Mirror symmetry for a cusp polynomial Landau-Ginzburg orbifold

Alexey Basalaev

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Cusp polynomial

Fix $A' = (a'_1, a'_2, a'_3) \in \mathbb{N}^3$ and any fixed $c \in \mathbb{C}^*$

$$f_{A'} := x_1^{a'_1} + x_2^{a'_2} + x_3^{a'_3} - c^{-1} x_1 x_2 x_3.$$

Set

$$\mu_{A'} := 2 + \sum_{i=1}^{3} \left(a'_i - 1\right), \quad \chi_{A'} := 2 + \sum_{i=1}^{3} \left(\frac{1}{a'_i} - 1\right).$$

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- $\chi_{A'} = 0$, simple–elliptic singularity $\widetilde{E}_6, \widetilde{E}_7$ of $\widetilde{E}_8$,
- $\chi_{A'} > 0$, *affine cusp singularity*,
- $\chi_{A'} < 0$, $\mathbf{x} = 0$ **is not** the only critical point.

In what follows assume $f_{A'}$ in a small neighborhood of $\mathbf{x} = 0$.

Geigle-Lenzing orbifold $\mathbb{P}^1_{A,\Lambda}$

Fix $A = (a_1, \dots, a_r) \in \mathbb{N}^r$.

Geigle-Lenzing orbifold $\mathbb{P}^1_{A,\Lambda}$

Fix $A = (a_1, \dots, a_r) \in \mathbb{N}^r$. Let $\Lambda = (\lambda_1, \dots, \lambda_r)$, $\lambda_k \in \mathbb{P}^1$, pairwise distinct

$$\lambda_1 = \infty,\ \lambda_2 = 0, \lambda_3 = 1.$$

Define

$$\mathbb{P}^1_{A,\Lambda} := \left[\left(\mathrm{Spec}(R_{A,\Lambda}) \backslash \{0\} \right) / \mathrm{Spec}(\mathbb{C} L_A) \right],$$

for

1. $R_{A,\Lambda} := \mathbb{C}[X_1, \dots, X_r] \,/ I_\Lambda$, with I_Λ generated by

$$X_i^{a_i} - X_2^{a_2} + \lambda_i X_1^{a_1}, \quad i = 3, \dots, r.$$

2. L_A is an abelian group, generated by r–letters $\vec{X}_i$, $i = 1, \dots, r$

$$L_A := \bigoplus_{i=1}^{r} \mathbb{Z}\vec{X}_i \,/ M_A\ ,$$

for being M_A generated by

$$a_i\vec{X}_i - a_j\vec{X}_j, \quad 1 \le i < j \le r.$$

Geigle-Lenzing orbifold, continued

$\mathbb{P}^1_{A,\Lambda}$ satisfies:

- topologically $\mathbb{P}^1$,
- has r points with non–trivial stabilizers $\mathbb{Z}/a_k\mathbb{Z}$,
- $\lambda_1, \dots, \lambda_r$ are positions of these points.

Example: Elliptic orbifolds

For A being $(3,3,3), (4,4,2), (6,3,2)$ or $(2,2,2,2)$, we have

$$\mathbb{P}^1_{A,\Lambda} \cong \mathcal{E}/G$$

for $\mathcal{E}$ – elliptic curve, G - finite group.

- $(3,3,3)$ — hexagonal lattice, rotation by $2\pi/3$,
- $(4,4,2)$ — square lattice, rotation by $\pi/2$,
- $(6,3,2)$ — hexagonal lattice, rotation by $2\pi/6$,
- $(2,2,2,2)$ — any lattice, hyperelliptic involution.

Mirror symmetry, non-equivariant

Theorem (Ebeling-Shiraishi-Satake-Takahashi, Milanov-Ruan-Shen)

For $A' = (a'_1, a'_2, a'_3) \in \mathbb{N}^2$, *the orbifold* $\mathbb{P}^1_{A',\Lambda}$ *is mirror to* $f_{A'}$.

