Real-valued semiclassical approximation for the asymptotics with complex-valued phases of the Hermitian type orthogonal polynomials.

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FORMULATION OF THE PROBLEM:

HERMITIAN TYPE ORTHOGONAL POLYNOMIALS

(P. Deift, A. I. Aptekarev, P. M. Bleher, A. Branquinho, A.R.Its,
A. B. J. Kuijlaars, T. Kriecherbauer, K. T.-R. McLaughlin, W. Van Assche,
S. Venakides, X. Zhou ....)

Recurrent equations

$$\mathrm{H}_{\mathrm{n}_1+1,\mathrm{n}_2}(\mathrm{z},\alpha) = \mathrm{z}\mathrm{H}_{\mathrm{n}_1,\mathrm{n}_2}(\mathrm{z},\alpha) + \alpha\mathrm{H}_{\mathrm{n}_1,\mathrm{n}_2}(\mathrm{z},\alpha) - \frac{1}{2}\left(\mathrm{n}_1\mathrm{H}_{\mathrm{n}_1-1,\mathrm{n}_2}(\mathrm{z},\alpha) + \mathrm{n}_2\mathrm{H}_{\mathrm{n}_1,\mathrm{n}_2-1}(\mathrm{z},\alpha)\right),$$

$$\mathrm{H}_{\mathrm{n}_1,\mathrm{n}_2+1}(\mathrm{z},\alpha) = \mathrm{z}\mathrm{H}_{\mathrm{n}_1,\mathrm{n}_2}(\mathrm{z},\alpha) - \alpha\mathrm{H}_{\mathrm{n}_1,\mathrm{n}_2}(\mathrm{z},\alpha) - \frac{1}{2}\left(\mathrm{n}_1\mathrm{H}_{\mathrm{n}_1-1,\mathrm{n}_2}(\mathrm{z},\alpha) + \mathrm{n}_2\mathrm{H}_{\mathrm{n}_1,\mathrm{n}_2-1}(\mathrm{z},\alpha)\right)$$

Initial data

$$\mathrm{H}_{0,0}(\mathrm{z},\alpha) = 1, \quad \mathrm{H}_{\mathrm{n},-1}(\mathrm{z},\alpha) = \mathrm{H}_{-1,\mathrm{n}}(\mathrm{z},\alpha) = 0, \mathrm{n} \in \mathbb{Z}^+, \mathrm{n} > 1$$

The aim:
to construct the asymptotics of dioganal polynomial $H_{n,n}(z,\alpha)$ as $n \to \infty$

Recurrent equations for $H_{n,n}(z,\alpha), H_{n,n-1}(z,\alpha)$

$$\begin{pmatrix} \mathrm{H}_{\mathrm{n,n}} \\ \mathrm{H}_{\mathrm{n+1,n}} \\ \mathrm{H}_{\mathrm{n+1,n+1}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \alpha \mathrm{n} & -\mathrm{n} & \mathrm{z}+\alpha \\ \alpha \mathrm{n}(\mathrm{z}-\alpha) & -\mathrm{nz} & \mathrm{z}^2-\alpha^2-\frac{1}{2}(2\mathrm{n}+1) \end{pmatrix} \begin{pmatrix} \mathrm{H}_{\mathrm{n-1,n-1}} \\ \mathrm{H}_{\mathrm{n,n-1}} \\ \mathrm{H}_{\mathrm{n,n}} \end{pmatrix}$$

Initial data

$$\mathrm{H}_{0,0}(\mathrm{z},\alpha) = 1, \quad \mathrm{H}_{1,0}(\mathrm{z},\alpha) = \mathrm{z}+\alpha, \quad \mathrm{H}_{1,1}(\mathrm{z},\alpha) = \mathrm{z}^2-\alpha^2-1/2$$

Introduce small artificial parameter $h = \dfrac{1}{n}$,

thus we are looking for asymptotics as $h \to +0$

Two approaches:

(1) based on the construction of decompositions of bases of homogeneous difference equations (A.I.Aptekarev's talk),

(2) "real semiclassics for asymptotics with complex-valued phases"

A lot of results: E.Hilb, M. Plancherel, W. Rotach, F. Olver, P.K.Suetin, S.P.Suetin, X. Zhou, Z. Wang, R. Wong, X-Sh. Wang, P. Deift, A.R.Its, A. B. J. Kuijlaars, T. Kriecherbauer, K. T.-R. McLaughlin, A.I.Aptekarev, D.M.Tulyakov, P. M. Bleher, A. Branquinho, W. Van Assche, S. Venakides, V.Yu.Novokshonov, D.R.Yafaev, I.T.Yakubov.....

Retreat: excursion to the polynomials defined recurrence equations of the second order

Uniform Plancherel-Rotach type asymptotics of Hermitian polynomials

$$\mathrm{H}_{\mathrm{n}+1}(\mathrm{z}) = 2\mathrm{z}\mathrm{H}_{\mathrm{n}}(\mathrm{z}) - 2\mathrm{n}\mathrm{H}_{\mathrm{n}-1}(\mathrm{z}),\ \mathrm{H}_0(\mathrm{z}) = 1,\ \mathrm{H}_1(\mathrm{z}) = 2\mathrm{z}$$

Following V.P.Maslov introduce the smooth function

$$\psi(x, z) : \psi(kh, z) = H_k(z),$$

the shift operators $\mathrm{e}^{\pm \mathrm{i}\hat{\mathrm{p}}}\psi(\mathrm{x}) = \psi(\mathrm{x} \pm \mathrm{h}), \quad \hat{\mathrm{p}} = -\mathrm{i}\mathrm{h}\frac{\partial}{\partial \mathrm{x}}$

and rewrite the difference equation in a pseudodifferential form

$$\left(\frac{1}{2}\mathrm{e}^{\mathrm{i}\hat{\mathrm{p}}} + \frac{\mathrm{x}}{\mathrm{h}}\mathrm{e}^{-\mathrm{i}\hat{\mathrm{p}}} - \frac{\mathrm{y}}{\sqrt{\mathrm{h}}}\right)\psi(\mathrm{x}, \mathrm{y}) = 0, \quad \mathrm{y} = \sqrt{\mathrm{h}}\mathrm{z}$$

with complex-valued symbol $\mathcal{H}(\mathrm{x}, \mathrm{p}; \mathrm{h}) = \frac{1}{2}\mathrm{e}^{\mathrm{i}\mathrm{p}} + \frac{\mathrm{x}}{\mathrm{h}}\mathrm{e}^{-\mathrm{i}\mathrm{p}} - \frac{\mathrm{y}}{\sqrt{\mathrm{h}}}$

$$f = h^{-x/2}\psi : \quad \hat{\mathbf{H}}f = 0, \quad \mathbf{H} = (\frac{1}{2} + x)\cos p - y + i(\frac{1}{2} - x)\sin p$$

General case:

$$a_n u_{n+1}(z) + b_n(z) u_n(z) + c_n u_{n-1}(z) = 0, \quad n = 0, 1, \ldots, N, \ldots, \quad z \in \mathbb{R},$$

$$b_n = b_n^0 + z b_n^1, \quad u_0(z) = v^0, u_1(z) = v^1 + v^2 z$$

Pseudodifferential equation for $f(x; z), \quad u_n = f(nh; z)$:

$$\alpha(x, h; z) \longrightarrow a_n,\ \beta(x, h; z) \longrightarrow b_n(z),\ \gamma(x, h; z) \longrightarrow c_n$$

$$\alpha(x, h; z) e^{i\hat{p}} f(x; z) + \beta(x, h; z) f(x; z) + \gamma(x, h; z) e^{-i\hat{p}} f(x; z) =$$

$$((\alpha + \gamma) \cos \hat{p} + \beta + i(\alpha - \gamma) \sin \hat{p}) f(x; z) = 0,$$

x is a variable and z is a parameter

WKB-anzatz: $f = \sum A_j(x; z, h) e^{\frac{i}{h}(S^j(x;z) + i\Phi^j(x;z))}$

Complex-valued Hamiltonians $\Longrightarrow$ complex-valued phases

Approaches with complex-valued phases:

WKB-method with transition to the complex plane:

second-order differential equations (Stokes,..., Olver,..., Fedoryuk,..., Shkalikov, Shafarevich, Alliluyeva, S. Stepin, S. Suetin...), above-barrier scattering (Pokrovsky...), "momentum" tunneling (Dobrokhotov, Shafarevich)
discrete Schrödinger equation (Buslaev, Fedotov, Klopp, Shetka ...)

Limitations:
1-D case, equations of the type of continuous and discrete Schrödinger equation

complex rays (Maslov $\Longrightarrow$ Khudyakov, Kravtsov-Orlov),

almost-analytic continuation for the case $\mathrm{Im} S \geq 0$
(Maslov, Kucherenko, Sjöstrand),
complex germ theory (Maslov)== approximate finding of complex phases and the corresponding geometry+linearization of complex Lagrangian manifolds and Hamiltonian systems (application to difference schemes – Maslov, Danilov)
special case (quadratic complex germ) – oscillatory approximation; Gaussian beams; V. M. Babich's asymptotic eigenfunctions concentrated in the vicinity of a closed geodesic (the relativistic Sokolov-Ternov electron in accelerators); trajectory-coherent states, etc.

purely imaginary phases $\mathrm{Re} S = 0$): tunneling problems and probabilistic problems, thermodynamics (*tropical mathematics*)

Our approach for asymptotics of polynomials

First, we explain the idea by the example of a second-order differential equation:

$$-h^2\alpha(x)\frac{d^2u}{dx^2} + h\beta(x)\frac{du}{dx} + \gamma(x)u = 0 \Leftrightarrow \alpha\hat{p}^2u + i\beta\hat{p}u + \gamma u = 0, \quad \hat{p} = -ih\frac{d}{dx}.$$

The symbol $\mathcal{H} = \alpha(x)p^2 + i\beta(x)p + \gamma(x)$ of this equation is complex.
A standard well-known replacement

$$u = w\psi, \quad w = e^{\frac{\Phi_0(x)}{h}}, \quad \Phi_0 = \int_{x_0}^{x} \frac{\beta}{2\alpha}dx$$

reduces original equation to the Schrödinger type equation with a real potential

$$\mathcal{H}_1(\hat{p}, x)\psi \equiv -h^2\frac{d^2\psi}{dx^2} + (\mathcal{V}_0(x) + h\mathcal{V}_1(x))\psi = 0,$$

here $\mathcal{H}_1 = p^2 + \mathcal{V}_0(x) + h\mathcal{V}_1(x)$, $\quad \mathcal{V}_0(x) = \frac{\beta^2}{4\alpha^2} + \frac{\gamma}{\alpha}$, $\quad \mathcal{V}_1(x) = -\frac{d}{dx}\left(\frac{\beta}{2\alpha}\right)$.

The equation for w: $\mathcal{H}_0(\hat{\xi}, x)w = 0$, $\hat{\xi} = h\frac{d}{dx}$ with real symbol $\mathcal{H}_0 = \xi - \frac{\beta}{2\alpha}$.

