Mirror Symmetry for quasi-smooth Calabi-Yau hypersurfaces in weighted projective spaces

Victor Batyrev
(joint work with Karin Schaller)

Eberhard Karls Universität Tübingen

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The purpose of the talk

B., K. Schaller, arXiv:2006.04465

To explain the combinatorial framework behind the Mirror Symmetry construction for quasi-smooth Calabi-Yau hypersurfaces in weighted projective spaces.

Discrepancies

X a normal irreducible quasi-projective $\mathbb{Q}$-Gorenstein algebraic variety. Take a resolution of singularities of X

$$\rho : Y \to X$$

with the exceptional locus $\bigcup_{i=1}^{r} D_i$ union of smooth irreducible divisors with only normal crossings.
$I := \{1, \ldots, r\}$

$$K_Y = \rho^* K_X + \sum_{i \in I} a_i D_i,$$

The rational numbers $a_i \in \mathbb{Q}$ $(i \in I)$ are called *discrepancies* of divisors D_i.

Singularities in MMP

Definition

Singularities of X are called at worst

- *terminal* if $a_i > 0$, $\forall i \in I$;
- *canonical* if $a_i \geq 0$, $\forall i \in I$;
- *log-terminal* if $a_i > -1$, $\forall i \in I$.

Canonical Calabi-Yau varieties

Definition

A d-dimensional smooth projective normal variety X with at worst Gorenstein canonical singularities is called *canonical Calabi-Yau* variety if

- the canonical divisor K_X is trivial;
- $h^i(X, \mathcal{O}_X) = 0$ $(0 < i < d)$.

Non-degenerate hypersurfaces in torus

Let $M \cong \mathbb{Z}^d$ be a lattice of rank d. We consider M as the lattice of characters of d-dimensional algebraic torus $\mathbb{T}_d \cong (\mathbb{C}^*)^d$.

Definition

A Laurent polynomial

$$f(\mathbf{t}) = \sum_{m \in A} a_m \mathbf{t}^m \in \mathbb{C}[M] \cong \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$$

with Newton polytope $\Delta = \mathrm{conv}(A) \subset M_{\mathbb{R}} := M \otimes \mathbb{R}$ is called *non-degenerate* if for any face $\Theta \preceq \Delta$ the affine hypersurface

$$Z_{f,\Theta} := \{ \sum_{m \in A \cap \Theta} a_m \mathbf{t}^m = 0 \} \subset \mathbb{T}_d.$$

is smooth. The non-degeneracy of $f(\mathbf{t})$ is a Zariski open condition on its coefficients $\{a_m\} \in \mathbb{C}^{|A \cap M|}$.

Canonical Fano polytopes

Definition

A d-dimensional lattice polytope $\Delta \subset M_{\mathbb{R}}$ is called *canonical Fano polytope*, if it contain exactly one lattice point p in its interior Δ°. For simplicity we assume that $p = 0 \in M$.

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Theorem (Khovanskiî, 1978)

The geometric genus p_g of a non-degenerate toric hypersurface Z_f defined by Laurent polynomial f with Newton polytope Δ equals $\Delta^{\circ} \cap M$. In particular, $p_g = 1$ (Calabi-Yau case) if and only if Δ is a canonical Fano polytope.

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Theorem

There exists a natural bijection between d-dimensional canonical Fano polytopes Δ up to $GL(d, \mathbb{Z})$-isomorphism and d-dimensional $\mathbb{Q}$-Gorenstein toric Fano varieties X_{Δ} with at worst canonical singularities up to isomorphism.

Canonical Fano polytopes

For any fixed dimension d there exist only finitely many d-dimensional canonical Fano polytopes up to a $GL(d, \mathbb{Z})$-isomorphism.

- There exists exactly one canonical Fano polytope of dimension 1: $\Delta = [-1, 1]$.
- There exist exactly 16 canonical Fano polytopes of dimension 2.
- There exist exactly $674, 688$ three-dimensional canonical Fano polytopes (Kasprzyk, 2010)
- The complete list of all 4-dimensional canonical Fano polytopes is still unknown.

