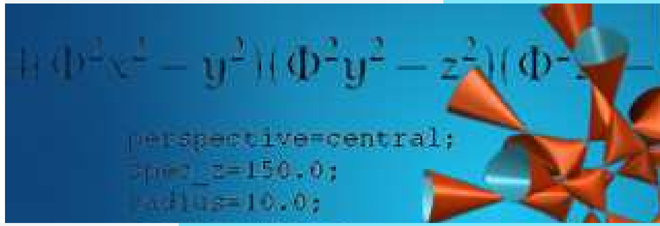
Spinorial description of G_2 and $\mathrm{SU}(3)$-manifolds



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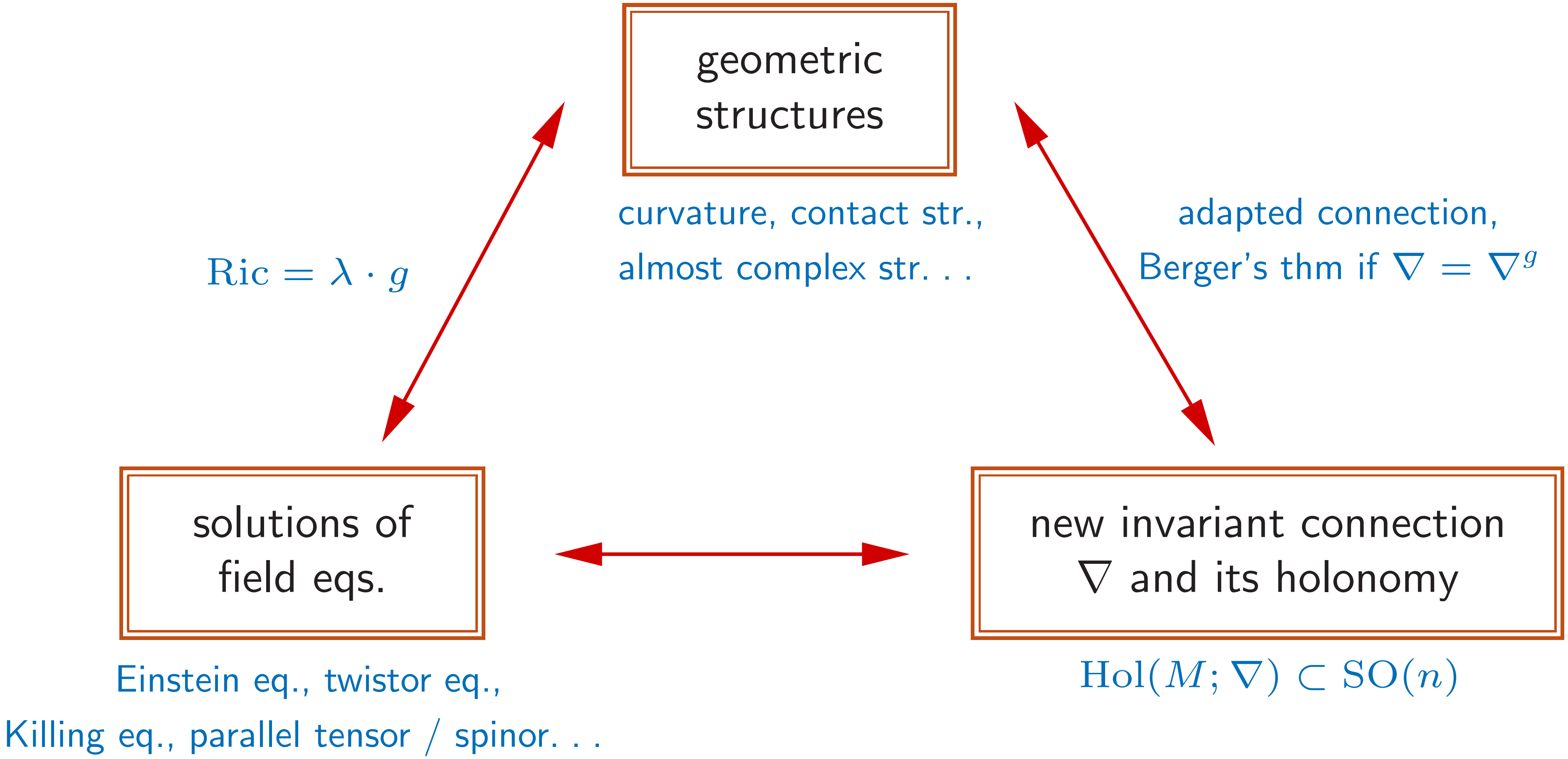
Relations between different objects on a Riemannian manifold (M^n, g):



