

# Gamma conjecture I for del Pezzo surfaces

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# Algebra and linear operator

- $\mathcal{A}$ :  $\mathbb{R}$ -algebra.

$\alpha \in \mathcal{A} \implies$  linear operator  $\hat{\alpha} : \mathcal{A} \longrightarrow \mathcal{A}; \beta \mapsto \alpha \cdot \beta$ .

## Question

What can we say about the eigenvalues of  $\hat{\alpha}$ ?

- ▶  $\mathcal{A} = QH^*(X)$ .
- ▶  $\alpha = c_1(X)$ .

# Conjectures

- Galkin-Golyshev-Iritani (Duke Math. J. 2016)

$X$ : Fano manifold.

▶ Conjecture  $\mathcal{O}$   $\longleftrightarrow$  Eigenvalue of  $\widehat{c}_1$  on  $QH^*(X)$ .

▶ Assume Conjecture  $\mathcal{O}$  first.

Gamma conjecture I  $\longleftrightarrow$  Asymptotic expansion of  $J_X$ .

▶ Assume the semisimplicity of quantum cohomology first.

Gamma conjecture II  $\longleftrightarrow$  Full exceptional collection in  $D_{\text{coh}}^b(X)$ .

- Hosono (2006), Abouzaid-Ganatra-Iritani-Sheridan (2018)

Gamma conjectures for Calabi-Yau manifolds.

# What is ...?

## Conjecture $\mathcal{O}$ (Galkin-Golyshev-Iritani)

Let  $X$  be a **Fano manifold**, and denote by  $\text{Spec}(\hat{c}_1)$  the set of eigenvalues of the following linear operator  $\hat{c}_1$  induced by quantum multiplication  $\star$ .

$$\hat{c}_1 : H^{\text{even}}(X, \mathbb{C}) \longrightarrow H^{\text{even}}(X, \mathbb{C}); \quad \beta \mapsto c_1(X) \star \beta|_{q=1}.$$

Then the following should hold.

- (i)  $\rho := \max_{\delta \in \text{Spec}(\hat{c}_1)} |\delta|$  **belongs to  $\text{Spec}(\hat{c}_1)$ ; mult.  $\rho = 1$ .**
- (ii)  $r := \max\{k \in \mathbb{Z} \mid \frac{c_1(X)}{k} \in H^2(X, \mathbb{Z})\}$ .  $\forall \delta \in \text{Spec}(\hat{c}_1)$  with  $|\delta| = \rho$ ,

$$(\delta/\rho)^r = 1.$$

# Conjecture $\mathcal{O}$ holds if ...

## Example

Conjecture  $\mathcal{O}$  holds if  $X = \mathbb{P}^1$ .

- $H^*(\mathbb{P}^1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}x$ , where  $x = P.D.[\mathbb{P}^0] \in H^2(\mathbb{P}^1, \mathbb{Z})$ .
- $c_1(\mathbb{P}^1) = 2x \implies r(\mathbb{P}^1) = 2$ .

# Conjecture $\mathcal{O}$ holds if ...

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- $c_1(\mathbb{P}^1) = 2x \implies r(\mathbb{P}^1) = 2$ .

- $QH^*(\mathbb{P}^1) = \mathbb{C}[x, q]/(x^2 - q)$

- ▶  $QH^{\text{ev}}(\mathbb{P}^1)|_{q=1} = \mathbb{C}[x]/(x^2 - 1) = \mathbb{C} \oplus \mathbb{C}x$

- ▶

$$\hat{c}_1 \begin{pmatrix} 1 \\ x \end{pmatrix} = 2x \star \begin{pmatrix} 1 \\ x \end{pmatrix} \Big|_{q=1} = \begin{pmatrix} 2x \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}$$

$$\implies \text{Spec}(\hat{c}_1) = \text{Spec} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \{2, -2\}$$

# Viewpoint from mirror symmetry

- Mirror symmetry

$X$ : Fano

$$f : \check{X} \rightarrow \mathbb{C}$$

Expectations:

- ▶  $QH^*(X) \cong Jac(f)$
- ▶  $\{\text{eigenvalues of } \hat{c}_1\} \longleftrightarrow \{\text{critical values of } f\}$
- ▶ ...
- ▶ Galkin-Golyshev-Iritani: "*Each Morse function of  $f$  associates a vanishing cycle, which gives an object of the Fukaya-Seidel category of  $f$ . Under homological mirror symmetry, the vanishing cycle associated to  $\rho$  should correspond to the structure sheaf  $\mathcal{O}$  of  $X$ .*"

# Gamma conjecture I: Gamma class

- Gamma function:  $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ .
- Gamma class:

$$\begin{aligned}\hat{r}_X &= \prod_{i=1}^N \Gamma(1 + x_i) \in H^{\text{ev}}(X, \mathbb{R}), \quad \text{where} \quad c(TX) = \prod_{i=1}^N (1 + x_i) \\ &= \exp \left( -c_{\text{eu}} c_1(X) + \sum_{k=2}^{\infty} (-1)^k (k-1)! \zeta(k) ch_k(TX) \right) \\ &\stackrel{\text{e.g.}}{=} \begin{cases} 1 - c_{\text{eu}} c_1(X), & \text{if } \dim X = 1 \\ 1 - c_{\text{eu}} c_1(X) + \zeta(2) ch_2(TX) + \frac{1}{2} c_{\text{eu}}^2 c_1^2(X), & \text{if } \dim X = 2. \end{cases}\end{aligned}$$