2-dimensional canonical Fano polytopes

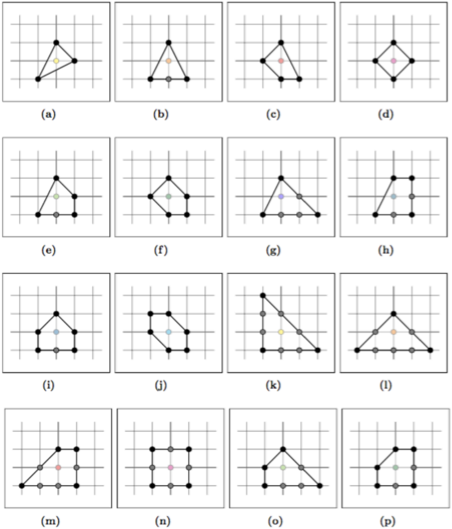


Figure: 2-dimensional canonical Fano polytopes

Source: Karin Schaller, *Stringy Invariants of Algebraic Varieties and Lattice Polytopes*, Ph.D. thesis, Eberhart Karls Univ. Tübingen, 2019.

Reflexive polytopes

Denote $N := \mathrm{Hom}(M, \mathbb{Z})$, $M_{\mathbb{R}} := M \otimes \mathbb{R}$, $N_{\mathbb{R}} := N \otimes \mathbb{R}$, and

$$\langle *, * \rangle \ : \ M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$$

the natural pairing.

Definition

A d-dimensional canonical Fano polytope $\Delta \subset M_{\mathbb{R}}$ is called *reflexive* if the *polar dual* polytope

$$\Delta^* := \{ y \in N_{\mathbb{R}} \ : \ \langle x, y \rangle \geq -1, \ \ \forall x \in \Delta \}$$

is also a canonical Fano polytope.

The combinatorial duality

If Δ is reflexive, then Δ^* is also reflexive and

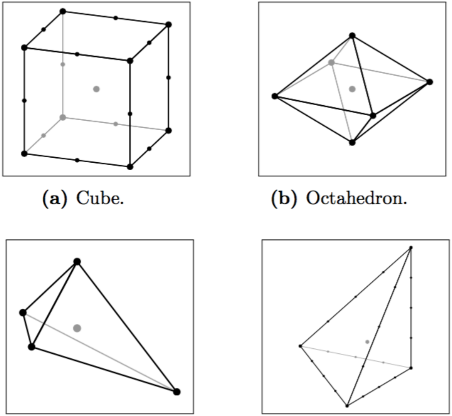
$$\boxed{(\Delta^*)^* = \Delta.}$$

There exists a natural 1-to-1 correspondence between k-dimensional faces $\theta \prec \Delta$ and $(d-k-1)$-dimensional faces $\theta^* \prec \Delta^*$:

$$\theta^* := \{y \in \Delta^* \,:\, \langle x, y\rangle = -1 \;\; \forall x \in \theta\}.$$

The combinatorial duality $\Delta \leftrightarrow \Delta^*$ perfectly agrees with the prediction of Mirror Symmetry for Calabi-Yau hypersurfaces in toric varieties $X \subset \mathbb{P}_\Delta$ and $X^* \subset \mathbb{P}_{\Delta^*}$.

Some Reflexive 3-polytopes



(a) Cube. (b) Octahedron. (c) 3-simplex. (d) Tetrahedron.

Figure: Some Reflexive 3-polytopes.

Source: Karin Schaller, *Stringy Invariants of Algebraic Varieties and Lattice Polytopes*, Ph.D. thesis, Eberhart Karls Univ. Tübingen, 2019.

Topological mirror symmetry tests

The Hodge numbers of two d-dimensional smooth Calabi-Yau varieties V and V^* that are mirror symmetric to each other must satisfy the equalities

$$h^{p,q}(V) = h^{d-p,q}(V^*)$$

for all p, q $(0 \leq p, q \leq d)$. In particular, the Euler number $\chi = \sum_{p,q}(-1)^{p+q}h^{p,q}$ must satisfy the equality

$$\chi(V) = (-1)^d \chi(V^*).$$

The stringy Euler number $\chi_{\mathrm{str}}(X)$

Definition

Let $\rho : Y \to X$ be a resolution and $K_Y = \rho^* K_X + \sum_{i \in I} a_i D_i$.
Define for any subset $J \subseteq I$:

$$D_\emptyset := Y, \ \ D_J := \bigcap_{j \in J} D_j \ \ (\emptyset \neq J \subseteq I).$$