Discrete case $\quad f(x,h) = e^{\frac{S_0(x,h)}{h}} g(x,h) = e^{\frac{S_0(x)}{h}} A_0(x,h) g(x,h).$

We use various Feynman-Maslov operator formulas in semi-classical approximation problems, such as commutation formulas for a pseudodifferential operator with a rapidly changing exponent $A(x)e^{\frac{i}{h}S(x)}$, the transition from generating operators to the product of their symbols, etc. It requires justification if $S(x)$ is a complex-valued function with $\mathrm{Im}S$ of indeterminate sign. In the case of general pseudo-differential operators, the justification of these formulas requires a complicated and delicate analysis, based in particular on the saddle point method. Moreover, these formulas are, generally speaking, may prove to be incorrect. In the theory of the complex Maslov germ, it is assumed that $\mathrm{Im}S \geq 0$, the corresponding formulas are approximate and work in a small neighborhood of the set $\mathrm{Im} = 0$. The situation here is *radically* different, the considerations used in the complex germ are not enough at all. But the class of operators here is also very narrow-we work with shift operators, and the corresponding formulas are easily proved using Taylor series expansion. Moreover, these considerations are transferred to the multidimensional case. Also such formulas

$$e^{\pm i\hat{p}}(f_1(x)f_2(x)) = f_1(x \pm h)f_2(x \pm h), \quad e^{\pm i\hat{p}}(F(f(x)) = (F(f(x \pm h))$$

are true (although they are not correct for general pseudodifferential operators)

Results

$$f(x,h) = e^{\frac{\mathcal{S}_0(x,h)}{h}} g(x,h) = e^{\frac{S_0(x)}{h}} (A_0(x) + O(h)) g(x,h)$$

$$S_0(x) = \frac{1}{2}\int \log\frac{\gamma(y,0)}{\alpha(y,0)}dy, \quad A_0(x) = \exp\left[\frac{1}{2}\int \frac{\partial}{\partial h}\log\frac{\gamma(y,h)}{\alpha(y,h)}\bigg|_{h=0} dy\right]$$

Discrete Schrödinger type equation

$$\hat{\mathcal{H}}_1 g(x,h) = \left(\cos\hat{p} + V(x,h)\right) g(x,h) = 0$$

$$V(x,h) = \frac{\beta(x,h)}{2\gamma(x,h)} e^{\frac{1}{1+e^{i\hat{p}}}\log\frac{\gamma(x,h)}{\alpha(x,h)}} = \frac{\beta(x,h)}{2\gamma(x,h)}\sqrt{\frac{\gamma(x-\frac{h}{2},h)}{\alpha(x-\frac{h}{2},h)}}(1+O(h^2))$$

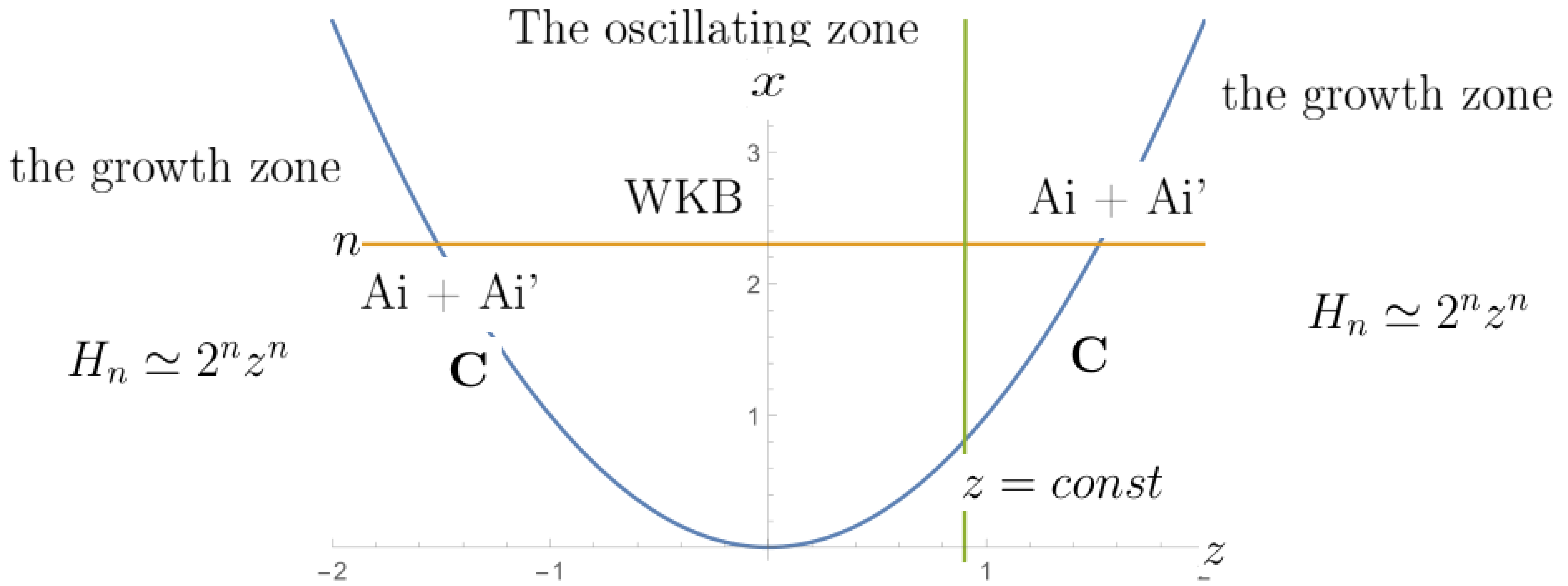
The structure of solutions for polynomials: then $V = V(x, z, h)$

The oscillating zone on the half plane $\{x > 0, z \in \mathbb{R}\colon |V(x, z, 0)| \le 1,$

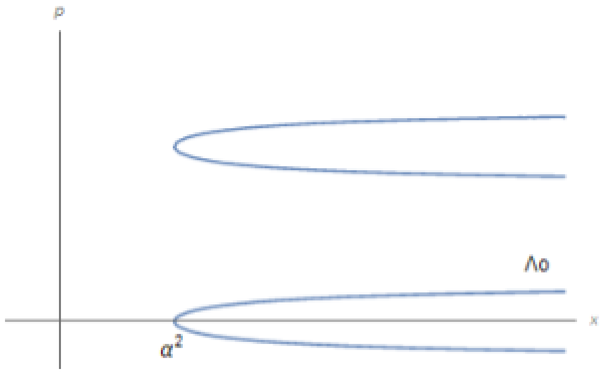
the boundary of the oscillating zone $\mathbf{C} = \{|V(x, z, 0)| = 1 \quad x = \mathcal{X}(z)\},$

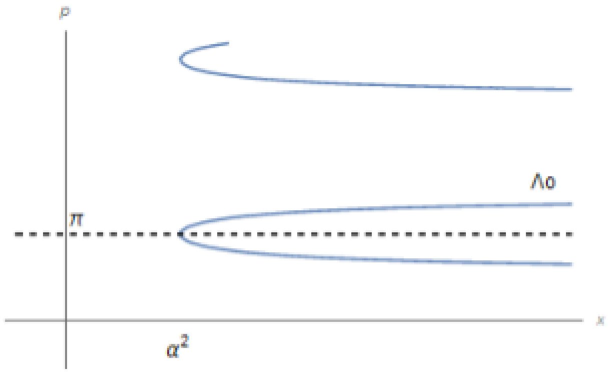
$\mathbf{C}$ be a set of turning points (a caustic) if $\left.\frac{\partial V(x,z,0)}{\partial x}\right|_{\mathbf{C}} \neq 0$

The the example of Hermitian polynomials



Lagrangian manifolds=invariant sets: $\Lambda_k = \{\cos p + V(x, z, 0) = 0,\ z = const\}$





$$V = -\frac{x}{\sqrt{y}},\ x = nh,\ y = \sqrt{h}z$$

The answer $g = q(z) K_{\Lambda_0} \mathcal{A}^0,\ z > 0, \quad g = q(z) K_{\Lambda_1} \mathcal{A}^1,\ z < 0,$
here K_{Λ_j} is the Maslov canonical operator,
analytical function $q(z)$ is the constant of integration

Boundary conditions: $H_n \simeq 2^n z^n$

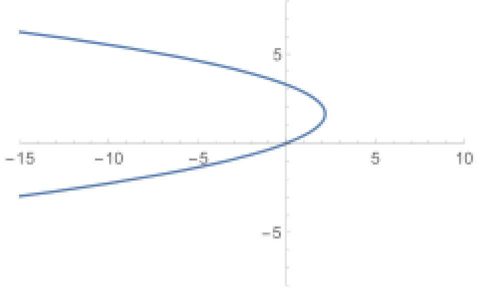
Using the Maslov canonical operator and new uniform formulas for its realisation in the wide neighborhood of simple caustics in the form of Airy functions (A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. V. Tsvetkova, Theoret. and Math. Phys., 2019) we obtain the generalised Plancherel-Rotach asymptotics (S. Yu. Dobrokhotov, A. V. Tsvetkova, Math. Notes, 2018)

x *is real and not complex*!

General approach:

The focal point and uniform asymptotics in the form of Airy function:

"naive" constructive approach



Let $x = X(\alpha^*)$ be nondegenerated focal point:

$$\frac{\partial X}{\partial \alpha}(\alpha^*) = 0, \quad \frac{\partial^2 X}{\partial \alpha^2}(\alpha^*) \neq 0.$$

$$\Lambda = \{p = P^* + P'^*(\alpha - \alpha^*) + O((\alpha-\alpha^*)^2), x = X^* + \frac{1}{2}X''^*(\alpha-\alpha^*)^2 + O((\alpha-\alpha^*)^3)\}$$

looks like a "horizontal" parabola.

Then for small $x - X^*$ one can show (at least on the physical level of rigor) that the asymptotic of the integral is expressed in terms of the **Airy function and its derivative**. This implies the following anzats:

$$\psi \approx e^{i\frac{Q(x)}{h}}\big(A_1(x,h)\mathrm{Ai}(\Phi(x,h)) + A_2(x,h)\mathrm{Ai}'(\Phi(x,h))\big),$$

here phases $Q(x)$, $\Phi(x,h)$ and amplitudes $A_j(x,h)$ are unknown functions,

Why Airy ′ ? $\quad \alpha^* = 0 \quad A(\alpha) = \frac{A(\alpha)+A(-\alpha)}{2} + \alpha\frac{A(\alpha)-A(-\alpha)}{2\alpha} = \underbrace{g_1(X)}_{\text{Airy}} + \underbrace{\alpha g_2(X)}_{\text{Airy}'}$

We can write for $\Phi(x) \ll -1$

$$\psi \approx \frac{e^{i\frac{Q(x)}{h}}}{\sqrt{\pi}}\left(\frac{A_1(x,h)}{\sqrt[4]{z}}\sin(z+\frac{\pi}{4})+A_2(x,h)\sqrt[4]{z}\cos(z+\frac{\pi}{4})\right), \quad z = \frac{2}{3}(-\Phi)^{3/2}.$$

From the other side the Maslov canonical operator gives:

$$\psi \approx a_+(x)e^{-i\frac{\pi}{2}}e^{\frac{iS_+(x)}{h}} + a_-(x)e^{\frac{iS_-(x)}{h}} =$$

$$e^{-\frac{i\pi}{4}}e^{\frac{i}{h}(S_+ + S_-)}(a_+ + a_-)\cos(\frac{S_- - S_+}{h} + \frac{\pi}{4}) + i(a_- - a_+)\sin(\frac{S_- - S_+}{h} + \frac{\pi}{4}),$$

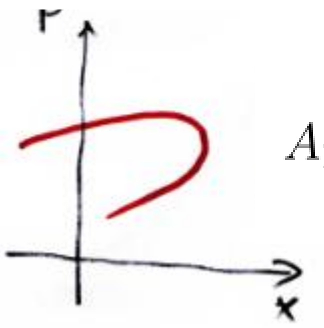
where

$$S_\pm = \int_{\alpha_0}^{\alpha_\pm(x)} P dX, \quad a_\pm = \frac{A(\alpha_\pm(x))}{\sqrt{|J(\alpha_\pm(x))}}$$

and $\alpha_\pm(x)$ are two solutions to the equation $X(\alpha) = x$.