N.B. $\nabla^g :=$ Levi-Civita connection

Observation:

- $\exists$ multitude of different spinorial field equations, related to different geometric structures and geometric questions

Goal:

- Uniform description of different types of spinor fields
- Applications

The Riemannian Dirac operator

(M^n, g): compact Riemannian spin mnfd, Σ: spin bdle (of dim. $2^{[n/2]}$)

Classical Riemannian Dirac operator D^g:

$$\text{Dfn :} \qquad D^g : \Gamma(\Sigma) \longrightarrow \Gamma(\Sigma), \quad D^g\psi := \sum_{i=1}^n e_i \cdot \nabla^g_{e_i}\psi$$

Properties:

- D^g is elliptic differential operator of first order, essentially self-adjoint on $L^2(\Sigma)$, pure point spectrum
- Of equal fundamental importance than the Laplacian
- In dimension 4: $\text{index}(D^g) = \sigma(M^4)/8$ [Atiyah-Singer, $\sim$ 1963]
- <u>S</u>chrödinger (1932), <u>L</u>ichnerowicz (1962): $$(D^g)^2 = \Delta + \frac{1}{4}\text{Scal}^g$$

$\sim$ "‘root of the Laplacian"’ for $\text{Scal}^g = 0$

Spinors and Riemannian eigenvalue estimates

SL formula $\Rightarrow$ EV of $(D^g)^2$: $\lambda \ \geq \ \frac{1}{4}\mathrm{Scal}^g_{\min}$

• optimal only for spinors with $\langle \Delta\psi, \psi\rangle = \|\nabla^g\psi\|^2 = 0$, i. e. parallel spinors

Thm. (M, g) has parallel spinors iff $\mathrm{Hol}_0(M) = \mathrm{SU}(n), \mathrm{Sp}(n), G_2, \mathrm{Spin}(7)$, and then $\mathrm{Ric}^g = 0$. [Wang, 1989]

Thm. Optimal EV estimate: $\lambda \geq \dfrac{n}{4(n-1)}\mathrm{Scal}^g_{\min}$ [Friedrich, 1980]

• "$=$" iff $\exists$ a Killing spinor (KS) ψ: $\nabla^g_X\psi = \mathrm{const}\cdot X\cdot\psi \quad \forall X$

Link to special geometries:

Thm. $\exists$ KS $\Leftrightarrow n = 5 : (M, g)$ is Sasaki-Einstein mnfd

$\Leftrightarrow n = 6 : (M, g)$ nearly Kähler mnfd

$\Leftrightarrow n = 7 : (M, g)$ nearly parallel G_2 mnfd

(similarly for other n) [Friedrich, Grunewald, Kath, 1985-90]

Killing spinors and submanifolds

Thm. Suppose (M,g) is Sasaki-Einstein ($n=5$), nearly Kähler ($n=6$), or nearly parallel G_2 (n=7). Then the metric cone

$$(\bar{M},\bar{g}) := (M \times \mathbb{R}^+, \tfrac{1}{4} r^2 g^2 + dr^2)$$

has a ∇^g-parallel spinor; in particular, it is Ricci-flat of Riemannian holonomy $\mathrm{SU}(3)$, G_2, resp. $\mathrm{Spin}(7)$. [Bryant 1987 $\rightsquigarrow$ B-Salamon 1989, Bär 1993 (+ Wang '89)]