The *stringy Euler number* of X is the rational number

$$\chi_{\mathrm{str}}(X) := \sum_{\emptyset \subseteq J \subseteq I} \chi(D_J) \prod_{j \in J} \left(\frac{1}{a_j + 1} - 1 \right)$$

$$= \sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|} \chi(D_J) \prod_{j \in J} \frac{a_j}{a_j + 1}.$$

(a product over $\emptyset$ is assumed to be 1)

Some properties of $\chi_{\mathrm{str}}(X)$

General remarks

- The rational number $\chi_{\mathrm{str}}(X)$ *does not depend* on the choice of a desingularization $\rho : Y \to X$. In particular, if X is smooth, then
$$\chi_{\mathrm{str}}(X) = \chi(X)$$
(we can take $\rho = id$).
- If $\rho : Y \to X$ is a *crepant* desingularization ($a_i = 0 \ \forall i \in I$), then
$$\chi_{\mathrm{str}}(X) = \chi(Y).$$
Examples: minimal desingularizations of ADE-singularities of surfaces.
- If X and X' are birational *K-equivalent*, then
$$\chi_{\mathrm{str}}(X) = \chi_{\mathrm{str}}(X').$$

Combinatorial formula for $\chi_{\mathrm{str}}(X)$

Theorem (B., Dais 1994)

Let Δ be a d-dimensional reflexive polytope. Then the stringy Euler number of a general CY hypersurface $X \subset \mathbb{P}_\Delta$ equals

$$\boxed{\chi_{\mathrm{str}}(X) = \sum_{k=1}^{d-2}(-1)^{k-1} \sum_{\substack{\theta \prec \Delta \\ \dim(\theta)=k}} Vol_k(\theta) \cdot Vol_{d-k-1}(\theta^*).}$$

If $X^* \subset \mathbb{P}^*_\Delta$ is a CY hypersurface corresponding to the dual polytope Δ^*, then

$$\boxed{\chi_{\mathrm{str}}(X) = (-1)^{d-1}\chi_{\mathrm{str}}(X^*).}.$$

$\chi_{\rm str}(X)$ for $\dim \Delta = 3$

If Δ is a 3-dimensional reflexive polytope, then

$$\chi_{\rm str}(X) = \sum_{\substack{\theta \prec \Delta \\ \dim(\theta)=1}} Vol_1(\theta) \cdot Vol_1(\theta^*) = 24.$$

$\chi_{\mathrm{str}}(X)$ for $\dim \Delta = 4$

If Δ is a 4-dimensional reflexive polytope, then

$$\chi_{\mathrm{str}}(X) = \sum_{\substack{\theta \prec \Delta \\ \dim(\theta)=1}} Vol_1(\theta) \cdot Vol_2(\theta^*) - \sum_{\substack{\theta \prec \Delta \\ \dim(\theta)=2}} Vol_1(\theta) \cdot Vol_2(\theta^*).$$

For quintic 3-folds X in $\mathbb{P}^4$:

$$\chi(X) = \chi_{\mathrm{str}}(X) = 10 \cdot (5 \cdot 1) - 10 \cdot (25 \cdot 1) = -200.$$

Weighted projective space $\mathbb{P}(w_0, \dots, w_d)$

Weight vector $\overline{w} := (w_0, \dots, w_d) \in \mathbb{Z}^{d+1}_{>0}$ is called well-formed if

$$\gcd(w_0, \dots, w_{i-1}, w_{i+1}, \dots, w_d) = 1 \;\; \forall i \in \{0, \dots, d\}.$$

A weighted projective space $\mathbb{P}(\overline{w})$ is the quotient of $\mathbb{C}^{d+1} \setminus \{0\}$ by $\mathbb{C}^*$-action

$$(z_0, \dots, z_d) \mapsto (\lambda^{w_0} z_0, \dots, \lambda^{w_d} z_d) \;\; \forall \lambda \in \mathbb{C}^*.$$

It is a d-dimensional toric variety which is a projective compactification of the d-dimensional algebraic torus $\mathbb{T}_{\overline{w}} \cong (\mathbb{C}^*)^{d+1}/\mathbb{C}^*$ whose group of characters is

$$N_{\overline{w}} = \{(u_0, \dots, u_d) \in \mathbb{Z}^{d+1} \mid \sum_{i=0}^{d} w_i u_i = 0\}.$$

