(Formal) Frobenius manifolds

$\eta = (\eta_{\alpha\beta})$ - symmetric, nondegenerate.

$\eta^{-1} = (\eta^{\alpha\beta})$

starts from cubic terms.

$F(t^1, \dots, t^N) \in \mathbb{C}[[t^1, \dots, t^N]]$.

$$\frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}$$

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^{\mu\nu} \frac{\partial^3 F}{\partial t^\nu \partial t^\gamma \partial t^\delta} = \beta \longleftrightarrow \gamma$$

$1 \le \alpha, \beta, \gamma, \delta \le N$

conformal dim.

$$\sum_{\alpha=1}^{N} ((1-q_\alpha) t^\alpha + r^\alpha) \frac{\partial F}{\partial t^\alpha} = (3-\gamma) F + (\text{quadr. terms})$$

$q_\alpha, r^\alpha \in \mathbb{C}$ $\quad q_1 = 0$.

Principal hierarchy

Differ. polynomials.

$v^1, \dots, v^N, v^1_x, \dots, v^N_x, \dots$

$v^\alpha = v^\alpha_0 \quad v^\alpha_x = v^\alpha_1 \quad v^\alpha_{xx} = v^\alpha_2 \dots$

$\mathcal{A}_{v^1, \dots, v^N} = \mathbb{C}[[v^\alpha]][v^\alpha_x, v^\alpha_{xx}, \dots]$

ε - formal variable.

$\hat{\mathcal{A}}_{v^1, \dots, v^N} = \mathcal{A}_{v^1, \dots, v^N}[[\varepsilon]]$

$\deg v^\alpha_d = d \quad \deg \varepsilon = -1$

$\hat{\mathcal{A}}^{[d]}_{v^1, \dots, v^N} = \{ f \in \hat{\mathcal{A}}_{v^1, \dots, v^N} \mid \deg f = d \}$

$$p^{[0]\alpha}_{\beta,b}(v^1,\dots,v^N) \in \mathbb{C}[[v^1,\dots,v^N]]$$

$$\begin{cases} p^{[0]\alpha}_{\beta,0} = \eta^{\alpha\mu} \dfrac{\partial^2 F}{\partial t^\mu \partial t^\beta}\Big|_{t^\gamma = v^\gamma} \\ \dfrac{\partial p^{[0]\alpha}_{\beta,d}}{\partial v^\gamma} = p^{[0]\mu}_{\beta,d-1} \dfrac{\partial p^{[0]\alpha}_{\mu,0}}{\partial v^\gamma}, \quad d \geq 1. \\ p^{[0]\alpha}_{\beta,d}(0) = 0 \end{cases}$$

Principal hierarchy

$$\frac{\partial v^\alpha}{\partial t^\beta_d} = \partial_x p^{[0]\alpha}_{\beta,d} = \frac{\partial p^{[0]\alpha}_{\beta,d}}{\partial v^\gamma} v^\gamma_x$$

Properties

the flows commute

Bihamiltonian str. $\quad \Theta^\alpha_{\beta,\gamma} := \eta^{\alpha\mu} \dfrac{\partial^3 F}{\partial t^\mu \partial t^\beta \partial t^\gamma}\Big|_{t=0} \in \mathbb{C}$

Tau-symmetry

Virasoro symmetries

In particular;

$$\begin{cases} \dfrac{\partial p^{[0]\alpha}_{\beta,d}}{\partial v^1} = p^{[0]\alpha}_{\beta,d-1}, \quad d \geq 0, \quad p^{[0]\alpha}_{\beta,-1} := \delta^\alpha_\beta \\ \sum \left((1-q_\alpha) v^\alpha + r^\alpha\right) \dfrac{\partial}{\partial v^\alpha} p^{[0]\alpha}_{\beta,d} = \\ \quad = (1+d-q_\alpha+q_\beta) p^{[0]\alpha}_{\beta,d} + r^\gamma \Theta^\mu_{\gamma\beta} p^{[0]\alpha}_{\mu,d-1} \end{cases}$$

Example

$N=1 \quad \eta_{1,1}=1 \quad F(t^1)=\frac{(t^1)^3}{6}$

$$P^{[0]1}_{1,d}=\frac{(v^1)^{d+1}}{(d+1)!}\overset{v^1=v}{=}\frac{v^{d+1}}{(d+1)!}$$

principal hierarchy

$$\frac{\partial v}{\partial t_d}=\frac{v^d}{d!}v_x \quad , d\geq 0.$$

Dispersionless KdV hierarchy.