This gives

$$Q = (S_- + S_+), \quad \Phi = -\frac{3(S_- - S_+)^{2/3}}{2h^{2/3}},$$

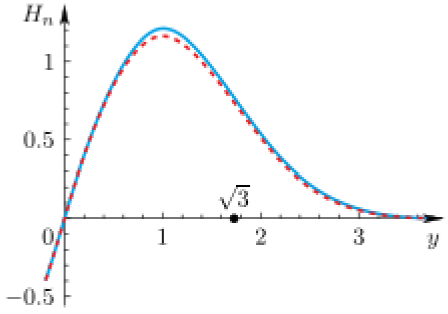


$$A_1 = \frac{ie^{-\frac{i\pi}{4}}}{\sqrt[6]{h}\sqrt{\pi}}(a_+ - a_-)\sqrt[3]{S_+ - S_-}, \quad A_2 = \frac{\sqrt[6]{h}e^{-\frac{i\pi}{4}}}{\sqrt{\pi}}\frac{(a_+ + a_-)}{\sqrt[3]{S_+ - S_-}}$$

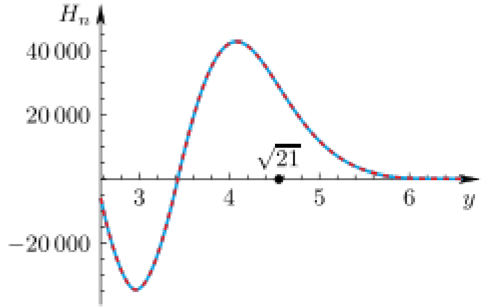
Uniform Plancherel-Rotach type formula for the Hermitian polinomials

$$H_n(z) \simeq \operatorname{sign}(z^n) e^{\frac{z^2}{2}} \sqrt{2\pi} \left(\frac{2n}{e}\right)^{\frac{n}{2}} \left|1 - \frac{z^2-1}{2n}\right|^{-\frac{1}{4}} |F_n(z)|^{\frac{1}{4}} \operatorname{Ai}\left(F_n(z)\right)$$

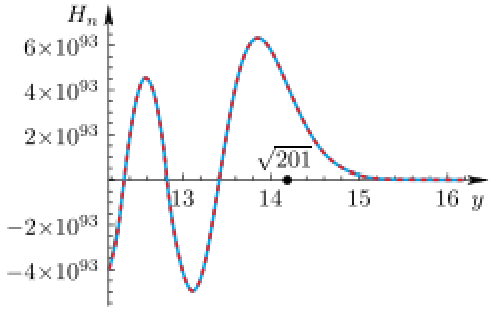
$$F_n(z) = \begin{cases} -\left(\frac{3}{2}\left(\frac{\pi}{4} + \frac{\pi}{2}n - \frac{|z|}{2}\sqrt{2n+1-z^2} - \frac{2n+1}{2}\arcsin\frac{|z|}{\sqrt{2n+1}}\right)\right)^{\frac{2}{3}} \\ z^2 \leq 2n+1, \\ \left(\frac{3}{2}\left(\frac{|z|}{2}\sqrt{z^2-2n-1} - \frac{2n+1}{2}\ln\left(\frac{|z|+\sqrt{z^2-2n-1}}{\sqrt{2n+1}}\right)\right)\right)^{\frac{2}{3}}, \\ z^2 > 2n+1. \end{cases}$$