Quasi-smooth Calabi-Yau hypersurfaces

Definition

A weight vector $\overline{w} \in \mathbb{Z}_{>0}^{d+1}$ is called transverse if the weighted projective space $\mathbb{P}(w_0, w_1, \dots, w_d)$ contains a quasi-smooth Calabi-Yau hypersurface X_w of degree $w = \sum_{i=0}^{d} w_i$ defined by a transverse weighted homogeneous polynomial $W \in \mathbb{C}[z_0, \dots, z_d]$, i.e., all partial derivatives $\partial W / \partial z_i$ $(0 \leq i \leq d)$ form a regular sequence in $\mathbb{C}[z_0, z_1, \dots, z_d]$.
A weighted homogeneous polynomial $W \in \mathbb{C}[z_0, z_1, \dots, z_d]$ is transverse if and only if $0 \in \mathbb{C}^{d+1}$ is the only singular point of the d-dimensional affine hypersurface $\{W = 0\} \subset \mathbb{C}^{d+1}$.

Gorenstein weighted projective spaces

Definition

A weight vector $\overline{w} \in \mathbb{Z}_{>0}^{d+1}$ is called Gorenstein if w_i divides $w = \sum_{i=0}^{d}$ for all $i \in \{0, \ldots, d\}$.
Every Gorenstein weight vector $\overline{w} \in \mathbb{Z}_{>0}^{d+1}$ is transverse, because one can choose a transverse weighted polynomial W in Fermat form:

$$W = \sum_{i=0}^{d} z_i^{w/w_i}.$$

Weight vectors with *IP*-property

Definition

A weight vector $\overline{w} \in \mathbb{Z}^{d+1}_{>0}$ is called to have IP-property if

$$\operatorname{conv}\{(u_0, \ldots, u_d) \in \mathbb{Z}^{d+1}_{\geq 0} \mid \sum_{i=0}^{d} w_i u_i = w\}$$

is a d-dimensional lattice polytope $\Delta(\overline{w})$ containing the lattice point $\mathbf{1} := (1, \ldots, 1)$ in its interior.

Any transverse weight vector $\overline{w} \in \mathbb{Z}^{d+1}_{>0}$ has *IP*-property (Skarke). Moreover, if $\overline{w} \in \mathbb{Z}^{d+1}_{>0}$ is an arbitrary weight vector with *IP*-property, then a general hypersurface $X_w \subset \mathbb{P}(\overline{w})$ is a canonical Calabi-Yau variety.

Classification of weight vectors for $d \leq 5$

For any fixed dimension $d = \dim \mathbb{P}(\overline{w})$, there exist only finitely many $IP(d)$ weight vectors $\overline{w} \in \mathbb{Z}_{>0}^{d+1}$ with IP-property. In particular, there exist only finitely many $T(d)$ transverse weight vectors and finitely many $G(d)$ Gorenstein weight vectors.

$d =$	2	3	4	5
$G(d)$	3	14	147	3,462
$T(d)$	3	95	7,555	1,100,055
$IP(d)$	3	95	184,026	322,383,760,930

Vafa's formula (1989)

Let $\overline{w} \in \mathbb{Z}^{d+1}$ be a transvers weight vector and let $X_{\overline{w}} \subset \mathbb{P}(\overline{w})$ be a quasi-smooth hypersurface defined by a transverse polynomial $W \in \mathbb{C}[z_0, \ldots, z_d]$. Then

$$\chi_{\mathrm{orb}}(X_w) = \frac{1}{w} \sum_{l,r=0}^{w-1} \prod_{\substack{0 \le i \le d \\ lq_i, rq_i \in \mathbb{Z}}} \left(1 - \frac{1}{q_i}\right).$$

In this formula, one denotes $q_i := \frac{w_i}{w}$ $(0 \le i \le d)$, and one assumes

$$\prod_{\substack{0 \le i \le d \\ lq_i, rq_i \in \mathbb{Z}}} \left(1 - \frac{1}{q_i}\right) = 1$$

if $lq_i, rq_i \notin \mathbb{Z}$ for all $i \in \{0, \ldots, d\}$.