Observe: Construction relies on existence of a Killing spinor

Thm. Let (M,g) be a spin manifold with a ∇^g-parallel spinor ψ, $N \subset M$ a codimension one hypersurface. Then $\varphi := \psi\big|_N$ is a *generalized Killing spinor* on N, i.e. $\nabla^g_X \varphi = A(X)\cdot\varphi$ for a symmetric endomorphism A (Weingarten map). [Friedrich 1998, Bär-Gauduchon-Moroianu 2005]

Observe: Generalizes the Weierstraß representation of minimal surfaces, based on ideas of Eisenhardt (1909)

Link to special geometries:

Thm. $\exists$ gen. KS $\Leftrightarrow n = 5 : (M, g)$ is hypo $\mathrm{SU}(2)$ mnfd

($\not\subset$ contact metric mnfds) [Conti-Salamon, 2007]

$\Leftrightarrow n = 6 : (M, g)$ half-flat $\mathrm{SU}(3)$ mnfd

$\Leftrightarrow n = 7 : (M, g)$ cocalibrated G_2 mnfd

Spin structures and topology in dimension 6 and 7

Observation:

Any 8-dimensional real vector bundle over a n-dimensional manifold ($n = 6, 7$) admits a section of length one

$\Rightarrow$ a 6-dim. oriented Riemannian manifold admits a spin structure iff it admits a reduction from $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$ to $\mathrm{SU}(3)$

$\Rightarrow$ a 7-dim. oriented Riemannian manifold admits a spin structure iff it admits a reduction from $\mathrm{Spin}(7)$ to G_2

Use this section to give a uniform spinorial description of $\mathrm{SU}(3)$-manifolds and G_2-manifolds!

Spin linear algebra in dimension 6 and 7

• In $n = 6, 7$, the spin representations are real and $2^3 = 8$-dimensional, they coincide as vector spaces, call it $\Delta := \mathbb{R}^8$.

$\underline{n = 6}$ [A-Fr-Chiossi-Höll, 2014]

• Δ admits a $\mathrm{Spin}(6)$-invariant cplx structure j (because $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$)

• any real spinor $0 \neq \phi \in \Delta$ decomposes Δ into three pieces,

$$\Delta = \mathbb{R} \cdot \phi \oplus \mathbb{R} \cdot j(\phi) \oplus \underbrace{\{X \cdot \phi \,:\, X \in \mathbb{R}^6\}}_{\cong \mathbb{R}^6, \text{ the base space}} \qquad (*)$$

• the following formula defines an orthogonal cplx str. on the last piece,

$$J_\phi(X) \cdot \phi \ := \ j(X \cdot \phi)$$

• the spinor defines a 3-form by $\psi_\phi(X, Y, Z) := -(X \cdot Y \cdot Z \cdot \phi, \phi)$.

Exa. Consider $\phi = (0, 0, 0, 0, 0, 0, 0, 1) \in \Delta = \mathbb{R}^8$. Then:

$$J_\phi = -e_{12} + e_{34} + e_{56}, \quad \psi_\phi = e_{135} - e_{146} + e_{236} + e_{245}.$$

Spin linear algebra in dimension 6 and 7

Thm. The following is a 1-1 correspondence: (well-known)

• $\mathrm{SU}(3)$-structures on $\mathbb{R}^6 \longleftrightarrow$ real spinors of length one $(\mathrm{mod}\,\mathbb{Z}_2)$,

$$\mathrm{SO}(6)/\mathrm{SU}(3) = \{\mathrm{SU}(3)\text{-structures on } \mathbb{R}^6\} = \mathbb{P}(\Delta) = \mathbb{RP}^7.$$

$\underline{n = 7}$

• any real spinor $0 \neq \phi \in \Delta$ decomposes Δ into two pieces,

$$\Delta = \mathbb{R}\cdot\phi \oplus \underbrace{\{X\cdot\phi \,:\, X \in \mathbb{R}^7\}}_{\cong\mathbb{R}^7,\text{ the base space}} \qquad (**)$$

• the spinor defines again a 3-form ψ_ϕ, which turns out to be *stable* (i. e. open GL-orbit); but no analogue of neither j nor J_ϕ

