Stokes phenomenon, reflection equations and Frobenius manifolds

Xiaomeng Xu
Peking University

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Outline

- Linear systems of ODEs with singularities, and their Stokes matrices.
- ($\hbar$-deformation) Universal solutions of Yang-Baxter and reflection equations (quantum groups and quantum symmetric pairs).
- A deformation of Frobenius manifolds.
- (ε-deformation) WKB approximation and canonical bases.

Solving differential equations via formal power series

Every thing is over $\mathbb{C}$.

Example

Consider the differential equation $z^2 f'(z) = f(z) - z$. *It has a formal solution* $f(z) = \sum_{n\geq 0} n! z^{n+1}$.

Idea: insert $\hat{f}(z) = \sum_{n\geq 0} f_n z^n$ into the equation, and compute f_n by comparing coefficients.

Problem: determine the radius of convergence of $\hat{f}$. Very often, one gets a divergent series.

Resummation of formal power series

A procedure to obtain a finite result from a divergent sum.

- Borel resummation (best known example):

Suppose $\hat{f}(z) = \sum f_k z^k$, with $|f_k| \leq C^k k!$. Formally we have

$$\hat{f} = \sum_{k=0}^{\infty} f_k z^k = \sum_{k=0}^{\infty} f_k \left(\int_0^{\infty(d)} e^{-t} t^k dt \right) \frac{z^k}{k!} = \int_0^{\infty(d)} e^{-t} \sum_{k=0}^{\infty} f_k \frac{(tz)^k}{k!} dt.$$

Example

Suppose $f(z) = \sum_{k \geq 0} k! z^k$:

- $\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$ *(analytically continued to* $t \leq 0$*).*
- *the resummation is* $\int_0^{-\infty} \frac{e^{-t}}{1-tz} dt = \frac{-1}{z} \cdot e^{\frac{-1}{z}} \cdot \Gamma\left(0, \frac{-1}{z}\right)$.

ODEs with second order poles

Consider the linear system on z-plane

$$\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{V}{z} \right) F,$$

where $F(z) \in \mathrm{gl}_n$, $u = \mathrm{diag}(u^1, ..., u^n)$ and $V \in \mathrm{gl}_n(\mathbb{C})$.

Unique formal fundamental solution:

$$\hat{F}(z) = \hat{H}(z) e^{-\frac{u}{z}} z^{[V]},$$

where $\hat{H}(z) = \mathrm{Id}_n + H_1 z + \cdots$ is a formal power series of matrices.

Problem: $|H_k| \sim k!$, the radius of convergence of $\hat{H}(z)$ is in general zero.

Canonical solutions

- Borel resummation (along a direction d):

$$\mathbb{BS}_d(\hat{H}) = \frac{1}{z}\int_0^{\infty(d)} e^{-\frac{t}{z}}\Big(\sum_{k\geq 0}\frac{H_k}{k!}t^k\Big)dt.$$

- Singular/Stokes directions $d = \arg(u_i - u_j)$.
- Stokes sectors are bounded by adjacent d's.

Theorem

In each Stokes sector R,

$$F_R(z) := \mathbb{BS}_R(\hat{H})(z)e^{-\frac{u}{z}}z^{[V]}$$

is the unique (therefore canonical) holomorphic solution with the asymptotics $F_R(z) \sim \hat{F}(z)$ at $z = 0$ within R.

- $F_R(z)e^{\frac{u}{z}}z^{-[V]} \sim \mathrm{Id}_n$ *at* $z = 0$ *within* R.

Stokes matrices

Take two opposite sectors $R_\pm$, and corresponding solutions $F_\pm$.

Definition

The **Stokes matrices** $S_\pm$ *of* $\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{V}{z}\right) F$ *are given by*

$$F_-(z) = F_+(z) \cdot S_+ \text{ in } R_-, \quad F_+(z) = F_-(z) \cdot S_- \text{ in } R_+.$$

Remark

1. The matrices measure the Stokes phenomenon (jump phenomenon of solutions) of the equation.
2. The Stokes matrices are complete invariants of the system (Riemann-Hilbert-Birkhoff correspondence).