# Gamma conjecture I: Givental's $J$ -function

Quantum connection on  $H^{\text{ev}}(X) \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$  : ( for  $\phi \in H^{2p}(X)$  )

$$\nabla_{z\partial_z} = z \frac{\partial}{\partial z} - \frac{1}{z} (c_1(X) \star_{\mathbf{q}=1}) + \mu, \text{ where } \mu(\phi) = (p - \frac{\dim X}{2})\phi$$

## Proposition

There exists a unique  $\text{End}(H^{\text{ev}}(X))$ -valued power series

$S(z) = id + S_1 z^{-1} + S_2 z^{-2} + \dots$  such that  $(z^A := \exp^{A \log z})$

$$\begin{cases} \nabla(S(z) z^{-\mu} z^{c_1(X)} \phi) = 0, & \forall \phi \in H^{\text{ev}}(X) \\ T(z) = z^\mu S(z) z^{-\mu} \text{ is regular at } z = \infty \text{ and } T(\infty) = id. \end{cases}$$

- Givental's  $J$ -function (restricted at  $c_1(X) \log t$  with  $t = z^{-1}$ ):

$$J_X(t) = z^{\frac{\dim X}{2}} (S(z) z^\mu z^{c_1(X)})^{-1} 1.$$

# Gamma conjectures

$$H^{\text{ev}}(X) \longleftrightarrow \{s : \mathbb{R}_{>0} \rightarrow H^{\text{ev}}(X) \mid \nabla s = 0\}$$
$$\alpha \mapsto (2\pi)^{\frac{-\dim X}{2}} S(z) z^{-\mu} z^{c_1(X)} \alpha.$$

- $\mathcal{A} := \{s : \mathbb{R}_{>0} \rightarrow H^{\text{ev}}(X) \mid \nabla s = 0, \quad \|e^{\rho/z} s(z)\| = O(z^{-m}) \text{ as } z \rightarrow +0 (\exists m)\}.$

## Proposition

*Assume conjecture  $\mathcal{O}$  holds. Then*

- $\dim \mathcal{A} = 1$ , so that  $\mathcal{A} = \mathbb{C}[S(z) z^{-\mu} z^{c_1(X)} A_X].$
- $J_X(z) = C z^{\frac{\dim X}{2}} e^{\frac{\rho}{z}} (A_X + O(z))$

# Gamma conjectures

- $\mathcal{A} := \{s : \mathbb{R}_{>0} \rightarrow H^{\text{ev}}(X) \mid \nabla s = 0, \quad \|e^{\rho/z} s(z)\| = O(z^{-m}) \text{ as } z \rightarrow +0 (\exists m)\}.$

## Proposition

Assume conjecture  $\mathcal{O}$  holds. Then  $J_X(z) = Cz^{\frac{\dim X}{2}} e^{\frac{\rho}{z}} (A_X + O(z))$

## Gamma Conjectures (Galkin-Golyshev-Iritani)

- Gamma Conjecture I:  $A_X = \hat{\Gamma}_X$  (assuming Conj.  $\mathcal{O}$ )
- Gamma Conjecture II:  $A_{X,\delta} = \hat{\Gamma}_X \text{Ch}(E_\delta)$  for some full exceptional collection  $\{E_\delta\}$  of  $\mathcal{D}_{\text{coh}}^b(X)$  (assuming semi-simplicity of  $QH^{\text{ev}}(X)$ )

Here  $\text{Ch}(E) := \sum_p (2\pi\sqrt{-1})^p \text{ch}_p(E).$

In particular  $\text{Ch}(\mathcal{O}_X) = 1.$

# Conjecture $\mathcal{O}$ holds if ...

## Theorem (Hu-Ke-L.-Yang)

Conjecture  $\mathcal{O}$  holds if  $X$  is a del Pezzo surface.

- Flag varieties  $G/P$  of arbitrary Lie type: Cheong-L. 2017.
  - ▶ Complex Grassmannians: Rietsch 2003, Galkin-Golyshev 2006.
  - ▶ Lagrangian and orthogonal Grassmannians: Cheong 2017.
- Fano 3-folds:
  - ▶ Picard rank one: Golyshev-Zagier 2016.
  - ▶ Bott-Samelson varieties: Withrow 2018.
- Horospherical varieties: L.-Mihalcea-Shifler 2017;  
Bones-Fowler-Schneider-Shifler 2018
- Fano complete intersections in  $\mathbb{P}^n$ : Galkin-Iritani 2015;  
Sanda-Shamoto 2017; Ke 2018.

# Gamma conjectures hold if ...

## Theorem (Hu-Ke-L.-Yang)

Gamma conjecture I holds if  $X$  is a del Pezzo surface.

