Multiple Orthogonal Polynomials with respect to Hermite weights: applications and asymptotics

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Joint project with

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(Ishlinsky Institute for Problems in Mechanics RAS)

and

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(Keldysh Institute of Applied Mathematics RAS)

Plan of the Talk

1. Introduction: definitions, properties and goals
2. Appications: Random matrices and Brownian bridges
3. Recurrences and Plancherel-Rotach type asymptotics
4. Expansions of bases of homogeneous difference equations
5. Spectral curve, statement of the results and disscusion

Introduction

Multiple Orthogonal Polynomials (MOPs)

- OPs P_n : $\int_{\mathbb{R}} x^k P_n(x) d\mu(x) = 0\,, \quad 0 \leq k < n := \deg P_n.$

  Recurrence relations $P_n(x) = x^n + \dots$:

  $$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x)\,.$$

- Input: $\vec{\mu} := (\mu_1, \dots, \mu_d)$ on $\mathbb{R}$, $\quad \vec{n} := (n_1, \dots, n_d) \in \mathbb{Z}_+^d$

  MOPs $P_{\vec{n}}$: $\int_{\mathbb{R}} x^k P_{\vec{n}}(x) d\mu_j(x) = 0\,, \quad 0 \leq k < n_j,$

  $$\deg P_{\vec{n}} \leqslant |\vec{n}| := \sum_{l=1}^{d} n_l, \qquad j = 1, \dots, d.$$

- Lattice recurrence relations $\quad P_{\vec{n}}(x) = x^{|\vec{n}|} + \dots$:

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  $$\vec{n} \in \mathbb{N}^d, \qquad j = 1, \dots, d.$$

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Multiple Hermite Polynomials

- Definition (d=2). For multi-index $\vec{n} = (n_1, n_2) \in \mathbb{Z}_+^2$, $H_{\vec{n}}(x)$, $\deg H_{\vec{n}} = |\vec{n}| =: (n_1 + n_2)$:

$$\begin{cases} \int\limits_{-\infty}^{+\infty} H_{\vec{n}}(x) x^{\nu} e^{-x^2 - ax}\, dx = 0, \quad \nu = 0, \dots, n_1 - 1, \\ \int\limits_{-\infty}^{+\infty} H_{\vec{n}}(x) x^{\nu} e^{-x^2 + ax}\, dx = 0, \quad \nu = 0, \dots, n_2 - 1, \end{cases} \qquad a \neq 0.$$

- Properties (for fixed $\vec{n}$): recurrences, differential equations, Rodrigues type formulas, integral representations, ...

  V. N. Sorokin, J. Soviet Math. 45 (1986), 1461–1499.

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Applications

Random matrices ensembles

- History of the subject
- $\{\mathrm{H}_n, \mu_n\}$: $M \in \mathrm{H}_n$, $\mu_n(dM)$
- Unitary ensembles (UE) : $\mu_n(dM) = \frac{1}{Z_n} e^{-n\mathrm{Tr}(V(M))} dM$.

$$dM := \prod_{j=1}^{n} dM_{jj} \prod_{1 \leqslant j < k \leqslant n} d\,\Re(M_{jk})\, d\,\mathrm{Im}(M_{jk}),$$

- Formula for the Joint probability density of e.v. distribution.
- Orthogonal, Normal, ... matrices ensembles
- Distribution of e.v., Global and Local regimes, Universality

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- Matrix elements are Gaussian random variables.

- Average characteristic polynomials is Hermite polynomial!

- Global regime of e.v. distribution –> Wigner Semicircle Law

- Local regime: Sinus kernel and Universality, ...

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Random matrices with external source and Perturbation by GUE Matrices

- $\{\mathrm{H}_n, \mu_n\}$: $\quad M \in \mathrm{H}_n\,, \quad \mu_n(dM) = \frac{1}{Z_n}\, e^{-n\mathrm{Tr}(V(M) - AM)}\, dM\,.$

- $V(x) = \frac{x^2}{2} \quad \Rightarrow \quad M = A + M_0,$
  where M_0 is a random matrix from GUE

- Density of joint probability distribution of e.v.

$$\frac{1}{\tilde{Z}_n} \prod_{1 \le j < k \le n} (\lambda_j - \lambda_k) \det\left(e^{n\lambda_j a_k}\right)_{j,k=1}^{n} \prod_{j=1}^{n} e^{-\frac{1}{2} n \lambda_j^2}.$$

where $a_1, \ldots, a_n$ are e.v. of A.