(a) $n = 1$

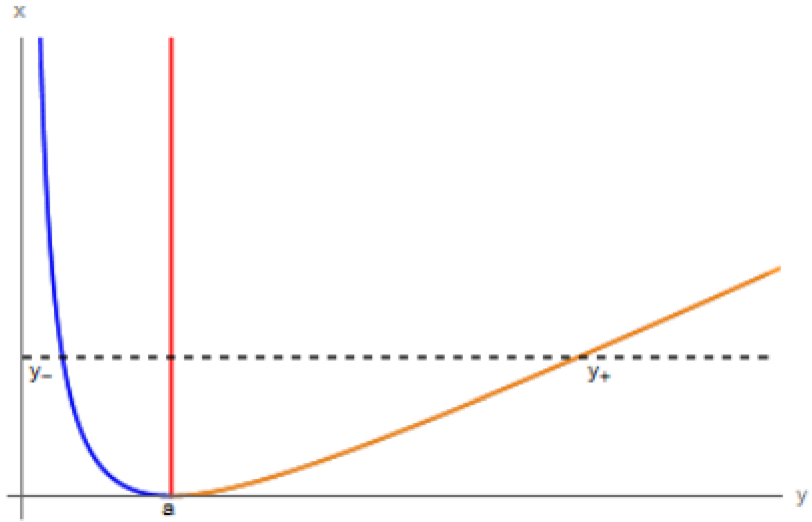


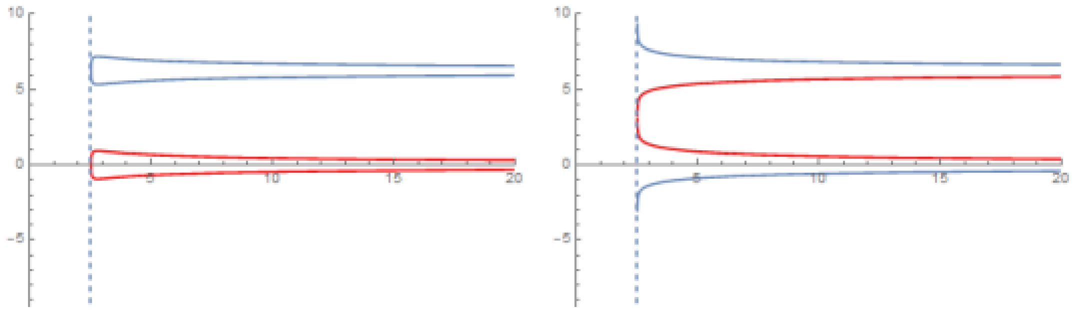
(b) $n = 10$

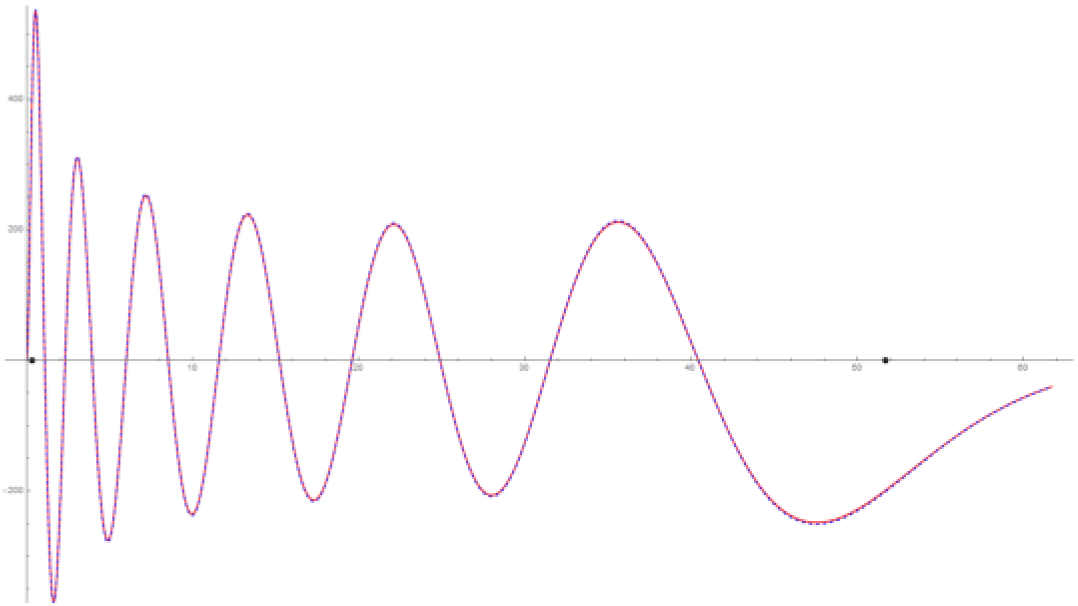


(c) $n = 100$

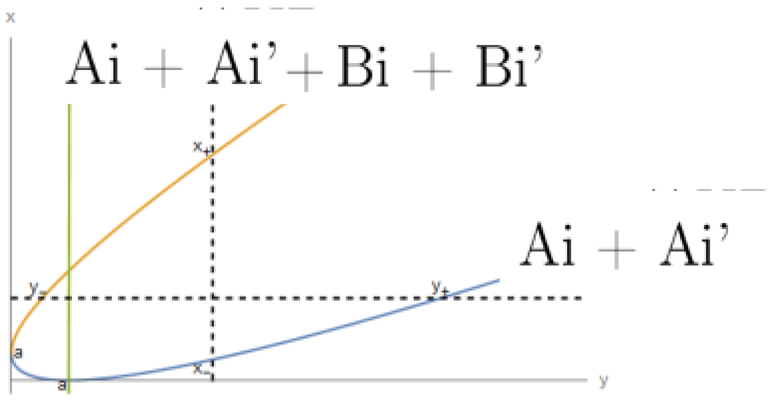
Generalized Laguerre polynomials

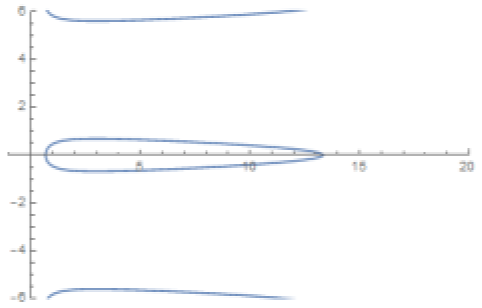




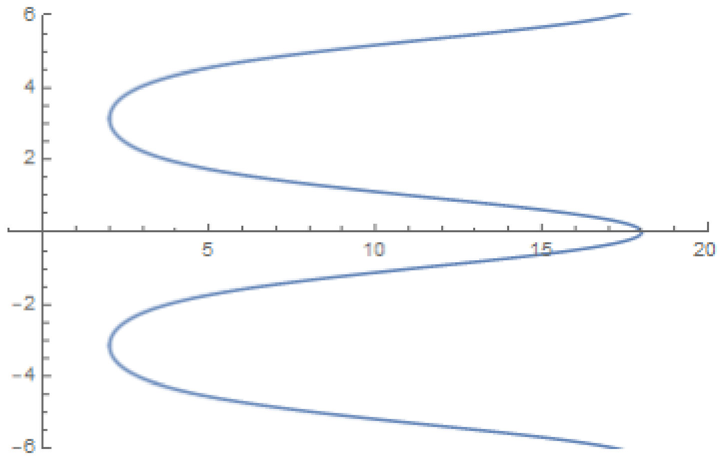


The Charlier Polynomials

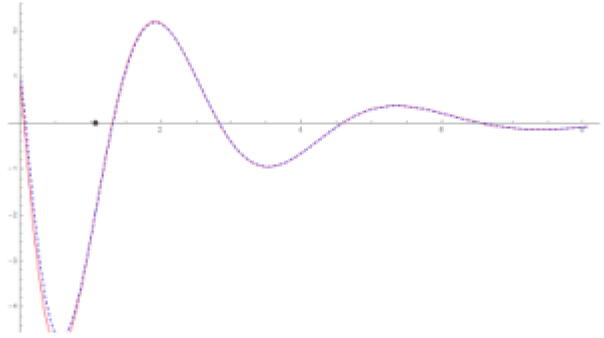


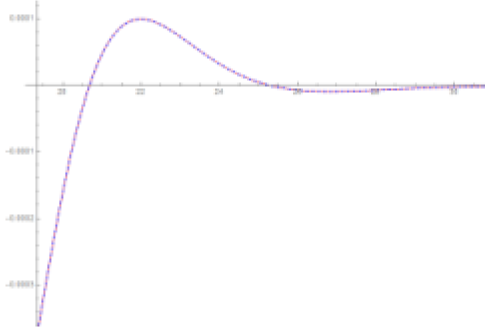


при $0 < y < a$



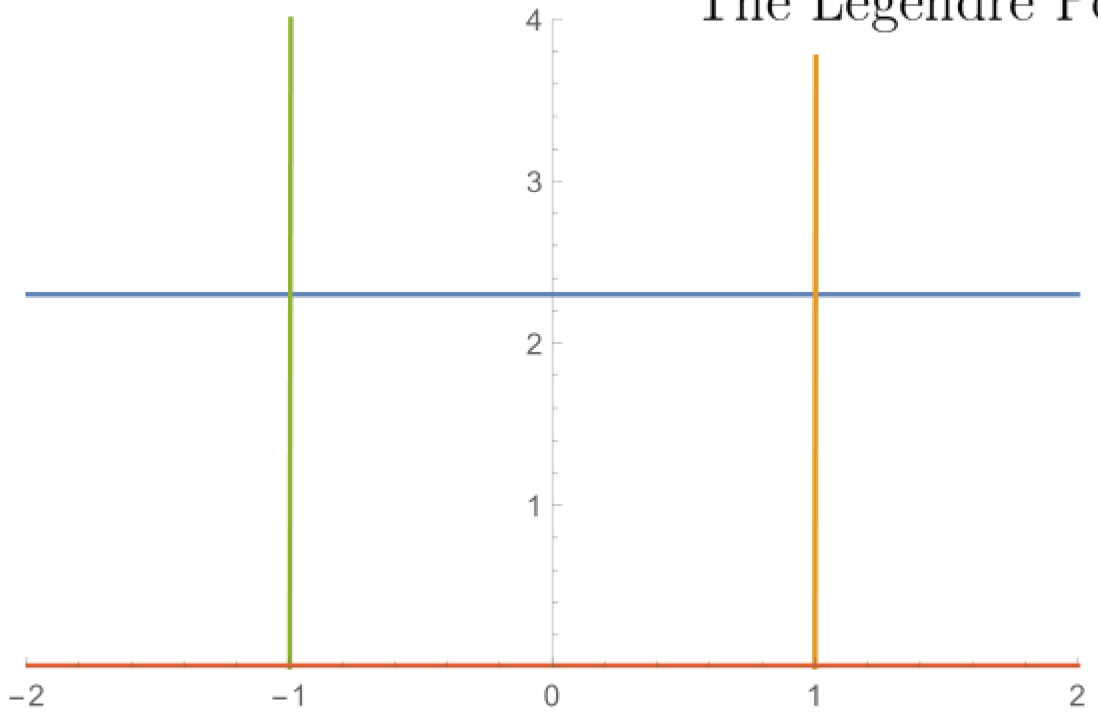
No the discrete Schrödinger operator,
no the Maslov canonical operator (or modified Maslov canonical operator)





The Chebyshev Polynomials

The Legendre Polynomials



Lagrangian manifolds=invariant sets:

$$\Lambda_k = \{\cos p - z) = 0,\ z = const\}$$

Trigonometric functions

The Bessel functions J_0, J_1

Recurrent equations for $H_{n,n}(z,\alpha), H_{n,n-1}(z,\alpha)$

$$\begin{pmatrix} \mathrm{H}_{n,n} \\ \mathrm{H}_{n+1,n} \\ \mathrm{H}_{n+1,n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \alpha n & -n & z+\alpha \\ \alpha n(z-\alpha) & -nz & z^2-\alpha^2-\frac{1}{2}(2n+1) \end{pmatrix} \begin{pmatrix} \mathrm{H}_{n-1,n-1} \\ \mathrm{H}_{n,n-1} \\ \mathrm{H}_{n,n} \end{pmatrix}$$

Initial data

$$\mathrm{H}_{0,0}(z,\alpha)=1, \quad \mathrm{H}_{1,0}(z,\alpha)=z+\alpha, \quad \mathrm{H}_{1,1}(z,\alpha)=z^2-\alpha^2-1/2$$

Introduce small artificial parameter $h=\dfrac{1}{n}$,

thus we are looking for asymptotics as $h \to +0$

Diagonal polynomials and pseudodifferential equations

$$\psi(\mathrm{nh};\mathrm{z},\alpha)=\mathrm{h}^{\mathrm{n}}\mathrm{H}_{\mathrm{n},\mathrm{n}}(\mathrm{z},\alpha),\qquad \theta(\mathrm{nh};\mathrm{z},\alpha)=\mathrm{h}^{\mathrm{n}}\mathrm{z}\mathrm{H}_{\mathrm{n},\mathrm{n}-1}(\mathrm{z},\alpha)$$

The system

$$\mathrm{e}^{\mathrm{i}\hat{\mathrm{p}}}\psi(\mathrm{x})=\mathrm{a}(\mathrm{y}-\mathrm{a})\mathrm{x}\mathrm{e}^{-\mathrm{i}\hat{\mathrm{p}}}\psi(\mathrm{x})-\mathrm{x}\theta(\mathrm{x})+\left(\mathrm{y}^2-\mathrm{a}^2-\mathrm{x}-\frac{\mathrm{h}}{2}\right)\psi(\mathrm{x})$$

$$\mathrm{e}^{\mathrm{i}\hat{\mathrm{p}}}\theta(\mathrm{x})=\mathrm{axye}^{-\mathrm{i}\hat{\mathrm{p}}}\psi(\mathrm{x})-\mathrm{x}\theta(\mathrm{x})+(\mathrm{y}+\mathrm{a})\mathrm{y}\psi(\mathrm{x}),$$

$$\mathrm{y}=\mathrm{z}\sqrt{\mathrm{h}},\ \mathrm{a}=\alpha\sqrt{\mathrm{h}},\ \hat{\mathrm{p}}=-\mathrm{ih}\frac{\partial}{\partial \mathrm{x}}$$

The scalar (formal) equation for $\psi(x):=\psi(x,y,a)$
(y,a are parameters)

$$\hat{\mathrm{H}}(\mathrm{x},\hat{\mathrm{p}};\mathrm{h})\psi(\mathrm{x})=$$
$$\left[\mathrm{e}^{\mathrm{i}\hat{\mathrm{p}}}+\mathrm{a}^2\mathrm{x}\mathrm{e}^{-\mathrm{i}\hat{\mathrm{p}}}+\mathrm{x}\mathrm{y}^2(\mathrm{e}^{\mathrm{i}\hat{\mathrm{p}}}+\mathrm{x})^{-1}-\left(\mathrm{y}^2-\mathrm{a}^2-\mathrm{x}-\frac{\mathrm{h}}{2}\right)\right]\psi(\mathrm{x})=0$$

The boundary condition (large z, y)

$$\mathrm{H}_{\mathrm{n,n}}(\mathrm{z},\alpha) = \mathrm{z}^{2\mathrm{n}} - \mathrm{n}\left(\alpha^2 + \mathrm{n} - \frac{1}{2}\right)\mathrm{z}^{2\mathrm{n}-2} + \mathrm{O}(\mathrm{z}^{2\mathrm{n}-4})$$

$$\psi(\mathrm{x};\mathrm{y},\mathrm{a}) \approx \mathrm{h}^{\frac{\mathrm{x}}{\mathrm{h}}}\left(\frac{\mathrm{a}}{\sqrt{\mathrm{h}}}\right)^{2\frac{\mathrm{x}}{\mathrm{h}}}\left(\mathrm{q}^{\frac{2\mathrm{x}}{\mathrm{h}}} - \frac{\mathrm{x}}{\mathrm{h}}\left(1 + \frac{\mathrm{x}}{\mathrm{a}^2} - \frac{\mathrm{h}}{2\mathrm{a}^2}\right)\mathrm{q}^{\frac{2\mathrm{x}}{\mathrm{h}}-2}\right), \quad \mathrm{q} = \frac{\mathrm{y}}{\mathrm{a}} = \frac{\mathrm{z}}{\alpha}$$

Ordered form

$$\hat{\mathcal{H}}\psi = \mathcal{H}(\overset{2}{\mathrm{x}}, \overset{1}{\hat{\mathrm{p}}}, \mathrm{y}, \mathrm{a}, \mathrm{h})\psi = 0$$

$$\mathcal{H}(\mathrm{x},\mathrm{p};\mathrm{h}) = \mathcal{H}_0(\mathrm{x},\mathrm{p};\mathrm{h}) + \mathrm{h}\mathcal{H}_1(\mathrm{x},\mathrm{p};\mathrm{h}) + \mathrm{O}(\mathrm{h}^2) = \mathrm{e}^{2\mathrm{ip}} + \mathrm{e}^{\mathrm{ip}}(\mathrm{a}^2 - \mathrm{y}^2 + 2\mathrm{x}) + (2\mathrm{a}^2\mathrm{x} + \mathrm{x}^2) + \mathrm{a}^2\mathrm{x}^2\mathrm{e}^{-\mathrm{ip}} + \mathrm{h}\left[\frac{\mathrm{e}^{\mathrm{ip}} + \mathrm{x}}{2} + \mathrm{e}^{\mathrm{ip}} + \mathrm{a}^2 + \frac{\mathrm{y}^2\mathrm{e}^{\mathrm{ip}}}{\mathrm{e}^{\mathrm{ip}} + \mathrm{x}}\right] + \mathrm{O}(\mathrm{h}^2).$$

Splitting a pseudo-differential equation for ψ

WKB-solutions $\quad \eta = \sum_{k} e^{\frac{iS_k(x,y,a)}{h}} A_k(x,y,a), \qquad S_k = \mathcal{S}_k + i s_k$

The Hamilton-Jacobi equations with complex-valued Hamiltonian

$$\mathcal{H}_0\left(x, \frac{\partial S_k}{\partial x}\right) = 0, \quad \mathcal{H}_0(x,p) = e^{-ip}\mathcal{R}(x, e^{ip})$$