Example: 2 by 2

We consider

$$\frac{dF}{dz} = \frac{1}{z^2}\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} t_1 & b_2 \\ b_1 & t_2 \end{pmatrix} F.$$

- ${}_1F_1(\alpha;\beta;z) := \sum_{n=0}^{\infty} \frac{\alpha^{(n)} z^n}{\beta^{(n)} n!}$, $\alpha^{(n)} := \alpha \cdots (\alpha + n - 1)$,

$$\begin{pmatrix} b_2 e^{u_1 z} \xi^{t_1+\beta} {}_1F_1(\beta, \beta-\alpha+1; -\xi) & b_2 e^{u_1 z} \xi^{t_1+\alpha} {}_1F_1(\alpha, \alpha-\beta+1; -\xi) \\ \beta e^{u_2 z} \xi^{t_1+\beta} {}_1F_1(-\alpha, \beta-\alpha+1; \xi) & \alpha e^{u_2 z} \xi^{t_1+\alpha} {}_1F_1(-\beta, \alpha-\beta+1; \xi) \end{pmatrix}$$

where $\xi = (u_1 - u_2)z$, and $\alpha = \lambda_1 - t_1$, $\beta = \lambda_2 - t_1$, λ_1, λ_2 eigenvalues of A. Using

$${}_1F_1(\alpha;\beta;z) \sim \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(-z)^{\alpha}(1+O(z)) + \frac{\Gamma(\beta)}{\Gamma(\alpha)} e^{-z} z^{-\alpha+\beta}(1+O(z))$$

- $S_+ = \begin{pmatrix} 1 & \frac{2\pi i b_2 (u_2-u_1)^{t_1-t_2}}{\Gamma(1-\lambda_1+t_1)\Gamma(1-\lambda_2+t_1)} \\ 0 & 1 \end{pmatrix}$

Part I

Stokes phenomenon, Yang-Baxter and reflection equations

Yang-Baxter and reflection equations

Definition

Let V and W be finite dimensional vector spaces. The Yang-Baxter and reflection equations for elements $R \in \mathrm{End}(V \otimes V)$, $K \in \mathrm{End}(W \otimes V)$ are

$$\begin{aligned} R^{12}R^{13}R^{23} &= R^{23}R^{13}R^{12} \in \mathrm{End}(V^{\otimes 3}), \\ K^{12}R^{32}K^{13}R^{32} &= R^{32}K^{13}R^{23}K^{12} \in \mathrm{End}(W \otimes V^{\otimes 2}). \end{aligned}$$

Here if $R = \sum X_a \otimes Y_a$, $R^{13} := \sum X_a \otimes 1 \otimes Y_a \in \mathrm{End}(V^{\otimes 3})$.

- The braid group B_n on $\mathbb{C}^\times$, $\pi_1^{S_n}((\mathbb{C}^\times)^n \setminus \{z_i \neq z_j\})$, has generators $\tau, b_1, \dots, b_{n-1}$ and relations

$$\tau b_1 \tau b_1 = b_1 \tau b_1 \tau, \qquad b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1},$$
$$b_i b_j = b_j b_i, \; |i-j| > 1, \qquad \tau b_i = b_i \tau, \; i \geq 2.$$

- Action of B_n on $W \otimes V^{\otimes n}$; $\tau \mapsto K^{12}, b_i \mapsto R^{i+1,i+2}$.