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Theorem (on global asymptotical regime)

- Given the ensemble of Hermitian random matrices:

$$\mu(dM) := \frac{1}{\tilde{Z}_n} e^{-n\,\mathrm{Tr}(\frac{1}{2}M^2 - AM)} dM\,, \qquad M \in H_n$$

$$A = \mathrm{diag}(\underbrace{a, \ldots, a}_{n/2}, \underbrace{-a, \ldots, -a}_{n/2}).$$

- Then density of limiting joint probability distribution of e.v. is

$$\rho(x) = \frac{1}{\pi} |\Im \xi(x)|\,,$$

where

$$\xi^3 - x\xi^2 - (a^2 - 1)\xi + xa^2 = 0.$$

- A. I. Aptekarev, P. M. Bleher and A. B. J. Kuijlaars, Large n limit of Gaussian random matrices with external source, Part II, Comm. Math. Phys. , 259 (2005), 367–389.

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Brownian bridjes

- n independent non-intersecting Brownian motions: $(t=0)\ s_1 < s_2 < \cdots < s_n \to (t=1)\ b_1 < b_2 < \ldots < b_n$
- Density of joint probability distribution of position at the moment $t \in (0,1)$

$$p_n(x_1,\ldots,x_n) = \frac{1}{C_n}\det(p(s_j,x_k;t))_{j,k=1}^n \det(p(x_j,b_k;1-t))_{j,k=1}^n$$

- $$p(x,y;t) = \sqrt{\frac{n}{2\pi t}}\, e^{-\frac{n(x-y)^2}{2t}}\,; \qquad \forall\, s_i \to 0 \Rightarrow$$

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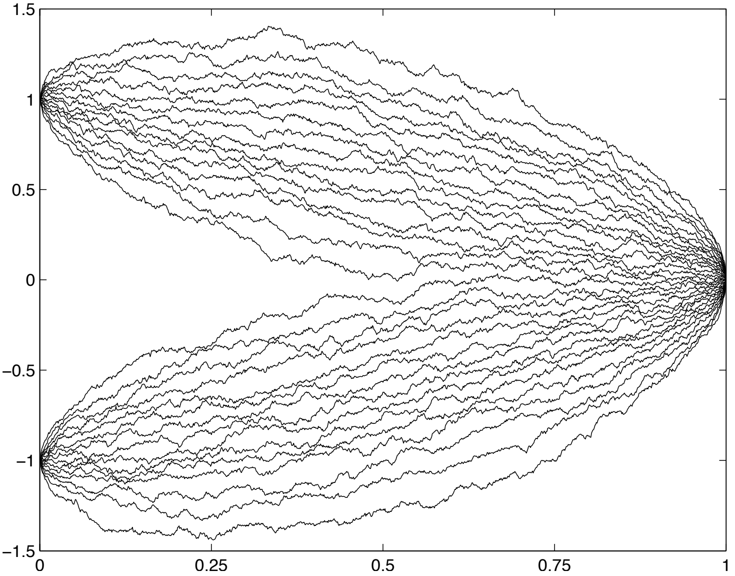
Browinian bridges



Recurrences and asymptotics of Hermite Polynomials

Plancherel-Rotach (PR) type asymptotics and approaches

- $d = 1$ Classical Hermite Polynomials $H_n(x)$:

  M. Plancherel, W. Rotach, Sur les valeures asymptotiques des polynomes d'Hermite, Commentarii Math. Helvetici, 1929, vol.1, 227–257.

  Scaled asymptotics of $H_n(x)$ (using integral representation):

$$n \to \infty \ : \qquad \frac{x}{\sqrt{n}} \ \in \ K \ \Subset \mathbb{C}.$$

- Modern approaches to global PR asymptotics:

  Steepest descend for Matrix Riemann-Hilbert Problem:

  P. Deift at all, A. Its and P. Bleher, ...

- Second goal of our Project:

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Plancherel-Rotach (PR) type asymptotics and approaches

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$$H_{n+1}(x) \;=\; 2x\,H_n(x) - 2n\,H_{n-1}(x)\,, \quad H_0 = 1\,,\; H_{-1} = 0 \quad n \in \mathbb{N}.$$

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- (1): –> Let $\mathfrak{F}_n(x) \;:=\; \frac{(x+\sqrt{x^2-2n})^{n-1/2}}{\sqrt{2}\;\sqrt[4]{x^2-2n}} \exp\left\{\frac{(x-\sqrt{x^2-2n})^2}{4}\right\}$

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$$E(z) = \frac{x^2}{4} + \frac{z^2}{8x^{2/3}} + \, ... \, + \mathcal{O}\big(\frac{|z|^{\frac{k+3}{2}}}{|x|^{\frac{2k}{3}}}\big),$$

$$h(z) := \frac{-z}{2^{2/3}} + \frac{1}{(2x)^{2/3}} + \frac{2^{1/3}z^2}{15x^{4/3}} + ... + \mathcal{O}\big(\frac{|z|_+^{\frac{k+2}{2}}}{|x|^{\frac{2k}{3}}}\big).$$