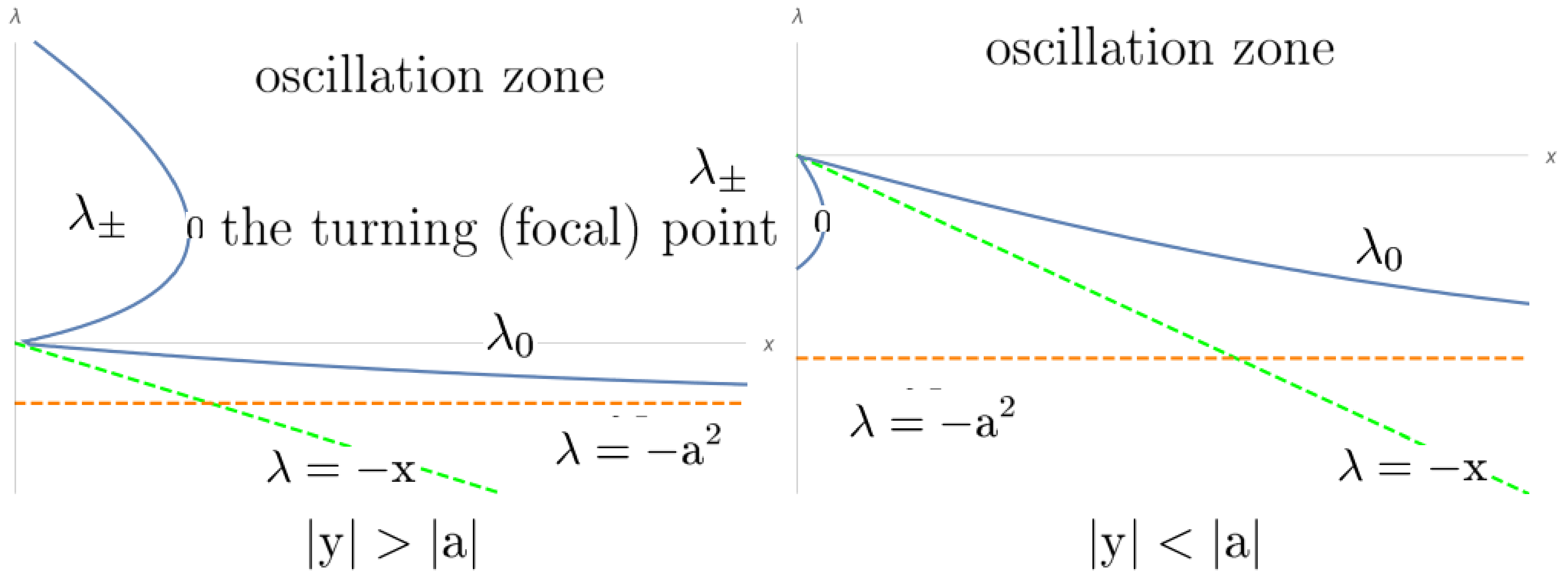
The algebraic curve

$$\mathcal{R}(x,\lambda,y,a) = \lambda^3 + \lambda^2(a^2 - y^2 + 2x) + \lambda(2a^2x + x^2) + a^2x^2 \equiv$$
$$(\lambda + x)^2(\lambda + a^2) - \lambda^2 y^2.$$

The phases and the structure of the roots

$$\lambda_0, \lambda_\pm \quad : \qquad \mathcal{R}(x, \lambda, y, a) = 0,$$

$$S_{0,\pm} = -i \int \log \lambda_{0,\pm}(x, y, a) dx$$



The stucture of the "fundamental" solution

The domain where $\lambda_0, \lambda_\pm$ are real is the "decreasing" zone of the solution (with a suitable weight multiplier)

The domain where $\lambda_\pm$ are complex is the is the zone of solution oscillations

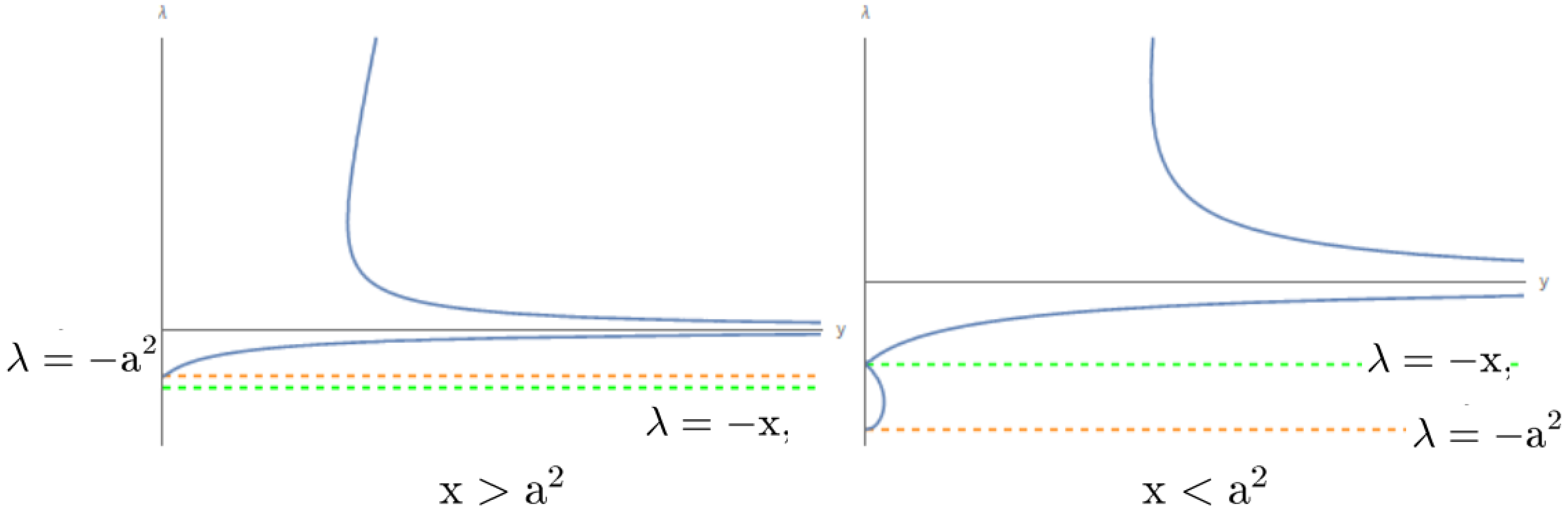
Appropriate representation outside of turning (focal) points:

$$\psi = A_0(x,y,a)e^{\frac{i\pi x}{h}}e^{\frac{\Phi_0(x,y,a)}{h}} + A_1(x,y,a)e^{\frac{\Phi_1(x,y,a)}{h}}\left(A_2(x,y,a)\sin\left(\frac{\Phi_2(x,y,a)}{h} + g(x,y,a)\right)\right)$$

Focal (turning) points (transition $\lambda_\pm$ from complex to real)

$$4\tilde{x}^3 - (12+q^2)\tilde{x}^2 + (20q^2+12)\tilde{x} - 4(1-q^2)^2 = 0, \quad \tilde{x} = x/a^2, q = y/a.$$

Roots on the plane (y, λ)



Factorization of the characteristic equation and the parametrisation $\lambda_\pm$ via λ_0

$$\mathcal{R}(x,\lambda) = \big(\lambda - \lambda_0(x)\big)(\lambda^2 - A(x)\lambda + B(x))$$

here

$$A(x;y,a) \equiv A(x) = \lambda_+ + \lambda_- = y^2 - a^2 - 2x - \lambda_0,$$

$$B(x;y,a) \equiv B(x) = \lambda_+\lambda_- = -\frac{a^2x^2}{\lambda_0} \equiv$$

$$x(2a^2 + x) + (a^2 + 2x - y^2)\lambda_0 + \lambda_0^2 > 0$$

$$\lambda_+ = \frac{A + \sqrt{A^2 - 4B}}{2}, \quad \lambda_- = \frac{A - \sqrt{A^2 - 4B}}{2}.$$

$\lambda_\pm$ are real if $A^2 - 4B \geq 0, \quad \lambda_- \leq \lambda_+,$ and are complex if $A^2 - 4B < 0$

Real-valued Hamiltonians

We put: $S_0 = -i\Phi_0 + \pi x, \qquad S_\pm = -i\Phi_1 \pm \Phi_2,$

$$e^{i\frac{\partial S_0}{\partial x}} - e^{i\pi}(-\lambda_0) = 0 \quad \Leftrightarrow \quad e^{\frac{\partial \Phi_0}{\partial x}} + \lambda_0 = 0$$

$$e^{\frac{\partial \Phi_1}{\partial x}} e^{\pm i\frac{\partial \Phi_2}{\partial x}} - A(x) + B(x) e^{-\frac{\partial \Phi_1}{\partial x}} e^{\mp i\frac{\partial \Phi_2}{\partial x}} = 0 \Rightarrow$$
$$e^{\frac{\partial \Phi_1}{\partial x}} - \sqrt{B} = 0,$$

$$2\sqrt{B(x)}\cos\left(\frac{\partial \Phi_2}{\partial x}\right) - A(x) = 0 \quad \Longleftrightarrow \quad \cos\left(\frac{\partial \Phi_2}{\partial x}\right) - \frac{A(x)}{2\sqrt{B(x)}} = 0$$