- $\Omega = \sum_{1\le i,j\le n} E_{ij} \otimes E_{ji} \in U(\mathrm{gl}_n)^{\otimes 2}$;
- $\Omega_{\mathfrak{k}} = \frac{1}{2}\sum_{1\le i,j\le n}(E_{ij} - E_{ji}) \otimes (E_{ji} - E_{ij}) \in U(\mathrm{so}_n)^{\otimes 2}$,
- $C_{\mathfrak{k}} = \frac{1}{2}\sum_{1\le i,j\le n}(E_{ij} - E_{ji})(E_{ji} - E_{ij}) \in U(\mathrm{so}_n) \subset U(\mathrm{gl}_n)$,
- $u = \mathrm{diag}(u_1, ..., u_n) \in \mathrm{gl}_n$ with distinct $u_i's$

For $V \in \mathrm{Rep}(\mathrm{gl}_n)$ and $W \in \mathrm{Rep}(\mathrm{so}_n)$, consider the equations for a $V^{\otimes 2}$-valued function $Y(z)$ and a $W \otimes V$-valued function $F(z)$

$$\frac{dY}{dz} = (\frac{u^{(1)}}{z^2} + h\frac{\Omega}{z}) \cdot Y,$$
$$\frac{dF}{dz} = (\frac{u^{(2)}}{z^2} + h\frac{2\Omega_{\mathfrak{k}} + C_{\mathfrak{k}}^{(2)}}{z}) \cdot F,$$

Theorem

For any u, the Stokes matrices $S_h(u) \in \mathrm{End}(V^{\otimes 2})$ and $K_h(u) \in \mathrm{End}(W \otimes V)$ satisfies YB and reflection equations.

Example: simplest case

Let us take gl_2, the natural representation V, thus

$$\frac{dF(z)}{dz} = (\frac{u}{z} + \frac{h\Omega}{z})F(z),$$

where $u = \mathrm{diag}(u_1, u_1, u_2, u_2)$, and $h\Omega = \begin{pmatrix} h & 0 & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & 0 & h \end{pmatrix}$.

We get

$$S_+ = \begin{pmatrix} e^h & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2i\sin(\pi h) & 1 & 0 \\ 0 & 0 & 0 & e^h \end{pmatrix}.$$

It is the evaluation of the universal R-matrix of quantum gl_2 on $V \otimes V$.

Universal setting

- The construction works for any $V \in \mathrm{Rep}(\mathrm{gl}_n)$ and $W \in \mathrm{Rep}(\mathrm{so}_n)$ in a functorial way.
- A universal setting:

$$\frac{dF}{dz} = \left(\frac{u^{(2)}}{z^2} + \hbar \frac{2\Omega_{\mathfrak{k}} + C_{\mathfrak{k}}^{(2)}}{z}\right) \cdot F,$$

where $F(z) \in U(\mathrm{so}_n)\hat{\otimes} U(\mathrm{gl}_n)[[\hbar]]$.

- Its Stokes matrix $K_\hbar(u) \in U(\mathrm{so}_n)\hat{\otimes} U(\mathrm{gl}_n)[[\hbar]]$ satisfy the reflection equation. (solutions in $W \otimes V[[\hbar]]$ for all W, V.)

Proposition (Gauge transformation)

As a function of $u = \mathrm{diag}(u_1, ..., u_n)$, the Stokes matrix $K_\hbar(u) \in U(\mathrm{so}_n)\hat{\otimes} U(\mathrm{gl}_n)[[\hbar]]$ satisfies

$$\frac{\partial K_\hbar}{\partial u_i} = \hbar \sum_{1 \le i < j \le n} \frac{[C_{\mathfrak{k},ij}^{(1)} + C_{\mathfrak{k},ij}^{(2)}, K_\hbar]}{u_i - u_j}, \quad \text{for } i = 1, ..., n.$$

Compatible differential equations

We introduce an IKZ system for a $U(\mathfrak{so}_n)\hat{\otimes}U(\mathfrak{gl}_n)[[\hbar]]$–valued function $F(z, u^1, ..., u^n)$:

$$\frac{\partial F}{\partial z} = \Big(\frac{u^{(2)}}{z^2} + \hbar\frac{2\Omega_{\mathfrak{k}}(u) + C_{\mathfrak{k}}^{(2)}}{z}\Big)\cdot F,$$
$$\frac{\partial F}{\partial u^i} = \Omega_i(z,u)\cdot F,$$

such that the canonical solutions of the first equation also satisfy the a linear system.