Thm. The following is a 1-1 correspondence: (well-known)

stable 3-forms ψ of fixed length, with isotropy $\subset \mathrm{SO}(7) \longleftrightarrow \ldots$ (as above),

$$\mathrm{SO}(7)/G_2 \;=\; \mathbb{P}(\Delta) \;=\; \mathbb{RP}^7.$$

Special almost Hermitian geometry

• $\mathrm{SU}(3)$ manifold (M^6, g, ϕ): Riemannian spin manifold (M^6, g) equipped with a global spinor ϕ of length one, j as before, J induced almost cplx str., ω its kähler form, ψ_ϕ induced 3-form, $\psi_\phi^J := J \circ \psi_\phi$.

Decomposition $(*)$ $\Rightarrow \exists_1$ 1-form η and endomorphism S s. t.

$$\nabla^g_X \phi = \eta(X) j(\phi) + S(X) \cdot \phi$$

η: "intrinsic 1-form", S: "intr. endomorphism" (indeed: $\Gamma = S \lrcorner \psi_\phi - \frac{2}{3}\eta \otimes \omega$)

This equation summarizes all spinor eqs. previously known in dim.6!

Thm. $(\nabla^g_X \omega)(Y, Z) = 2\psi_\phi^J(S(X), Y, Z)$ $(*)$

This generalizes the classical nK condition $\nabla^g_X \omega(X, Y) = 0\ \forall X, Y$.

A classical example: Hypersurfaces in $\operatorname{Im}\mathbb{O}$ [Calabi, 1958]

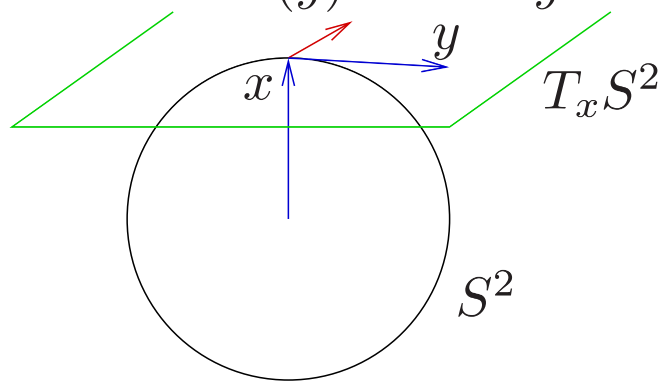
- M^6 a compact hypersurface in $\mathbb{R}^7$
- N: normal vector field
- K: shape operator (Weingarten map)
- Define $J \in \operatorname{End}(TM)$ by

$$J(Y) = N \times Y, \quad Y \in TM$$

- $J^2 = -\mathrm{Id}$ is a non integrable almost complex structure satisfying

for $S^2 \subset \mathbb{R}^3$:

$J(y) := x \times y$



$$\langle (\nabla^g_X J)(Y), Z\rangle \;=\; \langle K(X) \times Y, Z\rangle$$

This is exactly the more general eq. $(*)$ cited before

- For $M^6 = S^6$, $K = \mathrm{Id}$ and J makes it a nearly Kähler manifold: $\nabla^g_X J(X) = 0$ ($\Rightarrow$ Einstein)

There are 7 basic classes of $\mathrm{SU}(3)$-structures, called $\chi_1, \chi_{\bar{1}}, \chi_2, \chi_{\bar{2}}, \chi_3, \chi_4, \chi_5$.

[Chiossi-Salamon, 2002]

They are a refinement of the classical Gray-Hervella classification of $\mathrm{U}(3)$-structures. Write $\chi_{1\bar{2}4}$ for $\chi_1^+ \oplus \chi_2^- \oplus \chi_4$ etc.

Examples.

