no new intermediate vertices that belong to a superset of A with the least value $y(\cdot)$. \square

By Lemma 4.3.10, we have a right, for each cc-graph \mathcal{A} in \mathbf{K}_0^f , to add an information on *self-sufficient closure* $\overline{\mathcal{A}}$ in a graph $\mathcal{M} \in \mathbf{K}^f$. Clearly, such addition is not unique (depend on choice of \mathcal{M}) and generate, for each cc-graph, finitely many possibilities of c -isomorphism types of self-sufficient closures. This number is defined by possibilities for distributions of lengths of shortest routes between less than $k_{|T^*(A)|}$ elements, forming universes for c -isomorphism types of self-sufficient closures. If a cc-graph \mathcal{A} coincides with its self-sufficient closure, \mathcal{A} is called a *self-sufficient cc-graph*.

Below, we suppose that the record W of each cc-graph contains an information on its self-sufficient closure, and this information is in accordance with extensions and restrictions of cc-graphs.

An injective map $f : A \rightarrow B$ is called *cc-embedding* of cc-graph $\mathcal{A} = \langle A, Q_A, W_A \rangle$ into a cc-graph $\mathcal{B} = \langle B, Q_B, W_B \rangle$ (written $f : \mathcal{A} \rightarrow_{cc} \mathcal{B}$) if f is an embedding of the c -graph, that formed from \mathcal{A} by restriction of the record W_A to c -graph one, into the c -graph, that formed from \mathcal{B} by restriction of the record W_B to c -graph one, and such that the restriction of W_B to $f(A)$ coincides with the substitution to W_A of elements in $f(A)$ instead of correspondent elements of A .

Two cc-graphs, \mathcal{A} and \mathcal{B} , are called *cc-isomorphic* if there exists a cc-embedding $f : \mathcal{A} \rightarrow_{cc} \mathcal{B}$ with $f(A) = B$. Furthermore, the

map f is called a *cc-isomorphism* between \mathcal{A} and \mathcal{B} , and cc-graphs \mathcal{A} and \mathcal{B} are *cc-isomorphic copies*.

An injection $f : A \rightarrow N$ is called a *cc-embedding* of cc-graph \mathcal{A} into a digraph \mathcal{N} (written $f : \mathcal{A} \rightarrow_{\text{cc}} \mathcal{N}$) if f is a cc-embedding of \mathcal{A} into the cc-subgraph $f(\mathcal{A})$ of \mathcal{N} with the universe $f(A)$.

A cc-embedding f of cc-graph \mathcal{A} into a cc-graph \mathcal{B} is *strong* if $f(\mathcal{A}) \leq_{\text{cc}} \mathcal{B}$.

Let $\mathcal{A}, \mathcal{B} = \langle B, Q_{\mathcal{B}}, W_{\mathcal{B}} \rangle$, and $\mathcal{C} = \langle C, Q_{\mathcal{C}}, W_{\mathcal{C}} \rangle$ be self-sufficient cc-graphs, and $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$. The *free cc-amalgam* of cc-graphs \mathcal{B} and \mathcal{C} over \mathcal{A} (written $\mathcal{B} *_{\mathcal{A}} \mathcal{C}$) is the self-sufficient cc-graph $\langle B \cup C, Q_{\mathcal{B}} \cup Q_{\mathcal{C}}, W_{\mathcal{B}} \cup W_{\mathcal{C}} \cup W \rangle$, where W is the record on self-sufficiency of resulted cc-graph.

Notice, that, by definition, any free cc-amalgam is *cf-closed*.

Lemma 4.3.11. (*amalgamation lemma*). *The class \mathbf{K}_0^f has the cc-amalgamation property cc-(AP), i.e., for any strong cc-embeddings $f_0 : \mathcal{A} \rightarrow_{\text{cc}} \mathcal{B}$ and $g_0 : \mathcal{A} \rightarrow_{\text{cc}} \mathcal{C}$, where \mathcal{A}, \mathcal{B} , and \mathcal{C} are self-sufficient cc-graphs in \mathbf{K}_0^f , there exist a self-sufficient cc-graph $\mathcal{D} \in \mathbf{K}_0^f$ and strong cc-embeddings $f_1 : \mathcal{B} \rightarrow_{\text{cc}} \mathcal{D}$ and $g_1 : \mathcal{C} \rightarrow_{\text{cc}} \mathcal{D}$ such that $f_0 \circ f_1 = g_0 \circ g_1$.*

P r o o f. Without loss of generality, we may assume that $\mathcal{A} \leq_{\text{cc}} \mathcal{B}$, $\mathcal{A} \leq_{\text{cc}} \mathcal{C}$, $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$. We claim that cc-graph $\mathcal{D} = \mathcal{B} *_{\mathcal{A}} \mathcal{C}$ is as required. To this end, in view of the symmetry of the definition of free cc-amalgam, it suffices to show that $\mathcal{B} \leq_{\text{cc}} \mathcal{D}$ and $\mathcal{D} \in \mathbf{K}_0^f$.

Assume that $\mathcal{B} \not\leq_{\text{cc}} \mathcal{D}$. Then there exist some n_1, \dots, n_m new elements in $\Upsilon(\mathcal{D})$, not all in $\Upsilon(\mathcal{B})$, such that

$$n_1 + \dots + n_m < \alpha_1 \cdot e_1 + \alpha_2 \cdot (e_2 - e'_2) + \dots + \alpha_p \cdot (e_p - e'_p),$$

where $p = |T^*(\mathcal{B})|$; e_s and e'_s are values as in (4.9). Since elements, that contradict to self-sufficiency of \mathcal{B} in \mathcal{D} , belong to $\Upsilon(\mathcal{C})$, these elements, for an extension of \mathcal{A} by elements, via which elements of $B \setminus A$ are linked with new elements in C , violate the condition $\mathcal{A} \leq_{\text{cc}} \mathcal{C}$ in view of (*).

Indeed, if routes, connecting elements in B with elements in $C \setminus A$, contain compulsory furcations, then correspondent links are not used in the calculation of Δ_i . So, w.l.o.g., shortest routes S , connecting $a \in B \setminus A$ and $b \in C \setminus A$, doesn't contain compulsory furcations. Suppose now, that some value $\Delta_1 + \dots + \Delta_m$ is negative after adding of elements of $C \setminus A$ to some $B' \subseteq B$.

By definition of free amalgam, for any pair $(a, a') \in B \times (C \setminus A)$ of vertices, connected only by external shortest routes over $B' \cup (C \setminus A)$, one of the following conditions holds:

- (1) a belongs to A ;
- (2) a doesn't belong to A , and some external shortest (a, a') - or (a', a) -route over $B' \cup (C \setminus A)$ contains an intermediate vertex in A .

Denote by A' the set of all intermediate vertices of A described in (2). After adding the set A' and correspondent links to B' , the negative part $\sum_{k=1}^{\infty} \alpha_k \cdot e_k$ of value $\Delta_1 + \dots + \Delta_m$ increases in view of (*), and positive part is the same, since adding these minimal links with elements in $C \setminus A$ to parts in \mathcal{D} , we preserve non-furcation vertices. Then for the set $(B' \cap A) \cup A'$, some value $\Delta_1 + \dots + \Delta_m$ is also negative after adding of elements in $C \setminus A$. The last contradicts to the self-sufficiency of \mathcal{A} in \mathcal{C} .

Now, we claim that any cc-graph \mathcal{D} belongs to \mathbf{K}_0^f . It means that $y_p^\Delta(\mathcal{E}) \geq b_n^p$, where $\mathcal{E} \sqsubseteq \mathcal{D}$, n is a natural number, for which $y_p^\Delta(\mathcal{E}) = n - s \cdot \alpha_1$ with some $s \in \omega$, $k_p \leq n < k_{p+1}$.

Denote by \mathcal{E}_1 the cc-graph $\mathcal{E} \upharpoonright B$, by \mathcal{E}_2 the cc-graph $\mathcal{E} \upharpoonright C$, and by \mathcal{E}_3 the cc-graph $\mathcal{E} \upharpoonright A$. Suppose for definiteness, that for values $y_r^\Delta(\mathcal{E}_1) = l_1 - s_1 \cdot \alpha_1$ and $y_q^\Delta(\mathcal{E}_2) = l_2 - s_2 \cdot \alpha_1$, $k_r \leq l_1 < k_{r+1}$, $k_q \leq l_2 < k_{q+1}$, correspondent to value $y_p^\Delta(\mathcal{E})$, the inequality $l_1 \leq l_2$ holds, and w.l.o.g., $E_1 \not\subseteq A$ and $E_2 \not\subseteq A$. Notice, that values $\Delta_1 + \dots + \Delta_m$ are not negative (i.e., are positive) after adding to \mathcal{E}_2 of elements in $E_1 \setminus E_3$. By Lemma 4.3.5(2), the correspondent values $\Delta_1^q + \dots + \Delta_m^q$ are also positive. By Lemma 4.1.1(3), we conclude that

$$\text{sl}((l_2, y_q^\Delta(\mathcal{E}_2)), (n, y_q^\Delta(\mathcal{E}))) \geq \text{sl}((l_2, b_{l_2}^q), (n, b_n^q)). \quad (4.10)$$

Since \mathcal{E}_2 belongs to \mathbf{K}_0^f , the point $(l_2, y_q^\Delta(\mathcal{E}_2))$ is above the point $(l_2, b_{l_2}^q)$. Then, by inequality (4.10), we have that the point $(n, y_q^\Delta(\mathcal{E}))$ is above the point (n, b_n^q) , i.e., $y_q^\Delta(\mathcal{E}) \geq b_n^q$.

If $p = q$, then the required inequality $y_p^\Delta(\mathcal{E}) \geq b_n^p$ holds. Otherwise, we have $q < p$ and then $y_p^\Delta(\mathcal{E})$ differs from $y_q^\Delta(\mathcal{E})$ by less than $\alpha_{q+1} \cdot k_{q+1} \cdot (k_{q+1} - 1) < 2\varepsilon_{q+1}$, since \mathcal{E} contains less than $k_{q+1} \cdot (k_{q+1} - 1)$ arcs and records on shortest Q -routes. By Lemma 4.1.1(4), it follows that $y_p^\Delta(\mathcal{E}) \geq b_n^p$. \square

Lemmas 4.3.7–4.3.11 imply

Corollary 4.3.12. *The class $(\mathbf{T}_0^f; \leq'_{cc})$ is self-sufficient.*

Denote the $(\mathbf{T}_0^f; \leq'_{cc})$ -generic theory by T^f .

We claim that, upon adding a formula $\chi_{\bar{\Phi}}(\bar{A})$ such that $(T^f, \bar{A}) \vdash \chi_{\bar{\Phi}}(\bar{A})$ to every type $\bar{\Phi}(\bar{A}) \in \mathbf{T}_0^f$, we obtain a self-sufficient class $(\mathbf{T}_0; \leq''_{cc})$ having the uniform t -amalgamation property.

Indeed, in view of Remark 4.3.6, for every cc-graph $\mathcal{A} \in \mathbf{K}_0^f$ of cardinality p and every graph $\mathcal{M} \models T^f$, $\mathcal{A} \subseteq_{cc} \mathcal{M}$, the condition $\mathcal{A} \leq_{cc} \mathcal{M}$ is equivalent to inequalities $\Delta_1^p + \dots + \Delta_m^p > 0$ after extensions of subsets of A by elements such that not all of them belong to A . Since we have to check these conditions only for extensions of cardinalities bounded by $n_{\mathcal{A}}^7$ and it is sufficient to count links only by relations Q^1, \dots, Q^p , then the property of self-sufficiency $\mathcal{A} \leq_{cc} \mathcal{M}$ is definable by a formula $\chi_{\mathcal{A}}(X)$ of the graph language $\{Q\}$, where the set X of variables is bijective with the set A .

Let \mathcal{A} and \mathcal{B} be cc-graphs in \mathbf{K}_0^f , \mathcal{M} be a generic model of T^f , and $\mathcal{A} \leq_{cc} \mathcal{B} \leq_{cc} \mathcal{M}$. Denote by $\psi_{\mathcal{A},s}(X)$ (respectively, $\psi_{\mathcal{B},s}(X, Y)$) a formula that describes the $\{Q^1, \dots, Q^s\}$ -type of \mathcal{A} (of \mathcal{B}), where X and Y are disjoint sets of variables bijective with A and $B \setminus A$. Then, for every $s \geq |T(B)|$, the following formula is true in the model \mathcal{M} :

$$\forall X ((\chi_{\mathcal{A}}(X) \wedge \psi_{\mathcal{A},s}(X)) \rightarrow \exists Y (\chi_{\mathcal{B}}(X, Y) \wedge \psi_{\mathcal{B},s}(X, Y))).$$

The last relation implies the uniform t -amalgamation property for the class $(\mathbf{T}_0^f; \leq''_{cc})$ that we obtain from $(\mathbf{T}_0^f; \leq'_{cc})$ by adding the formulas to the types which establish the cardinality bounds and the $\{Q^1, \dots, Q^p\}$ -structures of the self-sufficient closures, as well as the formulas $\chi_{\mathcal{A}}(A)$ to the types of self-sufficient sets A .

Since, by Lemma 4.3.10, the theory T^f has the finite closure property, then, in view of Theorem 2.5.1, the following holds:

Theorem 4.3.13. *A $(\mathbf{T}_0^f; \leq''_{cc})$ -generic model \mathcal{M} is saturated. In addition, each finite set $A \subseteq M$ can be extended to its self-sufficient closure $\bar{A} \subseteq M$, and the type $\text{tp}_X(\bar{A})$ is deduced from the set $[\Phi(\bar{A})]_{\bar{A}}^{\bar{A}}$, where $\Phi(\bar{A})$ is the type in \mathbf{T}_0^f such that $\mathcal{M} \models \Phi(\bar{A})$.*

⁷The upper estimate for cardinalities of extensions depends only on doubled cardinality of A .

Let \mathcal{N} be an ω -saturated model of T^f .

Proposition 4.3.14. *For each finite set A in \mathcal{N} , we have $\text{acl}(A) = \overline{A}$.*

P r o o f. The inclusion $\text{acl}(A) \supseteq \overline{A}$ follows from the uniqueness of \overline{A} in any elementary extension of \mathcal{N} .

We now take an arbitrary element $b \in N \setminus \overline{A}$ and show that $b \notin \text{acl}(A)$. Denote by B some self-sufficient cc-subgraph of \mathcal{N} such that $\overline{A} \cup \{b\} \subseteq B$ and, for each vertex in $B_f \cap A$, a vertex of degree 1 is added to \mathcal{A} so that, for extended self-sufficient cc-graph \mathcal{A}' , the conditions $A'_f = B_f \cap A$ and $b \notin A'$ hold. In view of construction of generic model, there exist infinitely many cc-isomorphic pairwise disjoint copies of $B \setminus A'$ over A' , i.e., $b \notin \text{acl}(A)$. Thus, $\text{acl}(A) \subseteq \overline{A}$. \square

4. Stability of generic theory. The scheme of proof for stability of T^f will be rather different from the proof of stability of generic theories in Sections 4.1 and 4.2, since the specificity of generic construction, represented in previous Subsection, generates the structure with properties that don't allow to use standard arguments as in the work by J. T. Baldwin and N. Shi [37]. In particular, we don't define analogues of rank function and don't use them as a tool for the proof (these analogues, for instance, don't have the monotonicity property), and lead a direct analysis of saturated model, allowing to state countable separability (countable baseness) of finite sets from given closed sets and, as corollary, to find an upper estimate for number of types over a set that guarantees the stability of T^f .

Let \mathcal{N} be a sufficiently saturated model of T^f . Below, all considered cc-graphs \mathcal{A} are supposed to be parts of cc-subgraphs in \mathcal{N} , and all considered sets are subsets of N . Furthermore, self-sufficient (in \mathcal{N}) cc-graphs \mathcal{A} are denoted by their universes A , and the universes A are called *self-sufficient sets*.

A set $X \subseteq N$ is called *closed* (in \mathcal{N}) (denoted by $X \leq N$) if, for any finite set $A \subseteq X$, an inclusion $\overline{A} \subseteq X$ holds, i.e., in $T(N) \setminus \Upsilon(X)$, there are no any n new elements for which the inequality (4.9) is true.

Lemma 4.3.15. *Let X and Y be sets in \mathcal{N} . Then the following holds:*

- (1) *if $X \leq N$ and $Y \leq N$ then $X \cap Y \leq N$;*

- (2) there exists the least closed set $\overline{X} \supseteq X$; moreover, $\overline{X} = \bigcup \{\overline{A} \mid A \subseteq_{\text{fin}} X\}$, and $\overline{X} = \text{acl}(X)$;
(3) if $X \subset Y$ then $\overline{X} \subseteq \overline{Y}$.

The set \overline{X} is called the *intrinsic closure* of X (in \mathcal{N}).

P r o o f repeats the proof of Lemma 4.1.13. \square

A set V (in \mathcal{N}) is called a *free cc-amalgam* of closed sets X and Y over a set Z (written $X *_Z Y$) if $X \cup Y = V$, $X \cap Y = Z$ and there are no pairs $(a, b) \in V^2 \setminus (X^2 \cup Y^2)$ of vertices connected by arcs or by only external shortest routes over V .

We say that sets X and Y are *independent* over Z and write $X \downarrow_Z Y$ if $X' \cap Y' = \overline{Z}$, $X' \cup Y' = X' *_Z Y'$, and $X' \cup Y' \leq N$, where $X' = \overline{X} \cup \overline{Z}$ and $Y' = \overline{Y} \cup \overline{Z}$.

Lemma 4.3.16. *If X is a self-sufficient set, Y is a closed set, $Z = (\overline{X} \cup \overline{Z}) \cap Y$ and $X \downarrow_Z Y$, then the type $\text{tp}(X/Y)$ is uniquely determined by the type $\text{tp}(X/Z)$, a description of $(\overline{X} \cup \overline{Z}) \cup Y$ to be closed, and a type, describing the coincidence of $(\overline{X} \cup \overline{Z}) \cup Y$ with the free cc-amalgam $X *_Z Y$.*

P r o o f. By the construction of generic model, clearly, taking two complete types, $q_1(X)$ and $q_2(X)$, over Y such that these types contain $\text{tp}(X/Z)$, a description of $(\overline{X} \cup \overline{Z}) \cup Y$ to be closed, and a type, describing the coincidence of $(\overline{X} \cup \overline{Z}) \cup Y$ with the free amalgam $X *_Z Y$, there exists an automorphism of $|Y|^+$ -saturated model, mapping a realization of $q_1(X)$ to a realization of $q_2(X)$. \square

Lemma 4.3.17. *If X is a closed set and a is an element in \mathcal{N} not belonging to X then there exists at most countable closed subset $X' \subseteq X$ such that $\{a\} \downarrow_{X'} X$.*

P r o o f. Notice, that the element a is connected by arcs or only external shortest routes over X with at most countably many elements of X . Indeed, assuming that there are uncountably many such elements, there exist uncountably many elements in \overline{X} , connected with a by arcs or by only external shortest routes of the same length, and moreover, a is a common origin or a common endpoint of these routes. But an exceeding of number of routes of a certain length over their correspondent weight α_k means that a belongs to the closure of chosen elements in \overline{X} . It is impossible, since X is closed and $a \notin X$. Aforesaid arguments also show that each at most countable set, that disjoint with X , also has at most countably many links with X by arcs or external shortest routes.

Assume, that the required set X' doesn't exist. Then there is an uncountable set of pairwise different finite closed subsets $X_i \subseteq X$, for which self-sufficient closures $\overline{X_i \cup \{a\}}$ don't contained in $X \cup \{a\}$ and some elements in $\overline{X_i \cup \{a\}} \setminus X$ are connected with X_i by pairwise different (w.r.t. i) arcs or only external shortest routes. Moreover, w.l.o.g. sets $\overline{X_i \cup \{a\}}$ form cc-isomorphic cc-graphs, for which all correspondent (w.r.t. i) elements either coincide (*fixed*) or pairwise different (*mobile*). By definition of self-sufficient closure, the sets X_i have minimal, by inclusion, subsets X'_i , for which some extensions of $X'_i \cup \{a\}$ in $\overline{X_i \cup \{a\}}$ have exceedings of numbers of new links over numbers of new elements in accordance with the inequality (4.9), i.e., some sums $\Delta_1 + \dots + \Delta_m$, obtained via extensions of $X'_i \cup \{a\}$ by elements, not lying in X , are negative. Again, since the family of sets X'_i is uncountable, we may assume, that the sets $\overline{X'_i \cup \{a\}}$ are pairwise different and form cc-isomorphic cc-graphs. Thus, the sums $\Delta \Leftarrow \Delta_1 + \dots + \Delta_m$ coincide for all X'_i .

If sums Δ can be formed by fixed elements a_1, \dots, a_k in $\overline{X_i \cup \{a\}} \setminus X$ only, then, in view of $\overline{X_i \cup \{a\}}$ being cc-isomorphic copies, we can choose n sets X'_i with an exceeding of weighted number of links between elements in X'_i and fixed elements more than $2 \cdot k + 1$. This choice guarantees that the elements a_1, \dots, a_k and a belong to the self-sufficient closure of the union of these sets X_i . It contradicts the condition $a \notin X$.