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Theorem (Ebeling-Shiraishi-Satake-Takahashi, Milanov-Ruan-Shen)

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Dubrovin–Frobenius manifolds

$$M^{\zeta}_{f_{A'}} - \text{ of Saito theory with primitive form } \zeta,$$
$$M_{\mathbb{P}^1_{A',\Lambda}} - \text{ of Gromov–Witten theory.}$$

($\dim_{\mathbb{C}} = \mu_{A'} = a'_1 + a'_2 + a'_3 - 1$) are isomorphic

$$M^{\zeta}_{f_{A'}} \cong M_{\mathbb{P}^1_{A',\Lambda}}.$$

A' on both sides!

Mirror symmetry, equivariant — conjecture

Fix some $G \subseteq G_{f_{A'}}$ with

$$G_{f_{A'}} := \{g = \mathrm{diag}(g_1, g_2, g_3) \in \mathrm{GL}(3, \mathbb{C}) \mid f_{A'}(\mathbf{x}) = f_{A'}(g \cdot \mathbf{x})\}\,.$$

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Define

$$A := \left(\frac{a'_1}{|G/K_1|} * |K_1|, \frac{a'_2}{|G/K_2|} * |K_2|, \frac{a'_3}{|G/K_3|} * |K_3| \right),$$

- $K_i \subseteq G$ — maximal subgroup, fixing x_i,
- $*|K_i|$ = “repeat $|K_i|$ times”,
- 1’s should be dropped.

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Conjecture (Ebeling-Takahashi)

The orbifold $\mathbb{P}^1_{A,\Lambda}$ is mirror to $(f_{A'}, G)$.

Supported by computing Gabrielov/Dolgachev numbers.

Mirror symmetry, equivariant — theorem

Theorem (B-Takahashi)

- *There is a construction of Dubrovin–Frobenius manifold* $M^{\zeta}_{(f_{A'},G)}$ *of the pair* $(f_{A'}, G)$*.*
- $M^{\zeta}_{(f_{A'},G)} \cong M_{\mathbb{P}^1_{A,\Lambda}}$ *(no prime on the RHS).*

Corollary

$M^{\zeta}_{(f_{A'},G)}$ *is unique up to isomorphism.*

Examples: $\chi_{A'} = 0$

Denote $e[\alpha] := \exp(2\pi\sqrt{-1}\cdot\alpha)$.

A'	$f_{A'}$ type	G	A
$(3,3,3)$	$\widetilde{E}_6$	$\langle(e[0], e[\frac{1}{3}], e[\frac{2}{3}])\rangle = K_1$	$(3,3,3)$
$(4,4,2)$	$\widetilde{E}_7$	$\langle(e[0], e[\frac{1}{2}], e[\frac{1}{2}])\rangle = K_1$	$(4,4,2)$
$(4,4,2)$	$\widetilde{E}_7$	$\langle(e[\frac{1}{4}], e[\frac{3}{4}], e[0])\rangle = K_3$	$(2,2,2,2)$
$(4,4,2)$	$\widetilde{E}_7$	$\langle(e[0], e[\frac{1}{2}], e[\frac{1}{2}]), (e[\frac{1}{2}], e[0], e[\frac{1}{2}])\rangle$	$(2,2,2,2)$
$(6,3,2)$	$\widetilde{E}_8$	$\langle(e[\frac{1}{2}], e[0], e[\frac{1}{2}])\rangle = K_2$	$(3,3,3)$
$(6,3,2)$	$\widetilde{E}_8$	$\langle(e[\frac{1}{3}], e[\frac{2}{3}], e[0])\rangle = K_3$	$(2,2,2,2)$

$$A := \left(\frac{a'_1}{|G/K_1|} * |K_1|, \frac{a'_2}{|G/K_2|} * |K_2|, \frac{a'_3}{|G/K_3|} * |K_3|\right),$$