- nearly Kähler mnfds: class $\chi_{\bar{1}}$
- half-flat $\mathrm{SU}(3)$-mnfds: class $\chi_{\bar{1}\bar{2}3}$

Next: express Niejenhuis tensor, $d\omega, \delta\omega$ through ψ_ϕ^j, η, S – for example:

- $\delta\omega(X) = 2[(D\phi, Xj(\phi)) - \eta(X)]$ (χ_4 component)
- $N(X,Y,Z) = -2[\psi_\phi^J((J_\phi S + SJ_\phi)X, Y, Z) - \psi_\phi^J((J_\phi S + SJ_\phi)Y, X, Z)]$ ($\chi_{1\bar{1}2\bar{2}}$ component)

Thm. The classes of $\mathrm{SU}(3)$ str. are determined as follows:

class	description	dimension
χ_1	$S = \lambda \cdot J_\phi$, $\eta = 0$	1
$\chi_{\bar{1}}$	$S = \mu \cdot \mathrm{Id}$, $\eta = 0$	1
χ_2	$S \in \mathfrak{su}(3)$, $\eta = 0$	8
$\chi_{\bar{2}}$	$S \in \{A \in S^2_0(\mathbb{R}^6) \vert AJ_\phi = J_\phi A\}$, $\eta = 0$	8
χ_3	$S \in \{A \in S^2_0(\mathbb{R}^6) \vert AJ_\phi = -J_\phi A\}$, $\eta = 0$	12
χ_4	$S \in \{A \in \Lambda^2(\mathbb{R}^6) \vert AJ_\phi = -J_\phi A\}$, $\eta = 0$	6
χ_5	$S = 0$, $\eta \neq 0$	6

where $\lambda, \mu \in \mathbb{R}$. In particular S is symmetric and $\eta = 0$ if and only if the class is $\chi_{\bar{1}\bar{2}3}$.

The *symmetries of S* translate into a *differential eq. for ϕ*:

$$\begin{aligned} SJ_\phi = \pm J_\phi S \quad &\Longleftrightarrow \quad (J_\phi Y \nabla^g_X \phi, \phi) = \mp (Y \nabla^g_{J_\phi X} \phi, \phi), \\ S \text{ is } \pm\text{-symmetric} \quad &\Longleftrightarrow \quad (X \nabla^g_Y \phi, \phi) = \pm (Y \nabla^g_X \phi, \phi). \end{aligned}$$

Thm. The classification of $\mathrm{SU}(3)$ str. in terms of ϕ is given by
$(\lambda := \frac{1}{6}(D^g\phi, j(\phi))$, $\mu := -\frac{1}{6}(D^g\phi, \phi))$: (. . . and similarly for mixed classes)

class	spinorial equation
χ_1	$\nabla^g_X \phi = \lambda X j(\phi)$ for $\lambda \in \mathbb{R}$
$\chi_{\bar{1}}$	$\nabla^g_X \phi = \mu X \phi$ for $\mu \in \mathbb{R}$ (Killing sp.)
χ_2	$(J_\phi Y \nabla^g_X \phi, \phi) = -(Y \nabla^g_{J_\phi X} \phi, \phi)$, $(Y \nabla^g_X \phi, j(\phi)) = +(X \nabla^g_Y \phi, j(\phi))$, $\lambda = \eta = 0$
$\chi_{\bar{2}}$	$(J_\phi Y \nabla^g_X \phi, \phi) = +(Y \nabla^g_{J_\phi X} \phi, \phi)$, $(Y \nabla^g_X \phi, j(\phi)) = -(X \nabla^g_Y \phi, j(\phi))$, $\mu = \eta = 0$
χ_3	$(J_\phi Y \nabla^g_X \phi, \phi) = +(Y \nabla^g_{J_\phi X} \phi, \phi)$, $(Y \nabla^g_X \phi, j(\phi)) = +(X \nabla^g_Y \phi, j(\phi))$, and $\eta = 0$
χ_4	$(J_\phi Y \nabla^g_X \phi, \phi) = -(Y \nabla^g_{J_\phi X} \phi, \phi)$, $(Y \nabla^g_X \phi, j(\phi)) = -(X \nabla^g_Y \phi, j(\phi))$ and $\eta = 0$
χ_5	$\nabla^g_X \phi = (\nabla^g_X \phi, j(\phi))\, j(\phi)$

Corollary. On a 6-dim spin mnfd, $\exists$ spinor of constant length s. t.

$$D^g\phi \;=\; 0$$

iff admits a $\mathrm{SU}(3)$ structure of class $\chi_{2\bar{2}345}$ with $\delta\omega = -2\eta$.