Dispersive deformations, preserving the structures.

$w^1,\dots,w^N$

$$\frac{\partial w^\alpha}{\partial t^\beta_d}=\partial_x P^\alpha_{\beta,d}\ , \quad 1\leq\alpha,\beta\leq N, \quad d\geq 0.$$

$$P^\alpha_{\beta,d}\in\mathcal{A}^{[0]}_{w^1,\dots,w^N}$$

$$P^\alpha_{\beta,d}\Big|_{\varepsilon=0}=P^{[0]\alpha}_{\beta,d}\Big|_{v^\gamma=w^\gamma}$$

Example

For the previous example such a deformation is the full KdV hierarchy

$$\begin{cases} \frac{\partial w}{\partial t_0} = w_x \\ \frac{\partial w}{\partial t_1} = w w_x + \frac{\varepsilon^2}{12} w_{xxx} \\ \vdots \end{cases}$$

$\vdots$

open problem

$\Theta^{\alpha}_{g\gamma}$ define an algebra without nilpotents

Theorem (Dubrovin, Zhang)

For a semisimple F, if such a deformation exists, then it is unique. Moreover, it comes from a cohomological field theory.

$$\overline{\mathcal{M}}_{g,n} = \left\{ (C; p_1, \ldots, p_n) \;\middle|\; \begin{array}{l} C\text{-alg. curve with at most nodal sing.} \\ g(C) = g \\ p_1, \ldots, p_n \in C^{sm} \\ p_i \neq p_j \\ |\mathrm{Aut}(C; p_1, \ldots, p_n)| < \infty \end{array} \right\} / \text{isomorphism}$$

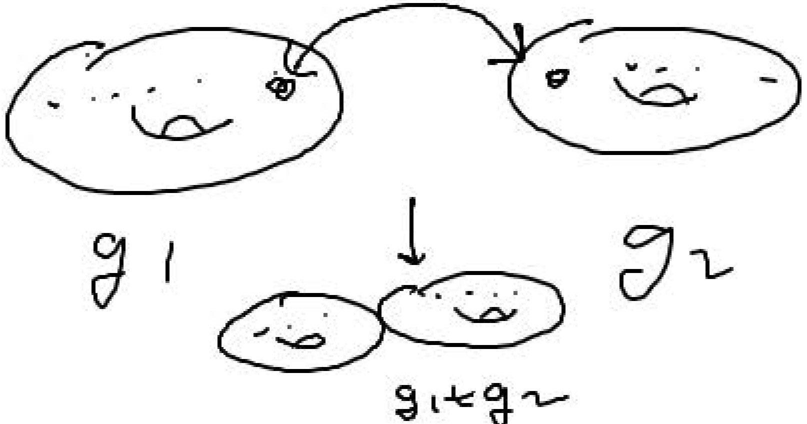
locally $\{xy=0\} \subset \mathbb{C}^2$

$\overline{\mathcal{M}}_{g,n}$ – complex orbifold, $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g-3+n$

Natural maps between the moduli spaces

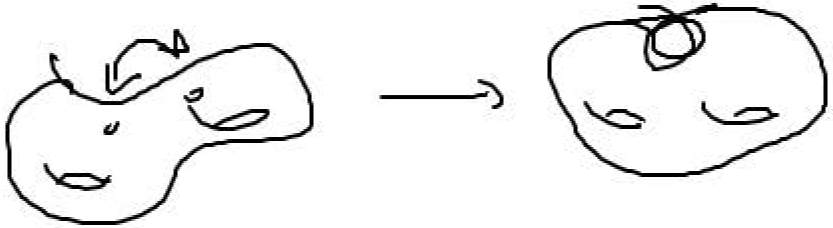
$\pi: \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$

glue two marked points



$gl: \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$

Similar map associated to



Cohomological field theory (CohFT)

Definition

V, $\dim V = N$

$\parallel$

$\langle e_1, \dots, e_N \rangle$

$\eta = (\eta_{\alpha\beta})$ - sym., nondeg

$c_{g,n} : V^{\otimes n} \to H^{even}(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$ -

- linear maps, $g, n \geq 0$, satisfying

1) $c_{g,n}$ is S_n-equivariant.