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$(4,4,2)$	$\widetilde{E}_7$	$\langle(\mathrm{e}[0],\mathrm{e}[\frac{1}{2}],\mathrm{e}[\frac{1}{2}])\rangle = K_1$	$(4,4,2)$
$(4,4,2)$	$\widetilde{E}_7$	$\langle(\mathrm{e}[\frac{1}{4}],\mathrm{e}[\frac{3}{4}],\mathrm{e}[0])\rangle = K_3$	$(2,2,2,2)$
$(4,4,2)$	$\widetilde{E}_7$	$\langle(\mathrm{e}[0],\mathrm{e}[\frac{1}{2}],\mathrm{e}[\frac{1}{2}]),$ $(\mathrm{e}[\frac{1}{2}],\mathrm{e}[0],\mathrm{e}[\frac{1}{2}])\rangle$	$(2,2,2,2)$
$(6,3,2)$	$\widetilde{E}_8$	$\langle(\mathrm{e}[\frac{1}{2}],\mathrm{e}[0],\mathrm{e}[\frac{1}{2}])\rangle = K_2$	$(3,3,3)$
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$$M^{\zeta}_{f_{3,3,3},K_1} \cong M_{\mathbb{P}^1_{3,3,3}}, \quad M^{\zeta}_{f_{6,3,2},K_2} \cong M_{\mathbb{P}^1_{3,3,3}},$$

$$M^{\zeta}_{f_{4,4,2},K_1} \cong M_{\mathbb{P}^1_{4,4,2}}.$$

Examples: $\chi_{A'} = 0$ continued

A'	$f_{A'}$ type	G	A	f_A type
$(3,3,3)$	$\widetilde{E}_6$	$\langle(\mathrm{e}[0], \mathrm{e}[\frac{1}{3}], \mathrm{e}[\frac{2}{3}])\rangle$	$(3,3,3)$	$\widetilde{E}_6$
$(4,4,2)$	$\widetilde{E}_7$	$\langle(\mathrm{e}[0], \mathrm{e}[\frac{1}{2}], \mathrm{e}[\frac{1}{2}])\rangle$	$(4,4,2)$	$\widetilde{E}_7$
$(4,4,2)$	$\widetilde{E}_7$	$\langle(\mathrm{e}[\frac{1}{4}], \mathrm{e}[\frac{3}{4}], \mathrm{e}[0])\rangle$	$(2,2,2,2)$	-
$(4,4,2)$	$\widetilde{E}_7$	$\langle(\mathrm{e}[0], \mathrm{e}[\frac{1}{2}], \mathrm{e}[\frac{1}{2}]),$ $(\mathrm{e}[\frac{1}{2}], \mathrm{e}[0], \mathrm{e}[\frac{1}{2}])\rangle$	$(2,2,2,2)$	-
$(6,3,2)$	$\widetilde{E}_8$	$\langle(\mathrm{e}[\frac{1}{2}], \mathrm{e}[0], \mathrm{e}[\frac{1}{2}])\rangle$	$(3,3,3)$	$\widetilde{E}_6$
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$$M^{\zeta}_{f_{3,3,3},K_1} \xrightarrow{Eq} M_{\mathbb{P}^1_{3,3,3}} \xrightarrow{nonEq} M^{\zeta}_{f_{3,3,3},\{\mathrm{id}\}},$$

$$M^{\zeta}_{f_{4,4,2},K_1} \xrightarrow{Eq} M_{\mathbb{P}^1_{4,4,2}} \xrightarrow{nonEq} M^{\zeta}_{f_{4,4,2},\{\mathrm{id}\}},$$

$$M^{\zeta}_{f_{6,3,2},K_2} \xrightarrow{Eq} M_{\mathbb{P}^1_{3,3,3}} \xrightarrow{nonEq} M^{\zeta}_{f_{3,3,3},\{\mathrm{id}\}}.$$

Examples: $\widetilde{E}_6$ and $\widetilde{E}_7$

Proposition(B-Takahashi)

The compositions

$$M^{\zeta}_{f_{3,3,3},K_1} \xrightarrow{nonEq \circ Eq} M^{\zeta}_{f_{3,3,3},\{\mathrm{id}\}}, \quad M^{\zeta}_{f_{4,4,2},K_1} \xrightarrow{nonEq \circ Eq} M^{\zeta}_{f_{4,4,2},\{\mathrm{id}\}}$$

restrict to the non–identical automorphism on the Dubrovin-Frobenius submanifold of $M^{\zeta}_{f_{3,3,3},\{\mathrm{id}\}}$ and $M^{\zeta}_{f_{4,4,2},\{\mathrm{id}\}}$ respectively.