Example: twistor spaces as $\mathrm{SU}(3)$-manifolds

- $M^6 = \mathbb{CP}^3$, $\mathrm{U}(3)/\mathrm{U}(1)^3$: twistor spaces of S^4 and $\mathbb{CP}^2$. Both carry metrics $g_t (t > 0)$ and two almost complex structures $\Omega^{\mathrm{K}}, \Omega^{\mathrm{nK}}$ such that

- $(M^6, g_{1/2}, \Omega^{\mathrm{nK}})$ is a nearly Kähler manifold
- $(M^6, g_1, \Omega^{\mathrm{K}})$ is a Kähler manifold

- $\exists$ two real linearly indep. global spinors ϕ_ε in Δ_6 ($\varepsilon = \pm 1$). **Both spinors** induce the same almost cplx structure J_ϕ ($\Leftrightarrow \Omega^{\mathrm{nK}}$)!

- For $t = 1/2$, ϕ_ε are Riemannian Killing spinors. For general t, define $S_\varepsilon : TM^6 \to TM^6$ by $S_\varepsilon = \varepsilon\sqrt{c} \cdot \mathrm{diag}\left(\frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{1-t}{2\sqrt{t}}, \frac{1-t}{2\sqrt{t}}\right)$.

Verify: $\nabla^g_X \phi_\varepsilon = S_\varepsilon(X)\phi_\varepsilon$, hence S_ε *is* the intr. endom. and $\eta = 0$.

- Class: $\chi_{\bar{1}\bar{2}}$ for $t \neq 1/2$, $\chi_{\bar{1}}$ for $t = 1/2$.

- For $t = 1$, ϕ_ε are Kählerian Killing spinors, but they *do not* induce the Kählerian cplx str. Ω^{K}! Thus, the Kählerian structure cannot be recovered from the pair of Kählerian Killing spinors (only a $\mathrm{U}(3)$-reduction).

Characteristic connections

For all classes, an adapted metric connection ∇ can be defined.

$$\text{torsion:} \quad T(X,Y,Z) \; := \; g(\nabla_X Y - \nabla_Y X - [X,Y], Z)$$

Special case: require $T \in \Lambda^3(M^n)$ ($\Leftrightarrow$ same geodesics as ∇^g)

$$\Rightarrow \quad g(\nabla_X Y, Z) \; = \; g(\nabla^g_X Y, Z) + \tfrac{1}{2} T(X,Y,Z)$$

• If existent, such a connection is unique and called the *'characteristic connection'* [Fr-Ivanov 2002, A-Fr-Höll 2013]

Thm. A spin manifold (M^6, g, ϕ) admits a characteristic connection ∇ iff it is of class $\chi_{1\bar{1}345}$ and $\eta = \frac{1}{4}\delta\omega$. It satisfies $\nabla\phi = 0$.

Corollary. Whenever ∇ exists,

$$\phi \in \ker D^g \iff T\phi = 0 \iff \text{the } \mathrm{SU}(3)\text{-class is } \chi_3 \text{(almost Hermitian).}$$

G_2 geometry

• G_2 manifold (M^7, g, ϕ): Riemannian spin manifold (M^7, g) equipped with a global spinor ϕ of length one, ψ_ϕ induced 3-form.

Decomposition $(**) \Rightarrow \exists_1$ endomorphism S s. t.

$$\boxed{\nabla^g_X \phi = S(X) \cdot \phi}$$

S: "intrinsic endomorphism" (indeed: $\Gamma = -\frac{2}{3} S \lrcorner \psi_\phi$)

Thm. $(\nabla^g_V \psi_\phi)(X, Y, Z) = 2 * \psi_\phi(S(V), X, Y, Z)$.

This generalizes the nearly parallel G_2 condition $\nabla \psi_\phi = d\psi_\phi = c * \psi_\phi$!

There are 4 basic classes of G_2-structures, called $\mathcal{W}_1, \ldots, \mathcal{W}_4$.