Corollary (Isomonodromy deformation)

The Stokes matrix $K_\hbar(u)$ of the first equation is locally constant.

- Global picture: braid group actions on $K_\hbar(u)$.
- Semiclassical limit: IKZ→Dubrovin connections;

$K_\hbar(u)$ →classical Stokes matrices; Givental twisted loop group, Dubrovin-Ugaglia Poisson structures.

Part III

Quantization of Frobenius manifolds

Frobenius manifolds

Introduced by Dubrovin as a geometrical formulation of

- Witten-Dijkgraaf-Verlinde-Verlinde equations,
- Family of 2d TFT (Cohomological field theory).

Atiyah-Segal axioms $\rightarrow$ a family of Frobenius algebras:

- Cohomology: $(H^\bullet(M), \wedge, g)$ is a Frobenius algebra.
- Quantum cohomology: $(H^\bullet(M), \star_m, g)$ is a family of Frobenius algebras, together with a natural grading.

Powerful tool: Frobenius manifolds include many information of enumerative questions.

Definition – Frobenius manifolds

A triple $(g, \circ, E)$ on a complex (manifold) M,

- g flat metric;
- $\circ$ bilinear product on $\mathcal{T}_M$;
- E conformal/Euler vector field,

which satisfy:

[1] $g(X \circ Y, Z) = g(X, Y \circ Z), \forall X, Y, Z \in \mathcal{T}_M$;

[2] $C : \mathcal{T}_M \to \mathrm{End}_{\mathcal{O}_M}(\mathcal{T}_M)$ defined by $C_X Y = X \circ Y$ is flat.

Picture: in flat coordinates $(t_1..t_n)$, [2] $\Rightarrow \exists$ function $P(t_1, ..., t_n)$

$$g(\frac{\partial}{\partial t^i} \circ \frac{\partial}{\partial t^j}, \ \frac{\partial}{\partial t^k}) = \frac{\partial^3 P}{\partial t^i \partial t^j \partial t^k}.$$

Example

- *q-ring of a pt:* $P(t_1) = \frac{1}{6} t_1^3$
- *q-ring of* $\mathbb{C}P^1$*:* $P(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + e^{t_2}$

Dubrovin connection and canonical coordinates

The data $(M, g, \circ, E)$ is encoded by a connection ∇ on $M \times \mathbb{C}$.

- z-extension of the Levi-Civita connection;
- Flatness of $\nabla \quad = \quad (M, \circ, g, E)$ is Frobenius.

Assumption: M is semisimple $= (T_u M, \circ) \cong \mathbb{C}^n$ at a point u.

Proposition (Dubrovin)

There exist unique coordinates $\{u^i\}$ at u such that

- $\frac{\partial}{\partial u^i} \circ \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^i}$;
- $E = \sum u^i \frac{\partial}{\partial u^i}$;
- *the metric is diagonal,* $g(u) = h_i(u)(du^i)^2$.

Local model: $\{h_i(u)\}$.

Dubrovin systems

In $(z, u^1, ..., u^n)$, the connection ∇ becomes a system of PDEs

$$\frac{\partial F}{\partial z} = \left(\frac{u}{z^2} + \frac{V(u)}{z}\right) F,$$
$$\frac{\partial F}{\partial u^i} = \left(\frac{E_{ii}}{z} + V_i(u)\right) F,$$

where $u = \mathrm{diag}(u^1, ..., u^n)$, $[V(u)]_{ij} := u_j \frac{\partial_j \sqrt{h_i}}{\sqrt{h_j}} - u_i \frac{\partial_i \sqrt{h_j}}{\sqrt{h_i}}$ is skew-symmetric.