Focal (turning) points $\quad A^2 = 4B$

Three real-valude Hamiltonians

$$H_0(x, \xi) = e^{\xi} + \lambda_0$$

Asymptotics with purely imaginary phases

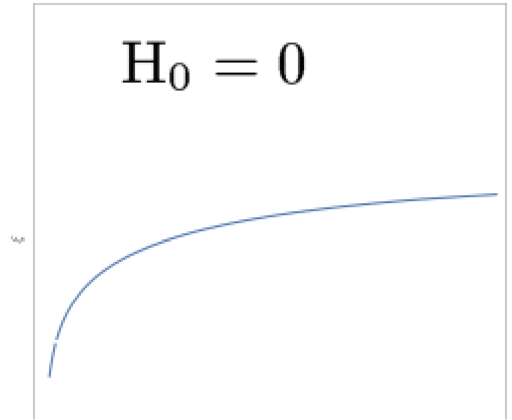
$$H_1 = e^{\xi} - \sqrt{B}$$

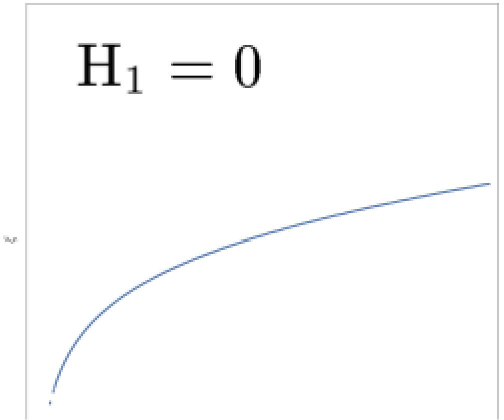
oscillating asymptotics

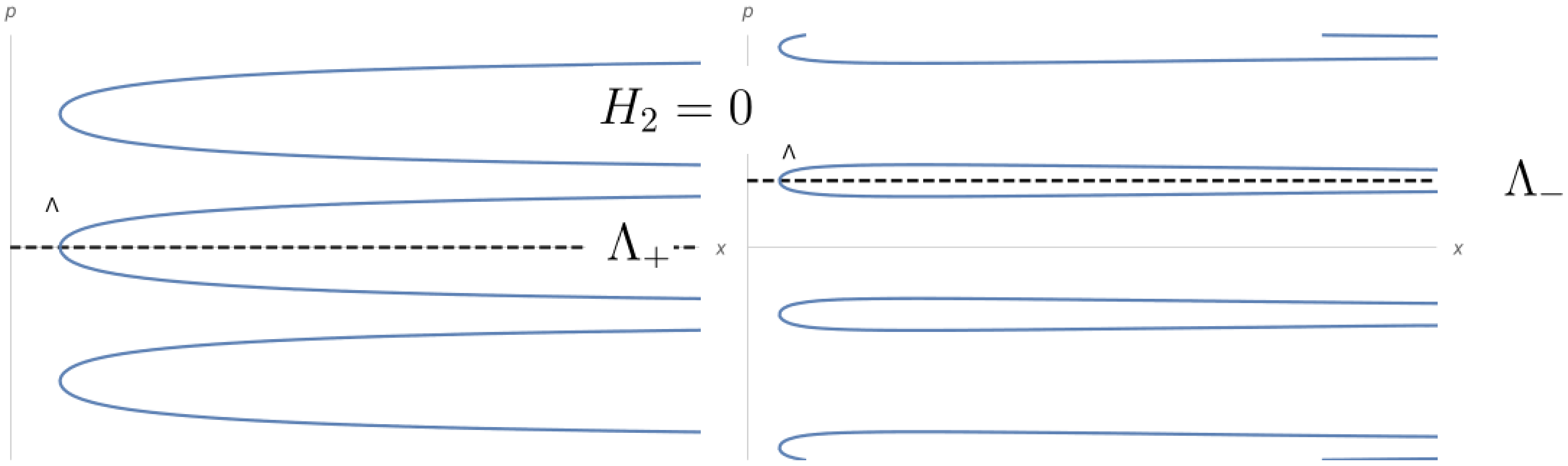
$$H_2 = \cos p - \frac{A(x)}{2\sqrt{B(x)}}$$

quntization $\xi \to \hat{\xi} = h\frac{\partial}{\partial x}, \quad p \to \hat{p} = -ih\frac{\partial}{\partial x} \quad + \textbf{corrections}$

Lagrangian manifolds (level- cut lines of the Hamiltonians)

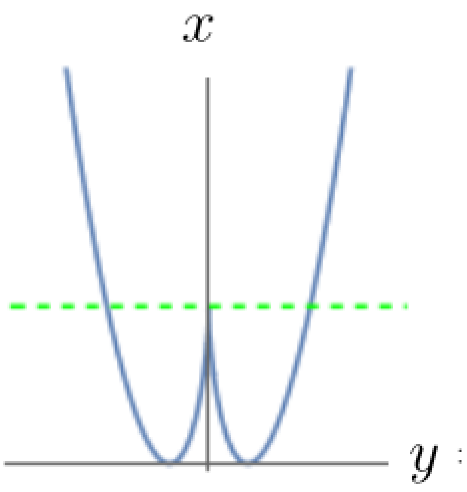






$|y| > |a|$ $|y| < |a|$

The set of turning points on the half-plane $x > 0, y \in \mathbb{R}$



$$\frac{A(x,y)}{2\sqrt{B(x,y)}} = \pm 1$$

Splitting of the solution and 3 reduced pseudodifferential equations

ansatz:
$$\psi = C_0(y,a)\Psi_0 + C_1(y,a)\Psi(x,y,a),$$
$$\Psi_0 = e^{\frac{i\pi x}{h}}\psi_0(x,y,a), \quad \Psi = \psi_1(x,y,a)\psi_2(x,y,a),$$

The aim: final equations

$$\big(e^{\hat{\xi}} + \lambda_0(x) + hV_0(x) + O(h^2)\big)\psi_0 = 0,$$
$$\big(e^{\hat{\xi}} - \sqrt{B(x)} + hV_1(x) + O(h^2)\big)\psi_1 = 0,$$
$$\big(\cos\hat{p} - \frac{A(x)}{2\sqrt{B(x)}} + hV_2(x) + O(h^2)\big)\psi_2 = 0.$$

$$\hat{\xi} = h\frac{\partial}{\partial x}$$

$$\hat{p} = -ih\frac{\partial}{\partial x}$$

requirement: V_0, V_1, V_2 are smooth real-valued

Splitting of the solution and 3 reduced pseudodifferential equations

$$\hat{\mathcal{H}}\psi = \mathcal{H}(\overset{2}{\mathrm{x}}, \overset{1}{\hat{\mathrm{p}}}, \mathrm{y}, \mathrm{a}, \mathrm{h})\psi = 0.$$

$$\mathcal{H}(\mathrm{x}, \mathrm{p}; \mathrm{h}) = \mathcal{H}_0(\mathrm{x}, \mathrm{p}; \mathrm{h}) + \mathrm{h}\mathcal{H}_1(\mathrm{x}, \mathrm{p}; \mathrm{h}) + \mathrm{O}(\mathrm{h}^2) = \mathrm{e}^{2\mathrm{ip}} + \mathrm{e}^{\mathrm{ip}}(\mathrm{a}^2 - \mathrm{y}^2 + 2\mathrm{x}) +$$
$$(2\mathrm{a}^2\mathrm{x} + \mathrm{x}^2) + \mathrm{a}^2\mathrm{x}^2\mathrm{e}^{-\mathrm{ip}} + \mathrm{h}\left[\frac{\mathrm{e}^{\mathrm{ip}} + \mathrm{x}}{2} + \mathrm{e}^{\mathrm{ip}} + \mathrm{a}^2 + \frac{\mathrm{y}^2\mathrm{e}^{\mathrm{ip}}}{\mathrm{e}^{\mathrm{ip}} + \mathrm{x}}\right] + \mathrm{O}(\mathrm{h}^2).$$

We act on the original equation with the operators (very useful technical trick)

$$(\mathrm{e}^{\mathrm{i}\hat{\mathrm{p}}} - \lambda_0)^{-1} \qquad \text{and} \qquad (\mathrm{e}^{\mathrm{i}\hat{\mathrm{p}}} - \mathrm{A}(\mathrm{x}) + \mathrm{B}(\mathrm{x})\mathrm{e}^{-\mathrm{i}\hat{\mathrm{p}}})^{-1}$$

this gives

$$\widehat{\mathcal{H}^0}\Psi_0 := [e^{i\hat{p}} - \lambda_0(x) + h\mathcal{L}_0(\overset{2}{x}, \overset{1}{e^{i\hat{p}}}) + O(h^2)]\Psi_0 = 0$$

$$\mathcal{L}_0(x,\lambda) = \Big[\frac{3\lambda + x + 2a^2}{2(\lambda - A + B\lambda^{-1})} + \frac{y^2\lambda}{(\lambda + x)(\lambda - A + B\lambda^{-1})} - \frac{2(1 - B\lambda^{-2})(\lambda + x)(\lambda + a^2)}{(\lambda - A + B\lambda^{-1})^2}\Big].$$

and

$$\widehat{\mathcal{H}^1}\Psi := [e^{i\hat{p}} - A(x) + B(x)e^{-i\hat{p}} + h\mathcal{L}_1(\overset{2}{x}, \overset{1}{e^{i\hat{p}}}) + O(h^2)]\Psi = 0,$$

$$\mathcal{L}_1 = \frac{3\lambda + x + 2a^2}{2(\lambda - \lambda_0)} + \frac{y^2\lambda}{(\lambda + x)(\lambda - \lambda_0)} - \frac{2(\lambda + x)(\lambda + a^2)}{(\lambda - \lambda_0)^2}$$

The construction of Ψ_0

$$\Psi_0 = e^{\frac{i\pi x}{h}} e^{\frac{\Phi_0(x)}{h}} A_0(x)$$

$$e^{\pm i\hat{p}} f(x) = f(x \pm h) = f(x) \pm h \frac{\partial f}{\partial x}(x) + \frac{h^2}{2} \frac{\partial^2 f}{\partial x^2}(x) + O(h^2)$$

$$e^{\pm i\hat{p}}(f_1(x) f_2(x)) = f_1(x \pm h) f_2(x \pm h).$$

and

$$\left[e^{i\hat{p}} - \lambda_0(x) + h\mathcal{L}_0(\overset{2}{x}, e^{i\overset{1}{\hat{p}}})\right] \left(e^{\frac{i\pi x}{h}} e^{\frac{\Phi_0(x)}{h}} A_0(x)\right) =$$

$$-e^{\frac{i\pi x}{h}} e^{\frac{\Phi_0(x+h)}{h}} A_0(x+h) + \left[-\lambda_0(x) + h\mathcal{L}_0(\overset{2}{x}, e^{i\overset{1}{\hat{p}}})\right] \left(e^{\frac{i\pi x}{h}} e^{\frac{\Phi_0(x)}{h}} A_0(x)\right) = -e^{\frac{i\pi x}{h}} e^{\frac{\Phi_0(x)}{h}}$$

$$\cdot \left[e^{\frac{\partial \Phi_0}{\partial x}} \left(1 + \frac{h}{2} \frac{\partial^2 \Phi_0}{\partial x^2}(x)\right) \left(A_0 + h \frac{\partial A_0}{\partial x}\right) + \lambda_0 A_0 - h\mathcal{L}_0(x, \lambda_0(x)) A_0 + O(h^2)\right] = 0.$$

this gives

$$\Psi_0 = \frac{e^{\frac{i\pi x}{h}} e^{\frac{\Phi_0(x)}{h}}}{\sqrt{-\lambda_0(x)}} \exp\left(- \int_{x_0}^{x} \frac{V_0(\xi)}{\lambda_0(\xi)} d\xi \right), \quad \Phi_0(x) = \int_{x_0}^{x} \log(-\lambda_0(\xi)) d\xi,$$

here x_0 is the integration constant

The construction of Ψ (an important ideological fact)

$$\Psi(x) = \psi_1(x)\psi_2(x),$$

where

$$\psi_1 = e^{\frac{\Phi_1(x)}{h}} A_1(x), \quad e^{\frac{\partial \Phi_1}{\partial x}} = \sqrt{B(x)} \quad \Longleftrightarrow \quad \Phi_1 = \frac{1}{2}\int_{x_1}^{x} \log(B(x))dx,$$

$A_1(x)$ is still unknown , x_1 is a constant

We seek the function ψ_2 in the form the Maslov canonical operator acting to the unknown function A_2 on $\Lambda_\pm$

$$\psi_2 = K_{\Lambda_\pm} A_2$$

Substitute function $\psi_1\psi_2$ into equation $\widehat{\mathcal{H}^1}\Psi = 0$

$$(e^{i\hat{p}} - A + Be^{-i\hat{p}})(\psi_1\psi_2) = e^{\frac{\Phi_1(x)}{h}} \Bigg[A_1 \left(e^{\frac{\partial\Phi_1}{\partial x}} e^{i\hat{p}} - A + Be^{-\frac{\partial\Phi_1}{\partial x}} e^{-i\hat{p}}\right) +$$

$$h\Bigg(e^{\frac{\partial\Phi_1}{\partial x}} \left(\frac{\partial A_1}{\partial x} + \frac{1}{2}\frac{\partial^2\Phi_1}{\partial x^2}A_1\right) e^{i\hat{p}} + e^{-\frac{\partial\Phi_1}{\partial x}} B \left(-\frac{\partial A_1}{\partial x} + \frac{1}{2}\frac{\partial^2\Phi_1}{\partial x^2}A_1\right) e^{-i\hat{p}}\Bigg] \psi_2 =$$

$$e^{\frac{\Phi_1(x)}{h}} \left[A_1 (2\sqrt{B}\cos\hat{p} - A) + h\left(A_1 \frac{\partial\sqrt{B}}{\partial x}\cos\hat{p} + 2i\frac{\partial A_1}{\partial x}\sqrt{B}\sin\hat{p}\right) + O(h^2)\right] \psi_2.$$