2) $\overline{\mathcal{M}}_{0,3} = pt$ $\quad c_{0,3}(\underset{\text{special}}{e_1} \otimes e_\alpha \otimes e_\beta) = \eta_{\alpha\beta} \in \mathbb{C} = H^*(\overline{\mathcal{M}}_{0,3})$

Homogeneity

$\deg c_{g,n}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}) + \pi_* c_{g,n+1}(\otimes e_{\alpha_i} \otimes r^\alpha e_\alpha) =$

$= \left(\sum q_{\alpha_i} + (g-1)\gamma\right) c_{g,n}(\otimes e_{\alpha_i})$ for some $q_\alpha, r^\alpha, \gamma$

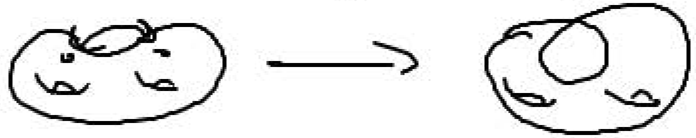
3) a) $gl : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$

$$gl^*\left(c_{g_1+g_2,n_1+n_2}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_{n_1+n_2}})\right) =$$

$$c_{g_1,n_1+1}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_{n_1}} \otimes e_\mu)\, \eta^{\mu\nu}\, c_{g_2,n_2+1}(e_\nu \otimes e_{\alpha_{n_1+1}} \otimes \cdots \otimes e_{\alpha_n})$$

$n = n_1 + n_2$

b) Similar property for



4) $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$

$\pi^* c_{g,n}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}) = c_{g,n+1}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n} \otimes e_1)$

CohFT $\longrightarrow$ system of PDEs.

t^α_d, $1 \le \alpha \le N$, $d \geq 0$, $t^\alpha_0 = t^\alpha$

<u>$\mathcal{F}(t^*_*, \varepsilon)$</u> $:= \sum_{g,n \geq 0} \frac{\varepsilon^{2g}}{n!} \left(\sum_{\substack{\alpha_1, \dots, \alpha_n \\ d_1, \dots, d_n}} \int_{\overline{\mathcal{M}}_{g,n}} c_{g,n}(e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n}) \psi_1^{d_1} \cdots \psi_n^{d_n} \right) t^{\alpha_1}_{d_1} \cdots t^{\alpha_n}_{d_n} \in \mathbb{C}[[t^*_*, \varepsilon]]$

$\psi_1, \dots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n})$

<u>Fact:</u> $F(t^1, \dots, t^N) := \mathcal{F}\Big|_{\substack{\varepsilon = 0 \\ t^*_{\geq 1} = 0}}$ gives a formal Frobenius manifold

<u>$(w^\alpha)^{top}(t^*_*, \varepsilon)$</u> $:= \eta^{\alpha\mu} \frac{\partial^2 \mathcal{F}}{\partial t^1 \partial t^\mu} \in \mathbb{C}[[t^*_*, \varepsilon]]$ $(w^\alpha)^{top}_n = \left(\frac{\partial}{\partial t^1}\right)^n (w^\alpha)^{top}$

Theorem (B.-Posthuma-Shadrin)

Assume F is semisimple.

(1) Then there exist unique

$P^{\alpha}_{\beta,d} \in \hat{\mathcal{A}}^{[0]}_{w^1,\dots,w^N}$

such that

$$\eta^{\alpha\mu}\frac{\partial^2 \mathcal{F}}{\partial t^{\mu}_0 \partial t^{\beta}_d} = P^{\alpha}_{\beta,d} \Big|_{w^{\gamma}_n = (w^{top})^{\gamma}_n}$$

(2) $P^{\alpha}_{\beta,d}$ give a deformation of the principal hierarchy satisfying

- flows commute
- tau-symmetry
- Hamiltonian
- Virasoro symmetries
- Bihamiltonian???