Proposition (1.Dubrovin)

- *Frobenius structure (local model)*
- *Flatness of* $g(u) = h_i(du^i)^2$
- *Compatibility of the system*
- *Isomonodromy equation for* $V(u)$ *(Hamiltonian system)*
- *Stokes matrices* $S(u)$ *are constant (thus invariants of Frobenius manifolds)*

The space of semisimple Frobenius manifolds of dimension n (three parameterization):

- $\mathrm{so_n}$: initial conditions of the equation for $V(u)$ at a point u_0 (However, no "natural" point u_0)
- Stokes matrices $S_+ \in U_+$ the set of upper traingular matrices with 1 along diagonal (due to isomonodromy).
- Manin's classification data.

Proposition (3. Dubrovin, Ugaglia)

The space U_+ carries a natural Poisson structure, and a braid group action, such that the Riemann-Hilbert-Birkhoff map $\nu(u_0) : \mathrm{so}_n \cong \mathrm{so_n}* \to U_+$ is Poisson.

- U_+ is called Dubrovin-Ugaglia Poisson space.
- Irregular Atiyah-Bott construction, by Boalch.

Example (n=3)

- $\mathrm{so}_n^* = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$, *the Poisson bracket is given by*

$$\{a, b\} = -2c,\ \{b, c\} = -2a,\ \{c, a\} = -2b.$$

- $U_+ = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$, *the Poisson bracket is given by*

$$\{x, y\} = xy - 2z,\ \{y, z\} = yz - 2x,\ \{z, x\} = zx - 2y.$$

- In coordinates, the RHB map is $x = x(a, b, c)$, $y = y(a, b, c)$ and $z = (a, b, c)$.
- "algebraically" characterize the Stokes matrices, via shift of argument algebras (in progress).

Twisted loop groups

- Set $\mathbb{H} = \mathbb{C}^n((z^{-1}))$ Laurent polynomials with coefficients in $\mathbb{C}^n$, equipped with a symplectic form $\omega(f(z), g(z)) := \mathrm{Res}_{z=0}\langle f(-z)g(z)\rangle$.

Proposition (2. Observation of Givental)

The canonical solutions $F(z)$ of the Dubrovin system are valued in the twisted loop group

$$L^{(2)}\mathrm{GL}_n := \{H \in \mathrm{End}(\mathbb{H}) \mid H(-z)^T H(z) = 1\}.$$

Proof. Since $d_z F = (\frac{u}{z^2} + \frac{V}{z})F$, then $d_z(F(-z)^T F(z)) = F(-z)^T(-\frac{u}{z^2} + \frac{V}{z})^T F(z) + F(-z)^T(\frac{u}{z^2} + \frac{V}{z})F(z)$. ■

Semiclassical limit

• Solutions of IKZ system are $U(\mathrm{so}_n)\hat{\otimes}U(\mathrm{gl}_n)[[\hbar]]$-valued functions over $(z, u_1, ..., u_n)$;

$$\frac{\partial F}{\partial z} = \Big(\frac{u^{(2)}}{z^2} + \hbar\frac{2\Omega_{\mathfrak{k}}(u) + C^{(2)}_{\mathfrak{k}}}{z}\Big)\cdot F,$$
$$\frac{\partial F}{\partial u^i} = \Omega_i(z,u)\cdot F.$$

• Semiclassical limit of group-like elements in (Hopf algebra) $U(\mathrm{so}_n)\hat{\otimes}U(\mathrm{gl}_n)[[\hbar]] \cong \hat{Sym}(\mathrm{so}_n)\hat{\otimes}U(\mathrm{gl}_n)[[\hbar]]$ is $GL_n[[so_n^*]]$.

• Solutions of Dubrovin systems GL_n-valued functions over $(z, u_1, ..., u_n)$, parameterized by $V(u_0) \in so_n^* \cong so_n$.

Theorem (Xu)

(a) The semiclassical limit of the IKZ system gives rise to Dubrovin systems.
(b) In particular, any solution $F(z;V)$ of the Dubrovin system has a $\hbar$-deformation $F_\hbar(z;V) = F(z;V) + F_1\hbar + F_2\hbar^2 + \cdots$.