If negative sums Δ are produced only by adding of correspondent mobile elements of $\overline{X_i \cup \{a\}} \setminus X$, then we chose n sets X_i , where $n \cdot \alpha > 2$, α is the least weight of shortest routes that used in calculation of Δ for mobile vertices. Adding fixed elements b_{ij} of $\overline{X_i \cup \{a\}} \setminus X$ to the union of sets X_i and mobile elements c_{ik} of $\overline{X_i \cup \{a\}} \setminus X$, connecting elements of $\overline{X_i \cup \{a\}}$ correspondent to the sum Δ by arcs or only external shortest routes, and not connecting mobile elements of different sets $\overline{X_i \cup \{a\}} \setminus X$ by arcs and external shortest routes, we obtain a cc-graph (being a free amalgam over a set of some elements of X and fixed elements including a), which has a negative sum $\Delta'_1 + \dots + \Delta'_{m'}$ after adding to the union of chosen n sets X_i elements b_{ij} , c_{ik} , and a . The last again contradicts the fact that a doesn't belong to the closed set X .

Thus, aforesaid uncountable family of finite sets X_i doesn't exist, and a required set X' can be chosen as the intersection of the set X and the closure Y of the at most countable union of at most countable sets X_i containing the element a , and also all possible elements in X , for which there are connections with elements in $Y \setminus X$ by arcs or only external shortest routes. \square

Theorem 4.3.18. *The theory T^f is stable, small and has the unique 1-type. The weight of this type is infinite.*

P r o o f. Theorem 4.3.13 implies that T^f is small. In order to prove the stability of T^f , we estimate the number of 1-types in $S(N)$, where \mathcal{N} is a model of T^f . Consider an arbitrary element a . By Lemma 4.3.17, there exists at most countable closed set $X \subseteq N$ such that $\overline{\{a\} \cup X} \cap N = X$ and $\{a\} \downarrow_X N$. By Lemma 4.3.16, the type $\text{tp}(a/N)$ is determined by $\text{tp}(a/X)$, a description of $\overline{\{a\} \cup X} \cup N$ to be closed, and a type describing that $\overline{\{a\} \cup X} \cup N$ coincides with the free amalgam $\overline{\{a\} \cup X} *_X N$. Therefore, in order to count the number of types in $S(N)$, it is sufficient to count the number of types in $S(X)$ and the number of ways of choosing the countable set X in N . Then

$$|S(N)| \leq 2^\omega \cdot |N|^\omega = |N|^\omega.$$

Consequently, T^f is stable.

By construction, every singleton $\{a\}$ is self-sufficient. In view of Theorem 4.3.13, this means that there exist the unique 1-type of T^f . Let \mathcal{A} be the cc-graph with the universe $\{a\}$, $\mathcal{A}_p = \{a, b_p\}$ be an universe of two-element self-sufficient cc-graph \mathcal{A}_p containing the Q^p -arc between a and b_p . Put $\mathcal{B}_1 = \mathcal{A}_1$, $\mathcal{B}_{p+1} = \mathcal{B}_p *_\mathcal{A} \mathcal{A}_{p+1}$, $\mathcal{B} = \bigcup_{p \in \omega \setminus \{0\}} \mathcal{B}_p$. By construction, \mathcal{B} forms a closed set in

some generic model \mathcal{M} of T^f . Since $\alpha_1 + \dots + \alpha_{p+1} < 1$ and $\{b_1, \dots, b_{p+1}\} \leq_{\text{cc}} \{a, b_1, \dots, b_{p+1}\} \leq_{\text{cc}} \mathcal{B} \leq M$, we have $\{b_{p+1}\} \downarrow_{\emptyset} \{b_1, \dots, b_p\}$. Therefore, $(b_p)_{p \in \omega \setminus \{0\}}$ is an infinite independent sequence of elements, where each b_p is dependent on a . \square

Since $\alpha_1 < \frac{1}{2}$, any two elements, a and b , are connected via an element c so that aQc and bQc , and also via an element d so that dQa and dQb . Thus, any model of T^f has the pairwise intersection property.

The uniqueness of 1-type implies the transitivity of automorphism group for any homogeneous model of T^f .

Since any two-element set $\{a, b\}$, where aQb , is self-sufficient, then, by Theorem 4.3.13, the formula $Q(x, y)$ is principal in T^f .

Since, by Proposition 4.3.14, $\text{acl}(\{a\}) = \{a\}$ for any element a in a model \mathcal{M} of T^f , we have

$$\text{acl}(\{a\}) \cap \bigcup_{n \in \omega} Q^n(\mathcal{M}, a) = \{a\}.$$

Comparing aforesaid properties with the definition of powerful digraph and using Theorem 4.3.18, we have the following:

Theorem 4.3.19. *Any generic model of T_f is a small, stable, powerful digraph.*

The following two assertions clarify the structure of the prime models \mathcal{M}_A over finite sets A .

Lemma 4.3.20. *If A and B are self-sufficient sets in \mathcal{N} and $A \leq_{cc} B$ then the type $\text{tp}(B/A)$ is isolated iff \mathcal{B} is a complete $\bigcup_{k=1}^{\infty} Q^k$ -graph over \mathcal{A} , i.e., any two elements, $a \in B$ and $b \in B \setminus A$, are connected by a Q^k -arc.*

P r o o f. Let Y be a set of variables bijective with $B \setminus A$. If \mathcal{B} is a complete $\bigcup_{k=1}^{\infty} Q^k$ -graph over \mathcal{A} , then Theorem 4.3.13 implies that the type $\text{tp}((B \setminus A)/A)$ is isolated by a principal formula of $\Phi(A, Y)$, where $\Phi(B)$ is a type in \mathbf{T}_0^f such that $\mathcal{N} \models \Phi(B)$. This formula exists in view of formula definability of self-sufficiency and the fact that records on existence of shortest routes between elements are finite. If \mathcal{B} is not complete $\bigcup_{k=1}^{\infty} Q^k$ -graph over \mathcal{A} then Theorem 4.3.13 implies that $\text{tp}((B \setminus A)/A)$ is isolated by the type $\Phi(A, Y)$, but is isolated by no finite part of this set. \square

Lemma 4.3.20 implies

Corollary 4.3.21. *Let A be a self-sufficient set in \mathcal{N} . The model \mathcal{M}_A is a complete $\bigcup_{k=1}^{\infty} Q^k$ -graph over A . The set of isomorphism types of prime models over finite sets coincides with the set of isomorphism types of the models \mathcal{M}_A , where A is a self-sufficient set.*

5. Powerful digraph with almost inessential ordered colorings. The aforesaid construction of stable generic powerful digraphs admits the following modification. Consider the class \mathbf{K}_0^f and expand each cc-graph \mathcal{A} in \mathbf{K}_0^f by all possible colorings $\text{Col} : A \rightarrow \omega \cup \{\infty\}$ such that if vertices $a, a' \in A$ are linked by (a, a') -route then $\text{Col}(a) \leq \text{Col}(a')$. We denote the expanded class by $\hat{\mathbf{K}}_0^f$. Notice, that countably many possibilities of colorings of vertices correspond to each nonempty cc-graph.

Since cc-graphs are contained in acyclic digraphs, the free amalgams of self-sufficient cc-graphs of $\hat{\mathbf{K}}_0^f$ are also self-sufficient cc-graphs in $\hat{\mathbf{K}}_0^f$, and by Lemma 4.3.11, the following holds:

Lemma 4.3.22. (*amalgamation lemma*). *The class $\hat{\mathbf{K}}_0^f$ has the cc-amalgamation property cc-(AP), i.e., for any strong cc-embeddings $f_0 : \mathcal{A} \rightarrow_{\text{cc}} \mathcal{B}$ and $g_0 : \mathcal{A} \rightarrow_{\text{cc}} \mathcal{C}$, where \mathcal{A}, \mathcal{B} , and \mathcal{C} are self-sufficient cc-graphs in $\hat{\mathbf{K}}_0^f$, there exist a self-sufficient cc-graph $\mathcal{D} \in \hat{\mathbf{K}}_0^f$ and strong cc-embeddings $f_1 : \mathcal{B} \rightarrow_{\text{cc}} \mathcal{D}$ and $g_1 : \mathcal{C} \rightarrow_{\text{cc}} \mathcal{D}$ such that $f_0 \circ f_1 = g_0 \circ g_1$.*

Therefore, the generic construction produces a generic stable saturated powerful digraph Γ^{cpg} , in which the coloring of elements is Q -ordered. Moreover, the type of any self-sufficient set A is defined by a formula describing its self-sufficiency, a set of formulas describing existence or absence of routes between elements. Since for graph restrictions, a preservation of information on interconnections of elements of self-sufficient sets means that complete types coincide, then the coloring of Γ^{cpg} is almost inessential.

Thus, the following holds:

Theorem 4.3.23. *There exists a small, stable, generic, powerful digraph $\Gamma = \langle X, Q \rangle$ with an almost inessential Q -ordered coloring.*

Denote the theory of generic powerful digraph Γ^{cpg} with the almost inessential ordered coloring by T^{cpg} .

Combining the proofs of Theorem 3.1.7 and Lemma 4.3.20, we obtain the following theorem, describing the types realized in a prime model of T^{cpg} , and in a prime model \mathcal{M}_{p_∞} over a realization of type $p_\infty(x)$ of infinite elements in color.

Theorem 4.3.24. (1) A type q of T^{cpg} is principal iff any two different elements, a_i and a_j , of any realization \bar{a} of q are linked by some (a_i, a_j) - or (a_j, a_i) -route, and all elements of realizations of q have finite colors.

(2) A type q of T^{cpg} is realized in \mathcal{M}_{p_∞} iff, for any realization \bar{a} of q , any two different elements, a_i and a_j , are linked by some (a_i, a_j) - or (a_j, a_i) -route and the following condition holds: if some elements of \bar{a} are finite in color, a_f is an element of finite color, being a common endpoint of routes connecting all elements of finite colors with a_f , and some elements of \bar{a} are infinite in color, a_∞ is an element of infinite color, being a common origin of routes, connecting all elements of infinite color with a_∞ , then there exists an (a_f, a_∞) -route.

§ 4.4. On expansions of powerful digraphs

In this Section, we investigate a possibility of expansion of structure of a stable powerful graph to a structure of stable Ehrenfeucht theory.

We claim, that the simplest form of expansion, proposed in Section 3 (an expansion by 1-inessential ordered coloring and locally graph \exists -definable multiplace relations guaranteing the realization-equivalence of all nonprincipal types), can not preserve the structure in the class of stable structures: a presence on 1-inessential ordered coloring with locally graph \exists -definable relations, that guaranteing the realization-equivalence of all nonprincipal types, implies an existence of formula in two free variables and with the strict order property.

1. Type unstable theories. The following concepts generalize the correspondent notions of Classification Theory [3], [22].

Let $q(\bar{x}, \bar{y})$ be some (not necessary complete) type of theory T , where \bar{x} and \bar{y} be disjoint tuples of variables; \mathcal{M} be a countably saturated model of T .

The type $q(\bar{x}, \bar{y})$ is called *unstable* or having the *order property* if there exist tuples $\bar{a}_n, \bar{b}_n, n \in \omega$, for which $\models q(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \leq j$.

We say that the type $q(\bar{x}, \bar{y})$ has the *independence property* if there exist tuples $\bar{a}_n, n \in \omega$, such that, for any set $w \subseteq \omega$, there exist a tuple \bar{b}_w for which $\models q(\bar{a}_n, \bar{b}_w) \Leftrightarrow n \in w$.

The type $q(\bar{x}, \bar{y})$ has the *strict order property* if there exist tuples $\bar{a}_n, n \in \omega$, such that $q(\bar{a}_n, \mathcal{M}) \not\subseteq q(\bar{a}_{n+1}, \mathcal{M})$ for every $n \in \omega$.

The theory T is called *type stable* if T doesn't have unstable types $q(\bar{x}, \bar{y})$. The theory T is *type unstable* or has the *type order property*, the *type independence property*, or the *type strict order property* if some type $q(\bar{x}, \bar{y})$ of T has the correspondent property.

Clearly, the order property, the independence property, and the strict order property for types generalize correspondent concepts for formulas (see [3], p. 342), and the type order property is implied by the type independence property and the type strict order property.

Let $\Gamma = \langle X, Q \rangle$ be a graph without loops, a be a vertex in Γ . The set $\nabla_Q(a) \equiv \bigcup_{n \in \omega} Q^n(a, \Gamma)$ (respectively, $\Delta_Q(a) \equiv \bigcup_{n \in \omega} Q^n(\Gamma, a)$) is called the *upper (lower) Q -cone* of a . We call the Q -cones $\nabla_Q(a)$ and $\Delta_Q(a)$ by *cones* and denote by $\nabla(a)$ and $\Delta(a)$, respectively, if a relation Q is fixed.

We have the following criterium for an inclusion of cones in another cones.

Proposition 4.4.1. *Fore any vertices a and b of graph Γ , the following conditions are equivalent:*

- (1) $\nabla(a) \supseteq \nabla(b)$;
- (2) $\Delta(a) \subseteq \Delta(b)$;
- (3) $a = b$ or the vertex a is reachable from b .

Now, we obtain two immediate corollaries.

Corollary 4.4.2. *For any sequence a_n , $n \in \omega$, of vertices in a graph Γ , the following conditions are equivalent:*

- (1) the upper cones $\nabla(a_n)$ form an infinite sequence properly increasing (decreasing) by inclusion;
- (2) the lower cones $\Delta(a_n)$ form an infinite sequence properly decreasing (increasing) by inclusion;
- (3) for any $n \in \omega$, the vertex a_n is reachable from (reaching to) a_{n+1} , and a_{n+1} is not reachable from (reaching to) a_n .

Corollary 4.4.3. *For any sequence a_n , $n \in \omega$, of vertices in an acyclic digraph Γ , the following conditions are equivalent:*

- (1) the upper cones $\nabla(a_n)$ form an infinite sequence properly increasing (decreasing) by inclusion;
- (2) the lower cones $\Delta(a_n)$ form an infinite sequence properly decreasing (increasing) by inclusion;
- (3) for any $n \in \omega$, the vertex a_n is reachable from (reaching to) a_{n+1} .

Clearly, the relations $x \notin \nabla(y)$ and $x \notin \Delta(y)$ are type definable. Thus, the following statement holds.

Proposition 4.4.4. *If Γ is an acyclic digraph with an infinite chain then the theory $\text{Th}(\Gamma)$ has the type strict order property.*

Since any powerful digraph is acyclic and any vertex has images and preimages, then we obtain the following:

Corollary 4.4.5. *For any powerful digraph Γ , the theory $\text{Th}(\Gamma)$ has the type strict order property.*

2. From the type to the formula strict order property.

Let $\Gamma = \langle X, Q \rangle$ be a graph with a transitive automorphism group, $\text{Col} : X \rightarrow \omega \cup \{\infty\}$ be an 1-inessential coloring of Γ , Δ be a set of formulas in the language $\{Q\} \cup \{\text{Col}_n \mid n \in \omega\}$. A relational expansion \mathcal{M} of the structure of colored graph $\langle \Gamma, \text{Col} \rangle$ is called *locally $(\Delta, 1, \text{Col})$ -definable* if, in expanded structure, the coloring Col is again 1-inessential and, for any new predicate symbol $R^{(m)}$, the formula $R(x, \bar{y}) \wedge \text{Col}_n(x)$ is equivalent to a Boolean combination of formulas of Δ . If Δ consists of \exists -formulas, then a locally $(\Delta, 1, \text{Col})$ -definable expansion is called *locally $(\exists, 1, \text{Col})$ -definable*.

Let Col be an 1-inessential Q -ordered coloring of a powerful digraph $\Gamma_{\text{pg}} = \langle X, Q \rangle$, $p_\infty(x)$ be a type of infinite elements in color. A locally $(\exists, 1, \text{Col})$ -definable expansion \mathcal{M} of the structure $\langle \Gamma_{\text{pg}}, \text{Col} \rangle$ is called *p_∞ -powerful* if the following conditions hold:

(1) the semi-isolation relation SI_{p_∞} on realizations of p_∞ is non-symmetric and it is witnessed by the formula $Q(x, y)$;

(2) p_∞ is a powerful type of the theory $T \equiv \text{Th}(\mathcal{M})$;

(3) each nonprincipal type $q(\bar{y}) \in S(T)$ is realized in \mathcal{M}_{p_∞} via some principal formula $R_q(a, \bar{y})$ (i.e., $R_q(a, \bar{y}) \vdash q(\bar{y})$), where R_q is a new language symbol, $\models p_\infty(a)$;

(4) if $\models R_q(a, \bar{b})$, $\models p_\infty(a)$, $\models p_\infty(b_i)$, where $b_i \in \bar{b}$, then there exists an (a, b_i) - Q -route.

Notice, that the properties (1)–(4) are satisfied in Morleyizations of structures of formula neighbourhoods of nonprincipal powerful types, from which structures of powerful digraphs can be taken (see proof of Proposition 1.4.2).

Recall (see proof of Proposition 1.3.4) that if lengths of shortest routes are bounded in a powerful digraph Γ_{pg} then the theory $\text{Th}(\Gamma_{\text{pg}})$ has the strict order property and, in particular, is unstable.

Let \mathcal{M} be a locally $(\exists, 1, \text{Col})$ -definable p_∞ -powerful expansion of a structure of powerful digraph $\Gamma_{\text{pg}} = \langle X, Q \rangle$ with unbounded lengths of shortest routes. Then there exists a type $q(y_1, y_2, y_3) \in S^3(\text{Th}(\mathcal{M}))$ for which the first coordinate of any realization is finite in color, second and third coordinates are infinite in colors, and these realizations are not linked by routes in digraph. Consider a formula

$$\varphi(x, y_1) \equiv \exists y_2 \exists y_3 R_q(x, y_1, y_2, y_3).$$

Since the relation R_q is locally $(\exists, 1, \text{Col})$ -definable, the realizations a and b_1 (where $\models \varphi(a, b_1)$, $\models p_\infty(a)$) are mutually non-reachable in Γ_{pg} , and the lengths of shortest routes are unbounded, then the formula $\varphi(x, y_1)$ defines a binary relation that is not definable by a formula in the language of colored digraph. Moreover, since, for a graph structure with a transitive automorphism group, \exists -formulas defined only bounded lengths of shortest routes, the conditions $\models p_\infty(a_1)$ and $\models Q(a_1, a_2)$ imply

$$\models \forall y (\varphi(a_1, y) \rightarrow \varphi(a_2, y)) \wedge \exists y (\neg \varphi(a_1, y) \wedge \varphi(a_2, y)). \quad (4.11)$$

Since elements a_1 and a_2 realizes the same type p_∞ , the relation (4.11) implies the strict order property.

Thus the following theorem holds, in accordance of which the preservation of 1-inessentiality for coloring of powerful digraph is inconsistent with the stability of locally $(\exists, 1, \text{Col})$ -definable structure of expanded theory, having a nonprincipal powerful type p_∞ .

Theorem 4.4.6. *For any locally $(\exists, 1, \text{Col})$ -definable p_∞ -powerful expansion \mathcal{M} of a structure of powerful digraph, the theory $\text{Th}(\mathcal{M})$ has the strict order property.*

A mechanism of obtaining of the strict order property via an expansion of structure of powerful digraph, represented in the proof of Theorem 4.4.6, illustrates a transformation of the type strict order property, for a theory of powerful digraph, to the formula strict order property, generated by specificity of the expansion.

§ 4.5. Description of features of generic construction of stable Ehrenfeucht theories

In the following four Sections, we prove a strengthening of the basic result of Chapter 3 (Theorem 3.4.1) stating the realizability in the class of stable theories of all admissible parameters described in Characterizing Theorem 1.1.13. This strengthening, in particular, represent a positive solution of the Lachlan problem on existence of stable Ehrenfeucht theory.

Recall that, in Chapter 3, the construction of Ehrenfeucht theories with three countable models is formed by the following components:

- the nonprincipal powerful type $p_\infty(x)$ with the structure consisting of elements of color Col_∞ and obtained by inessential ordered coloring Col of all elements in colors belonging to $\omega \cup \{\infty\}$;

- the binary relation Q defining the powerful digraph with unbounded lengths of shortest Q -routes on the structure of $p_\infty(x)$ and on any its neighbourhood and, as corollary, defining partial orders on these structures via transitive closures $\text{TC}(Q)$;

- the countable set of binary relations R_q and R'_q , $R'_q = R_q^{-1}$, guaranteing the corealization amalgamation (the coincidence of prime models over realizations of $p_\infty(x)$ if these realizations are connected by R_q) and defining the equivalence relation such that the equivalence classes form connected components w.r.t. $R^* \Leftrightarrow \bigcup_q R_q \cup R'_q$; moreover, the connected components, w.r.t. R^* ,

are partially ordered by the relation $\text{TC}(Q)$ (two components, C_1 and C_2 , are connected by $\text{TC}(Q)$ if some their representatives are connected by $\text{TC}(Q)$), and elements of each connected component are pairwise incomparable w.r.t. $\text{TC}(Q) \setminus \text{id}_X$;

- multiplace predicates $R_{\mathbf{A}}$ guaranteing realization-equivalence of $p_\infty(x)$ with all nonprincipal types.