Take into account the equality

$$e^{\frac{\Phi_1(x)}{h}} \left(A_1\mathcal{L}_1(x, e^{\frac{\partial\Phi_1}{\partial x}} e^{i\hat{p}}) + O(h)\right) \psi_2 =$$

$$e^{\frac{\Phi_1(x)}{h}} A_1 \left[\mathcal{L}_1^{Re}(x, \cos\hat{p}) + i\sin\hat{p}\,\mathcal{L}_1^{Im}(x, \cos\hat{p}) + O(h)\right] \psi_2 = 0$$

this gives

$$e^{\frac{\Phi_1(x)}{h}} \left[A_1 \left(2\sqrt{B} \cos \hat{p} - A \right) + h \left(A_1 \frac{\partial \sqrt{B}}{\partial x} \cos \hat{p} \; + 2i \frac{\partial A_1}{\partial x} \sqrt{B} \sin \hat{p} + \right.\right.$$

$$\left.\left. A_1 \mathcal{L}_1^{\mathrm{Re}} \left(x, \frac{A}{2\sqrt{B}} \right) + i A_1 \mathcal{L}_1^{\mathrm{Im}} \left(x, \frac{A}{2\sqrt{B}} \right) \sin \hat{p} \right) + O(h^2) \right] \psi_2 = 0,$$

$$2 \frac{\partial A_1}{\partial x} \sqrt{B} + A_1 \mathcal{L}_1^{\mathrm{Im}} \left(x, \frac{A}{2\sqrt{B}} \right) = 0$$

Imaginary part $= 0 \quad \Longrightarrow$

$$A_1(x; y, a) = \exp \left[- \int_{x_1}^{x} \frac{1}{2\sqrt{B(\xi; y, a)}} \mathcal{L}_1^{\mathrm{Im}} \left(\xi, \frac{A}{2\sqrt{B}}(\xi; y, a) \right) d\xi \right]$$

$$\text{Real part} = 0 \quad \Longrightarrow$$

$$\left(\cos\hat{p} - \frac{A(x)}{2\sqrt{B(x)}} + hV_2(x) + O(h^2)\right)\psi_2 = 0,$$

$$V_2(x;y,a) = \frac{A}{8B\sqrt{B}}\frac{\partial B}{\partial x}(x;y,a) + \frac{1}{2\sqrt{B(x;y,a)}}\mathcal{L}_1^{\mathrm{Re}}\left(x, \frac{A}{2\sqrt{B}}(x;y,a)\right).$$

and

$$\psi_2 = \sqrt{\pi}\left(\frac{v_1}{\sqrt[6]{h}}\,\mathrm{Ai}\left(\mathrm{sign}(x_\pm^* - x)\left(\frac{3\Phi_2^\pm}{2h}\right)^{2/3}\right) + \sqrt[6]{h}\,v_2\mathrm{Ai}'\left(\mathrm{sign}(x_\pm^* - x)\left(\frac{3\Phi_2^\pm}{2h}\right)^{2/3}\right)\right).$$

(A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. V. Tsvetkova, Uniform asymptotic solution in the form of an Airy function for semiclassical bound states in one-dimensional and radially symmetric problems, Theoret. and Math. Phys., 201:3 (2019), 1742-1770)

Phases and amplitudes

$$J(x;y,a)=\sqrt{\left|1-\frac{A^2(x;y,a)}{4B(x;y,a)}\right|},\quad g_\pm(x;y,a)=\pm\int_{x_\pm^*}^{x}\frac{V_2(x;y,a)}{\sqrt{\left|1-\frac{A^2(x;y,a)}{4B(x;y,a)}\right|}}dx.$$

for $x>x_\pm^*$

$$\Phi_2^\pm(x;y,a)=\int_{x_\pm^*}^{x}\arccos\left(\pm\frac{A(x;y,a)}{2\sqrt{B(x;y,a)}}\right)dx,$$

$$v_1(x;y,a)=\frac{\cos g^\pm}{\sqrt{J}}\left(\frac{3\Phi_2^\pm}{2}\right)^{1/6},\quad v_2(x;y,a)=-\frac{\sin g^\pm}{\sqrt{J}}\left(\frac{3\Phi_2^\pm}{2}\right)^{-1/6}$$

for $0<x<x_\pm^*$

$$\Phi_2^\pm(x;y,a)=\int_{x_\pm^*}^{x}\log\left(\pm\frac{A(x)}{2\sqrt{B(x)}}+\frac{1}{2\sqrt{B(x)}}\sqrt{A^2(x)-4B(x)}\right)dx,$$

$$v_1(x;y,a)=\frac{\cosh g^\pm}{\sqrt{J}}\left(\frac{3\widetilde{\Phi}_2^\pm}{2}\right)^{1/6},\quad v_2(x;y,a)=\frac{\sinh g^\pm}{\sqrt{J}}\left(\frac{3\widetilde{\Phi}_2^\pm}{2}\right)^{-1/6}$$

Finding constants of integration

$$\psi = C_0(y,a)\Psi_0 + C_1(y,a)\Psi(x,y,a),$$

$$\Psi_0 = e^{\frac{i\pi x}{h}}\psi_0(x,y,a), \quad \Psi = \psi_1(x,y,a)\psi_2(x,y,a),$$

Parametrization (how one can use obtained asymptotic formulas)

Parametrization via root λ_0

$$x = -\lambda_0\left(1 + \frac{|y|}{\sqrt{a^2+\lambda_0}}\right)$$

Parametrization via artificial parameter μ

$$a^2 + \lambda_0 = a^2\mu^2.$$

$$\lambda_0 = a^2(\mu^2 - 1), \quad x = a^2(1-\mu^2)\left(1 + \frac{|q|}{\mu}\right), \quad q = \frac{y}{a} \equiv \frac{z}{\alpha}$$

Comparison for large $q = \frac{z}{\alpha}$ coefficients for the highest degree $\mathrm{h}^{\frac{\mathrm{x}}{\mathrm{h}}}\left(\frac{\mathrm{a}}{\sqrt{\mathrm{h}}}\right)^{\frac{2\mathrm{x}}{\mathrm{h}}}\mathrm{q}^{\frac{2\mathrm{x}}{\mathrm{h}}}$

of the polynomial $H_{n,n}$ and the function $\psi = \mathrm{C}_0(\mathrm{y},\mathrm{a})\Psi_0 + \mathrm{C}_1(\mathrm{y},\mathrm{a})\Psi(\mathrm{x},\mathrm{y},\mathrm{a})$,

We have $$\mu(\mathrm{x},\mathrm{q}) \sim 1 - \frac{\mathrm{x}}{2\mathrm{a}^2\mathrm{q}} + \frac{4\mathrm{a}^2\mathrm{x}+\mathrm{x}^2}{8\mathrm{a}^4|\mathrm{q}|^2} - \frac{\mathrm{a}^2\mathrm{x}+\mathrm{x}^2}{2\mathrm{a}^4|\mathrm{q}|^3}, \quad \mathrm{q}\to\infty,$$

$$\mu^*(\mathrm{q}) \sim \frac{4}{|\mathrm{q}|} - \frac{32}{|\mathrm{q}|^3}.$$

and $\mathrm{q}\to\infty$

$$e^{\frac{\Phi_0(\mu)}{\mathrm{h}}} \sim e^{\frac{\mathrm{x}}{\mathrm{h}}\log\left(\frac{\mathrm{x}}{\mathrm{q}}\right)}e^{-\mathrm{x}-\frac{\mathrm{x}}{\mathrm{q}}} = \mathrm{h}^{\frac{\mathrm{x}}{\mathrm{h}}}(\mathrm{q})^{-\frac{\mathrm{x}}{\mathrm{h}}}e^{\frac{\mathrm{x}}{\mathrm{h}}\left(\log\frac{\mathrm{x}}{\mathrm{h}}-1-\frac{1}{\mathrm{q}}\right)} \quad \Longrightarrow \quad C_0 \approx 0$$

$$\Psi_1 \sim \mathrm{h}^{\frac{\mathrm{x}}{\mathrm{h}}}\left(\frac{\mathrm{a}|\mathrm{q}|}{\sqrt{\mathrm{h}}}\right)^{\frac{2\mathrm{x}}{\mathrm{h}}}\exp\left[-\frac{\mathrm{a}^2}{2\mathrm{h}}(|\mathrm{q}|-1)^2 + \frac{1}{4}\log\frac{1}{2(1+|\mathrm{q}|)}\right]$$

and $$\mathrm{C}_1(\mathrm{a},\mathrm{q}) = \exp\left[\frac{\mathrm{a}^2}{2\mathrm{h}}(|\mathrm{q}|-1)^2 - \frac{1}{4}\log\frac{1}{2(1+|\mathrm{q}|)}\right]$$

Parametric form

При $n \to \infty$

$$\mathrm{H}_{\mathrm{n,n}}(\mathrm{z}, \alpha) \approx (\mathrm{sign}(\mathrm{z}^2 - \alpha^2))^{\mathrm{n}} \sqrt{2\pi} \mathrm{e}^{\frac{\mathrm{z}^2}{2}} \left(\frac{\mu(|\alpha|\mu + |\mathrm{z}|)^2}{2\alpha^2\mu^3 + |\alpha \mathrm{z}|(1+\mu^2)} \right)^{1/4} \cdot$$
$$\exp\left[\frac{\alpha^2}{2} \left(\frac{(1-\mu^2)(|\alpha|\mu + |\mathrm{z}|)}{|\alpha|\mu} \log\left(\mathrm{B}\left(\mu; \frac{\mathrm{z}}{\alpha}\right)\right) + \mu^2 - \frac{2|\mathrm{z}|}{|\alpha|\mu} \right) \right] \cdot$$
$$\left(\mathrm{v}_1^{\pm}\left(\mu; \frac{\mathrm{z}}{\alpha}\right) \mathrm{Ai}\left(\mathrm{sign}(\mathrm{x}_{\pm}^* - \mathrm{x}) \left(\frac{3\Phi_2^{\pm}\left(\mu; \frac{\mathrm{z}}{\alpha}\right)}{2} \right)^{2/3} \right) + \right.$$
$$\left. \mathrm{v}_2^{\pm}\left(\mu; \frac{\mathrm{z}}{\alpha}\right) \mathrm{Ai}'\left(\mathrm{sign}(\mathrm{x}_{\pm}^* - \mathrm{x}) \left(\frac{3\Phi_2^{\pm}\left(\mu; \frac{\mathrm{z}}{\alpha}\right)}{2} \right)^{2/3} \right) \right), \qquad (68)$$