Orbifold Euler number

Theorem,(Ono and Roan 1993)

Let $X_w \subset \mathbb{P}(\overline{w})$ be a quasi-smooth Calabi-Yau hypersurface X_w of degree $w = \sum_{i=0}^{d} w_i$ defined by a general transverse polynomial W and let $S^{2d+1} \subseteq \mathbb{C}^{d+1} \setminus \{0\}$ be the unit sphere. Consider the compact smooth $(2d-1)$-dimensional real manifold $S_w := S^{2d+1} \cap \{W = 0\}$ together with the S^1-fibration $S_w \to X_w$ obtained from the Seifert S^1-fibration $S^{2d+1} \to \mathbb{P}(w_0, w_1, \ldots, w_d)$. Then the S^1-equivariant K-groups $K^i_{S^1}(S_w)$ $(i = 0, 1)$ have finite rank and

$$\mathrm{rank}\, K^0_{S^1}(S_w) - \mathrm{rank}\, K^1_{S^1}(S_w) = \frac{1}{w} \sum_{l,r=0}^{w-1} \prod_{\substack{0 \le i \le d \\ lq_i, rq_i \in \mathbb{Z}}} \left(1 - \frac{1}{q_i}\right).$$

In particular, the right hand side of the above equality is an integer.

The Laurent polynomial of Givental-Hori-Vafa

The dual to $\mathbb{T}_{\overline{w}}$ is the d-dimensional algebraic torus:

$$\mathbb{T}^*_{\overline{w}} := \{(x_0, x_1, \ldots, x_d) \in (\mathbb{C}^*)^{d+1} \mid \prod_{i=0}^{d} x_i^{w_i} = 1\} \subset (\mathbb{C}^*)^{d+1}$$

with the lattice of characters

$$M_{\overline{w}} = \mathbb{Z}^{d+1}/\mathbb{Z}(\overline{w}).$$

If x_i $(0 \leq i \leq d)$ is the standard basis of characters of $(\mathbb{C}^*)^{d+1}$, the the sum $\sum_{i=0}^{d} x_i$ is a regular function on $\mathbb{T}^*_{\overline{w}}$, a Laurent polynomial $f^0_{\overline{w}}(\mathbf{t})$ that we call Givental-Hori-Vafa polynomial of the weighted projective space $\mathbb{P}(w_0, w_1, \ldots, w_d)$.

The Newton polytope of $f^0_{\overline{w}}(\mathbf{t})$

The Newton polytope of $f_{\overline{w}}(\mathbf{t})$ is a d-dimensional simplex $\Delta := \mathrm{conv}(v_0, v_1, \ldots, v_d)$ with lattice vertices $v_0, v_1, \ldots, v_d$ spanning the lattice $N_{\overline{w}}$ and satisfying the relation $\sum_{i=0}^{d} w_i v_i = 0$.

Example

Let $\overline{w}$ be a sequence of weights $w_0, w_1, \ldots, w_d$ such that $w_0 = 1$. Then there is an isomorphism $M_w \cong \mathbb{Z}^d$ such that the lattice vectors $v_1, \ldots, v_d \in \mathbb{Z}^d$ can be chosen as the standard $\mathbb{Z}$-basis and $v_0 = (-w_1, \ldots, -w_d)$. Then the Laurent polynomial $f^0_w \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ has the form

$$f^0_w(\mathbf{t}) = \sum_{i=0}^{d} \mathbf{t}^{v_i} = \frac{1}{t_1^{w_1} \cdots t_d^{w_d}} + t_1 + \cdots + t_d.$$

The Laurent polynomial of Givental-Hori-Vafa

Example

If all weights w_i are equal 1, we obtain the well-known polynomial

$$f^0(\mathbf{t}) = \frac{1}{t_1 \cdots t_d} + t_1 + \cdots + t_d$$

for usual d-dimensional projective space. It describes Landau-Ginzburg mirror of $\mathbb{P}^d$.

The Newton polytope of $f_w^0(\mathbf{t})$

The Newton polytope of the Giventatl-Hori-Vafa polynomial $f_w(\mathbf{t})$ is the lattice simplex $\Delta_{\overline{w}}$ with lattice vertices $v_0, v_1, \dots, v_d \in M_w$ generating the lattice M_w and satisfying the relation

$$\sum_{i=0}^{d} w_i v_i = 0.$$

The origin $0 \in M$ is an interior lattice point of $\Delta_{\overline{w}}$. It is easy to show that the Laurent polynomial $f_w^0(\mathbf{t})$ is non-degenerate.