Quantum Prosition 1.

We have seen

Proposition (1. Dubrovin)

[1] *Frobenius structure (local model)*
[2] *Flatness of* $g(u) = h_i(du^i)^2$
[3] *Compatibility of the Dubrovin system*
[4] *Isomonodromy deformation*
[5] *Stokes matrices* $S(u)$ *are constant.*

and

Proposition (Xu)

[3′] *Compatibility of the IKZ system*
[4′] *q-Isomonodromy deformation*
[5′] *q-Stokes matrices* $K_\hbar(u)$ *are constant.*

Quantum Proposition 2.

Proposition (2. Observation of Givental)

The solution $F(z;V)$ of the Dubrovin system parameteried by $V = V(u_0) \in so_n$ is valued in the twisted loop group

$$L^{(2)}\mathrm{GL}_n := \{F \in \mathrm{End}(\mathbb{H}) \mid F(-z)^T F(z) = 1\}.$$

In the quantum setting

Proposition

For any $V \in so_n$, the solution $F_\hbar(z;V) = F(z;V) + O(\hbar)$ of the IKZ equation is valued in the $\hbar$-deformed twisted loop group

$$L^{(2)}\mathrm{GL}_n[[\hbar]] := \{F_\hbar \in \mathrm{End}(\mathbb{H}[[\hbar]]) \mid F_\hbar(-z)^T F_\hbar(z) = 1\}.$$

Proof. Differentiating $F_\hbar(-z)^T F_\hbar(z)$ to get 0. ∎

- $\hbar$-deformation.

Quantum Proposition 3.

Proposition (3. Dubrovin, Ugaglia)

The space U_+ of semisimple Frobenius manifolds of dimension n carries a natural Poisson structure.

A quantum analog is

Proposition

The fact that the q-Stokes matrices $S_\hbar$, $K_\hbar$ satisfy YB and reflection equations gives the commutative diagram

$$\begin{tikzcd}
\boxed{IKZ\ system} \arrow[r, "q-Stokes\ matrices"] \arrow[d, "\hbar=0"'] & \boxed{Quantum\ symmetric\ pair\ B(so_n) \subset U_\hbar(gl_n)} \arrow[d, "\hbar=0\ (known)"'] \\
\boxed{Dubrovin\ systems} \arrow[r, "Stokes\ matrices"] & \boxed{Dubrovin\text{-}Ugaglia\ Poisson\ space\ U_+}
\end{tikzcd}$$

Geometric structures

We have seen

Proposition (Dubrovin)

[1] *Frobenius structure (local model)*
[2] *Flatness of* $g(u) = h_i(du^i)^2$
[3] *Compatibility of the Dubrovin system*
[4] *Isomonodromy equation for* $V(u)$ *(Hamiltonian system)*
[5] *Stokes matrices* $S(u)$ *are constant.*

and

Proposition (Xu)

[3′] *Compatibility of the IKZ system*
[4′] *Q-Isomonodromy equation for* $\Omega(u)$
[5′] *Q-Stokes matrices* $S_\hbar(u)$ *are constant.*

Q: What about [1] and [2]?

Deformed metric– Darboux-Egoroff picture (in progress)

- Given $(M, g, \circ, E)$, Darboux-Egoroff picture is the local model in canonical coordinates.

Recall that from $V(u)$ to the metric $g(u) = h_i(du^i)^2$, we solve the Darboux-Egoroff system

$$V(u) := \Big[\frac{\partial_j \sqrt{h_i}}{\sqrt{h_j}},\ u\Big].$$

- IKZ system $\Rightarrow \Omega_{\mathfrak{k}}(u) = V(u) + \hbar V_1(u) + o(\hbar)$.
- Plug into (2): a deformation $g_\hbar(u) = g(u) + \hbar g_1(u) + o(\hbar)$

Remark

The deformed metric $g_\hbar$ is not flat.