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$$M^{\zeta}_{f_{3,3,3},K_1} \xrightarrow{nonEq\circ Eq} M^{\zeta}_{f_{3,3,3},\{\mathrm{id}\}}, \quad M^{\zeta}_{f_{4,4,2},K_1} \xrightarrow{nonEq\circ Eq} M^{\zeta}_{f_{4,4,2},\{\mathrm{id}\}}$$

restrict to the non–identical automorphism on the Dubrovin-Frobenius submanifold of $M^{\zeta}_{f_{3,3,3},\{\mathrm{id}\}}$ and $M^{\zeta}_{f_{4,4,2},\{\mathrm{id}\}}$ respectively.

These automorphism are equivalent to the following relations in the rings of quasimodular forms:

$\theta_2(q^2)^2 + \theta_3(q^2)^2 = \theta_3(q)^2,$

$$\frac{1}{2}\left(E_2(q) - 9E_2\left(q^9\right)\right) + 3\left(\theta_2\left(q^6\right)\theta_2\left(q^{18}\right) + \theta_3\left(q^6\right)\theta_3\left(q^{18}\right)\right)^2$$

$$+\,\theta_2\left(q^2\right)^2\theta_2\left(q^6\right)^2 + 2\theta_2\left(q^2\right)\theta_3\left(q^2\right)\theta_3\left(q^6\right)\theta_2\left(q^6\right) + \theta_3\left(q^2\right)^2\theta_3\left(q^6\right)^2 = 0$$

$$-\frac{6\eta\left(q^9\right)^3}{\eta\left(q^3\right)} + \theta_2\left(q^6\right)\left(\theta_2\left(q^2\right) - \theta_2\left(q^{18}\right)\right) + \theta_3\left(q^6\right)\left(\theta_3\left(q^2\right) - \theta_3\left(q^{18}\right)\right) = 0$$

Dubrovin–Frobenius manifold of $(f_{A'}, G)$ - 1

- $M^{\infty}_{f_{A'},G}$ has the flat coordinates

  $$v_{\mathrm{id},k},\ 1 \le k \le \mu_A, \quad \text{and} \quad v_{g_i^l,k},\ 1 \le l \le n_i - 1,\ 1 \le k \le a_i - 1,$$

  for g_i - generator of $K_i \subseteq G$, $n_i := |K_i|$.
  The vector $\partial/\partial v_{\mathrm{id},1}$ is the unit,
- $M^{\infty}_{f_{A'},G}$ is quasihomogeneous of conformal dimension 1 with respect to the Euler field

  $$E = v_{\mathrm{id},1}\frac{\partial}{\partial v_{\mathrm{id},1}} + \sum_{i=1}^{3}\sum_{l=1}^{n_i-1}\sum_{k=1}^{a_i-1}\frac{a_i - k}{a_i} v_{g_i^l,k}\frac{\partial}{\partial v_{g_i^l,k}} + \chi_A \frac{\partial}{\partial v_{\mathrm{id},\mu_A}}$$

- potential of $M^{\infty}_{f_{A'},G}$ has a series expansion in $v_{\mathrm{id},1}, \dots, v_{\mathrm{id},\mu_A - 1}$, $v_{g_i^l,k}$ and $\exp(|G| v_{\mathrm{id},\mu_A})$,
- the tangent space $T_0 M^{\infty}_{f_{A'},G}$, restricted at $v_{\mathrm{id},1} = \dots = v_{\mathrm{id},\mu_A - 1} = 0$ and $v_{g_i^l,k} = 0$, is isomorphic to $\overline{\mathcal{A}^*}(f_{A'}, G)$ as Frobenius $\mathbb{C}[c]$–algebra with $c = \exp(|G| v_{id,\mu_A})$

Dubrovin–Frobenius manifold of $(f_{A'}, G)$ - 2

- for every fixed $1 \le i \le 3$ the potential of $M^{\infty}_{f_{A'},G}$ is invariant under the change of the variables $v_{g_i^l,k} \to g_i^l v_{g_i^l,k}$, $1 \le l \le n_i - 1$.