It is not difficult to see that aforesaid attributes, possibly with the local pairwise intersection property w.r.t. Q or with degenerated relations R^* and $R_{\mathbf{A}}$, are represented in any complete theory with three countable models.

The strategy of construction of required stable theories will be rather different from the strategy of construction of Ehrenfeucht theories in Chapter 3, in view of the following circumstances.

As shown in previous Section, using an 1-inessential coloring with consequent expansion of structure of powerful digraph by coordinated predicates $R_{\mathbf{A}}$, realizing all nonprincipal types, we lead the structure out the class of stable structures. Therefore, before the construction of $R_{\mathbf{A}}$ -expansions, we expand the structure of powerful digraph with an almost inessential ordered coloring by pairwise disjoint symmetric binary relations $P_{i,n}$, $i, n \in \omega$, each of which is disjoint with $\text{TC}(Q)$ and connects only elements of color, that equals to the second index, with elements of more colors, so that the following (P, Q) -intersection property: for any elements a_1, \dots, a_k of infinite color, any elements b_1, \dots, b_l of finite colors m_1, \dots, m_l , respectively, and any color $\sigma > \max\{m_1, \dots, m_l\}$, there exists an element c of color σ , for which

$$\models \bigwedge_{j=1}^k Q^{i_j}(c, a_j) \wedge \bigwedge_{j=1}^l P_{r_j, m_j}(c, b_j)$$

holds with some $i_1, \dots, i_k, r_1, \dots, r_l \in \omega \setminus \{0\}$.

Predicates $P_{i,n}$, being constructed in a stable generic structure on a base of linear prerank function, allow, preserving the graph structure and the stability of theory, to realize, in a prime model \mathcal{M}_{p_∞} over a realization of p_∞ , all nonprincipal types via $R_{\mathbf{A}}$ -expansions, for which

$$\exists \bar{y} \setminus y_j R_{\mathbf{A}}(x, \bar{y}) \equiv Q^n(x, y_j) \wedge \bigwedge_{m < n} \neg Q^m(x, y_j)$$

or

$$\exists \bar{y} \setminus y_j R_{\mathbf{A}}(x, \bar{y}) \equiv P_{i,n}(x, y_j).$$

The control of balance between numbers of elements and numbers of links, satisfied via a prerank function and guaranteeing the stability of theory, will be fulfilled by a *fusion* of generic model of powerful graph and of generic model in the language $\{P_{i,n}^{(2)} \mid n \in \omega\}$.

Various nature of closures of finite sets in R^* - and Q -structures (route-closures for forests and self-sufficient closures w.r.t. prerank functions for structures of generic powerful digraphs) has difficulty for checking the satisfaction of the finite closure property in the fusions of these structures.

To obtain a common nature of closures for finite sets in fusions of R^* - and Q -structures (of self-sufficient closures w.r.t. prerank functions, reducing to a common prerank function), we define connected components w.r.t. R^* , being variants of Hrushovski construction for a stable non-superstable countably categorical structure (see [81]). We introduce symmetric binary relations R_j in these connected components, playing the role of relations R_q . The possibility of simultaneous control of number of R_j -links, $P_{i,n}$ -links, and of number of links w.r.t. Q -routes via a common prerank function allows to bound the cardinality of self-sufficient closure for any finite set by a common function, dependent only on cardinality of that set. Thus, the type-definability of self-sufficient closures for a fusion of R^* -, $P_{i,n}$ -, and Q -structures will be guaranteed, and the finite closure property, conditioned by a stepped special system of closures, will be satisfied. A presence of the finite closure property, in one's turn, will allow to state that the generic models are saturated and required theories are stable.

§ 4.6. Stable graph extensions of colored powerful digraphs

Denote by Γ the stable generic powerful digraph $\Gamma^{\text{cpg}} = \langle X, Q \rangle$ with the almost inessential Q -ordered coloring Col , described in Section 4.3.

Consider the class $\hat{\mathbf{K}}_0^f$, defined in Section 4.3 for the construction of colored graph Γ , and expand structures in $\hat{\mathbf{K}}_0^f$ by symmetric binary predicates $P_{i,n}$, $i, n \in \omega$, connecting only vertices a of color n with vertices b of colors more than n and such that there are no (a, b) - Q -routes.

Like operations T_c and T_c^* on the class of cc-graphs, we define operations T_c and T_c^* on the class of expanded structures. Furthermore, obtaining correspondent structures without furcations, any pair of vertices of $P_{i,n}$ is interpreted by one edge, connecting elements in $J_0 \cup J_1$.

Consider now a monotonically decreasing sequence of positive real numbers α_k , defined for the prerank function $y^1(\cdot)$ in Section 4.3 as weights of pairs of vertices, connected by shortest Q -routes of length k . Put elements $\alpha_m^Q \doteq \alpha_{2m}$ to weights of pair of vertices, connected by shortest Q -routes of length m , and elements $\alpha_m^P \doteq \alpha_{2m+1}$ to weights of $P_{i,n}$ -edges, where m is the value $c(i, n)$ of Cantor enu-

merating function (see, for instance, [26], p. 137). After aforesaid renutations, we define the *prerank function* $y(\cdot)$, for structures \mathcal{A} from $\hat{\mathbf{K}}_0^f$, expanded by $P_{i,n}$, with the rule:

$$y(\mathcal{A}) = 2 \cdot |A_f| + |A_{nf}| - \sum_{k=1}^{\infty} \alpha_k^Q \cdot e_k^Q(\mathcal{A}) - \sum_{k=0}^{\infty} \alpha_k^P \cdot e_k^P(\mathcal{A}),$$

where $e_1^Q(\mathcal{A})$ is the number of Q -arcs in \mathcal{A} ; $e_k^Q(\mathcal{A})$, for $k \geq 2$, is the number of pairs $(a, a') \in A^2$, connected only by external shortest (a, a') - Q -routes of length k and such that there are no (a, a') - Q -routes of length k with \mathcal{A} -external \mathcal{A} -compulsory furcations; $e_k^P(\mathcal{A})$ is the number of $P_{i,n}$ -edges in \mathcal{A} , where $k = c(i, n)$.

A p -*approximation* of the prerank function $y(\cdot)$ is the function $y_p(\cdot)$ assigning a real to every structure \mathcal{A} from $\hat{\mathbf{K}}_0^f$, expanded by $P_{i,n}$, with the rule:

$$y_p(\mathcal{A}) = 2 \cdot |A_f| + |A_{nf}| - \sum_{k=1}^p \alpha_k^Q \cdot e_k^Q(\mathcal{A}) - \sum_{k=0}^p \alpha_k^P \cdot e_k^P(\mathcal{A}).$$

Denote by $\mathbf{K}_1^{f,P}$ the class of all structures \mathcal{A} from $\hat{\mathbf{K}}_0^f$, expanded by predicated $P_{i,n}$ and satisfying conditions $y_1(\mathcal{A}') \geq b_n^1$ for any structure $\mathcal{A}' \sqsubseteq \text{cfc}(\mathcal{A})$, where $n = |T(\mathcal{A}')|$ and \sqsubseteq the relation “to be a part of structure”. For the structure \mathcal{A} in $\mathbf{K}_1^{f,P}$, we write $\mathcal{A} \in \mathbf{K}_{p+1}^{f,P}$, where $p > 0$, iff $\mathcal{A} \in \mathbf{K}_p^{f,P}$ and $y_p^\Delta(\mathcal{A}') \geq b_n^p$ for any structure $\mathcal{A}' \sqsubseteq \text{cfc}(\mathcal{A})$, where $y_p^\Delta(\mathcal{A}')$ is a minimal value $y_p(T_c(\mathcal{A}_0)) + \Delta_1 + \dots + \Delta_m$, $\Delta_i = y_p(T_c((\mathcal{A}_{i+1})_{T_c^*(\mathcal{A}_i)})) - y_p(T_c^*(\mathcal{A}_i))$, $\mathcal{A}_0 \subseteq_{cc} \mathcal{A}_1 \subseteq_{cc} \dots \subseteq_{cc} \mathcal{A}_m = \mathcal{A}'$, n is a natural number, for which $y_p^\Delta(\mathcal{A}') = n - s \cdot \alpha_1$ with some $s \in \omega$, $k_p \leq n < k_{p+1}$.

Put $\mathbf{K}_0^{f,P} \Leftarrow \bigcap_{p=1}^{\infty} \mathbf{K}_p^{f,P}$ and denote by $\mathbf{K}^{f,P}$ the class of all graphs in the language $\Sigma_P \Leftarrow \{\text{Col}_n \mid n \in \omega\} \cup \{Q\} \cup \{P_{i,n} \mid i, n \in \omega\}$, for which each finite *substructure* (i.e., a subgraph with an information on lengths of shortest Q -routes and on compulsory furcations) belongs to the class $\mathbf{K}_0^{f,P}$.

Let \mathcal{A} be a finite substructure of a graph (respectively, of a finite substructure) \mathcal{M} (of a graph) belonging to $\mathbf{K}^{f,P}$. We say that \mathcal{A} is a *selfsufficient substructure* of the graph (respectively, of the finite

structure) \mathcal{M} and write $\mathcal{A} \leq \mathcal{M}$ if for any finite structures $\mathcal{A}' \sqsubseteq \mathcal{A}$, $\mathcal{B}' \sqsubseteq \mathcal{M}$, $\mathcal{A}' = \mathcal{A}_0 \subseteq_{cc} \mathcal{A}_1 \subseteq_{cc} \dots \subseteq_{cc} \mathcal{A}_m = \mathcal{B}'$, the inequality $\Delta_1 + \dots + \Delta_m < 0$, where $\Delta_i = y(T_c((\mathcal{A}_{i+1})_{T_c^*(\mathcal{A}_i)})) - y(T_c^*(\mathcal{A}_i))$, implies $\mathcal{B}' \sqsubseteq \mathcal{A}$. If $\mathcal{A} \leq \mathcal{M}$ and \mathcal{M} is a finite structure, then \mathcal{A} is called a *strong substructure* of \mathcal{M} .

Denote by $\mathbf{T}_0^{f,P}$ the class of types correspondent to all structures in $\mathbf{K}_0^{f,P}$ and equip that class by the relation \leq' , where $\Phi(A) \leq' \Psi(B) \Leftrightarrow \mathcal{A} \leq \mathcal{B}$.

Repeating arguments for the class $(\mathbf{T}_0^f; \leq'_{cc})$, we state that $(\mathbf{T}_0^{f,P}; \leq')$ is a self-sufficient generic class, for which, after adding to types of necessary formulas, describing self-sufficient closures, the uniform t -amalgamation property is satisfied. This fact implies that a $(\mathbf{T}_0^{f,P}; \leq')$ -generic model is ω -saturated and realizes all types $\Phi(X)$ correspondent to types $\Phi(A)$ in $\mathbf{T}_0^{f,P}$, and also the $(\mathbf{T}_0^{f,P}; \leq')$ -generic theory $T^{f,P}$ is stable. Since $\alpha_k < \frac{1}{2}$ and $\lim_{k \rightarrow \infty} \alpha_k = 0$, the theory $T^{f,P}$ has the (P, Q) -intersection property.

Since any two-element set $\{a, b\}$ with $(a, b) \in \bigcup_{i,n \in \omega} P_{i,n}$ are self-sufficient, the type $\text{tp}(a \hat{=} b)$ is defined by colors of endpoints of the edge $[a, b]$, i.e., all formulas $P_{i,n}(a, x)$, where $\text{Col}(a) > n$, are principal. The same property is preserved for formulas $Q(a, x)$, where $\text{Col}(a) = \infty$. Thus, the following two theorems hold:

Theorem 4.6.1. *A $(\mathbf{T}_0^{f,P}; \leq')$ -generic model \mathcal{M} is saturated. In addition, each finite set $A \subseteq M$ can be extended to its self-sufficient closure $\bar{A} \subseteq M$, and the type $\text{tp}_X(\bar{A})$ is deducible from the set $[\Phi(\bar{A})]_{\bar{A}}^{\bar{A}}$, where $\Phi(\bar{A})$ is a type in $\mathbf{T}_0^{f,P}$, for which $\mathcal{M} \models \Phi(\bar{A})$.*

Theorem 4.6.2. *The theory $T^{f,P}$ is stable, small, possesses the (P, Q) -intersection property and has countably many 1-types, and each 1-type is defined by color of any its realization. Every formula $P_{i,n}(a, x)$, where $\text{Col}(a) > n$, and also every formula $Q(b, x)$, for $\text{Col}(b) = \infty$, are principal.*

Lemma 4.6.3. *If A and B are self-sufficient sets in a model $\mathcal{N} \models T^{f,P}$ and $A \leq B$, then the type $\text{tp}(B/A)$ is isolated iff \mathcal{B} is a complete $\bigcup_{k,n \in \omega} (Q^k \cup P_{k,n})$ -graph over \mathcal{A} , i.e., any two different*

elements, $a \in B$ and $b \in B \setminus A$, are connected by an Q^k -arc or some $P_{i,n}$ -edge.

P r o o f. Let Y be a set of variables bijective with $B \setminus A$. If \mathcal{B} is a complete $\bigcup_{k=1}^{\infty} (Q^k \cup P_{k,n})$ -graph over \mathcal{A} , then, by Theorem 4.6.1, the type $\text{tp}((B \setminus A)/A)$ is isolated by a principal formula of type $\Phi(A, Y)$, where $\Phi(B)$ is a type in $\mathbf{T}_0^{f,P}$ such that $\mathcal{N} \models \Phi(B)$. This formula exists in view of type definability of self-sufficiency, by the condition for connections elements via $P_{i,n}$ -edges, and by the finiteness of records on existence of shortest Q -routes between elements. If \mathcal{B} is not complete $\bigcup_{k=1}^{\infty} (Q^k \cup P_{k,n})$ -graph over \mathcal{A} , then Theorem 4.6.1 implies that the type $\text{tp}((B \setminus A)/A)$ is isolated by the type $\Phi(A, Y)$, but is isolated by no finite part of this set. \square

Lemma 4.6.3 implies

Corollary 4.6.4. *Let A be a self-sufficient set in a model $\mathcal{N} \models T^{f,P}$. A prime model \mathcal{M}_A over A is a complete $\bigcup_{k,n \in \omega} (Q^k \cup P_{k,n})$ -graph over A . The set of isomorphism types of prime models over finite sets coincides with the set of isomorphism types of the models \mathcal{M}_A , where A are self-sufficient sets and \mathcal{M}_A are complete $\bigcup_{k,n \in \omega} (Q^k \cup P_{k,n})$ -graphs over A .*

§ 4.7. Stable theories with nonprincipal powerful types

In this Section, we describe a modification of generic construction in Section 3.2, allowing to create stable theories with nonprincipal types as expansions of the theory $T^{f,P}$ of powerful digraph $\langle X, Q \rangle$ with ordered coloring Col and binary predicates $P_{i,n}$, guaranteeing that the (P, Q) -intersection property holds.

Fix a theory $T^{f,P}$ ⁸ and define an expansion of $T^{f,P}$ with the same ordered coloring, so that the type $p_{\infty}(x)$ of infinite in color elements becomes powerful.

Below, in order to turn p_{∞} into a powerful type preserving the uniqueness of nonprincipal complete 1-type over \emptyset , for every type $q(y_2, \dots, y_k)$ not contained in any (p_{∞}, y_1) -principal type $r(y_1, y_2, \dots, y_k)$, we introduce a new k -ary predicate R_q so that the

⁸Recall, that $T^{f,P}$ is uniquely defined by a special sequence of irrational numbers α_k , $0 < \alpha_{k+1} \ll \alpha_k < \frac{1}{2}$, $k \in \omega \setminus \{0\}$.

set $\{R_q(y_1, \dots, y_k)\} \cup p_\infty(y_1)$ is consistent and

$$\{R_q(y_1, \dots, y_k)\} \cup p_\infty(y_1) \vdash q(y_2, \dots, y_k).$$

With this goal in mind, we renumber the set \mathbf{q} of all types $q(y_1, \dots, y_k)$ for tuples \bar{a}_q with mutually distinct coordinates for which the sets of elements of \bar{a}_q are self-sufficient and models $\mathcal{M}_{\bar{a}_q}$ are not isomorphic to a prime model or to \mathcal{M}_{p_∞} : $\mathbf{q} = \{q_m(y_1, \dots, y_{k_m}) \mid m \in \omega\}$. In this event, by Theorem 4.6.1, $q_m(y_1, \dots, y_{k_m})$ are defined by *isomorphism types* $\mathbf{A}_m \equiv \Psi_m(Y_m)$ (where $\Psi_m(A_m) \in \mathbf{T}_0^{f,P}$) of their realizations \bar{a}_m . In what follows, therefore, the types $q_m(y_1, \dots, y_{k_m})$ will be identified with the isomorphism types \mathbf{A}_m .

For any type $q_m(\bar{y}) \in \mathbf{q}$ and correspondent type \mathbf{A}_m , we fix a set $\Phi_{\mathbf{A}_m}(\bar{y})$ of formulas $\varphi_n(\bar{y})$, $n \in \omega$, isolating $q_m(\bar{y})$ and describing the following:

- (a) finite colors of elements in \bar{a}_m , and also negations of colors less than n , for elements in \bar{a}_m having the infinite color;
- (b) an existence and lengths of shortest Q -routes connecting elements of \bar{a}_m ;
- (c) an absence of Q -routes of lengths less than n , connecting elements of \bar{a}_m if elements are not linked by Q -routes;
- (d) $P_{i,m}$ -edges connecting elements in \bar{a}_m , and also an absence of $P_{j,n'}$ -edges, with $c(j, n') < n$, connecting elements in \bar{a}_m if elements are not connected by $P_{i,m}$ -edges;
- (e) the self-sufficiency of the set of elements \bar{a}_m .

Now, consider an isomorphism type \mathbf{A} of an arbitrary tuple $\bar{a} = (a_1, \dots, a_k)$ of self-sufficient set $A = \{a_1, \dots, a_k\}$ of cardinality k , which is not in \mathcal{M}_{p_∞} . Denote by $\max_{\mathbf{A}}$ the maximal finite color of elements in A if such elements exist, and put $\max_{\mathbf{A}} \equiv 0$ if all elements of A are infinite in color. Choose for each element a_j of finite color n_j a predicate P_{i_j, n_j} such that

$$\sum_{j \in \{j' \mid \text{Col}(a_{j'}) < \omega\}} \alpha_{c(i_j, n_j)}^P < 1 - 2 \cdot \alpha_1^Q.^9$$

⁹Such choice of predicates is admissible in view of the (P, Q) -intersection property and the condition $\lim_{i \rightarrow \infty} \alpha_{c(i, n_j)}^P = 0$. We choose these elements to preserve the self-sufficiency under adding to A an element connected with elements a_j of finite colors by P_{i_j, n_j} -edges and with elements a_j of infinite color by Q -routes.

We define $(k + 1)$ -ary relations $R_{\mathbf{A}}$ as follows:

$$(1) \vdash \left(\exists \bar{y} (R_{\mathbf{A}}(x, \bar{y}) \wedge \varphi_m(\bar{y})) \leftrightarrow \bigwedge_{i \leq \max_{\mathbf{A}}} \neg \text{Col}_i(x) \right) \wedge \\ \wedge \left(\exists \bar{y} (R_{\mathbf{A}}(x, \bar{y}) \wedge \varphi_n(\bar{y})) \leftrightarrow \bigwedge_{i \leq n} \neg \text{Col}_i(x) \right), \quad m \leq \max_{\mathbf{A}} \leq n;$$

(2) for any $n > \max_{\mathbf{A}}$, the formula $R_{\mathbf{A}}(x, y_1, \dots, y_k) \wedge \text{Col}_n(x)$ is equivalent to a conjunction of $\varphi_n(\bar{y}) \wedge \neg \varphi_{n+1}(\bar{y})$ and the formula describing the following properties:

(a) if $\langle a_{i_1}, \dots, a_{i_r} \rangle$ (where $k_1 < \dots < k_r$) is a tuple of all elements a_i of infinite color in \bar{a} , and $\langle a_{j_1}, \dots, a_{j_s} \rangle$ (where $j_1 < \dots < j_s$) is a tuple of all elements a_j of finite colors in \bar{a} , then there exist elements z_0, \dots, z_{r-1} such that $z_{r-1} = y_{k_r}$, $Q(z_{m-1}, z_m) \wedge Q(z_{m-1}, y_{i_m})$, $m = 1, \dots, r-1$, $x = z_0$, and $P_{i_{j_l}, n_{j_l}}(x, y_{j_l})$, $l = 1, \dots, s$;

(b) a structure consisting of elements $x, y_1, \dots, y_k, z_1, \dots, z_{r-1}$ contains no edges with numbers $c(\cdot, \cdot) \leq n$ and Q -arcs other than the edges and Q -arcs specified in (a) and in the description of \mathbf{A} for the elements \bar{y} ; nor does it contain external shortest Q -routes of length at most n but for the external shortest Q -routes connecting elements of \bar{y} and the elements described in \mathbf{A} ; moreover, the aforesaid structure forms a self-sufficient set.¹⁰

By definition, predicates $R_{\mathbf{A}}$ refine the graph structure while not increasing sets of binary relations defined by projections such as $\exists y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_k R_{\mathbf{A}}(x, \bar{y})$.