[Fernandez-Gray, 1982]

Thm. The classes of G_2 structures are determined as follows:

class	description	dimension
$\mathcal{W}_1$	$S = \lambda\,\mathrm{Id}$	1
$\mathcal{W}_2$	$S \in \mathfrak{g}_2$	14
$\mathcal{W}_3$	$S \in S^2_0\mathbb{R}^7$	27
$\mathcal{W}_4$	$S \in \{V \lrcorner \Psi_\phi \mid V \in \mathbb{R}^7\}$	7

In particular, S is symmetric if and only if $S \in \mathcal{W}_{13}$ and skew iff it belongs to $\mathcal{W}_{24}$.

Corollary. Let (M^7, g, ϕ) be a Riemannian spin manifold with unit spinor ϕ. Then ϕ is harmonic

$$D^g\phi = 0$$

iff the underlying G_2-structure is of class $\mathcal{W}_{23}$.

Thm. The basic classes of G_2-manifolds described in terms of ϕ:

($\lambda := -\frac{1}{7}(D^g\phi, \phi) : M \to \mathbb{R}$ is a real function and $\times$ the cross product relative to Ψ_ϕ)

class	spinorial equation
$\mathcal{W}_1$	$\nabla^g_X \phi = \lambda X \phi$ (Killing spinor)
$\mathcal{W}_2$	$\nabla^g_{X\times Y}\phi = Y\nabla^g_X\phi - X\nabla^g_Y\phi + 2g(Y, S(X))\phi$
$\mathcal{W}_3$	$(X\nabla^g_Y\phi, \phi) = (Y\nabla^g_X\phi, \phi)$ and $\lambda = 0$
$\mathcal{W}_4$	$\nabla^g_X\phi = XV\phi + g(V, X)\phi$ for some $V \in TM^7$
$\mathcal{W}_{12}$	$\nabla^g_{X\times Y}\phi = Y\nabla^g_X\phi - X\nabla^g_Y\phi + g(Y, S(X))\phi - g(X, S(Y))\phi - \lambda(X \times Y)\phi$
$\mathcal{W}_{13}$	$(X\nabla^g_Y\phi, \phi) = (Y\nabla^g_X\phi, \phi)$
$\mathcal{W}_{14}$	$\exists V, W \in TM^7 : \nabla^g_X\phi = XVW\phi - (XVW\phi, \phi)$
$\mathcal{W}_{23}$	$S\phi = 0$ and $\lambda = 0$, or $D^g\phi = 0$
$\mathcal{W}_{24}$	$(X\nabla^g_Y\phi, \phi) = -(Y\nabla^g_X\phi, \phi)$

Example: 7-dim. $3(\alpha,\delta)$-Sasaki mnfds

Dfn. An *almost 3-contact metric manifold* is a Riemannian manifold (M^{4n+3}, g) endowed with 3 almost contact structures (ϕ_i, ξ_i, η_i), $i = 1, 2, 3$ s.t. g is compatible with each a.c.str. and

$$\begin{aligned} \varphi_k = \varphi_i\varphi_j - \eta_j \otimes \xi_i = -\varphi_j\varphi_i + \eta_i \otimes \xi_j, \\ \xi_k = \varphi_i\xi_j = -\varphi_j\xi_i, \quad \eta_k = \eta_i \circ \varphi_j = -\eta_j \circ \varphi_i, \end{aligned} \tag{1}$$

for any even permutation (i, j, k) of $(1, 2, 3)$.

• $TM = \mathcal{H} \oplus \mathcal{V}$, where $\quad \mathcal{H} := \bigcap_{i=1}^{3} \mathrm{Ker}(\eta_i), \quad \mathcal{V} := \langle \xi_1, \xi_2, \xi_3 \rangle.$

Dfn. An a. 3-c. m. m. M will be called a 3-(α,δ)-*Sasaki manifold* if

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k$$

for every even permutation (i, j, k) of $(1, 2, 3)$, where $\alpha \in \mathbb{R}^*, \delta \in \mathbb{R}$.