  It follows from this axiom together with extended Jac *axiom that the product of* $M^{\infty}_{f_{A'},G}$ *is* G*–graded. In particular,*

$$M^{\infty}_{f_{A'},\mathrm{id}} := M^{\infty}_{f_{A'},G} \mid_{v_{g_i^l,k}=0} .$$

  is a Frobenius submanifold.
- there is a Frobenius manifold isomorphism $(M^{\infty}_{f_{A'}})^G \cong M^{\infty}_{f_{A'},\mathrm{id}}$.
- for every group automorphism $\phi : G \to G$, the potential of $M^{\infty}_{f_{A'},G}$ is invariant with respect to the change of the variables $v_{g_i^l,k} \to v_{\phi(g_i^l),k}$.

Examples of potentials - 1

$A' = (2,2,4)$ and $G = K_1 \Rightarrow A = (2,2,2)$.

$$
\begin{aligned}
\mathcal{F}_{f_{A'},K_1}(\mathrm{v}) = & \frac{q^8}{8} + \left(\frac{1}{2} v_{g^2,1}^2 + \frac{v_{g,1}^2}{2} + \frac{v_{\mathrm{id},x}^2}{4} \right) q^4 \\
& + \left(\frac{1}{2} v_{g,1}^2 v_{\mathrm{id},x} - \frac{1}{2} v_{g^2,1}^2 v_{\mathrm{id},x} \right) q^2 \\
& - \frac{1}{96} v_{g^2,1}^4 - \frac{1}{16} v_{g,1}^2 v_{g^2,1}^2 - \frac{v_{g,1}^4}{96} - \frac{v_{\mathrm{id},x}^4}{192} \\
& + v_{\mathrm{id},1} \left(\frac{1}{4} v_{g^2,1}^2 + \frac{v_{g,1}^2}{4} + \frac{v_{\mathrm{id},x}^2}{8} \right) + \frac{1}{2} v_{\mathrm{id},1}^2 v_{\mathrm{id},5}, \quad q = \exp(v_{\mathrm{id},5}).
\end{aligned}
$$

with coordinates $v_{\mathrm{id},1}, v_{\mathrm{id},x}, v_{\mathrm{id},5} = v_{\mathrm{id},\mu'}$ and $v_{g,1}, v_{g^2,1}$.

Examples of potentials - 2

$A' = (4,4,2)$ and $G = K_3 \Rightarrow A = (2,2,2,2)$.

$$\mathcal{F}^{\infty}_{(\widetilde{E}_7,K_3)}(\mathrm{v}) = -\frac{w}{96}\left(3v_{g,1}v_{g^2,1} + 2v_{g^0,1}^2\right)^2$$
$$+\frac{x^2}{288}\left(12\left(v_{g^2,1}^3 + v_{g,1}^3\right)v_{g^0,1} + 9v_{g,1}^2v_{g^2,1}^2 + 8v_{g^0,1}^4\right)$$
$$+\frac{y^2}{144}\left(-18v_{g,1}v_{g^2,1}v_{g^0,1}^2 - 6\left(v_{g^2,1}^3 + v_{g,1}^3\right)v_{g^0,1} - 9v_{g,1}^2v_{g^2,1}^2 - 2v_{g^0,1}^4\right)$$
$$+ v_{\mathrm{id},1}\left(\frac{1}{2}v_{g,1}v_{g^2,1} + \frac{v_{g^0,1}^2}{3}\right) + \frac{1}{2}v_{\mathrm{id},1}^2 v_{\mathrm{id},6}.$$

with

$$x(q) = \left(\theta_3(q^4)\right)^2,\ y(q) = \left(\theta_2(q^8)\right)^2, \quad q = \exp(v_{\mathrm{id},6}) \tag{1}$$
$$w(q) = \frac{1}{3}\left(E_2(q^4) - 2E_2(q^8) + 4E_2(q^{16})\right). \tag{2}$$

Dubrovin-Frobenius manifold: at the origin

Assume the $\mu_{A'}$–dimensional vector spaces

$$\mathrm{Jac}(f_{A'}) := \mathcal{O}_{\mathbb{C}^3,0}/(\partial_{x_1} f_{A'}, \partial_{x_2} f_{A'}, \partial_{x_3} f_{A'}),$$
$$\Omega_{f_{A'}} := \Omega^3(\mathbb{C}^3)/(df_{A'} \wedge \Omega^2(\mathbb{C}^3)).$$