We claim that with the above-mentioned refinement of the graph structure (via relations $R_{\mathbf{A}}$) in hand, the required expansion may be realized by newly constructing the generic model from the finite structures expanded by finite records saying of positive links between elements via intermediate elements through projections of the relations $R_{\mathbf{A}}$.

We start our construction by describing the class \mathbf{K}_1^* of finite structures endowed with finite records holding of interrelations of the elements satisfying conditions (1) and (2). Since the desired generic model expands a $(\mathbf{T}_0^{f,P}; \leq')$ -generic model, we assume that every finite structure \mathcal{A} , which enters \mathbf{K}_1^* and is restricted to the graph language $\{Q\} \cup \{P_{i,n} \mid i, n \in \omega\}$ with coloring Col , form

¹⁰This means, in particular, that types of sets A have unique extensions to types including elements x, z_1, \dots, z_{r-1} , where x satisfies p_∞ .

a structure with the record $W_{\mathcal{A}}$ and belonging to $\mathbf{K}_0^{f,P}$. Moreover, introduction of the relations $R_{\mathbf{A}_m}$ requires that the record $W_{\mathcal{A}}$ is added positive information on interrelations of the elements w.r.t. projections $\exists y_{l_1}, \dots, y_{l_t} R_{\mathbf{A}_m}(x, \bar{y})$ in accordance with item (2).

Before we end to define structures in the class \mathbf{K}_1^* , we observe the following. As shown in Theorem 4.6.1, a type in $\mathbf{T}_0^{f,P}$ of every self-sufficient structure \mathcal{A} determines type of the set A in generic model. In defining every relation $R_{\mathbf{A}}$, the belonging of every tuple $a \hat{=} \bar{a}$ to that relation is specified either by a principal formula describing relations between the elements in prime model or by a sequence of formulas (see item (2)) which locally describe the absence of links between some elements of \bar{a} via edges or Q -routes, while keeping the links between the element a and the elements of \bar{a} fixed in length.

The last description, as noted above for binary relations, depends directly on the interrelations between colors of approximations a^n (in prime model) of an element a (these approximations are called *sources*) and lengths of shortest Q -routes and numbers of P -edges between appropriate elements of approximations \bar{a}^n (in prime model) of a tuple \bar{a} (these approximations are called *successors*). Namely, if the color number of a source a^n does not exceed (unbounded as $n \rightarrow \infty$) lengths of shortest Q -routes (numbers of P -edges) between elements of successors \bar{a}^n , then the relation $R_{\mathbf{A}}$ holds, provided that the elements z_1, \dots, z_k described in item (2) are in hand. But if the color number of a^n is greater than is some (unbounded as $n \rightarrow \infty$) length of shortest Q -routes (number of P -edges) between elements of \bar{a}^n , then $R_{\mathbf{A}}$, under the same conditions, will fail. Below, the relation, which the color number of a source a^n associates with a collection of pairwise non-bounded (as $n \rightarrow \infty$) lengths of shortest Q -routes and numbers of P -edges between elements of a successor $\bar{a}^{n''}$, is for brevity referred to as *CLN-correlation*.

Since, in generic theory, any type of $S(\emptyset)$ is expandable to a type in $S(\emptyset)$ of some self-sufficient set, described by colors of elements, lengths of shortest Q -routes, numbers of P -edges, and the condition of self-sufficiency of set, the CLN correlation may be characterized by formulas $\rho(x, y_i, y_j)$, which express the lengths of shortest (x, y_i) - and (x, y_j) - Q -routes and numbers of P -edges between x and y_i, y_j , as well as correlations between colors of elements a^n and lengths of shortest Q -routes (numbers of P -edges) connecting a_i^n and a_j^n .

Thus, the formulas $\rho(x, y_i, y_j)$ are ternary indicators for positive or negative entries of the formulas $\exists y_{l_1}, \dots, y_{l_t} R_{\mathbf{A}_m}(x, \bar{y})$ in types of the theory for the generic model in question. We add these indicators to the descriptions of types \mathbf{A}_m in defining the relations $R_{\mathbf{A}_m}$ themselves (the formulas $\rho(y_i, y_j, y_k)$ or add their negations conjunctively to $\varphi_n(\bar{y})$) and to the general descriptions of the types of tuples. The adding of ρ , in this instance, extends the very language determined by the type \mathbf{A}_m with the formulas $\rho^\delta(y_i, y_j, y_k)$, $\delta \in \{0, 1\}$, added. This extended language remains countable due to there being finitely many versions of adding positive formulas ρ to every type.

Ultimately we specify that the class \mathbf{K}_1^* consists of all finite structures of the language

$$\Sigma_1 \equiv \{\text{Col}_n \mid n \in \omega\} \cup \{Q^{(2)}\} \cup \{P_{i,n}^{(2)} \mid i, n \in \omega\} \cup \{R_{\mathbf{A}_m} \mid m \in \omega\} \cup \{\rho^{(3)} \mid \rho(x, y_i, y_j) \text{ characterizes some CLN-correlation}\},$$

which are obtained from structures belonging to $\mathbf{K}_0^{f,P}$ by adding the relations ρ , and the relations $R_{\mathbf{A}_m}$ consistent with item (2) (with the types \mathbf{A}_m extended by ρ), and all possible admissible formulas $\rho(a_i, a_j, a_k)$.

Finite structures \mathcal{A} with their records $W_{\mathcal{A}}$ forming the class \mathbf{K}_1^* are called c_1 -structures. Denote by \mathbf{K}_1 the class of all models of the language Σ_1 , whose every finite subset forms a c_1 -structure in \mathbf{K}_1^* .

The concept of a c_1 -embedding $f : \mathcal{A} \rightarrow_{c_1} \mathcal{B}$ for c_1 -structures \mathcal{A} and \mathcal{B} , under which an appropriate record $W_{\mathcal{A}}$ ($W_{f(\mathcal{A})} = W_{\mathcal{B}} \upharpoonright f(\mathcal{A})$) is preserved, is a natural generalization of the concept of the cc-embedding. Thereby we also define the concept of a c_1 -embedding $f : \mathcal{A} \rightarrow_{c_1} \mathcal{N}$ of a c_1 -structure \mathcal{A} into a model \mathcal{N} in \mathbf{K}_1 .

Two c_1 -structures, \mathcal{A} and \mathcal{B} , are said to be c_1 -isomorphic if there exists a c_1 -embedding $f : \mathcal{A} \rightarrow_{c_1} \mathcal{B}$ with $f(\mathcal{A}) = \mathcal{B}$.

A self-sufficiency relation \leq_1 in \mathbf{K}_1^* inherits the relation \leq in $\mathbf{K}_0^{f,P}$, i.e., for any c_1 -structures \mathcal{A} and \mathcal{B} in \mathbf{K}_1^* the following holds:

$$(\mathcal{A} \leq_1 \mathcal{B}) \Leftrightarrow (\mathcal{A} \upharpoonright \Sigma_P \leq \mathcal{B} \upharpoonright \Sigma_P).$$

Theorem 4.7.1. *There exists a saturated $(\mathbf{K}_1^*; \leq_1)$ -generic model \mathcal{M} of a stable theory satisfying the following conditions:*

(a) *if \mathcal{A} and \mathcal{B} are c_1 -isomorphic self-sufficient c_1 -structures in \mathcal{M} , then $\text{tp}_{\mathcal{M}}(\mathcal{A}) = \text{tp}_{\mathcal{M}}(\mathcal{B})$;*

(b) the restriction of \mathcal{M} to the language Σ_P is a $\mathbf{K}_0^{f,P}$ -generic model;

(c) the theory $\text{Th}(\mathcal{M})$ has countably many 1-types, and each 1-type is defined by color of any its realization; the type $p_\infty(x)$ of elements infinite in color is the unique nonprincipal 1-type, and its own weight is infinite;

(d) every formula $R_{\mathbf{A}}(a, \bar{y})$, where $\models p_\infty(a)$, is principal, and the c_1 -isomorphism type of any realization of the formula $R_{\mathbf{A}}(a, \bar{y})$ coincides with the c_1 -isomorphism type \mathbf{A} ;

(e) every formula $R_{\mathbf{A}}(x, \bar{a})$, where \mathbf{A} is the c_1 -isomorphism type of tuple \bar{a} , is principal, and each realization of $R_{\mathbf{A}}(x, \bar{a})$ is a realization of the type p_∞ .

P r o o f of existence of saturated $(\mathbf{K}_1^*; \leq_1)$ -generic model \mathcal{M} in \mathbf{K}_1 , satisfying (a) and (b), almost word for word repeats the proof of Theorem 4.6.1. Furthermore, the presence of new predicates $R_{\mathbf{A}_m}$ and ρ is not involved in count of values of the prerank function, since these predicates refine correspondent graph structures of $\mathbf{K}_0^{f,P}$.

The proof of stability of generic theory $\text{Th}(\mathcal{M})$ and of item (c) is similar the proof of Theorem 4.6.2.

Repeating the proof of items (d) and (e) in Theorem 3.2.3, we prove correspondent items. \square

Denote by T_1 the theory $\text{Th}(\mathcal{M})$ of $(\mathbf{K}_1^*; \leq_1)$ -generic model \mathcal{M} .

Since each type of T_1 over the empty set is a subtype of type, that defined by record of some self-sufficient c_1 -structure, and, for any type q of c_1 -structure not in a prime model, there exists a principal formula $\exists y_1, \dots, y_t R_{\mathbf{A}}(a, \bar{y})$ (where $\models p_\infty(a)$), for which $\exists y_1, \dots, y_t R_{\mathbf{A}}(x, \bar{y})(a, \bar{y}) \vdash q$, then all types of T_1 are realized in the model \mathcal{M}_{p_∞} . Thus, the type $p_\infty(x)$ is powerful.

Theorem 4.7.1(e) implies that, for any nonprincipal type $q(\bar{y})$ of T_1 , the type p_∞ is realized in the model \mathcal{M}_q . Therefore, every nonprincipal type is powerful. Moreover, the introduction of predicates $R_{\mathbf{A}}$ allows, for every tuple \bar{a} , having some c_1 -isomorphism type \mathbf{A}_m , to find a realization a of $p_\infty(x)$ such that $\models R_{\mathbf{A}_m}(a, \bar{a})$, and so, in view of Proposition 1.1.3, the model $\mathcal{M}_{\bar{a}}$ coincides with a model \mathcal{M}_a . Thus, all prime model over tuples, realizing nonprincipal types, are isomorphic to \mathcal{M}_{p_∞} , and the following theorem holds.

Theorem 4.7.2. *There exists a small stable theory T_1 expanding the theory $T^{f,P}$ and satisfying the condition $|\text{RK}(T_1)| = 2$.*

§ 4.8. Stable theories with three countable models

General principles for expanding T_1 , which lead to the construction of a small stable theory T with the condition $|\text{RK}(T)| = 2$ and the property (CEP), coincides with the same principals, that described in Section 3.3, but have reservations reflected in Section 4.5. For construction of required stable theory T , we shall use corealization amalgams, that, in difference of construction in Chapter 3, will be obtained automatically, in view of self-sufficiency of two-element graphs, containing new edges, and by the fact that weights of new edges are unboundedly small. Furthermore, the theory T will be varied on dependence of alternating weights α_k^Q for lengths of shortest Q -routes, weights α_k^P for $P_{i,n}$ -edges, and weights α_k^R of R_j -edges.

Denote by $\mathbf{K}_{-1}^{f,P,R}$ the class of all finite structures \mathcal{A} (including empty structure) of language

$$\Sigma_{P,R} \equiv \{\text{Col}_n^{(1)} \mid n \in \omega\} \cup \{Q^{(2)}\} \cup \{P_{i,n}^{(2)} \mid i, n \in \omega\} \cup \{R_j^{(2)} \mid j \in \omega\}$$

satisfying the following conditions:

(a) the structure $\mathcal{A} \upharpoonright \Sigma_P$ belongs to $\mathbf{K}_0^{f,P}$;

(b) relations R_j are symmetric, irreflexive, pairwise disjoint and connect only the same vertices in color; moreover, connected $\bigcup_{j \in \omega} R_j$ -components are strictly Q -ordered, i.e., elements of the same $\bigcup_{j \in \omega} R_j$ -component are not connected by Q -routes, and if vertices a_1 and a_2 are linked by $\bigcup_{j \in \omega} R_j$ -route, vertices b_1 and b_2 are linked by $\bigcup_{j \in \omega} R_j$ -route, and there exists an (a_1, b_1) - Q -route, then there are no (b_2, a_2) - Q -routes.

Now, take a monotonically decreasing sequence of positive real numbers α_k , define for the prerank function $y^1(\cdot)$ in Section 4.3 as weights of pairs of vertices, connected by shortest Q -routes of length k . Put elements $\alpha_m^Q \equiv \alpha_{3m}$ to weights of pair of vertices, connected by shortest Q -routes of length m , elements $\alpha_m^P \equiv \alpha_{3m+1}$ to weights of $P_{i,n}$ -edges, where $m = c(i, n)$, and elements $\alpha_m^R \equiv \alpha_{3m+2}$ to weights of R_m -edges. After aforesaid renotations, we define the prerank function $y(\cdot)$ for structures \mathcal{A} in $\mathbf{K}_{-1}^{f,P,R}$, by the rule:

$$y(\mathcal{A}) = 2 \cdot |A_f| + |A_{\text{nf}}| - \sum_{k=1}^{\infty} \alpha_k^Q \cdot e_k^Q(\mathcal{A}) - \sum_{k=0}^{\infty} \alpha_k^P \cdot e_k^P(\mathcal{A}) - \sum_{k=0}^{\infty} \alpha_k^R \cdot e_k^R(\mathcal{A}),$$

where $e_1^Q(\mathcal{A})$ is the number of Q -arcs in \mathcal{A} ; $e_k^Q(\mathcal{A})$, for $k \geq 2$, is the number of pairs $(a, a') \in A^2$, connected only by external shortest (a, a') - Q -routes of length k and such that there are no (a, a') - Q -routes of length k with \mathcal{A} -external \mathcal{A} -compulsory furcations; $e_k^P(\mathcal{A})$ is the number of $P_{i,n}$ -edges in \mathcal{A} , where $k = c(i, n)$; $e_k^R(\mathcal{A})$ is the number of R_k -edges in \mathcal{A} .

A p -approximation of the prerank function $y(\cdot)$ is the function $y_p(\cdot)$ assigning a real to every structure $\mathcal{A} \in \mathbf{K}_{-1}^{f,P,R}$ by the rule:

$$y_p(\mathcal{A}) = 2 \cdot |A_f| + |A_{nf}| - \sum_{k=1}^p \alpha_k^Q \cdot e_k^Q(\mathcal{A}) - \sum_{k=0}^{\infty} \alpha_k^P \cdot e_k^P(\mathcal{A}) - \sum_{k=0}^p \alpha_k^R \cdot e_k^R(\mathcal{A}).$$

Analogous to operations T_c and T_c^* on the class of cc-graphs, we define operations T_c and T_c^* on the class $\mathbf{K}_{-1}^{f,P,R}$, where each pair of vertices belonging to $P_{i,n}$ of R_m is interpretable in graphs without furcations by one edge, connecting elements in $J_0 \cup J_1$.

Denote by $\mathbf{K}_1^{f,P,R}$ the class of all structures \mathcal{A} in $\mathbf{K}_{-1}^{f,P,R}$ such that $y_1(\mathcal{A}') \geq b_n^1$ for any structure $\mathcal{A}' \sqsubseteq \text{cfc}(\mathcal{A})$, where $n = |T(\mathcal{A}')|$. For a structure \mathcal{A} in $\mathbf{K}_1^{f,P,R}$, we write $\mathcal{A} \in \mathbf{K}_{p+1}^{f,P,R}$, for $p \geq 1$, iff $\mathcal{A} \in \mathbf{K}_p^{f,P,R}$ and $y_p^\Delta(\mathcal{A}') \geq b_n^p$ for any part \mathcal{A}' of structure $\text{cfc}(\mathcal{A}) \in \mathbf{K}_1^{f,P,R}$, where $y_p^\Delta(\mathcal{A}')$ is a minimal value $y_p(T_c(\mathcal{A}_0)) + \Delta_1 + \dots + \Delta_m$, $\Delta_i = y_p(T_c((\mathcal{A}_{i+1})_{T_c^*(\mathcal{A}_i)})) - y_p(T_c^*(\mathcal{A}_i))$, $\mathcal{A}_0 \subseteq_{\text{cc}} \mathcal{A}_1 \subseteq_{\text{cc}} \dots \subseteq_{\text{cc}} \mathcal{A}_m = \mathcal{A}'$, n is a natural number, for which $y_p^\Delta(\mathcal{A}') = n - s \cdot \alpha_1$ with some $s \in \omega$, $k_p \leq n < k_{p+1}$.

Put $\mathbf{K}_0^{f,P,R} \equiv \bigcap_{p=1}^{\infty} \mathbf{K}_p^{f,P,R}$, and denote by $\mathbf{K}^{f,P,R}$ the class of all graphs in the language $\Sigma_{P,R}$, for which each finite *substructure* (i.e., a subgraph together with an information on lengths of shortest Q -routes and on compulsory furcations) belongs to $\mathbf{K}_0^{f,P,R}$.

Let \mathcal{A} be a finite substructure of graph (respectively, of finite substructure) \mathcal{M} (of a graph) belonging $\mathbf{K}^{f,P,R}$. We say that \mathcal{A} is a *self-sufficient substructure* of \mathcal{M} and write $\mathcal{A} \leq \mathcal{M}$ if for any finite structures $\mathcal{A}' \sqsubseteq \mathcal{A}$, $\mathcal{B}' \sqsubseteq \mathcal{M}$, $\mathcal{A}' = \mathcal{A}_0 \subseteq_{\text{cc}} \mathcal{A}_1 \subseteq_{\text{cc}} \dots \subseteq_{\text{cc}} \mathcal{A}_m = \mathcal{B}'$, the inequality $\Delta_1 + \dots + \Delta_m < 0$, where $\Delta_i = y(T_c((\mathcal{A}_{i+1})_{T_c^*(\mathcal{A}_i)})) - y(T_c^*(\mathcal{A}_i))$, implies $\mathcal{B}' \subseteq \mathcal{A}$. If $\mathcal{A} \leq \mathcal{M}$, and \mathcal{M} is a finite structure, then \mathcal{A} is called a *strong substructure* of \mathcal{M} .

Denote by $\mathbf{T}_0^{f,P,R}$ the class of types correspondent to all structures in $\mathbf{K}_0^{f,P,R}$, and equip it by the relation \leq' , where $\Phi(A) \leq' \Psi(B) \Leftrightarrow \mathcal{A} \leq \mathcal{B}$.

Repeating the arguments for the class $(\mathbf{T}_0^f; \leq'_{cc})$, we state that $(\mathbf{T}_0^{f,P,R}; \leq')$ is a self-sufficient generic class such that, adding to types of formulas describing self-sufficient closures, we have the uniform t -amalgamation property. It implies that a $(\mathbf{T}_0^{f,P,R}; \leq')$ -generic model is ω -saturated and realizes all types $\Phi(X)$, correspondent to types $\Phi(A)$ in $\mathbf{T}_0^{f,P,R}$, and the $(\mathbf{T}_0^{f,P,R}; \leq')$ -generic theory $T^{f,P,R}$ is stable. Since two-elements sets $\{a, b\}$ with $(a, b) \in \bigcup_{j \in \omega} R_j$ are self-sufficient,

the type $\text{tp}(a \hat{=} b)$ is defined by color of edge $[a, b]$ and by color of any its endpoint, i.e., all formulas $R_j(a, x)$ are principal. Using the proof of Theorem 4.3.18, we obtain that each type in $S^1(\emptyset)$ has the infinite own weight. Thus, the following theorem holds.