Main theorem

Theorem (B., Schaller, 2020)

Let $\overline{w} = (w_0, w_1, \dots, w_d)$ be a weight vector with *IP*-property. Let $\mathbb{T}_{\overline{w}}$ be d-dimensional algebraic torus with the lattice of characters $M_{\overline{w}} := \mathbb{Z}^{d+1}/\mathbb{Z}\overline{w}$. Denote by $v_0, v_1, \dots, v_d$ in the lattice points $M_{\overline{w}}$ obtained from the standard basis of $\mathbb{Z}^{d+1}$. Then any non-degenerate affine hypersurface $Z_w \subseteq \mathbb{T}_{\overline{w}}$ defined by a Laurent polynomial with Newton polytope $\Delta^*_{\overline{w}} = \mathrm{conv}(v_0, \dots, v_d)$ admits a Calabi-Yau compactification $X^*_{\overline{w}}$ and its stringy Euler number equals

$$\chi_{\mathrm{str}}(X^*_w) = (-1)^{d-1} \frac{1}{w} \sum_{l,r=0}^{w-1} \prod_{\substack{0 \le i \le d \\ lq_i, rq_i \in \mathbb{Z}}} \left(1 - \frac{1}{q_i}\right),$$

where $q_i = \frac{w_i}{w}$ $(i \in I)$. In particular,

$$\boxed{\chi_{\mathrm{str}}(X^*_w) = (-1)^{d-1} \chi_{\mathrm{orb}}(X_w),}$$

if $\overline{w}$ is transverse.

The above theorem supports the following:

Mirror Construction

Let $\overline{w} \in \mathbb{Z}_{>0}^{d+1}$ be a transverse weight vector. Then mirrors of quasi-smooth Calabi-Yau hypersurfaces $X_w \subset \mathbb{P}(w_0, w_1, \ldots, w_d)$ can be obtained as Calabi-Yau compactifications of non-degenerate affine hypersurfaces $Z_{\overline{w}} \subset \mathbb{T}^*_{\overline{w}}$ defined by Laurent polynomials f with the Newton polytope

$$\Delta^*_{\overline{w}} = \mathrm{conv}(v_0, v_1, \ldots, v_d).$$

Invertible polynomials

Definition

A transverse polynomial $W \in \mathbb{C}[z_0, \ldots, z_d]$ is called invertible if its Newton polytope is a d-dimensional simplex with vertices

$$\nu_0, \nu_1, \ldots, \nu_d \in \{\mathbb{Z}_{\geq 0}^{d+1} \cap \sum_{i=0}^{d} w_i u_i = w\}$$

and

$$W(\mathbf{z}) = \sum_{i=0}^{d} \mathbf{z}^{\nu_i}.$$

Bergulnd-Huebsch-Krawitz mirror construction

If a transverse weight vector $\overline{w}$ admits an invertible polynomial W, then the Bergulnd-Huebsch-Krawitz mirror construction suggests an invertible homogenization W' of the Givental-Hori-Vafa Laurent polynomial $f^0_{\overline{w}}$ as a G-invariant transverse Calabi-Yau hypersurface in another weighted projective space $\mathbb{P}(\overline{w'})$, where G is a finite abelian diagonal group G. The quotient $\{W' = 0\}/G$ is a mirror Calabi-Yau compactification of the affine Givental-Hori-Vafa hypersurface $Z^0_{\overline{w}}$. We remark that the choice of an invertible polynomial W is not unique. Different choices of W define different Calabi-Yau compactification of the same affine hypersurface $Z^0_{\overline{w}}$.

It is easy to see that the above mirror constuction is a generalization of Berglund-Huebsch-Krawitz mirror construction for quasi-smooth Calabi-Yau varieties defined by arbitrary transverse polynomials W.

Connection to reflexive polytopes

The proposed mirror construction for quasi-smooth Calabi-Yau hypersurfaces in weighted projective spaces is different from the one based on the duality for reflexive polytopes.

Reflexive Simplex Δ_1 and its dual $\Delta_1^* = [\Delta_1^*]$

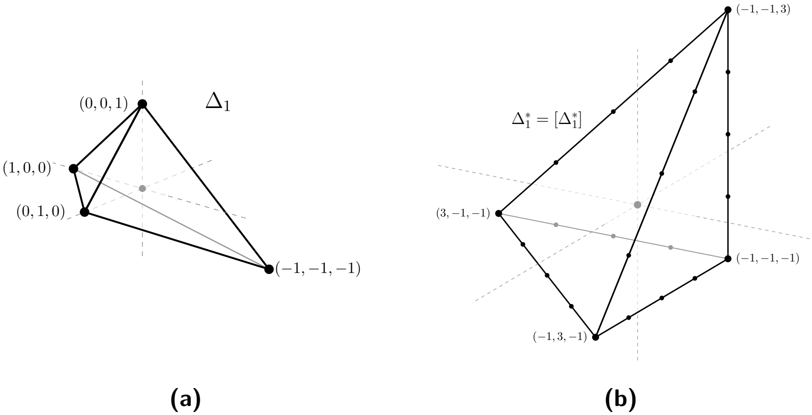


Figure: Reflexive Simplex Δ_1 and its dual $\Delta_1^* = [\Delta_1^*]$.