• When $\alpha = \delta = 1$, we have a 3-contact metric manifold, and hence a 3-Sasaki manifold by a theorem of Kashiwada

• Quat. Heisenberg groups are examples with $\delta = 0$

• Known: A 3-Sasaki mnfd is always Einstein and has 3 Riemannian Killing spinors

• each a.c. structure η_i induces a characteristic connection ∇^i, but $\nabla^1 \neq \nabla^2 \neq \nabla^3$?!?

Thm. Let ^{7} be a 3-(α, δ)-Sasaki manifold. There exists a cocalibrated G_2-structure with char. connection ∇ with parallel spinor ψ with the properties:

• ∇ preserves $\mathcal{V}$ and $\mathcal{H}$, and $\nabla T = 0$

• ψ and $\xi_i \cdot \psi$ are generalized Riemannian Killing spinors on M^7

[A-Dileo, 2019]

Observe: Only known example of gKS where endom. has three different eigenvalues

Application: cone constructions

• How to construct G_2-str. of any class on cones over $\mathrm{SU}(3)$-manifolds?

Start with (M^6, g, ϕ) with intrinsic torsion (S, η). Choose a function $h = h_1 + ih_2 : I \to S^1$ and define by

$$\phi_t \;:=\; h(t)\phi := h_1(t)\phi + h_2(t)j(\phi)$$

a new family of $\mathrm{SU}(3)$-structures on M^6 depending on $t \in I$.

Conformally rescale the metric by some function $f : I \to \mathbb{R}_+$ and consider $M_t^6 := (M^6, f(t)^2 g, \phi_t)$. Intrinsic torsion of M_t^6 : $(\frac{h^2}{f} S, \eta)$.

Dfn. *spin cone* over M^6: $(\bar{M}^7, \bar{g}) = (M^6 \times I, f^2(t)g + dt^2)$ with spinor ϕ_t.

Exa. Suppose we want $\bar{M}^7$ to be a nearly parallel G_2-manifold: need h'/h constant, so $h(t) = \exp(i(ct + d))$, $c, d \in \mathbb{R}$.
Easiest: sine cone $(M^6 \times (0, \pi), \sin(t)^2 g + dt^2, e^{it/2}\phi)$ [Fernández-Ivanov-Muñoz-Ugarte, 2008; Stock, 2009]

• Similarly, we can construct G_2-manifolds of any desired pure class (construction really uses the spinor!).

To conclude:

Obtained a uniform description of all possible defining spinorial differential eqs. on 6-dim. $\mathrm{SU}(3)$-manifolds and G_2-manifolds, generalizing Killing spinors, generalized Killing spinors, quasi-Killing spinors [Friedrich-Kim, 2000]. . .

So far, all spinors encountered are *generalized Killing spinor with torsion (gKST)*, i. e.

$$\nabla\phi = A(X)\cdot\phi$$

for some endomorphism $A : TM^6 \to TM^6$; but the same eq. can be expressed in different ways.

- Not the differential eq. is the basic object, but rather the G-structure!

References

Ilka Agricola, Simon G. Chiossi, Thomas Friedrich, Jos Höll, *Spinorial description of* $\mathrm{SU}(3)$*- and* G_2*-manifolds*, J. Geom. Phys. 98 (2015), 535-555.

Ilka Agricola, Jos Höll, *Cones of* G*-manifolds and Killing spinors with skew torsion*, Ann. Mat. Pura Appl. 194 (2015), 673-718.

Ilka Agricola, Thomas Friedrich, 3*-Sasakian manifolds in dimension seven, their spinors and* G_2*-Structures*, J. Geom. Phys. 60 (2010), 326-332.

Ilka Agricola, Giulia Dileo, *Generalizations of 3-Sasakian manifolds and skew torsion*, Adv. Geom. 20 (2020), 331-374.

Another application:

Ilka Agricola, Julia Becker-Bender, Hwajeong Kim, *Twistorial eigenvalue estimates for generalized Dirac operators with torsion*, Adv. Math. 243 (2013), 296-329.

* * *

Ilka Agricola, *The Srní lectures on non-integrable geometries with torsion* with an appendix by Mario Kassuba [pdf-file], Arch. Math. 42 (2006), 5-84.