Theorem 4.8.1. *The theory $T^{f,P,R}$ is stable, small, and has countably many 1-types, each of which is defined by color of every its realization and has the infinite own weight. The type $p_\infty(x)$, defined by infinite color, is the unique nonprincipal 1-type. Every formula $P_{i,n}(a, x)$, where $\text{Col}(a) > n$, $Q(b, x)$, where $\text{Col}(b) = \infty$, and $R_j(c, x)$ is principal.*

Since $\lim_{k \rightarrow \infty} \alpha_k^R = 0$, any self-sufficient set A in a generic model, containing two realizations, a_1 and a_2 , of the type $p_\infty(x)$, such that a_1 and a_2 are not connected by Q -routes or by edges, can be transformed to a self-sufficient set B in the same model, having the same structure as A with the unique difference, namely, with an R_j -edge, connecting elements b_1 and b_2 that correspond to a_1 and a_2 . Since formulas $R_j(b_1, x)$ and $R_j(x, b_2)$ are principal, the transformation from A to B can be interpreted as the corealization amalgam of models \mathcal{M}_{a_1} and \mathcal{M}_{a_2} over the type $\text{tp}(A)$. Thus, the structure of graph language $\{R_j \mid j \in \omega\}$ plays in a generic model the same role that acyclic undigraphs in generic models of Ehrenfeucht theories, described in Chapter 3.

Similarly Corollary 4.6.4, we have

Proposition 4.8.2. *Let A be a self-sufficient set in a model of the theory $T^{f,P,R}$. The model \mathcal{M}_A is a complete $\bigcup_{k,n \in \omega} (Q^k \cup P_{k,n} \cup$*

$R_k)$ -graph over A , i.e., any two different elements, $a \in M_A$ and $b \in M_A \setminus A$, are connected by a Q^k -arc or an edge. The set of isomorphism types of prime models over finite sets coincides with the set of isomorphism types of models \mathcal{M}_A , where A are self-sufficient sets and \mathcal{M}_A are complete $\bigcup_{k,n \in \omega} (Q^k \cup P_{k,n} \cup R_k)$ -graphs over A .

Define the class \mathbf{K}_2^* of finite structures in language

$$\Sigma_2 = \Sigma_{P,R} \cup \{R_{\mathbf{A}_m} \mid m \in \omega\} \cup$$

$$\cup \{\rho^{(3)} \mid \rho(x, y_i, y_j) \text{ characterizes some CLN-correlation}\},$$

equipped by records on mutual connections of elements, satisfying the conditions (1) and (2) in previous Section, where structures of $\mathbf{K}_0^{f,P,R}$ are taken instead of structures in $\mathbf{K}_0^{f,P}$. The self-sufficiency relation \leq_2 for the class \mathbf{K}_2^* naturally inherits the self-sufficiency relation \leq for the class $\mathbf{K}_0^{f,P,R}$.

Denote by \mathbf{K}_2 the class of all models in the language Σ_2 , for which each finite subset forms a structure in \mathbf{K}_2^* .

The concept of c_2 -embedding $f : \mathcal{A} \rightarrow_{c_2} \mathcal{B}$ for structures \mathcal{A} and \mathcal{B} in \mathbf{K}_2^* preserving the correspondent record $W_{\mathcal{A}}$ ($W_{f(\mathcal{A})} = W_{\mathcal{B}} \upharpoonright f(\mathcal{A})$) naturally generalizes aforesaid concepts of c -embeddings. Thus, the notion of c_2 -embedding $f : \mathcal{A} \rightarrow_{c_2} \mathcal{N}$ of c_2 -structure \mathcal{A} into a model \mathcal{N} in \mathbf{K}_2 is also defined.

Two structures, \mathcal{A} and \mathcal{B} , are called c_2 -isomorphic if there exists a c_2 -embedding $f : \mathcal{A} \rightarrow_{c_2} \mathcal{B}$ with $f(\mathcal{A}) = \mathcal{B}$.

Theorem 4.8.3. *There exists a saturated $(\mathbf{K}_2^*; \leq_2)$ -generic model \mathcal{M} of a stable theory, satisfying the following conditions:*

(a) *if \mathcal{A} and \mathcal{B} are c_2 -isomorphic self-sufficient structures in \mathcal{M} , then $\text{tp}_{\mathcal{M}}(\mathcal{A}) = \text{tp}_{\mathcal{M}}(\mathcal{B})$;*

(b) *the restriction of \mathcal{M} to the language $\Sigma_{P,R}$ is a $\mathbf{K}_0^{f,P,R}$ -generic model;*

(c) *the theory $\text{Th}(\mathcal{M})$ has countably many 1-types, each of which is defined by color of any its realization and has the infinite own weight; the type $p_{\infty}(x)$, defined by infinite color, is the unique non-principal 1-type;*

(d) *every formula $P_{i,n}(a, x)$, where $\text{Col}(a) > n$, $Q(b, x)$, where $\text{Col}(b) = \infty$, and $R_j(c, x)$ is principal;*

(e) *every formula $R_{\mathbf{A}}(a, \bar{y})$, where $\models p_{\infty}(a)$, is principal, and the c_2 -isomorphism type of any realization of formula $R_{\mathbf{A}}(a, \bar{y})$ coincides with the c_2 -isomorphism type \mathbf{A} ;*

(f) *every formula $R_{\mathbf{A}}(x, \bar{a})$, where \mathbf{A} is the c_2 -isomorphism type of \bar{a} , is principal, and every realization of the formula $R_{\mathbf{A}}(x, \bar{a})$ is a realization of p_{∞} .*

P r o o f consists in obvious combination of the proofs of Theorems 4.7.1 and 4.8.1. \square

Denote the $(\mathbf{K}_2^*; \leq_2)$ -generic theory by T_2 . Clearly, T_2 is an expansion of $T^{f,P,R}$.

Repeating the proof of Theorem 4.7.2 and using Theorem 4.8.3, we state the following:

Theorem 4.8.4. *The theory T_2 satisfies the condition $|\text{RK}(T_2)| = 2$.*

Theorem 4.8.5. *There is a limit model of the theory T_2 over the type p_∞ which is unique up to isomorphism.*

P r o o f. The existence of a limit model follows from Proposition 1.1.8 and Corollary 1.1.9, in view of the semi-isolation relation SI_{p_∞} being nonsymmetric w.r.t. $Q(x, y)$. In order to prove that our limit model is unique, it suffices to show that any limit model \mathcal{M} over p_∞ is saturated.

Let $\mathcal{M}_{a_n} \models p_\infty(a_n)$, $n \in \omega$, be any elementary chain over p_∞ whose union coincides with the limit model \mathcal{M} . Notice, that by generic construction, this chain can be chosen so that each principal formula $\varphi_n(a_{n+1}, x)$, for which $\models \varphi_n(a_{n+1}, a_n)$, $n \in \omega$, is equivalent to a formula $R_j(a_{n+1}, x)$ or a formula $Q^k(a_{n+1}, x) \wedge \neg Q^{k-1}(a_{n+1}, x)$, $k \geq 1$. If $\models R_j(a_{n+1}, a_n)$, then, in models \mathcal{M}_{a_n} and $\mathcal{M}_{a_{n+1}}$, connected components w.r.t. $\bigcup_{i \in \omega} R_j$, containing elements a_n and a_{n+1}

simultaneously, form a complete graph, and if there are no links from a_n to sequential elements by relations Q^k , then the model \mathcal{M} is prime over some element in the $\bigcup_{i \in \omega} R_j$ -component of connectivity, containing a_n , i.e., \mathcal{M} is not a limit model. Thus, the sequence $(a_n)_{n \in \omega}$ has infinitely many transitions from a_n to a_{n+1} satisfying

$\models Q^{k_n}(a_{n+1}, a_n) \wedge \neg Q^{k_n-1}(a_{n+1}, a_n)$. Now notice, that for elements a_n and a_{n+1} , satisfying $\models Q^{k_n}(a_{n+1}, a_n) \wedge \neg Q^{k_n-1}(a_{n+1}, a_n)$, the $\bigcup_{j \in \omega} R_j$ -components of connectivity in the models \mathcal{M}_{a_n} and $\mathcal{M}_{a_{n+1}}$,

containing elements a_n and a_{n+1} respectively, form a complete graph w.r.t. $\bigcup_{k \in \omega} (Q^k \cup R_k)$. Therefore, the sequence $(a_n)_{n \in \omega}$ contains an

infinite subsequence, in which all elements are pairwise connected by Q -routes. Condensing this subsequence by elements of shortest Q -routes, containing neighbouring elements of subsequence, we

obtain an elementary chain of prime models $(\mathcal{M}_{b_n})_{n \in \omega}$, for which $\mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_{b_n}$ and $\models Q(b_{n+1}, b_n)$, $n \in \omega$.

Now, take an arbitrary 1-type $q(x, \bar{c}) \in S(\bar{c})$, where \bar{c} is a tuple in M , and argue to show that $q(x, \bar{c})$ is realized in \mathcal{M} . Indeed, the tuple \bar{c} belongs to some model \mathcal{M}_{b_n} , and, by definition of $R_{\mathbf{A}}$, for some c_2 -isomorphism type \mathbf{A} , containing realizations of the type $q(x, \bar{y})$, and for some $n' \geq n$, there exists an element $d \in M_{b_{n'}}$ such that some projection $\exists z_{i_1}, \dots, z_{i_m} R_{\mathbf{A}}(d, \bar{z})$ is realized by \bar{c} . Since the formula $R_{\mathbf{A}}(d, \bar{z})$ is principal, there exists a tuple $\bar{d} \in M$, realizing that formula and extending \bar{c} . By the choice of the c_2 -isomorphism type \mathbf{A} , some coordinate of \bar{d} realizes the type $q(x, \bar{c})$. Since the chosen type q is arbitrary, \mathcal{M} is saturated. \square

By Corollary 1.1.15 and Theorems 4.8.4 and 4.8.5, we obtain the following:

Theorem 4.8.6. *There exists a stable Ehrenfeucht theory T satisfying the condition $I(T, \omega) = 3$.*

§ 4.9. Realizations of basic characteristics of stable Ehrenfeucht theories

We show that, similar Theorem 3.4.1, all possible basic characteristics of Ehrenfeucht theories are realized out in the class of stable theories.

Theorem 4.9.1. *For any finite preordered set $\langle X; \leq \rangle$ with the least element x_0 and the greatest class \widetilde{x}_1 in the ordered factor set $\langle X; \leq \rangle / \sim$ w.r.t. \sim (where $x \sim y \Leftrightarrow x \leq y$ and $y \leq x$), and for any function $f : X / \sim \rightarrow \omega$ satisfying the conditions $f(\widetilde{x}_0) = 0$, $f(\widetilde{x}_1) > 0$ for $|X| > 1$, and $f(\widetilde{y}) > 0$ for $|\widetilde{y}| > 1$, there exist a stable theory T and an isomorphism $g : \langle X; \leq \rangle \xrightarrow{\sim} \text{RK}(T)$ such that $\text{IL}(g(\widetilde{y})) = f(\widetilde{y})$ for any $\widetilde{y} \in X / \sim$.*

P r o o f. Let $\langle X; \leq \rangle$ and f be as above. Without loss of generality, we may assume that $|X| > 1$. We fix a numbering $\nu : |X| \rightarrow X$ such that $\nu(m) < \nu(n)$ and $\nu(m) \not\prec \nu(n)$ imply $m < n$, and in correspondence with every \sim -class is an interval in $|X|$. Consider a theory T_{-1} of unary predicates $S_1, \dots, S_{|X|-1}$ forming a partition on $|X|-1$ infinite classes with inessential coloring $\text{Col} : M \rightarrow \omega \cup \{\infty\}$ such that $\vdash \exists^{\geq \omega} (P_i(x) \wedge \text{Col}_n(x))$, $i = 1, \dots, |X| - 1$, $n \in \omega$.

Clearly, using a construction above, we can expand the theory T_{-1} by symmetric binary predicates $P_{i,n,i'}$, $i, n \in \omega$ (where the third index i' means, that elements of color exceeding n are connected with elements of color n , belonging to $S_{i'}$), to a small stable generic theory T_0 with weights $\alpha_{c(c(i,n),i')}^P$ of edges, for which any finite set of finite elements in color is connected by binary predicated with some infinite elements in color, belonging relations $S_1, \dots, S_{|X|-1}$. Therefore, adding predicates $R_{\mathbf{A}}$ with structures of powerful digraphs, we reduce the construction of required domination preorder \leq_{RK} to the statement of such preorder for 1-types of infinite elements in color, defined by formulas $S_1(x), \dots, S_{|X|-1}(x)$.

We claim that there exists a stable expansion T of T_0 with an isomorphism $g : \langle X, \leq \rangle \xrightarrow{\sim} \text{RK}(T)$ such that:

(i) $g(\nu(i)) = \mathbf{M}_{p_i}$, where \mathbf{M}_{p_i} is an isomorphism type of the prime model \mathcal{M}_{p_i} over a realization of the type $p_i(x)$ in $S^1(\emptyset)$, for $p_i(x)$ being isolated by the set $\{S_i(x) \wedge \neg \text{Col}_n(x) \mid n \in \omega\}$ of formulas, $i = 1, \dots, |X| - 1$, and $p_1(x), \dots, p_{|X|-1}(x)$ are all non-principal 1-types over \emptyset in the variable x ;

(ii) $\text{IL}(g(\tilde{y})) = f(\tilde{y})$ for any $\tilde{y} \in X/\sim$.

We construct $T = \bigcup_{i < |X|} T_i$ by induction so as to respect the numbering ν . Assume T_0, \dots, T_{k-1} are already constructed and $\nu(k), \nu(k+1), \dots, \nu(k+l)$ form a \sim -class.

If $f(\nu(k)) = 0$, then $l = 0$, and we define T_k by expanding the language of T_{k-1} by new binary predicate symbols R_{ki} (where the class $\nu(k)$ covers $\nu(i)$, $i \neq 0$) so as to satisfy the following conditions:

(1) each predicate R_{ki} is injectively connected with some weight $\alpha_{k'}$ bounding via a linear prerank function $y(\cdot)$ (of form pointed out in Section 4.8) the number of R_{ki} -links in dependence on the cardinality of given finite set;

(2) $R_{ki}(a, y)$ is a principal formula and $R_{ki}(a, y) \vdash p_i(y)$ for any $a \models p_k$;

(3) for any $a, b \models p_i$, there are infinitely many elements $c \models p_k$ and infinitely many d not realizing types $p_1(x), \dots, p_{|X|-1}$ such that

$$\models R_{ki}(c, a) \wedge R_{ki}(c, b) \wedge R_{ki}(d, a) \wedge R_{ki}(d, b);$$

moreover, $c \models p_k$ and $\models R_{ki}(c, a)$ imply that a does not semi-isolate c .

Clearly, (1)–(3) can be realized so that the model \mathcal{M}_{p_k} of T_k has a unique realization of the type p_k and hence $\text{IL}(g(\nu(k))) = 0 = f(\nu(k))$. In addition, \mathcal{M}_{p_k} , by induction, will realize all types p_i dominated by p_k , that is, satisfying the relation $\nu(i) \leq \nu(k)$. A presence of prerank function allows on a base of arguments in previous Sections to state that the theory T_k is small and stable.

Suppose $f(\nu(k)) = r > 0$. We define T_k^0 by expanding the language of T_{k-1} by new binary predicate symbols R_{ki} (where $\nu(k)$ covers $\nu(i)$, $i \neq 0$) satisfying (1)–(3), and by binary predicate symbols R'_{ij} (where $\nu(i), \nu(j) \in \nu(k)$) with the following conditions:

(4) each predicate R_{ki} and R'_{ij} is injectively connected with some weight $\alpha_{k'}$ bounding via a linear prerank function $y(\cdot)$ (of form pointed out in Section 4.8) the number of R_{ki} - and R'_{ij} -links in dependence on the cardinality of given finite set;

(5) $R'_{ij}(a, y)$ is a principal formula and $R'_{ij}(a, y) \vdash p_j(y)$ for any $a \models p_i$;

(6) for any $a, b \models p_j$, there are infinitely many elements $c \models p_i$ and infinitely many d not realizing types $p_1(x), \dots, p_{|X|-1}$, with

$$\models R'_{ij}(c, a) \wedge R'_{ij}(c, b) \wedge R'_{ij}(d, a) \wedge R'_{ij}(d, b);$$

moreover, $c \models p_i$ and $\models R'_{ij}(c, a)$ imply that a does not semi-isolate c ;

(7) for any elements a and b not realizing formulas $S_{k+l+1}(x), \dots, S_{|X|-1}(x)$, there exist infinitely many elements $c \models p_j$ such that $\models R'_{ij}(a, c) \wedge R'_{ij}(b, c)$;

(8) the relation $R'_k = \bigcup_{\nu(i), \nu(j) \in \nu(k)} R'_{ij}$ forms a digraph isomorphic

to some digraph Γ^{cpg} , and, moreover, lengths of shortest Q -routes are injectively and monotonically connected with weights $\alpha_{k'}$.

As above, predicates R_{ki} guarantee that types p_i are dominated by p_k for $\nu(i) < \nu(k)$ and $\nu(i) \not\sim \nu(k)$, while relations R'_{ij} ensure that p_i and p_j are domination equivalent and that models \mathcal{M}_{p_i} and \mathcal{M}_{p_j} are non-isomorphic, for $i \neq j$. A presence of prerank function with weights $\alpha_{k'}$ also allows to state the smallness and the stability of theory, constructing in this step.

Now, similarly to the conditions (1) and (2) specified in Section 4.7 with Q replaced by R'_k , we extend the language by predicate

symbols $R_{\mathbf{A}}$ so that types p_i , $i = k, \dots, k+l$, dominate all types $q(\bar{x}) \in S(T_k)$ with $S_j(x_i) \notin q(\bar{x})$ for $j > k+l$, the types q in question that are not dominated by p_i , $i < k$, dominate all types p_k, \dots, p_{k+l} , and the models \mathcal{M}_q are isomorphic to \mathcal{M}_{p_k} .

For the condition $\text{IL}(g(\nu(k))) = r$ to be satisfied, on the structure of realizations of p_k , we define a graph structure with binary relations R_1'', \dots, R_r'' such that:

(9) each predicate R_i'' is injectively connected with some weight $\alpha_{k'}$ bounding via a linear prerank function $y(\cdot)$ (of form pointed out in Section 4.8) the number of R_i'' -links in dependence on the cardinality of given finite set;

(10) $R_i''(a, y)$ is a principal formula and $R_i''(a, y) \vdash p_k(y)$ for any $a \models p_k$, $i \leq r$;

(11) for any $a, b \models p_k$, there are infinitely many elements $c \models p_k$ and infinitely many d not realizing types $p_1(x), \dots, p_{|X|-1}$ for which

$$\models R_i''(c, a) \wedge R_i''(c, b) \wedge R_i''(d, a) \wedge R_i''(d, b);$$

moreover, $\models R_i''(c, a)$ and $c \models p_j$ imply that a does not semi-isolate c ;

(12) relations R_i'' form digraphs isomorphic to some digraphs Γ^{CPg} , where lengths of shortest routes are injectively and monotonically connected with weights α_k .

If \mathcal{M}_a and \mathcal{M}_b are prime models over realizations a and b of p_k , respectively, such that $\models R_i''(a, b)$ and $\mathcal{M}_b \prec \mathcal{M}_a$, then we call \mathcal{M}_a an R_i'' -extension of the model \mathcal{M}_b . An elementary chain $(\mathcal{M}_s)_{s \in \omega}$ over p_k is called an R_i'' -chain if \mathcal{M}_{s+1} is an R_i'' -extension of \mathcal{M}_s for any s .

If \mathcal{M}_a and \mathcal{M}_b are prime models over realizations a and b of p_i , respectively, with $\nu(i) \sim \nu(k)$, such that $\models R_{ii}'(a, b)$ and $\mathcal{M}_b \prec \mathcal{M}_a$, then the model \mathcal{M}_a is called an R_{ii}' -extension of \mathcal{M}_b .

Similarly to the conditions (a) and (b) given in Section 4.8, we extend the language by symbols R_j so as to satisfy the following:

(13) each predicate R_j is injectively connected with some weight $\alpha_{k'}$ bounding via a linear prerank function $y(\cdot)$ (of form pointed out in Section 4.8) the number of R_j -links in dependence on the cardinality of given finite set;

(14) for any limit model \mathcal{M} over a type p_{k+i} , $0 \leq i \leq l$, there is a relation R_j'' such that \mathcal{M} is the union of an R_j'' -chain $(\mathcal{M}_s)_{s \in \omega}$ over the type p_k ;

(15) limit models \mathcal{M}_1 and \mathcal{M}_2 over p_k are equivalent iff there is a predicate R_i'' such that \mathcal{M}_1 and \mathcal{M}_2 are unions of R_i'' -chains but are not unions of R_j'' -chains, for $j > i$.