Approach

System of homogeneous difference equations

Starting point : Recurrence relations

$$
\begin{cases}
H_{(n+1,m)}(x) = (-a+x)H_{(n,m)}(x) - \dfrac{n+m}{2}H_{(n,m-1)}(x) - anH_{(n-1,m-1)}(x), \\
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H_{0,0} := 1, \quad H_{1,0} := x-a, \quad H_{1,1} := x^2 - a^2 - \dfrac{1}{2}.
\end{cases}
$$

- PR $(n=m)$ corresponding growth $N \to \infty$:

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\frac{n}{N} \in K \Subset \mathbb{R}, \qquad \frac{x}{\sqrt{N}} \in \tilde{K} \Subset \mathbb{C}, \qquad \frac{a}{\sqrt{N}} \in \tilde{\tilde{K}} \Subset \mathbb{R},
$$

- Homogeneous difference problem: $\overrightarrow{H_{n+1}} = \mathcal{A}_n \overrightarrow{H_n}$:

$$
\mathcal{A}_n(x) = \begin{pmatrix} x^2 - a^2 - n - \frac{1}{2} & -xn & -a^2 n \\ x & -n & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \overrightarrow{H_n} = \begin{pmatrix} H_{(n,n)} \\ H_{(n,n-1)} - aH_{(n-1,n-1)} \\ H_{(n-1,n-1)} \end{pmatrix}
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Basis of solutions – Outline of the approach

- System of homogeneous difference equations:

$$\overrightarrow{H_{n+1}} = \mathcal{A}_n \overrightarrow{H_n}$$

- "Diagonalizator" V_n:

$$V_{n+1}^{-1} \mathcal{A}_n V_n \equiv \mathrm{diag}[V_{n+1}^{-1} \mathcal{A}_n V_n] =: D_n,$$

- Basis of the solutions:

$$B_n := V_n \prod_{k=k_0}^{n-1} D_n =: V_n \Pi_n$$

Indeed: $\mathcal{A}_n B_n = V_{n+1} V_{n+1}^{-1} \mathcal{A}_n V_n \Pi_n = B_{n+1}.$

- Goal: Find asymptotical expansion for V_n, then for Π_n.

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Statements of the results

Spectral curve and parametrization

Recall. $N \gg 1$, P-R growth of the parameters of $H_{(n,n)}(x)$:

$$\frac{n}{N} \in K \Subset \mathbb{R}, \qquad \frac{x}{\sqrt{N}} \in \tilde{K} \Subset \mathbb{C}, \qquad \frac{a}{\sqrt{N}} \in \tilde{\tilde{K}} \Subset \mathbb{R},$$

We have $\overrightarrow{H_{n+1}} = \mathcal{A}_n \overrightarrow{H_n}$. Eigen values $\{\Lambda_j\}_{j=1}^3$ of $\mathcal{A}_n$:

$$\Lambda : \quad \Lambda^3 + (a^2 + 2n - x^2)\Lambda^2 + (2a^2 n + n^2)\Lambda + a^2 n^2 = 0. \qquad (*)$$

is the spectral curve (4 branch points $\to \triangle$, δ, $\triangle_\pm$).

Another form:

$$(\Lambda + n)^2(\Lambda + a^2) - x^2\Lambda^2 = 0 \quad \Leftrightarrow \quad (\Lambda + a^2) = \frac{x^2\Lambda^2}{(\Lambda + n)^2} =: s^2 x^2.$$

It defines a parametrization of the algebraic curve:

$$\Lambda = (s^2x^2 - a^2), \qquad n = \frac{1}{s}(1 - s)(s^2x^2 - a^2).$$

Statement of the result

- Fix $\varepsilon > 0$, $(n \gg 1)$ and denote

$$\mathfrak{H}_n(x,a) = \Lambda_1^{n+1/2} e^{x^2(1-s)^2} \sqrt{\frac{s}{x^2 s^2(2s-1) - a^2}}, \quad s = \frac{\Lambda_1}{\Lambda_1 + n}.$$

- a) Zone of the growth (no zeros):

$$\Omega_1 := \{x : \operatorname{dist}(x, \triangle) > n^{\varepsilon}\} \cap \left\{x : \operatorname{dist}(x, 0) > n^{\varepsilon+1/3}\right\}, \quad n \in [\varepsilon a^2, (1-\varepsilon)a^2],$$

$$\Omega_2 := \{x : \operatorname{dist}(x, \triangle_+ \cup \triangle_+) > n^{\varepsilon}\}, \qquad n > (1+\varepsilon)a^2,$$

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- b) Zone of the osculation (accomulation of zeros), $x \in \mathbb{R}$:

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Thank you
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