In particular

$$\frac{\partial P^{\alpha}_{\beta,d}}{\partial w^1} = P^{\alpha}_{\beta,d-1}, \quad d \geq 0$$

$$\left(\sum(1-q_\alpha)w_n^\alpha\frac{\partial}{\partial w_n^\alpha}+r^\alpha\frac{\partial}{\partial w^\alpha}+\underbrace{\frac{1-r}{2}\varepsilon\frac{\partial}{\partial\varepsilon}}\right)p^\alpha_{\beta,d}$$

$$=(1+d-q_\alpha+q_\beta)p^\alpha_{\beta,d}+r^\gamma\Theta^\mu_{\gamma\beta}p^\alpha_{\mu,\ell-1}$$

Flat F-manifold

$F^\alpha(t^1,\dots,t^N)$, $1\le\alpha\le N$

$$\begin{cases}\dfrac{\partial^2F^\alpha}{\partial t^1\partial t^\beta}=\delta^\alpha_\beta\\[2ex]\dfrac{\partial^2F^\alpha}{\partial t^\beta\partial t^\mu}\dfrac{\partial^2F^\mu}{\partial t^\gamma\partial t^\delta}=\beta\leftrightarrow\gamma\end{cases}$$

For a Frob. md.

$F^\alpha=\eta^{\alpha\mu}\dfrac{\partial F}{\partial t^\mu}$

$$\sum((1-q_\alpha)t^\alpha+r^\alpha)\frac{\partial F^\beta}{\partial t^\alpha}=(2-q^\beta)F^\beta+\begin{pmatrix}\text{linear}\\\text{terms}\end{pmatrix}$$

no conformal dimension

The construction of the principal hierarchy is the same.

$$\frac{\partial v^\alpha}{\partial t^\beta_d}=\partial_x p^{(0)}_{\beta,d}$$

Dispersive deformations

$$\frac{\partial w^\alpha}{\partial t^\beta_d} = \partial_x P^\alpha_{\beta,d} \mp \partial_x \left(P^{(\omega)\alpha}_{\beta,d}\Big|_{v=w} + \varepsilon(\ldots) + \ldots \right)$$

satisfying

$$\frac{\partial P^\alpha_{\beta,d}}{\partial w^1} = P^\alpha_{\beta,d-1}$$

$$\left(\sum (1-q_\mu) w^\mu_n \frac{\partial}{\partial w^\mu_n} + r^\mu \frac{\partial}{\partial w^\mu} + \tilde{\delta}\, \varepsilon \frac{\partial}{\partial \varepsilon}\right) P^\alpha_{\beta,d} =$$

$$= (1 + d - q_\alpha + q_\beta) P^\alpha_{\beta,d} + r^\mu \Theta^\nu_{\mu\beta} P^\alpha_{\nu,\beta-1}$$

for some $\tilde{\delta}$.

Arsie-Lorenzoni:

$$\begin{cases} F^1(t^1,t^2) = \dfrac{(t^1)^2}{2} - \dfrac{(c+1)(2c+1)}{6}(t^2)^4 \\ F^2(t^1,t^2) = t^1 t^2 - \dfrac{2}{3}\left(\dfrac{3}{4} + c\right)(t^2)^3 \end{cases}$$

$q_1 = 0 \quad q_2 = \frac{1}{2} \quad r^1 = r^2 = 0$

$c \in \mathbb{C}$

1) there are nontrivial dispersive deformations
2) If a deformation is non trivia

2) If a deformation is nontrivial, then

$$\tilde{\delta} \in \left\{ \frac{-2c-1}{2}, \frac{2c+2}{2} \right\}.$$

Questions

1) How to construct a deformation?

2) How to explain the discrete choice of $\tilde{\delta}$?

Example

$$\frac{\partial}{\partial t^1_0} = \partial_x$$

principal hierarchy

$$\begin{cases} \dfrac{\partial v^1}{\partial t^2_0} = \partial_x \left(-\dfrac{(c+1)(2c+1)2}{3} (v^2)^3 \right) \\ \dfrac{\partial v^2}{\partial t^2_0} = \partial_x \left(v^1 - 2\left(\frac{3}{4}+c\right)(v^2)^2 \right) \end{cases}$$

Joint work with Arsie, Lorenzoni, Rossi:

1) For semisimple flat F-manifolds we constructed deformations

2) We explained how to determine the choice of $\tilde{\delta}$ from

the geometry of the
flat F-manifold: there is
a tuple of conformal dimensions
hiden in the flat F-manifold.
(constructed by Arsie-Lorenzoni)