Notice, that (14) and (15) are realized via predicates R_j “saying” the following:

(a) every R_{kk}' -extension \mathcal{M}_a of \mathcal{M}_b contains an R_1'' -extension and vice versa;

(b) for any i , $\nu(i) \sim \nu(k)$, $i \neq k$, and for any finite elementary chain $\mathcal{M}_{a_1}, \dots, \mathcal{M}_{a_s}$, $a_1, \dots, a_s \models p_i$, there exist realizations b_1, \dots, b_{s-1} of p_k such that the sequence $\mathcal{M}_{a_1}, \mathcal{M}_{b_1}, \dots, \mathcal{M}_{b_{s-1}}, \mathcal{M}_{a_s}$ is also an elementary chain;

(c) if \mathcal{M}_{b_0} and \mathcal{M}_{b_1} are R_s'' -extensions of \mathcal{M}_{a_n} , q is the type of a tuple $\langle a_0, a'_0, \dots, a_n, a'_n, a''_n, b_0, b_1 \rangle$ of elements which realize type p_k and are such that $\mathcal{M}_{a_{i+1}}$ is an R_{t_i}'' -extension of a model $\mathcal{M}_{a'_i}$, equal to \mathcal{M}_{a_i} , $\models R_{t_i}''(a_{i+1}, a'_i)$, \mathcal{M}_{b_0} and \mathcal{M}_{b_1} are R_{t_n}'' -extensions of the model \mathcal{M}_{a_n} , equal to $\mathcal{M}_{a'_n}$ and to $\mathcal{M}_{a''_n}$, $\models R_{t_n}''(b_0, a'_n) \wedge R_{t_i}''(b_1, a''_n)$, and elements b_0 and b_1 are not connected by (b_0, b_1) - Q - or (b_1, b_0) - Q -routes, then \mathcal{M}_{b_0} contains a corealization amalgam $\mathcal{M}_{b_0} *_q \mathcal{M}_{b_1}$;

(d) if $\mathcal{M}_{a_1}, \dots, \mathcal{M}_{a_s}$ is a finite elementary chain such that the model $\mathcal{M}_{a_{j+1}}$ is an R_{i_j}'' -extension of \mathcal{M}_{a_j} , $j = 1, \dots, s-1$, and $\max\{i_1, \dots, i_{s-2}\} < i_{s-1}$, then \mathcal{M}_{a_s} contains an $R_{i_{s-1}}''$ -extension of \mathcal{M}_{a_1} and vice versa.

With the expansions exercised above we obtain a small stable theory $T_k = T_{k+1} = \dots = T_{k+l}$ such that types p_k, \dots, p_{k+l} are domination equivalent, $\mathcal{M}_{p_k}, \dots, \mathcal{M}_{p_{k+l}}$ are pairwise non-isomorphic, and the number of limit models over p_k, \dots, p_{k+l} is equal to $f(\nu(\bar{k}))$.

Proceeding further with this process, at step $|X| - 1$, we obtain a small stable theory $T = T_{|X|-1}$ and an isomorphism $g : \langle X; \leq \rangle \xrightarrow{\sim} \text{RK}(T)$ such that $g(\nu(0))$ is an isomorphism type of the prime model for T , $g(\nu(m))$ is an isomorphism type of \mathcal{M}_{p_m} , $1 \leq m \leq |X| - 1$, and $\text{IL}(g(\tilde{y})) = f(\tilde{y})$ for any $\tilde{y} \in X/\sim$. \square

Chapter 5

HYPERGRAPHS OF PRIME MODELS AND DISTRIBUTIONS OF COUNTABLE MODELS OF SMALL THEORIES

In the final Chapter, we consider a family of hypergraphs of prime models for an arbitrary small theory and represent a mechanism of structure descriptions of models of theories by these families. Thus, in particular, we verify the key role of graph-theoretic constructions in creation of aforesaid examples of Ehrenfeucht theories.

We shall realize constructions on a base of models \mathcal{M}^{eq} , that, as shown, for instance, in the work by P. Tanović [194], allow in many cases to reduce general enough structure situations to classes of known objects.

Recall the notion of model \mathcal{M}^{eq} [22]. Let \mathcal{M} be a model. We add to the language $\Sigma(\mathcal{M})$ of \mathcal{M} all possible unary predicate symbols P_E , correspondent to \emptyset -definable equivalence relations $E(\bar{x}, \bar{y})$, and all possible $(l(\bar{x}) + 1)$ -ary predicate symbols π_E . The obtained language $\Sigma(\mathcal{M})^{\text{eq}}$ is the language of \mathcal{M}^{eq} . The universe M^{eq} of \mathcal{M}^{eq} consists of all possible E -classes. Moreover, the elements of \mathcal{M} are considered as $=$ -classes forming the *home sort* of elements, and interpretations of symbols in $\Sigma(\mathcal{M})$ coincide with their interpretations in \mathcal{M} . All other elements in M^{eq} form *imaginary sorts*. Each predicate P_E consists of all possible E -classes, and each predicate π_E is a graph of function mapping tuples of elements of the home sort in E -classes containing these tuples.

The theory of \mathcal{M}^{eq} doesn't depend on choice of model of given theory and is denoted by T^{eq} .

§ 5.1. Hypergraphs of prime models

Recall that a *hypergraph* is a pair (X, Y) of sets, where Y is a subset of the set $\mathcal{P}(X)$ being the set of all subsets of X .

Let \mathcal{M} be a model of *small* theory T . Denote by $\overline{H}(\mathcal{M})$ the set of all subsets in the universe M of \mathcal{M} such that these subsets are universes of prime, over tuples $\bar{a} \in M$, elementary submodels $\mathcal{M}_{\bar{a}}$ of \mathcal{M} , namely, $\overline{H}(\mathcal{M}) = \{M_{\bar{a}} \mid \mathcal{M}_{\bar{a}} \text{ is a prime, over a tuple } \bar{a} \in M, \text{ elementary submodel of } \mathcal{M}\}$. The pair $(M, \overline{H}(\mathcal{M}))$ is called a *hypergraph of all prime models* of \mathcal{M} .

Clearly, a tuple \bar{b} belong to a prime model $\mathcal{M}_{\bar{a}} \in \overline{H}(\mathcal{M})$ iff there exists an elementary submodel $\mathcal{M}_{\bar{b}} \in \overline{H}(\mathcal{M})$ of $\mathcal{M}_{\bar{a}}$. Thus, the inclusion relation on the set $\overline{H}(\mathcal{M})$ defines a relation that consists of all pairs (\bar{a}, \bar{b}) of tuples in M for which types $\text{tp}(\bar{b}/\bar{a})$ are principal.

Example 5.1.1. (1) For an infinite model \mathcal{M} of empty language, the $(M, \overline{H}(\mathcal{M}))$ includes all countable subsets of M .

(2) For a model \mathcal{M} of dense linear order without endpoints, the hypergraph $(M, \overline{H}(\mathcal{M}))$ consists of all possible universes of countable dense linearly ordered subsets without endpoints.

(3) A model \mathcal{M} of the Ehrenfeucht theory with three countable models, constructed on dense linearly ordered set without endpoints and with constants c_k , $k \in \omega$, such that $c_k < c_{k+1}$, generates a hypergraph $(M, \overline{H}(\mathcal{M}))$ consisting of all possible universes of countable dense linearly ordered subsets without endpoints, each of which includes a dense linear order for every interval $(-\infty; c_0)$, $(c_k; c_{k+1})$, $k \in \omega$.

(4) For a model \mathcal{M} of linear space over a field, the hypergraph $(M, \overline{H}(\mathcal{M}))$ is constructed by all possible subspaces of finite or countable dimensions. \square

Since the operation $(\cdot)^{\text{eq}}$ preserves the smallness of theory and naturally extends prime models over tuples to prime models over elements, below, for convenience, we shall often consider models \mathcal{M}^{eq} instead of \mathcal{M} , and we include in hypergraphs of prime models only universes of prime models over elements in \mathcal{M}^{eq} . Considered set of universes of prime models is denoted by $H(\mathcal{M}^{\text{eq}})$.

Notice, that the following notions and arguments work for models, in which every prime model over a tuple, being an elementary

submodel, is prime over an element. For instance, any prime model over \emptyset and any model of a theory with nonprincipal 1-types, where all nonprincipal types are powerful, have this property.

We claim, that hypergraphs $(M^{\text{eq}}, H(\mathcal{M}^{\text{eq}}))$ of countable models \mathcal{M} allow to define that given theory T is countably categorical.

Indeed, the primeness of \mathcal{M}^{eq} over an element is equivalent the property that the set M^{eq} belongs to $H(\mathcal{M}^{\text{eq}})$. Since every countable model of countably categorical theory is prime, and every small theory, being not countably categorical, has a countable saturated model that is not prime over any tuple, we have the following:

Proposition 5.1.1. *A theory T is countably categorical iff, for any countable model $\mathcal{M} \models T$, the set M^{eq} belongs to $H(\mathcal{M}^{\text{eq}})$.*

If given theory is not countably categorical then, as shown in aforesaid examples, structures $(M^{\text{eq}}, H(\mathcal{M}^{\text{eq}}))$ are poorly informative in general situation from classification point of view, since equivalence classes, relative isomorphism types of hypergraphs, are too wide w.r.t. types of mutual definability by formulas of given structures, and these structures weakly take into account specificities of given structures.

At the same time, for some situations, properties of hypergraphs $(M^{\text{eq}}, H(\mathcal{M}^{\text{eq}}))$ may be useful. For instance, a presence of minimal prime models is reflected in hypergraph by minimal sets, and structures with nonsymmetric semi-isolation relations imply an existence in hypergraphs of infinite \subseteq -chains without endpoints. For similar situations, the problem of investigation for interrelation of classes of structures and of classes of correspondent hypergraphs is expected to be perspective.

Notice the following useful property of hypergraphs $(M^{\text{eq}}, H(\mathcal{M}^{\text{eq}}))$ for countable models \mathcal{M} that is implied from a representation (see the proof of Proposition 1.1.7) of any countable model of small theory by a union of elementary chain of prime models, and marks all hypergraphs of countable models of small theories among all possible hypergraphs on countable universes.

Proposition 5.1.2. *Any countable set M^{eq} is a countable union of a \subseteq -chain of sets in $H(\mathcal{M}^{\text{eq}})$.*

§ 5.2. HPKB-Hypergraphs and theorem on structure of type

To obtain an additional classification information on models of given theory T , we add to the hypergraph $(M^{\text{eq}}, H(M^{\text{eq}}))$ a *kernel-function*

$$\ker_{\mathcal{M}} : H(M^{\text{eq}}) \rightarrow \mathcal{P}(M^{\text{eq}}),$$

mapping by the following rule:

$$\ker_{\mathcal{M}}(M_a) \hat{=} \{b \in M_a \mid \mathcal{M}_b = \mathcal{M}_a\}.$$

A set $\ker_{\mathcal{M}}(M_a)$ is called a *kernel* a model \mathcal{M}_a (w.r.t. \mathcal{M}).

Proposition 5.2.1. *The kernel of any elementary submodel \mathcal{M}_a of M^{eq} consists of all elements $b \in M_a$ connected with a by formulas $\varphi_b(x, y)$ for which $\models \varphi_b(a, b)$ and formulas $\varphi_b(a, y)$ and $\varphi_b(x, b)$ are principal.*

P r o o f. If aforesaid formula $\varphi_b(x, y)$ doesn't exist, then $\text{tp}(a/b)$ is a nonprincipal type and the equality $\mathcal{M}_b = \mathcal{M}_a$ is impossible. If we have such a formula $\varphi_b(x, y)$, then take an arbitrary tuple \bar{c} of elements in \mathcal{M}_a . By definition of prime model, there exists a principal formula $\psi(a, y, \bar{z})$ for which $\models \psi(a, b, \bar{c})$. Then the formula $\varphi_b(x, b) \wedge \psi(x, b, \bar{z})$ is principal and isolates the type $\text{tp}(a \hat{\ } \bar{c}/b)$. Thus, \mathcal{M}_a realizes only principal types over b and $\mathcal{M}_a = \mathcal{M}_b$. \square

In view of Proposition 5.2.1, for any countable prime model \mathcal{M} over \emptyset , we have $\ker_{\mathcal{M}}(M^{\text{eq}}) = M^{\text{eq}}$. Then this equality is true for any countable model of countably categorical theory T and, moreover, by Proposition 5.1.1, the following Corollary holds.

Corollary 5.2.2. *A theory T is countably categorical iff, for any countable model $\mathcal{M} \models T$, the kernel of any elementary submodel \mathcal{M}_a of M^{eq} equals to M_a .*

Now, for a model M^{eq} , we consider all possible formulas $\varphi(x, y)$ such that $\vdash \varphi(x, y) \rightarrow \neg(x \approx y)$ and there exist elements $a \in M^{\text{eq}}$, for which $\varphi(a, y)$ are principal formulas. For any such a formula $\varphi(x, y)$, we define a binary relation $R_\varphi \hat{=} \{(a, b) \mid M^{\text{eq}} \models \varphi(a, b)\}$. If $(a, b) \in R_\varphi$, the pair (a, b) is called a φ -arc. If $\varphi(a, y)$ is a principal formula then φ -arc (a, b) is said to be *principal*. If, in addition, $\varphi(x, b)$ is principal, then the set $[a, b] \hat{=} \{(a, b), (b, a)\}$ is called

a *principal φ -edge*. φ -Arcs and φ -edges are called, respectively, *arcs* and *edges* if we say about fixed or some formula $\varphi(x, y)$. If (a, b) is a principal arc and (b, a) is not a principal arc then (a, b) is called *irreversible*.

A model $\vec{\mathcal{M}}^{\text{eq}}$ with an universe M^{eq} and all possible binary relations R_φ is called a *principal graph* of \mathcal{M}^{eq} .

Proposition 5.2.3. *Projections of all possible binary relations R_φ define all sets of realizations of 1-types of T^{eq} that realized in \mathcal{M}^{eq} .*

P r o o f. Let $p(x)$ be an arbitrary 1-type of T^{eq} that is realized in \mathcal{M}^{eq} . Since T is small, there exists a principal formula $\varphi_0(y)$ of T . Hence, there exists a formula $\psi(x, y)$ and a realization a of $p(x)$ in \mathcal{M}^{eq} such that $\psi(a, y)$ is a principal formula and $\psi(a, y) \vdash \varphi_0(y)$. Moreover, for $\psi(x, y)$, we can choose any formula $\chi(x, y)$ of form $\psi(x, y) \wedge \varphi(x)$, where $\varphi(x)$ is an arbitrary formula of $p(x)$. Therefore, every realization of set X of all aforesaid formulas $\exists y R_\psi(x, y)$ is a realization of $p(x)$. \square

Thus, besides the *principal* binary structure, the model $\vec{\mathcal{M}}^{\text{eq}}$ contains the structure of all 1-types that realized in \mathcal{M}^{eq} .

Recall [25], that an undigraph without loops is called *complete* if any two different vertices are connected by edge.

Proposition 5.2.1 implies that the kernel of any prime model \mathcal{M}_a with all possible principal edges forms a complete undigraph, and any vertex $b \in M_a$, not belonging to $\ker_{\mathcal{M}}(M_a)$, is a common end-point of all possible principal arcs (c, b) , where $c \in \ker_{\mathcal{M}}(M_a)$, and none of (b, c) is a principal arc. In particular, for any countable model \mathcal{M} of countably categorical theory, the kernel $\ker_{\mathcal{M}}(M^{\text{eq}})$ forms the complete graph on the set M^{eq} , and this property represents one more characterization of countable categoricity.

For a hypergraph $(M^{\text{eq}}, H(\mathcal{M}^{\text{eq}}))$ with a kernel-function $\ker_{\mathcal{M}}$, kernels $\ker_{\mathcal{M}}(M_a)$ of prime models \mathcal{M}_a over realizations a of a fixed 1-type $p(x)$ form maximal connected graphs on the set M^{eq} . These graphs correspond to connected components C of undigraphs with colored edges that composed by all possible principal edges. Restrictions of C to the set of realizations of the type $p(x)$ form graphs with colored edges, and having saturated model \mathcal{M} , these graphs are called *kernel-undigraphs over the type p* . Furthermore, by definition, automorphism groups of kernel-undigraphs are transitive and theories of these undigraphs are small.

In view of Compactness Theorem, a kernel-undigraph on a set of realizations of type $p(x)$ in a model \mathcal{M}^{eq} , correspondent to a saturated model \mathcal{M} , is unique iff the structure of the kernel-undigraph, restricted to $p(\mathcal{M}^{\text{eq}})$, has finitely many 2-types and $p(x)$ is principal. Otherwise, there are infinitely many such kernel-undigraphs. Since the automorphism group of structure of a type, obtained from a saturated model, is transitive, all kernel-undigraphs over $p(x)$ are pairwise isomorphic.

Among known kernel-undigraphs, we note the following, playing an essential role in constructions of Ehrenfeucht theories:

(a) complete α -element undigraphs, where kernel-undigraphs are obtained replacing each element, in a dense linear order without endpoints with a countable chain of constant, by an equivalence class containing α pairwise incomparable elements (see Section 1.4);

(b) acyclic undigraphs with countably many colors of edges and infinitely many edges of each color that are incident to every vertex (see Section 3.3);

(c) Hrushovski — Herwig undigraphs [81] (small stable undigraphs with infinite weight and coloring of edges by countably many colors) that used in a solution of the Lachlan problem (see Section 4.8);

(d) models of cubic theories [184].

Kernel-undigraphs in the examples (a), (b), and (d) are complete. Saturated Hrushovski — Herwig undigraphs form kernel-undigraphs with infinite diameters relative unions of all entered binary relations.

Now, we consider links between kernel- undigraphs realized by irreversible principal arcs. It is pertinent to retrace these links using the following object that include aforesaid components.

For a given model \mathcal{M} , a *hypergraph of prime models with a kernel-function and a principal binary structure* or a *HPKB-hypergraph* is a quadruple

$$\mathcal{H}(\mathcal{M}) \equiv \left(M^{\text{eq}}, H(\mathcal{M}^{\text{eq}}), \ker_{\mathcal{M}}, \Sigma \left(\vec{\mathcal{M}}^{\text{eq}} \right) \right),$$

where $\Sigma \left(\vec{\mathcal{M}}^{\text{eq}} \right)$ is the set of all binary relations in $\vec{\mathcal{M}}^{\text{eq}}$.

We are going to show, how the structures of HPKB-hypergraphs $\mathcal{H}(\mathcal{M})$ entirely, and irreversible principal arcs, in particular, reflect structural properties of given theory T .

At first, we notice that if a model \mathcal{M} is isomorphically embedded in a model \mathcal{N} then the HPKB-hypergraph $\mathcal{H}(\mathcal{M})$ is isomorphically embedded in the HPKB-hypergraph $\mathcal{H}(\mathcal{N})$. Therefore, in particular, if \mathcal{M} is saturated (homogeneous, universal) then $\mathcal{H}(\mathcal{M})$ has a special structure that also can be considered as *saturated (homogeneous, universal)* HPKB-hypergraph among all HPKB-hypergraphs of models of given theory. Semi-isolation relations on sets of realizations of types (and their properties) are also transformed to semi-isolation relations (and correspondent properties) on sets of realizations of 1-types.

We claim, that having irreversible arcs, connecting realizations of complete 1-type $p(x)$, the semi-isolation relation SI_p is nonsymmetric (in given structure as well as in correspondent HPKB-hypergraph).

Indeed, let (a, b) be a $\varphi(x, y)$ -principal arc and the pair (b, a) is not a principal arc, where a and b are realizations of the type p . Suppose on contrary, that SI_p is symmetric. Then the considered formula φ can be chosen so that $\varphi(x, b) \vdash p(x)$. Since the theory is small, some pair (b, a') is a principal arc, where $\models \varphi(a', b)$. Take a formula $\psi(x, y)$, for which $\models \psi(a', b)$ and $\psi(x, b)$ is principal, and an automorphism f mapping a' to a . Then f maps b to an element b' such that $\models \varphi(a, b') \wedge \psi(a, b')$. Since the formula $\varphi(a, y)$ principal, there exist an automorphism fixing a and mapping b' to b . Hence, $\models \varphi(a, b) \wedge \psi(a, b)$ and (b, a) is a principal arc. Obtained contradiction means, that SI_p is nonsymmetric and this property is guaranteed by any irreversible principal arc (a, b) with endpoints realizing $p(x)$. Furthermore, since, for any principal type, the semi-isolation relation on the set of its realizations is symmetric, a presence of irreversible principal arc (a, b) implies that the type $p(x)$ is not isolated.

Aforesaid arguments show that irreversible principal arcs on the set of realizations of the type p link different kernel-undigraphs over p . Since semi-isolation relations are transitive, these arcs define a relation of partial order \leq on the set of kernel-undigraphs over p , and each kernel-undigraph belongs to an infinite \leq -chain.

Thus, the following *p-decomposition theorem* holds.

Theorem 5.2.4. (1) *A structure of realizations of any complete 1-type $p(x)$ over \emptyset in a HPKB-hypergraph $\mathcal{H}(\mathcal{M})$ is composed by pairwise disjoint kernel-undigraphs over $p(x)$ and by a partial order \leq on the set of these kernel-undigraphs that defined by irreversible principal arcs.*

(2) A kernel-undigraph on the set of realizations of type $p(x)$ in the model \mathcal{M}^{eq} , correspondent to countable saturated model \mathcal{M} , is unique if the structure of kernel-undigraph, restricted to $p(\mathcal{M}^{\text{eq}})$, has finitely many 2-types and the type $p(x)$ is principal. Otherwise, there are infinitely many such kernel-undigraphs. All kernel-undigraphs over $p(x)$ are pairwise isomorphic.