$$\mathrm{z}(\mu) = \pm \left(\frac{\mathrm{n}\mu}{|\alpha|(1-\mu^2)} - |\alpha|\mu \right) \qquad \mu \in (0, 1]$$

$$\mathrm{A}(\mu;\mathrm{q}) = \alpha^2 \frac{(\mu + |\mathrm{q}|)}{\mu}\left(\mu^2 - 2 + |\mathrm{q}|\mu\right),$$

$$\mathrm{B}(\mu;\mathrm{q}) = \alpha^4 \frac{(\mu + |\mathrm{q}|)^2}{\mu^2}(1-\mu^2),$$

$$\Phi_2^{\pm}(\mu;\mathrm{q}) = \begin{cases} \int_{\mu^*}^{\mu} \arccos\left|\frac{\mathrm{A}}{2\sqrt{\mathrm{B}}}(\mu;\mathrm{q})\right| \left(-\frac{\alpha^2(2\mu^3+\mu^2|\mathrm{q}|+|\mathrm{q}|)}{\mu^2}\right) \mathrm{d}\mu, & \mathrm{x} \geq \mathrm{x}^*, \\ \int_{\mu^*}^{\mu} \log\left(\left|\frac{\mathrm{A}}{2\sqrt{\mathrm{B}}}\right| + \frac{\sqrt{\mathrm{A}^2-4\mathrm{B}}}{2\sqrt{\mathrm{B}}}\right) \left(-\frac{\alpha^2(2\mu^3+\mu^2|\mathrm{q}|+|\mathrm{q}|)}{\mu^2}\right) \mathrm{d}\mu, & \mathrm{x} < \mathrm{x}^*, \end{cases}$$

$$\mathrm{v}_1^{\pm}(\mu;\mathrm{q}) = \frac{1}{\sqrt{\mathrm{J}(\mu;\mathrm{q})}} \left(\frac{3\Phi_2^{\pm}(\mu;\mathrm{q})}{2}\right)^{1/6} \begin{cases} \cos\left(\int_{\mu^*}^{\mu(\mathrm{x})} \pm\mathcal{V}_2(\mu;\mathrm{q})\mathrm{d}\mu\right), \ \mathrm{x} \geq \mathrm{x}^*, \\ \cosh\left(\int_{\mu^*}^{\mu(\mathrm{x})} \pm\mathcal{V}_2(\mu;\mathrm{q})\mathrm{d}\mu\right), \ \mathrm{x} < \mathrm{x}^*, \end{cases}$$

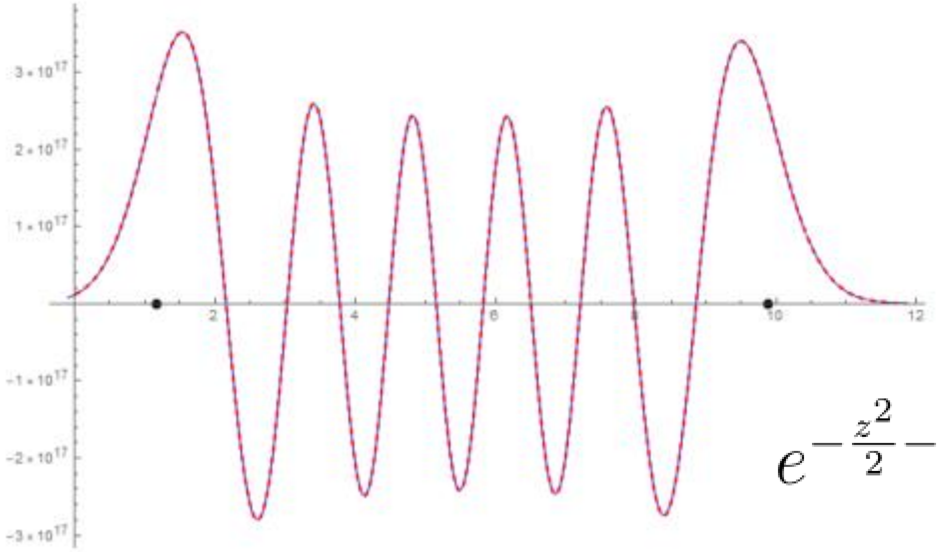
$$\mathrm{v}_2^{\pm}(\mu;\mathrm{q}) = \frac{1}{\sqrt{\mathrm{J}(\mu;\mathrm{q})}} \left(\frac{3\Phi_2^{\pm}(\mu;\mathrm{q})}{2}\right)^{-1/6} \begin{cases} -\sin\left(\int_{\mu^*}^{\mu(\mathrm{x})} \pm\mathcal{V}_2(\mu;\mathrm{q})\mathrm{d}\mu\right), & \mathrm{x} \geq \mathrm{x}^*, \\ \sinh\left(\int_{\mu^*}^{\mu(\mathrm{x})} \pm\mathcal{V}_2(\mu;\mathrm{q})\mathrm{d}\mu\right), & \mathrm{x} < \mathrm{x}^*, \end{cases}$$

$$\mu^*(|\alpha|\mu^* + |\mathrm{z}|)^2 - 4|\alpha\mathrm{z}| = 0$$

$$\mathcal{V}_2(\mu;\mathrm{q}) = \frac{\alpha^6(\mu+|\mathrm{q}|)^3}{8\mu^3(2\mu^3+|\mathrm{q}|+\mu^2|\mathrm{q}|)\sqrt{\mathrm{B}^3(\mu)}\mathrm{J}(\mu;\mathrm{q})} \cdot$$

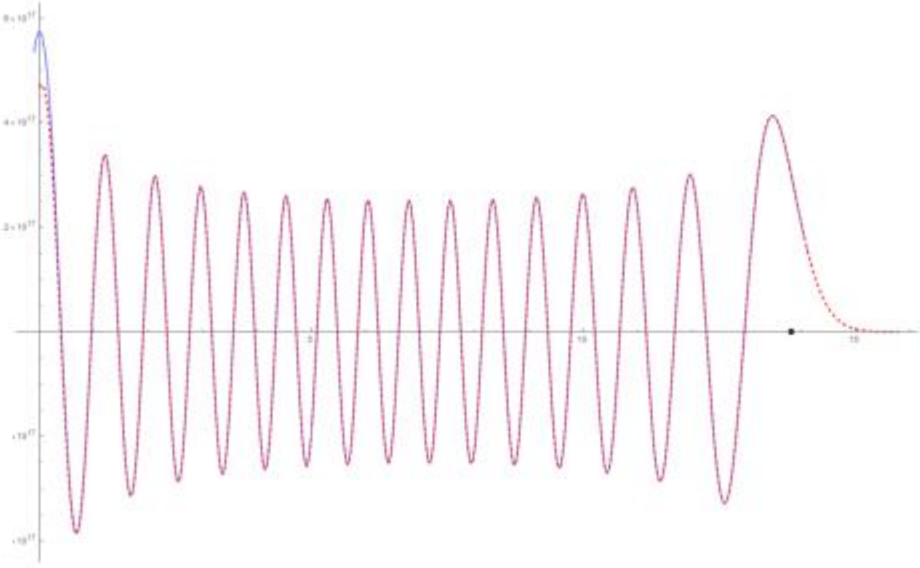
$$\cdot\left(4\mu^6+9\mu|\mathrm{q}|-12\mu^3|\mathrm{q}|+3\mu^5|\mathrm{q}|-|\mathrm{q}|^2-4\mu^2|\mathrm{q}|^2+\mu^4|\mathrm{q}|^2\right),$$

$$\mathrm{J}(\mu;\mathrm{q}) = \sqrt{\left|\frac{\mu(\mu(|\mathrm{q}|+\mu)^2-4|\mathrm{q}|)}{4(1-\mu^2)}\right|}.$$



a = 5, n = 10

$$e^{-\frac{z^2}{2}-\Phi_1}H_{n,n}(z,\alpha)$$



a = 5, n = 30

Relationship with 3-d order ODE for $u = H_{n_1,n_2}$

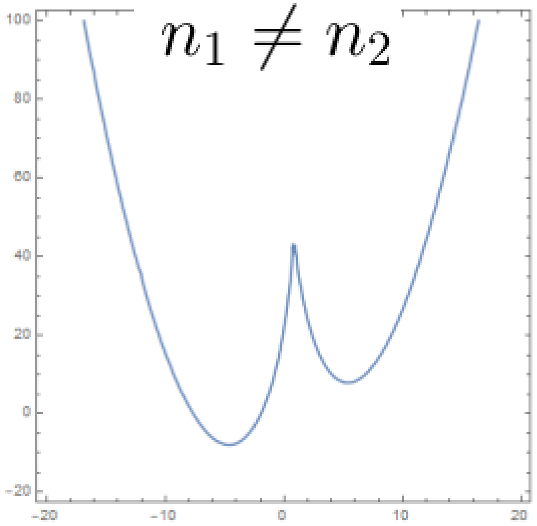
(A.Aptekarev, A. Branquinho, W. Van Assche, 2003)

$$\frac{d^3u}{dz^3} - 4z\frac{d^2u}{dz^2} + (4z^2 - 4\alpha^2 + 2(n_1+n_2-1))\frac{du}{dz} - 4(z(n_1+n_2) - \alpha(n_1-n_2))u = 0$$

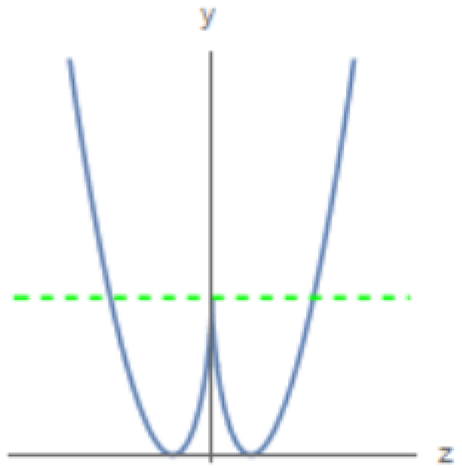
the oscillating factor

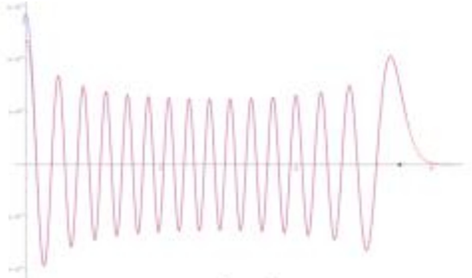
Ansatz $$u = C_0\Psi(z,\alpha,n_1,n_2) + C_1\psi_1(z,\alpha,n_1,n_2)\psi_2(z,\alpha,n_1,n_2)$$

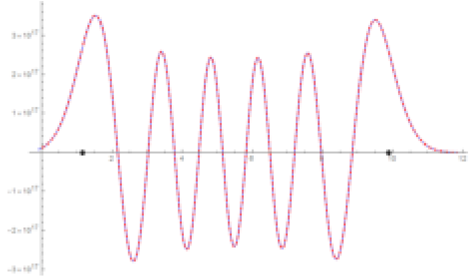
Double-well problem for the oscillating factor



$n_1 = n_2$




a = 5, n = 30


a = 5, n = 10

THANK YOU FOR YOUR ATTENTION!