Fix nowhere vanishing $\zeta \in \Omega_{f_{A'}}$,

$$\mathrm{Jac}(f_{A'}) \cong \Omega_{f_{A'}}, \ [\phi(x)] \mapsto [\phi(x)\zeta].$$

Poincare residue pairing

$$\eta(\omega_1, \omega_2) := \mathrm{res}_{\mathbb{C}^3} \frac{\phi_1\phi_2 \cdot \zeta}{\partial_{x_1} f_{A'} \cdot \partial_{x_2} f_{A'} \cdot \partial_{x_3} f_{A'}}, \quad \text{for } \omega_k = [\phi_k \zeta].$$

$\Rightarrow$ Frobenius algebra $(\mathrm{Jac}(f_{A'}), \circ, \eta)$: $\eta(u \circ v, w) = \eta(u, v \circ w)$.

Deforming the origin

Consider the unfolding

$$F_{A'}(\mathrm{x};\mathrm{s}',s'_{\mu_{A'}}) := x_1^{a'_1}+x_2^{a'_2}+x_3^{a'_3}-(s_{\mu_{A'}})^{-1}\cdot x_1x_2x_3+s_1\cdot 1+\sum_{i=1}^{3}\sum_{j=1}^{a'_i-1} s_{i,j}\cdot x_i^j.$$

Assume $c = s_{\mu_{A'}}$ and $s_\bullet \in \mathcal{S} \subset \mathbb{C}^{\mu_{A'}}$.

$$\mathcal{O}_{\mathcal{C}} := \mathcal{O}_{\mathbb{C}^3\times\mathcal{S},0}/(\partial_{x_1}F_{A'},\partial_{x_2}F_{A'},\partial_{x_3}F_{A'}),$$

Let $\mathcal{X} := \mathbb{C}^3 \times \mathcal{S}$ and $p : \mathcal{X} \to \mathcal{S}$ the projection on $\mathcal{S}$.

$$\Omega_{F_{A'}} := p_*\Omega^3_{\mathbb{C}^3/\mathcal{S}}/dF_{A'} \wedge p_*\Omega^2_{\mathbb{C}^3/\mathcal{S}}.$$

Fix nowhere vanishing $\zeta \in \Omega_{F_{A'}}$

$$\eta(\omega_1,\omega_2) := \mathrm{res}_{\mathbb{C}^3}\frac{\phi_1\phi_2\cdot\zeta}{\partial_{x_1}F_{A'}\cdot\partial_{x_2}F_{A'}\cdot\partial_{x_3}F_{A'}}, \quad \text{for } \omega_k = [\phi_k\zeta].$$

Dubrovin-Frobenius manifold of a Saito theory

$\exists$ isomorphism (fixed by ζ)

$$\mathcal{T}_{\mathcal{S}} \to p_* \mathcal{O}_{\mathcal{C}} \to \Omega_{F_{A'}}.$$

Denote by $\circ$ and $\eta = \eta(s)$ the product and pairing on $\mathcal{T}_{\mathcal{S}}$.

Theorem (K.Saito, M.Saito)

There is a choice of $\zeta \in \Omega_{F_{A'}}$, s.t. η is flat. In particular, there flat coordinates $t_1, \dots, t_{\mu_{A'}}$, s.t. the components of η are constants. Moreover, there is $\mathcal{F}^{\zeta}_{f_{A'}} = \mathcal{F}^{\zeta}_{f_{A'}}(t)$, s.t.

$$\eta_{\alpha,\beta} = \eta(\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}) = \frac{\partial^3 \mathcal{F}^{\zeta}_{f_{A'}}}{\partial t^1 \partial t^\alpha \partial t^\beta}$$

and the product $\circ$ reads

$$\frac{\partial}{\partial t^\alpha} \circ \frac{\partial}{\partial t^\beta} = \sum_{\sigma,\delta=1}^{\mu_{A'}} \frac{\partial^3 \mathcal{F}^{\zeta}_{f_{A'}}}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \eta^{\gamma\delta} \frac{\partial}{\partial t^\delta}.$$