(3) The partial order \leq is identical iff there are no irreversible principal arcs, connecting realizations of $p(x)$, and iff the semi-isolation relation SI_p is symmetric. If the partial order \leq is not identical then each kernel-undigraph belongs to an infinite \leq -chain.

Examples of aforesaid partial orders \leq w.r.t. irreversible principal arcs are extracted from the following known examples of structures with nonsymmetric semi-isolation relations:

(1) the Ehrenfeucht examples of structures on dense linear orders with countably many constants that ordered by increase;

(2) the Peretyat'kin examples of structures on infinite branched trees with countably many constants that ordered by increase [137], [138];

(3) the structure of free directed pseudoplane with 1-inessential ordered coloring (see Example 1.2.3);

(4) unstable and stable generic Ehrenfeucht structures, represented in Chapters 3 and 4.

§ 5.3. Graph connections between types

Now, we consider graph links between structures of different types.

Since, for any n -type that realized in \mathcal{M} , there exists a correspondent 1-type, realized in \mathcal{M}^{eq} by names of realizations of the n -type, the Rudin — Keisler preorder \leq_{RK} on the set of types realized in \mathcal{M} , is interpretable as the Rudin — Keisler preorder $\leq_{\text{RK}}^{\text{eq}}$ on the set of 1-types realized in \mathcal{M}^{eq} , and in turn, it is transformed in the structure of HPKB-hypergraph $\mathcal{H}(\mathcal{M})$.

In particular, it means, that a realizability in \mathcal{M} of a powerful type (or, that is equivalent, a presence in \mathcal{M} of greatest \leq_{RK} -class for the theory $\text{Th}(\mathcal{M})$) means that there exists a powerful 1-type realizable in $\mathcal{H}(\mathcal{M})$, i.e., there exists the greatest $(\leq_{\text{RK}}^{\text{eq}} \cap \geq_{\text{RK}}^{\text{eq}})$ -class for the set of 1-types realized in $(M^{\text{eq}}, \Sigma(\vec{\mathcal{M}}^{\text{eq}}))$, being the HPKB-hypergraph $\mathcal{H}(\mathcal{M})$ is universal.

Thus, the following Theorem and its immediate Corollary hold.

Theorem 5.3.1. *For any Rudin — Keisler preorder \leq_{RK} on the set of types of a small theory, there exists an universal HPKB-hypergraph $\mathcal{H}(\mathcal{M})$, for which a set with a preorder $\leq_{\text{RK}}^{\text{eq}}$, isomorphic to the set with the preorder \leq_{RK} , is interpretable on the set of all realizations of 1-types.*

Corollary 5.3.2. *A preorder \leq_{RK} of a theory T has a greatest $(\leq_{\text{RK}}^{\text{eq}} \cap \geq_{\text{RK}}^{\text{eq}})$ -class (a nonempty set of powerful types) iff there exists a greatest $(\leq_{\text{RK}}^{\text{eq}} \cap \geq_{\text{RK}}^{\text{eq}})$ -class for a set of 1-types, realized in $(M^{\text{eq}}, \Sigma(\vec{M}^{\text{eq}}))$, where \mathcal{M} is an universal model of T .*

Connections between kernel-undigraphs of 1-types p and q by the Rudin — Keisler preorder are realized by principal edges (where the types p and q are domination-equivalent and models \mathcal{M}_p and \mathcal{M}_q are isomorphic) or by irreversible principal arcs. If elements a and b of kernel-undigraphs Γ_p and Γ_q over p and q , respectively, are connected only by irreversible principal arcs (a, b) , then models \mathcal{M}_p and \mathcal{M}_q are non-isomorphic, and the domination-equivalence of the types p and q means that there exists a kernel-undigraph Γ'_p over p such that some element c in Γ'_p belongs to a principal arc (b, c) .

Hence, on the set of *principal connected components* C , i.e., of connected components formed by principal edges, we can introduce a partial order \leq defined by irreversible principal arcs. As for chains of kernel-undigraphs, \leq -chains of principal connected components in the model \mathcal{M}^{eq} , correspondent to saturated model \mathcal{M} , can be either singletons (for countably categorical theory) or infinite.

Indeed, in view of Compactness Theorem, if there are two principal connected components then there are infinitely many of them. On the other hand, any n -element set A , formed by elements in n different principal connected components, belongs to a prime model \mathcal{M}_A over A . Therefore, \mathcal{M}^{eq} contains an element c (a name for a tuple of all elements in A) that is a common initial vertex of n irreversible principal arcs with endpoints in A . Hence, any element in \mathcal{M}^{eq} is an endpoint of some irreversible principal arc, and, using these arcs, one can form an infinite \leq -chain of principal connected components. Moreover, an existence of aforesaid elements c implies that the set of principal connected components with the partial order \leq is downward directed.

Thus, similar to Theorem 5.2.4, the following *decomposition theorem* on structure, connecting kernel-undigraphs over different types, holds.

Theorem 5.3.3. (1) *The structure of set of realization of any complete 1-type $p(x)$ over \emptyset in HPKB-hypergraph $\mathcal{H}(\mathcal{M})$ is formed by pairwise disjoint principal connected components and by a partial order \leq on the set of principal connected components, that defined by irreversible principal arcs.*

(2) *A principal connected component in the model \mathcal{M}^{eq} , correspondent to countable saturated model \mathcal{M} , is unique if the structure \mathcal{M} is countably categorical. Otherwise, there are infinitely many principal connected components. All principal connected components, containing realizations of the same type, are pairwise isomorphic.*

(3) *The partial order \leq is identical iff there are no irreversible principal arcs, and iff the theory is countably categorical. If the partial order \leq is not identical then each principal connected component belongs to an infinite \leq -chain. The set of principal connected components with the partial order \leq is downward directed.*

§ 5.4. Limit models

A model \mathcal{M} is called *limit* if \mathcal{M} is not prime over tuples and $\mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_n$ for some elementary chain $(\mathcal{M}_n)_{n \in \omega}$ of prime models over tuples.

Using the proof of Proposition 1.1.7, we have

Proposition 5.4.1. *Any countable model \mathcal{M} of small theory T is represented as the union of an elementary chain $(\mathcal{M}_{\bar{a}_i})_{i \in \omega}$ of prime models over tuples \bar{a}_i .*

In view of Proposition 5.4.1 the following theorem holds.

Theorem 5.4.2. *Every countable model, being not prime over tuples, of small theory T is limit. Universes of countable models of T are represented in HPKB-hypergraphs $\mathcal{H}(\mathcal{M})$ by marked sets or by unions of countable chains of sequential properly included marked sets M_n , $n \in \omega$. Moreover, for the second case, elements of kernels $\ker_{\mathcal{M}}(M_n)$, $n \in \omega$, are connected with elements of $\ker_{\mathcal{M}}(M_k)$, $k < n$, by irreversible principal arcs.*

As noticed in Propositions 1.1.7 and 3.5.2, if a Rudin — Keisler preorder in $\text{RK}(T)$ is finite, then any countable model \mathcal{M} of T , being not prime over tuples, is limit over a type p .

In general situation, limit models can not be formed by prime models \mathcal{M}_n over realizations of the same type, i.e., kernels of models

\mathcal{M}_n , after transforming in \mathcal{M}^{eq} , can be only inside structures of realizations of types $p_{nk}(x)$, $k \in \omega$, such that $p_{n_1k_1} \neq p_{n_2k_2}$ for $n_1 \neq n_2$.

Limit models \mathcal{M} and \mathcal{N} are called *equivalent* (written $\mathcal{M} \sim \mathcal{N}$) if there exist elementary chains $(\mathcal{M}_n)_{n \in \omega}$ and $(\mathcal{N}_n)_{n \in \omega}$ of prime models over tuples, satisfying the following conditions:

$$(1) \mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_n, \mathcal{N} = \bigcup_{n \in \omega} \mathcal{N}_n;$$

(2) there exist constant expansions $\mathcal{M}'_{n+1} = \langle \mathcal{M}_{n+1}, c \rangle_{c \in M'_n}$ and $\mathcal{N}'_{n+1} = \langle \mathcal{N}_{n+1}, c \rangle_{c \in N'_n}$, $n \in \omega$, $\mathcal{M}'_0 = \mathcal{M}_0$, $\mathcal{N}'_0 = \mathcal{N}_0$, such that $\mathcal{M}'_{n+1} \simeq \mathcal{N}'_{n+1}$, $n \in \omega$.

The following Proposition is obvious.

Proposition 5.4.3. *If models \mathcal{M} and \mathcal{N} are limit, then $\mathcal{M} \simeq \mathcal{N}$ iff $\mathcal{M} \sim \mathcal{N}$.*

Now, consider hierarchies of limit models by degree of saturation, similar to hierarchies of prime models w.r.t. Rudin — Keisler preorders, allowing to estimate the number of pairwise non-isomorphic limit models.

Denote by $\mathbf{EEL}(T)$ the set \mathbf{LM} of isomorphism types of limit models of theory T with the relation \preceq of elementary embeddability.

Clearly, any nonempty set $\mathbf{EEL}(T)$ is preordered. Since the theory T is small, $\mathbf{EEL}(T)$ contains an element (the isomorphism type of countable saturated model), in which all isomorphism types of limit models are elementary embeddable. At the same time, it is possible that the structure $\mathbf{EEL}(T)$ doesn't have minimal $(\preceq \cap \succeq)$ -equivalence classes. Furthermore, generally speaking, such “greatest” element is not unique. Moreover, for instance, in known Ehrenfeucht theories, saturated models are elementary embeddable even in prime models over realizations of powerful types.

Thus, the number $I_l(T)$ of pairwise non-isomorphic limit models of small theory T has a lower bound $|\mathbf{EEL}(T)/(\preceq \cap \succeq)|$, i.e., the number of $(\preceq \cap \succeq)$ -equivalence classes. For some cases, the estimate

$$I_l(T) \geq |\mathbf{EEL}(T)/(\preceq \cap \succeq)| \tag{5.1}$$

is not improvable, for instance, when $I(T, \omega) = 3$. In some examples of Chapters 3 and 4, the inequality (5.1) is strict. The *problem* of characterization of small theories with $I_l(T) = |\mathbf{EEL}(T)/(\preceq \cap \succeq)|$ is still open.

Aforesaid arguments show, that the hierarchy of saturation of limit models, constructed on a base of set $EEL(T)$ with preorders of elementary embeddability is weakly connected with the number $I_l(T)$.

We introduce a refinement of $EEL(T)$ -hierarchy, for which the connections between isomorphism types of models are realized on the base of correspondence between sequences of principal connected components, and prime models over some elements of these components form given limit models.

We say that a limit model \mathcal{M} is *elementary embeddable in coordination* in a limit model \mathcal{N} if there exists an elementary embedding f of \mathcal{M} into \mathcal{N} mapping some tuples $\bar{a}_n \in M$ to correspondent tuples $f(\bar{a}_n) \in N$, $n \in \omega$, such that $\mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_{\bar{a}_n}$, $\mathcal{N} = \bigcup_{n \in \omega} \mathcal{N}_{f(\bar{a}_n)}$.

Denote by $CEEL(T)$ the set \mathbf{LM} of isomorphism types of limit models of theory T with the relation \preceq_c of elementary embeddability in coordination.

Similar to $EEL(T)$ -hierarchy, the $CEEL(T)$ -hierarchy allows to find a lower bound for the number $I_l(T)$:

$$I_l(T) \geq |CEEL(T)/(\preceq_c \cap_c \succeq)|. \quad (5.2)$$

The following example shows that, generally speaking, similar to mutual elementary embeddability, a presence of mutual elementary embeddability in coordination of models doesn't guarantee that models are isomorphic, i.e., the inequality (5.2) also can be strict.

Example 5.4.1. Take a countable model \mathcal{M} of connected acyclic digraph $\langle M; Q \rangle$, for which correspondent undigraph is a tree and any element has infinitely many images and infinitely many preimages (see Example 1.2.3). Expand this structure by 1-inessential coloring so that, on the set of realizations of the type $p_\infty(x)$, the semi-isolation relation is nonsymmetric, namely, if $\models p_\infty(a)$ and $\models Q(a, b)$ then a semi-isolates b by $Q(a, y)$, and b doesn't semi-isolate a . Now, we expand the structure by predicates $R_n^{(n+2)}$, $n \in \omega$, satisfying the following conditions:

- (1) $R_0 = Q$;
- (2) if $n \in \omega \setminus \{0\}$, $\models p_\infty(a_n)$, and $\models Q(a_{i+1}, a_i)$, $i = 0, \dots, n-1$, then

$$R_n(a_n, \dots, a_0, y) \vdash R_{n-1}(a_{n-1}, \dots, a_0, y)$$

and, for realizations of $p_\infty(x)$ in $Q(\mathcal{M}, a_{n-1})$, the formula $R_n(x, a_{n-1}, \dots, a_0, y)$ defines on the set $R_{n-1}(a_{n-1}, \dots, a_0, \mathcal{M})$ infinitely many equivalence classes such that each class is infinite.

Take a nonprincipal type $q(x)$ containing all formulas $R_n(a_n, \dots, a_0, x)$, where $(a_n)_{n \in \omega}$ is a sequence of realizations of p_∞ and $\models Q(a_{n+1}, a_n)$, $n \in \omega$. There exist mutually elementary embeddable in coordination, non-isomorphic limit models defined by the sequence $(a_n)_{n \in \omega}$ and such that the type $q(x)$ has different numbers of realizations. \square

A cause for mutually elementary embeddable in coordination limit models to be non-isomorphic, that is found on possibility of differently many realizations of types q (and, as corollary, on non-homogeneity of limit models with marked constants a_n), is, in essence, unique, since an absence of such types allows step-by-step to extend finite partial isomorphisms between mutually elementary embeddable in coordination limit models. A possibility of such extensions can be observed on examples of limit models of Ehrenfeucht theories in works [137], [138], [198], and of generic Ehrenfeucht theories in Chapters 3 and 4. Moreover, in generic examples, isomorphisms of limit models are constructed on a base of existence in these models of realizations of the same types over finite sets, guaranteed by special predicates $R_{\mathbf{A}}(x, \bar{y})$.

§ 5.5. λ -Model hypergraphs

In this Section, we define a link of HPKB-hypergraph structure $\mathcal{H}(\mathcal{M})$ for a countable saturated model \mathcal{M} of a theory T with the number $I(T, \omega)$ of pairwise non-isomorphic countable models of T . We also point out relations allowing to expand HPKB-hypergraphs such that the number of pairwise structural objects, defined by expanded HPKB-hypergraphs equals to $I(T, \omega)$.

Two complete 1-types $q_1(x)$ and $q_2(x)$ over \emptyset , that realized in a HPKB-hypergraph $\mathcal{H}(\mathcal{M})$, are called *p-equivalent* (written $q_1 \sim_p q_2$) if some principal connected component contains realizations of $q_1(x)$ and $q_2(x)$.

Denote by $I_p(T, \omega)$ the number of pairwise non-isomorphic countable models of theory T , each of which is prime over a tuple.

Proposition 5.5.1. *The number $I_p(T, \omega)$ coincides with the number of \sim_p -equivalence classes of T .*

P r o o f. Since the model \mathcal{M} is saturated, each prime model over a tuple \bar{a} is represented by some 1-type that is realized by the name $c_{\bar{a}}$ of \bar{a} in \mathcal{M}^{eq} . On the other hand, prime models, represented by \sim_p -equivalent types, are isomorphic, since having a route, consisting of principal edges and linking realizations of 1-types q_1 and q_2 , any model is prime over a realization of q_1 iff it is prime over a realization of q_2 . Thus, $I_p(T, \omega)$ equals to the number of pairwise \sim_p -non-equivalent complete 1-types over \emptyset that realized in $\mathcal{H}(\mathcal{M})$. \square

Since, in view of Theorem 5.4.2, each limit model of theory T is represented as a union of countable chain of structures of sequential properly included marked sets of HPKB-hypergraph $\mathcal{H}(\mathcal{M})$, a presence of isomorphism of limit models implies that correspondent limit structures in $\mathcal{H}(\mathcal{M})$ are isomorphic. Hence, by Proposition 5.4.1, the number $I(T, \omega)$ has a lower bound, being the sum of number of \sim_p -equivalence classes and of number $I_l(\mathcal{H}(\mathcal{M}))$ of pairwise non-isomorphic limit structures in $\mathcal{H}(\mathcal{M})$:

$$I(T, \omega) \geq I_p(T, \omega) + I_l(\mathcal{H}(\mathcal{M})). \quad (5.3)$$

This estimate is not improvable on graph level, since graph structures of \mathcal{M} are transformed in $\mathcal{H}(\mathcal{M})$.

Now, consider possibilities of expansions of HPKB-hypergraph $\mathcal{H}(\mathcal{M})$, allowing to obtain the equality in (5.3) under a substitution of expanded HPKB-hypergraph instead of $\mathcal{H}(\mathcal{M})$.

In view of Proposition 5.4.3, a presence of isomorphism of limit models \mathcal{M} and \mathcal{N} is witnessed by the possibility of *co-ordinated* extensions of prime models, represented in HPKB-hypergraphs $\mathcal{H}(\mathcal{M})$ and $\mathcal{H}(\mathcal{N})$. This coordination is obtained by introduction in HPKB-hypergraphs of correspondences between tuples of element and their names in \mathcal{M}^{eq} , as well as some structure of formula-definable binary relations, via which all formula-definable relations of given structure are formula-definable. Thus, we obtain the equality in (5.3) for expanded HPKB-hypergraphs.

Now, we show that language tools, used in Chapters 3 and 4 for constructions of realizations of basic characteristics of Ehrenfeucht theories, are minimal for interpretations of saturated models \mathcal{M}^{eq} by HPKB-hypergraphs, generating these characteristics.

Indeed, unary predicates $\text{Col}_n(x)$, $n \in \omega$, guarantee a presence of nonprincipal 1-types, that are realized in \mathcal{M}^{eq} . A binary predicate

$Q(x, y)$ (it is admissible to have at most countably many formula-independent family of predicates $Q_n(x, y)$, $n \in \omega$) forms a nonsymmetric semi-isolation relation on the set of realizations of powerful type $p_\infty(x)$, isolated by the set of formulas $\neg \text{Col}_n(x)$, $n \in \omega$. The family $\{R_j \mid j \in \omega\}$ of binary predicates form kernel-undigraphs on the set of realizations of $p_\infty(x)$, ordered by Q in accordance with Theorems 5.2.4 and 5.3.3, and predicates $R_{\mathbf{A}}(x, \bar{y})$ guarantee the corealization of the powerful type $p_\infty(x)$ with other powerful types.

Now, let T be an arbitrary theory with exactly three pairwise non-isomorphic models. Recall, that all nonprincipal types of T are powerful and, considering a model \mathcal{M}^{eq} instead of saturated model \mathcal{M} , we fix and denote by $p_\infty(x)$ a nonprincipal 1-type, realized in \mathcal{M}^{eq} . For a set of formulas $\varphi_n(x)$, $n \in \omega$, isolating $p_\infty(x)$, where $\{\varphi_n(x) \mid n \in \omega\} \vdash p_\infty(x)$, we put the formulas $\varphi_n(x) \wedge \neg \varphi_{n+1}(x)$ by $\text{Col}_n(x)$, and by $Q_n(x, y)$, $n \in \omega$, the countable family of formulas, each of which witnesses on non-symmetry of semi-isolation relation SI_{p_∞} and is principal after a substitution of any realization of p_∞ instead of first coordinate. Moreover, we require that, for any two realizations a and b of $p_\infty(x)$, there exists a realization c of $p_\infty(x)$ and formulas $Q_i(x, y)$, $Q_j(x, y)$ such that $\models Q_i(c, a) \wedge Q_j(c, b)$, and, thus, the local pairwise intersection property holds. For predicates R_j , $j \in \omega$, we take all possible binary formulas $\psi(x, y)$ consistent with $p_\infty(x) \cup p_\infty(y)$, for which all realizations on the structure of the type $p_\infty(x)$ are principal edges and form kernel-undigraphs. Finally, we take as $R_{\mathbf{A}}(x, \bar{y})$, for any nonprincipal 1-type $q(y)$ realized in \mathcal{M}^{eq} , all possible formulas $\chi_q(x, y)$ consistent with $p_\infty(x) \cup q(y)$, connecting realizations of $p_\infty(x)$ with realizations of $q(y)$ only by principal edges. Adding aforesaid predicates to the hypergraph $(M^{\text{eq}}, H(M^{\text{eq}}))$ as well as structure that guarantees the uniqueness of limit model, we obtain a *three-model hypergraph*.

In view of Corollary 1.1.15, the following theorem holds.

Theorem 5.5.2. *For any small theory T , the following conditions are equivalent:*

- (1) $I(T, \omega) = 3$;
- (2) T has a three-model hypergraph.