Deformed product– WDVV picture (in progress)

• Given $(M, g, \circ, E)$, WDVV picture is the local model in flat coordinates $\{t^i\}$ of g.

Recall that the potential condition means $\exists\ P(t^1, ..., t^n)$ s.t.

$$g(\frac{\partial}{\partial t^i} \circ \frac{\partial}{\partial t^j},\ \frac{\partial}{\partial t^k}) = \frac{\partial^3 P}{\partial t^i \partial t^j \partial t^k}.$$

• Deformed metric $g_\hbar$ (LHS) $\Rightarrow$ Deformed potential $P_\hbar$ (RHS)

If we insist to use flat metric $g \Rightarrow$ deformed product

$$g(\frac{\partial}{\partial t^i} \circ_\hbar \frac{\partial}{\partial t^j},\ \frac{\partial}{\partial t^k}) = \frac{\partial^3 P_\hbar}{\partial t^i \partial t^j \partial t^k}.$$

• A deformed $(M, g, \circ_\hbar, E)$ and potential $P_\hbar$. But $\circ_\hbar$ is not associative.

Part III

WKB approximation of Stokes matrices and canonical bases

WKB approximation

Introduce a parameter ε,

$$\varepsilon\frac{dF}{dz} = (\frac{u}{z^2} + \frac{V}{z})F$$

and consider the Stokes matrix $S_+(u, V, \varepsilon)$.

Goal: study the asymptotics of $S_+(u, V, \varepsilon)$ as $\varepsilon \to 0$.
Motivation: Hitchin systems (shift of argument algebras), tropicalization, cluster structures on the space of Stokes matrices.
Result: (with Alekseev) The WKB approximation

$$S(V, u, \varepsilon) = (S_{ij}(V, u, \varepsilon)) \sim (q_{ij}(V, u)e^{\frac{p_{ij}(V,u)}{\varepsilon}}(1{+}O(\varepsilon))), \ as\ \varepsilon \to 0,$$

where $\mathrm{Re}(p_{ij}(V, u))$ and $\mathrm{Arg}(q_{ij}(V, u))$, as functions on $V \in \mathrm{gl_n} \cong \mathrm{gl_n^*}$, are integrable systems, whose geometric quantization gives gl_n-crystals.

Example (2 by 2 case)

$$S(A,u,\varepsilon)=\begin{pmatrix} e^{\frac{t_1}{2\varepsilon}} & \frac{\frac{b_2}{\varepsilon}(\frac{u_2-u_1}{\varepsilon})^{\frac{t_1-t_2}{2\pi i\varepsilon}}}{\Gamma(1-\frac{\lambda_1-t_1}{2\pi i\varepsilon})\Gamma(1-\frac{\lambda_2-t_1}{2\pi i\varepsilon})} \\ 0 & e^{\frac{t_2}{2\varepsilon}} \end{pmatrix}$$

where $\lambda_1\le\lambda_2$ *are the eigenvalues of* $A=\begin{pmatrix} t_1 & b_2 \\ b_1 & t_2 \end{pmatrix}$.

Using $log(\Gamma(\frac{z}{\varepsilon}))\sim -\frac{z}{\varepsilon}log(\varepsilon)+\frac{z}{\varepsilon}log(z)-\frac{z}{\varepsilon}+\frac{1}{2}log(2\pi\varepsilon)-\frac{1}{2}log(z)$.

- $S_{11}\sim e^{\frac{t_1}{2\varepsilon}}$;
- $S_{12}\sim b_2 e^{\frac{\lambda_2}{2\varepsilon}}(1+O(\varepsilon))$;
- $S_{22}\sim e^{\frac{\lambda_1+\lambda_2-t_1}{2\varepsilon}}$.

• 1. t_1,λ_1,λ_2 Gelfand-Zeitlin integrable systems (Hitchin systems); 2. Gelfand-Zeitlin bases.

• In general case, one should use the closure of Stokes matrices, the cluster charts on Stokes matrices, WKB approximation of isomonodromy connections.

Thank you very much!