To my knowledge:

- Complex Grassmannians: Galkin-Golyshev-Iritani 2016.
- Fano 3-folds of Picard rank one: Golyshev-Zagier 2016.
- Fano complete intersections in  $\mathbb{P}^n$  ( $r \geq 2$ ): Sanda-Shamoto 2017.
- Toric Fano manifolds that satisfy conjecture  $\mathcal{O}$ : Galkin-Iritani.

## Remark (Gamma conjecture II holds if)

- $X$  is a complex Grassmannian (Galkin-Golyshev-Iritani).
- $X$  is a toric Fano manifold (Fang-Zhou).

# Approach to Conjecture $\mathcal{O}$ in good cases

- Kaoru Ono: *Conjecture  $\mathcal{O}$  (i) follows from Perron-Frobenius theorem, if  $\hat{c}_1$  is represented by an irreducible non-negative matrix.*
  - ▶ This may happen in good cases, e.g. when  $H^{\text{ev}}(X)$  has a basis of "positive" algebraic cycles.
- Conjecture  $\mathcal{O}$  (ii) would probably hold automatically if  $r = 1$ .

# Perron-Frobenius theorem

## Definition

A nonnegative matrix  $M$  is called **reducible** if the induced operator has a nontrivial invariant coordinate subspace, i.e., if

$$M = Q \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} Q^T \quad \text{for some permutation matrix } Q.$$

**irreducible = not reducible**

## Theorem (Perron(1907)-Frobenius(1912))

- 1 Every **irreducible nonnegative** matrix  $M$  has a real eigenvalue  $\delta_0$  of multiplicity one such that  $\delta_0 \geq |\delta|$  for all  $\delta \in \text{Spec}(M)$ .
- 2 All eigenvalues  $\delta$  with  $|\delta| = \delta_0$  are simple, and precisely the solutions of  $\delta_0^h - \delta^h = 0$  for some  $h \in \mathbb{Z}_{>0}$ .  
(inv.by  $\xRightarrow{e^{\frac{2\pi\sqrt{-1}}{r}}}$   $r|h$ )

# Del Pezzo surfaces = 2-dim. Fano manifolds

$$c_1(X_k) = -K_{X_k} = 3H - E_1 - \cdots - E_k$$

Fano index  $r = 1$  (since  $\langle -K_{X_r}, E_1 \rangle = 1$ ).

- Good:  $QH^*(X_k)$  is well studied.
- **NOT** good:  $\hat{c}_1$  is unlikely given by a nonnegative matrix.



# Key point for Conjecture $\mathcal{O}$ for del Pezzo surfaces

## Theorem (Generalized Perron-Frobenius Theorem)

Suppose that a real matrix  $M = (m_{ij})$  satisfies the following properties:

- ①  $\sum_j m_{ij} > 0$  for any  $i$ ;
- ②  $M^k$  is an irreducible nonnegative matrix for some  $k$ .

Then  $M$  has a real eigenvalue  $\delta_0$  of multiplicity one such that  $\delta_0 \geq |\delta|$  for all  $\delta \in \text{Spec}(M)$ .

## Remark

Generalized Perron-Frobenius Theorem is also applicable if

- $X$  is the Bott-Samelson resolution of  $F\ell_3$ .
- $X$  is the blowup of  $\mathbb{P}^4$  at a point.

# Approach to Gamma conjecture I by Galkin-Iritani

- Gamma conjecture I holds for a toric Fano manifold if the mirror Landau-Ginzburg potential  $f$  satisfies the  $B$ -analogue of conjecture  $\mathcal{O}$  (together with the coincidence with the conifold point of  $f|_{\text{realpart}}$ ).
- Gamma conjecture I is compatible with the **quantum Lefschetz principle** (whenever it is applicable), since so is Givental's  $J$ -function.

# Del Pezzo surfaces = 2-dim. Fano manifolds

Either  $\mathbb{P}^1 \times \mathbb{P}^1$ ,  $\mathbb{P}^2$  or one of the following:

- $X_1$ : degree  $(1, 1)$  hypersurface in  $\mathbb{P}^1 \times \mathbb{P}^2$ ;
- $X_2$ : complete intersection of divisors of degree  $(1, 0, 1)$  and  $(0, 1, 1)$  in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ ;
- $X_3$ : complete intersection of two divisors of degree  $(1, 1)$  in  $\mathbb{P}^2 \times \mathbb{P}^2$ ;
- $X_4$ : complete intersection of four hyperplanes in Grassmannian  $G(2, 5) \subset \mathbb{P}^9$  (embedded by Plücker);
- $X_5$ : complete intersection of two quadrics in  $\mathbb{P}^4$ ;
- $X_6$ : cubic surface in  $\mathbb{P}^3$ ;
- $X_7$ : hypersurface of degree 4 in  $\mathbb{P}(2, 1, 1, 1)$ ;
- $X_8$ : hypersurface of degree 6 in  $\mathbb{P}(3, 2, 1, 1)$ .

# Key ingredient for $\Gamma$ -conjecture I for $X_7, X_8$

## Theorem

*Conjecture O and Gamma conjecture I hold for weighted projective spaces  $\mathbb{P}(1, w_1, \dots, w_N)$ .*

Thank you!!...