As known (see Theorem 1.1.13 and Figure 1.1(b)), there are three pairwise disjoint cases for theories with four countable models:

- (1) two-element Rudin — Keisler preorder and two limit models over powerful types;

(2) three-element Rudin — Keisler preorder with two maximal elements and the unique limit model;

(3) three-element chain, forming the Rudin — Keisler preorder, and the unique limit model.

Aforesaid situations also can be characterized in terms of hypergraphs and can be represented by *four-model hypergraphs*.

Continuing analysis of cases in accordance with Theorem 1.1.13 w.r.t. Rudin — Keisler preorder and distribution functions of number of limit models for theories having n countable models, we construct *n -model hypergraphs* as above, allowing to characterize theories T with $I(T, \omega) = n$.

The process of construction of hypergraphs, preserving given number of pairwise non-isomorphic countable models, can be continued for infinite cardinalities. It is achieved by similar introduction in hypergraphs $(M^{\text{eq}}, H(M^{\text{eq}}))$ of definable information on Rudin — Keisler preorders and on number of limit models. Thus, we have the notion of *λ -model hypergraph*, allowing to characterize theories T with $I(T, \omega) = \lambda$ as the following generalization of Theorem 5.5.2.

Theorem 5.5.3. *For any small theory T , the following conditions are equivalent:*

- (1) $I(T, \omega) = \lambda$;
- (2) T has a λ -model hypergraph.

§ 5.6. Distributions of prime and limit models

Recall that, in Chapter 3, a classification of elementary complete theories with finite Rudin — Keisler preorders is represented, and all possible Rudin — Keisler preorders in small theories are described. At present Subsection, we describe possibilities of distributions of numbers of limit (not necessary over fixed types) models of small theories and, thus, represent structural possibilities for numbers of countable models of small theories. Below all considered theories are supposed to be small.

We define analogues of countable categoricity and of Ehrenfeuchtness for prime models over tuples and limit models.

A theory T is called *p -categorical* (respectively, *l -categorical*, *p -Ehrenfeucht*, and *l -Ehrenfeucht*) if $I_p(T, \omega) = 1$ (respectively, $I_l(T) = 1$, $1 < I_p(T, \omega) < \omega$, $1 < I_l(T) < \omega$).

Clearly, that T is p -categorical iff T countably categorical, T is p -Ehrenfeucht iff the structure $\text{RK}(T)$ finite and has at least two elements, and T is p -Ehrenfeucht with $1 \leq I_l(T) < \omega$ iff T is Ehrenfeucht.

Notice, that just an absence of p -Ehrenfeuchtness allowed in the series of papers to prove an absence of Ehrenfeucht theories in the classes of superstable [109], 1-based [145], pseudo-superstable [197], supersimple [105] theories, as well as for theories without the strict order property and interpreting infinite sets of pairwise different constants [193].

Theorem 3.5.3 shows that, for p -Ehrenfeucht theories, the number of countable models is defined by the number of prime models over tuples and by the distribution function IL of numbers of limit models over types.

Consider the class of l -categorical theories. Clearly, all such theories are not p -categorical and the uniqueness of limit model implies that it is saturated. Therefore, a union of any elementary chain of prime models over tuples is again a prime model over a tuple or forms a limit saturated model. Hence, by definition of saturated model, the theory is l -categorical iff, for any elementary chain $(\mathcal{M}_{\bar{a}_i})_{i \in \omega}$ of prime models over tuples \bar{a}_i , for which the union is a limit model, and for any tuple $\bar{b} \in M_{\bar{a}_i}$, every type $q(x) \in S^1(\bar{b})$ is realized in some model $\mathcal{M}_{\bar{a}_j}$, $j \geq i$.

One more criterium of l -categoricity is based on the property that it is impossible to construct a limit model \mathcal{M} as a union of elementary chain of prime models \mathcal{M}_{q_n} over types q_n , where $q_n \leq_{\text{RK}} q_{n+1}$, $n \in \omega$, for which there exists a type q with $q_n \leq_{\text{RK}} q$ and $q \not\leq_{\text{RK}} q_n$, $n \in \omega$. An absence of \mathcal{M} is implied by that saturated models are limit and realize q , while q is omitted in \mathcal{M} . Hence, if such limit models don't exist then the l -categoricity is equivalent to the uniqueness up to isomorphism of non-prime countable model, in which all types of given theory are realized.

Thus, the following theorem, representing characterizations of l -categoricity, holds.

Theorem 5.6.1. *For any small, non- ω -categorical theory T , the following conditions are equivalent:*

- (1) T is l -categorical;
- (2) any limit model of T is saturated;
- (3) for any elementary chain $(\mathcal{M}_i)_{i \in \omega}$ of prime models over tuples with the union forming a limit model of T , and for any tuple $\bar{b} \in M_i$, every type $q(x) \in S^1(\bar{b})$ is realized in some model \mathcal{M}_j , $j \geq i$;
- (4) the union of any elementary chain of prime models \mathcal{M}_{q_n} over types q_n , $n \in \omega$, is a prime model over a type or the unique, up to isomorphism, limit model, in which all types of T are realized.

Notice, that all theories T with $I(T, \omega) = 3$ are l -categorical.

Also every theory is l -categorical if isomorphism types of countable models are defined by dimensions in $\omega \cup \{\infty\}$ and finite values correspond to prime models over tuples. In particular, in accordance with [33], any ω_1 -categorical non- ω -categorical theory is l -categorical. Since l -categorical theories don't have non-one-element \sim_{RK} -classes, not equal to \leq_{RK} -greatest \sim_{RK} -classes (see Corollary 1.1.10), the classes of theories with three countable models and of theories with dimensions represent two disjoint possibilities for Rudin — Keisler preorders in l -categorical theories:

(1) a finite or countable Rudin — Keisler preorder with bounded lengths of \leq_{RK} -chains of \sim_{RK} -classes such that all \sim_{RK} -classes, not equal to the greatest class, are one-element;

(2) a countable Rudin — Keisler preorder with unbounded lengths of \leq_{RK} -chains of \sim_{RK} -classes, where each class is one-element and each element $\mathbf{M} \in \text{RK}(T)$ is related by $\mathbf{M} \leq_{\text{RK}} \mathbf{M}'$ with some element \mathbf{M}' of every infinite \leq_{RK} -chain.

Notice also, that since all prime and countable saturated models are homogeneous, all l -categorical theories are *almost homogeneous*, i.e., each countable model, expanded by some finite set of constants, is homogeneous.

For further investigation of an arbitrary small theory T and the counting the number of its countable models (in particular, for an investigation of l -Ehrenfeuchtness), we suppose that the preordered set $\text{RK}(T)$ is countable, and, in view of Theorem 3.6.1, is upward directed and has the least element. Since the possible numbers λ of limit models over types are described in Theorem 3.5.3 ($\lambda \in \{\omega, \omega_1, 2^\omega\}$), we consider the possibilities for numbers of limit models that are not limit over types.

By definition, any limit model \mathcal{M} is a union of elementary chain $(\mathcal{M}_i)_{i \in \omega}$ of prime models \mathcal{M}_i over types q_i . Moreover, same types can repeat infinitely many times (that corresponds to the limit model over a type) or the chain $(\mathcal{M}_i)_{i \in \omega}$ contains an infinite subchain of pairwise non-isomorphic models \mathcal{M}_{i_j} that are prime over pairwise different types q_{i_j} , and this subchain can not be condensed to an elementary chain $(\mathcal{N}_k)_{k \in \omega}$ (where $\mathcal{M} = \bigcup_{k \in \omega} \mathcal{N}_k$) of prime models

over tuples such that the last chain contains infinitely many prime models over a fixed type.

Any \leq_{RK} -sequence, i.e., a sequence of nonprincipal types $(q_n)_{n \in \omega}$ with $q_n \leq_{\text{RK}} q_{n+1}$, $n \in \omega$, can define some number $\lambda((q_n)_{n \in \omega}) \in \{\omega, \omega_1, 2^\omega\}$ of limit models $\mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_{q_n}$, where $(\mathcal{M}_{q_n})_{n \in \omega}$ is an

elementary chain of prime models over realizations of types q_n . Furthermore, the dependence between numbers $\lambda((q_n)_{n \in \omega})$ for different \leq_{RK} -sequences $(q_n)_{n \in \omega}$ is conditioned by the following equivalence. Two \leq_{RK} -sequences $(q_n)_{n \in \omega}$ and $(q'_n)_{n \in \omega}$ are called *equivalent* $((q_n)_{n \in \omega} \sim (q'_n)_{n \in \omega})$ if there exists their common \leq_{RK} -subsequence $(q''_n)_{n \in \omega}$.

Notice, that, generally speaking, there are no direct dependencies between numbers $\lambda((q_n)_{n \in \omega})$ and $\lambda((q'_n)_{n \in \omega})$ for equivalent \leq_{RK} -sequences $(q_n)_{n \in \omega}$ and $(q'_n)_{n \in \omega}$, since limit models, defined by \leq_{RK} -sequence $(q_n)_{n \in \omega}$, can be not defined by \leq_{RK} -sequence $(q'_n)_{n \in \omega}$, and vice versa. Only the following relations hold:

(1) $\lambda((q_n)_{n \in \omega}) \leq \lambda((q'_n)_{n \in \omega})$, where $(q'_n)_{n \in \omega}$ is a \leq_{RK} -subsequence of $(q_n)_{n \in \omega}$;

(2) $\lambda((q_n)_{n \in \omega}) = \lambda((q'_n)_{n \in \omega})$, where, for $(q_n)_{n \in \omega}$ and $(q'_n)_{n \in \omega}$, there exist numbers k and m such that $\mathcal{M}_{q_{k+n}} \simeq \mathcal{M}_{q'_{m+n}}$, since some n .

Indeed, by definition, any limit model over \leq_{RK} -sequence $(q_n)_{n \in \omega}$ is limit over any its subsequence $(q'_n)_{n \in \omega}$, but not vice versa. Furthermore, if for sequences $(q_n)_{n \in \omega}$ and $(q'_n)_{n \in \omega}$, there exist numbers k and m such that $\mathcal{M}_{q_{k+n}} \simeq \mathcal{M}_{q'_{m+n}}$, since some n , then any limit model over $(q_n)_{n \in \omega}$ is limit over $(q'_n)_{n \in \omega}$ and vice versa, since any elementary chain $(\mathcal{M}_{\bar{a}_n})_{n \in \omega}$ of prime models over tuples can be extended to an elementary chain $(\mathcal{M}_{\bar{b}_n})_{n \in \omega}$, where $\mathcal{M}_{\bar{a}_n} = \mathcal{M}_{\bar{b}_{n+l}}$, $n \in \omega$, $\text{tp}(\bar{b}_i) \leq_{\text{RK}} \text{tp}(\bar{b}_{i+1})$, $i = 0, \dots, l-1$.

The first relation follows that the continuum of \leq_{RK} -sequences $(q_n)_{n \in \omega}$, each of which is not a proper subsequence of other ones and define a limit model, that is not limit over other sequences, implies continuum many pairwise non-isomorphic models.

The following concept allows to formulate a sufficient condition for continual number of limit models.

A theory T is called *l-rich* if there exist nonprincipal types q_n , $n \in \omega$, over which prime models \mathcal{M}_{q_n} are pairwise non-isomorphic and such that, for any sequence $(n_k)_{k \in \omega}$, there exists an elementary chain of models \mathcal{M}_{n_k} , isomorphic to $\mathcal{M}_{q_{n_k}}$, with the union forming a limit model.

Notice, that all types q_n , in the definition of *l-rich* theory, are RK-equivalent, and the correspondent \sim_{RK} -class in $\text{RK}(T)$ is countable.

Proposition 5.6.2. *If T is a rich theory then $I_l(T) = 2^\omega$.*

P r o o f. By definition of l -rich theory, it suffices to show, that there are continuum many pairwise non-equivalent \leq_{RK} -sequences, formed by types q_n . We partition the set of naturals, indexing types q_n , on two parts X_0 and X_1 , and again each of these parts on countably many parts $X_i^{j,k}$, $i = 0, 1, j, k \in \omega$. Now, considering sequences of types q_n with indexes from $X_{i_0}^{0,k_0}, X_{i_1}^{1,k_1}, \dots, X_{i_m}^{m,k_m}, \dots$ such that different sequences of q_n correspond to different sets $X_{i_m}^{m,k_m}$, we have that these sequences are pairwise disjoint and subindexes code all possible binary sequences. Thus, the number of non-equivalent \leq_{RK} -sequences of types q_n is continual. \square

Notice that, in view of Theorem 3.5.3 and Proposition 5.6.2, there are three cases of generation of continual number of limit models over \sim_{RK} -classes:

- (1) continuum of limit models over some type in given \sim_{RK} -class;
- (2) continuum of limit models over some \leq_{RK} -sequence of pairwise different types in given \sim_{RK} -class;
- (3) continuum of limit models over pairwise different \leq_{RK} -sequences of pairwise different types in given \sim_{RK} -class.

In view of Morley Theorem [127], numbers of limit models over given \leq_{RK} -sequences of nonprincipal types \mathbf{q} (consisting of same or pairwise different types) can be varied in the set $\omega \cup \{\omega, \omega_1, 2^\omega\}$. Thus, the number $I_l(T)$ is represented as the sum $\sum_{\mathbf{q}} \text{IL}_{\mathbf{q}}$, where

$\text{IL}_{\mathbf{q}}$ is the number of limit models related to \leq_{RK} -sequence \mathbf{q} and not related to restrictions of \mathbf{q} , that used in count of $I_l(T)$.

Similar Corollary 1.1.10 using the proof of Proposition 1.1.8, we have the following:

Lemma 5.6.3. *If \mathbf{q} is a \leq_{RK} -sequence of types $q_n, n \in \omega$, and $(\mathcal{M}_{q_n})_{n \in \omega}$ is an elementary chain such that any co-finite subchain doesn't consist of pairwise isomorphic models, then $\text{IL}_{\mathbf{q}} \geq 1$.*

We say that a family \mathbf{Q} of \leq_{RK} -sequences \mathbf{q} of types *represents* a \leq_{RK} -sequence \mathbf{q}' of types if any limit model over \mathbf{q}' is limit over some $\mathbf{q} \in \mathbf{Q}$.

In view of aforesaid, we have the following Theorem, that generalize Theorem 3.5.3 for theories with arbitrary Rudin — Keisler pre-orders and describes all possible distributions of numbers of count-

able models of theory in dependence on given sequences of nonprincipal types.

Theorem 5.6.4. *Any small theory T satisfies the following conditions:*

- (a) *the structure $\text{RK}(T)$ is upward directed and has the least element \mathbf{M}_0 (the isomorphism type of prime model of T), $\text{IL}(\widetilde{\mathbf{M}}_0) = 0$;*
- (b) *if \mathbf{q} is a \leq_{RK} -sequence of nonprincipal types q_n , $n \in \omega$, such that each type q of T is related by $q \leq_{\text{RK}} q_n$ for some n , then there exists a limit model over \mathbf{q} ; in particular, $I_l(T) \geq 1$ and the countable saturated model is limit over \mathbf{q} , if \mathbf{q} exists;*
- (c) *if \mathbf{q} is a \leq_{RK} -sequence of types q_n , $n \in \omega$, and $(\mathcal{M}_{q_n})_{n \in \omega}$ is an elementary chain such that any co-finite subchain doesn't consist of pairwise isomorphic models, then there exists a limit model over \mathbf{q} ;*
- (d) *if $\mathbf{q}' = (q'_n)_{n \in \omega}$ is a subsequence of \leq_{RK} -sequence \mathbf{q} , then any limit model over \mathbf{q} is limit over \mathbf{q}' ;*
- (e) *if $\mathbf{q} = (q_n)_{n \in \omega}$ and $\mathbf{q}' = (q'_n)_{n \in \omega}$ are \leq_{RK} -sequences of types such that for some $k, m \in \omega$, since some n , any types q_{k+n} and q'_{m+n} are related by $\mathcal{M}_{q_{k+n}} \simeq \mathcal{M}_{q'_{m+n}}$, then any model \mathcal{M} is limit over \mathbf{q} iff \mathcal{M} is limit over \mathbf{q}' .*

Moreover the following decomposition formula holds:

$$I(T, \omega) = |\text{RK}(T)| + \sum_{\mathbf{q} \in \mathbf{Q}} \text{IL}_{\mathbf{q}},$$

where $\text{IL}_{\mathbf{q}} \in \omega \cup \{\omega, \omega_1, 2^\omega\}$ is the number of limit models related to the \leq_{RK} -sequence \mathbf{q} and not related to extensions and to restrictions of \mathbf{q} that used for the counting of all limit models, and the family \mathbf{Q} of \leq_{RK} -sequences of types represents all \leq_{RK} -sequences, over which limit models exist.

In view of Theorem 5.6.4, finite values $\sum_{\mathbf{q} \in \mathbf{Q}} \text{IL}_{\mathbf{q}}$ characterize the class of l -Ehrenfeucht theories.

By Theorem 5.6.4, similar to theories with finite Rudin — Keisler preorders, small theories are classified by Rudin — Keisler preorders and distribution functions of numbers of limit (not necessary over fixed types) models.

Recall, that Theorem 3.5.1 represents all possible realizations of theories with finite Rudin — Keisler preorders w.r.t. these preorders and distribution functions of numbers of limit models, where values of functions belong to $\omega \cup \{\omega, 2^\omega\}$.

The following Theorem generalizes Theorem 3.5.1 for arbitrary Rudin — Keisler preorders and correspondent distribution functions of numbers of limit models, where values of functions belong to $\omega \cup \{\omega, 2^\omega\}$.

Theorem 5.6.5. *Let $\langle X, \leq \rangle$ be a finite or countable upward directed preordered set with a least element x_0 , $f : Y \rightarrow \omega \cup \{\omega, 2^\omega\}$ be a function with the set Y of all \leq_0 -sequences, i.e. of sequences in $X \setminus \{x_0\}$, forming all \leq -chains, and satisfying the following conditions:*

- (a) $f(y) \geq 1$ if for any $x \in X$ there exists some x' in the sequence y such that $x \leq x'$;
- (b) $f(y) \geq 1$ if any co-finite subsequence of y doesn't contain of pairwise equal elements;
- (c) $f(y) \leq f(y')$ if y' is a subsequence of y ;
- (d) $f(y) = f(y')$ if $y = (y_n)_{n \in \omega}$ and $y' = (y'_n)_{n \in \omega}$ are sequences such that there exist some $k, m \in \omega$ for which $y_{k+n} = y'_{m+n}$ since some n .

Then there exists a small theory T and an isomorphism

$$g : \langle X, \leq \rangle \xrightarrow{\sim} \text{RK}(T)$$

such that any value $f(y)$ is equal to the number of limit models over \leq_{RK} -sequence $(q_n)_{n \in \omega}$, correspondent the \leq_0 -sequence $y = (y_n)_{n \in \omega}$, where $g(y_n)$ is the isomorphism type of the model \mathcal{M}_{q_n} , $n \in \omega$.

P r o o f consists of a modification of generic constructions in the proofs of Theorems 3.4.1 and 3.6.1 satisfying the following conditions:

- (1) any element $x \in X \setminus \{x_0\}$ is represented by an 1-type q_x that defined by the infinite color and unary predicate P_x ;
- (2) every prime model over a tuple is prime or isomorphic to a model \mathcal{M}_{q_x} ;
- (3) given \leq_{RK} -interconnection between types is formed by a countable family of binary predicates;
- (4) given distribution of numbers of limit models is defined by sequences of binary predicates, identified via calculus of identities (see Section 3.5) using corealization amalgams.

Since numbers of limit models may vary depending on sequences of pairwise different types $\mathbf{q} = (q_n)_{n \in \omega}$, considering identities in Section 3.5 to obtain given number of limit models over \mathbf{q} , we have to use only identities connecting words of same lengths.

To obtain n countable models, it suffices to use the following system of identities:

$$n - 1 \approx m,$$

$m \geq n$, and

$$n_0 n_1 \dots n_s \approx \underbrace{n_s \dots n_s}_{s+1 \text{ times}},$$

$\max\{n_0, n_1, \dots, n_{s-1}\} < n_s$, reducing all sequences in ω^ω to n constant sequences.

Considering the system

$$n_0 n_1 \dots n_s \approx \underbrace{n_s \dots n_s}_{s+1 \text{ times}},$$

$\max\{n_0, n_1, \dots, n_{s-1}\} < n_s$,

$$n_0 n_1 \dots n_s \approx n_0(n_0 + 1) \dots (n_0 + s),$$

$n_0 + s \leq n_s$,

$$n_0 n_1 \dots n_s \approx n_0(n_0 + 1) \dots (n_0 + t) \underbrace{(n_0 + t) \dots (n_0 + t)}_{s-t \text{ times}},$$

$n_0 + t = n_s$, $t > 0$, $s > t$, of identities, we get a theory with ω countable models over \mathbf{q} , where all sequences in ω^ω are reducible to constant or diagonal sequences. \square

The constructions of Chapter 4 allow to create realizations for the proof of Theorem 5.6.5 in the class of stable theories.

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