Source: Karin Schaller, *Stringy Invariants of Algebraic Varieties and Lattice Polytopes*, Ph.D. thesis, Eberhart Karls Univ. Tübingen, 2019.

Almost Reflexive Simplex Δ_2 and its dual $\Delta_2^* > [\Delta_2^*]$

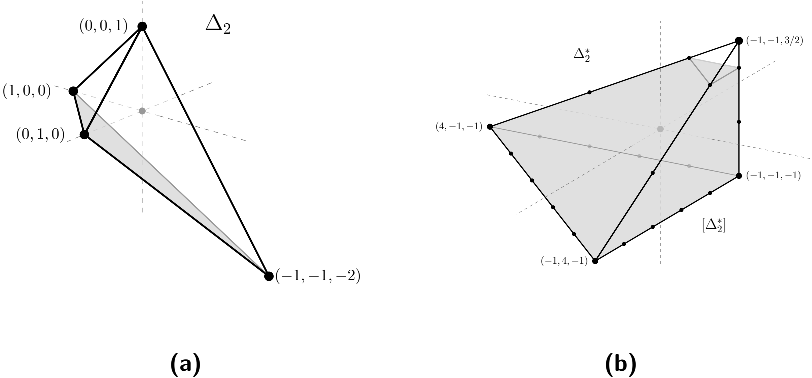


Figure: Almost Reflexive Simplex Δ_2 and its dual $\Delta_2^* > [\Delta_2^*]$.

Source: Karin Schaller, *Stringy Invariants of Algebraic Varieties and Lattice Polytopes*, Ph.D. thesis, Eberhart Karls Univ. Tübingen, 2019.

Example 1

Take the weight vector $\overline{w} := (1, 1, 6, 14, 21)$. It has *IP*-property, but it is not transverse. A general hypersurface $X_{43} \subseteq \mathbb{P}(1, 1, 6, 14, 21)$ is a canonical Calabi-Yau variety, but it is not quasi-smooth. The Newton polytope of $X_{43} \subset \mathbb{P}(1, 1, 6, 14, 21)$ is reflexive. Therefore, X_{43} is birational to a smooth CY 3-fold Y with $h^{1,1}(Y) = 21$, $h^{2,1}(Y) = 273$, and $\chi(Y) = -504$.
On the other hand, the affine hypersurface $Z_{\overline{w}} \subseteq (\mathbb{C}^*)^4$

$$\frac{1}{t_1 t_2^6 t_3^{14} t_4^{21}} + t_1 + t_2 + t_3 + t_4 = 0.$$

admits a Calabi-Yau compactification $X^*_{\overline{w}}$ with the stringy Euler number

$$\chi_{\rm str}(X^*_{\overline{w}}) = 506 \neq 504 = -\chi_{\rm str}(X_{43}) = -\chi(Y).$$

Therefore, $X^*_{\overline{w}}$ is not a mirror of X_{43}.

Example 2

Take the weight vector $\overline{w} := (1, 1, 2, 4, 5)$. It has *IP*-property, but it is not transverse. A general hypersurface $X_{13} \subseteq \mathbb{P}(1, 1, 2, 4, 5)$ is a canonical Calabi-Yau variety, but it is not quasi-smooth. Consider the affine hypersurface $Z_{\overline{w}} \subseteq (\mathbb{C}^*)^4$

$$\frac{1}{t_1 t_2^2 t_3^4 t_4^5} + t_1 + t_2 + t_3 + t_4 = 0.$$

It admits a Calabi-Yau compactification $X^*_{\overline{w}}$. However, the stringy Euler number

$$\chi_{\mathrm{str}}(X^*_{\overline{w}}) = \frac{1032}{5} \notin \mathbb{Z}.$$

Therefore, $X^*_{\overline{w}}$ has no mirror at all.

Thank you !