Non-diagonalisable Hydrodynamic Type Systems, Integrable by Tsarev's Generalised Hodograph Method

Maxim V. Pavlov

Lebedev Physical Institute

The Talk is based on joint works with my friends and colleagues:
E.V. Ferapontov, G.A. El, A.M. Kamchatnov, V.B. Taranov, S.P. Tsarev, S.A. Zykov

21 October 2021

Beijing-Novosibirsk Seminar on Geometry and Mathematical Physics"

Tsarev's Generalised Hodograph Method

If the hydrodynamic type system $u_t = V(\mathbf{u})u_x$ has a commuting flow $u_y = W(\mathbf{u})u_x$, where $V(\mathbf{u})$ and $W(\mathbf{u})$ are $N \times N$ matrices (the commutativity conditions $u_{ty} = u_{yt}$ impose differential constraints on V and W),

Tsarev's Generalised Hodograph Method

If the hydrodynamic type system $u_t = V(\mathbf{u})u_x$ has a commuting flow $u_y = W(\mathbf{u})u_x$, where $V(\mathbf{u})$ and $W(\mathbf{u})$ are $N \times N$ matrices (the commutativity conditions $u_{ty} = u_{yt}$ impose differential constraints on V and W), then the matrix relation

$$W(\mathbf{u}) = I\,x + V(\mathbf{u})\,t,$$

where I is the $N \times N$ identity matrix, defines an implicit solution $u^k(x, t)$.

Tsarev's Generalised Hodograph Method

If the hydrodynamic type system $u_t = V(\mathbf{u})u_x$ has a commuting flow $u_y = W(\mathbf{u})u_x$, where $V(\mathbf{u})$ and $W(\mathbf{u})$ are $N \times N$ matrices (the commutativity conditions $u_{ty} = u_{yt}$ impose differential constraints on V and W), then the matrix relation

$$W(\mathbf{u}) = I\,x + V(\mathbf{u})\,t,$$

where I is the $N \times N$ identity matrix, defines an implicit solution $u^k(x,t)$.

Tsarev's Generalised Hodograph Method

If the hydrodynamic type system $u_t = V(\mathbf{u})u_x$ has a commuting flow $u_y = W(\mathbf{u})u_x$, where $V(\mathbf{u})$ and $W(\mathbf{u})$ are $N \times N$ matrices (the commutativity conditions $u_{ty} = u_{yt}$ impose differential constraints on V and W), then the matrix relation

$$W(\mathbf{u}) = I\, x + V(\mathbf{u})\, t,$$

where I is the $N \times N$ identity matrix, defines an implicit solution $u^k(x, t)$. If the matrix function $W(\mathbf{u})$ depends on N arbitrary functions of a single variable, the above algebraic formula determines a general solution of the hydrodynamic type system $u_t = V(\mathbf{u})u_x$.

Tsarev's Generalised Hodograph Method

If the hydrodynamic type system $u_t = V(\mathbf{u})u_x$ has a commuting flow $u_y = W(\mathbf{u})u_x$, where $V(\mathbf{u})$ and $W(\mathbf{u})$ are $N \times N$ matrices (the commutativity conditions $u_{ty} = u_{yt}$ impose differential constraints on V and W), then the matrix relation

$$W(\mathbf{u}) = I\,x + V(\mathbf{u})\,t,$$

where I is the $N \times N$ identity matrix, defines an implicit solution $u^k(x,t)$. If the matrix function $W(\mathbf{u})$ depends on N arbitrary functions of a single variable, the above algebraic formula determines a general solution of the hydrodynamic type system $u_t = V(\mathbf{u})u_x$.
Then such systems we call integrable by Tsarev's Generalised Hodograph Method.

A Nijenhuis tensor

Recall that, given an affinor $V_k^i(\mathbf{u})$, its Haantjes tensor is defined by the formula

$$H_{jk}^i = N_{pr}^i V_j^p V_k^r - N_{jr}^p V_p^i V_k^r - N_{rk}^p V_p^i V_j^r + N_{jk}^p V_r^i V_p^r,$$

where (here $\partial_p \equiv \partial/\partial u^p$)

$$N_{jk}^i = V_j^p \partial_p V_k^i - V_k^p \partial_p V_j^i - V_p^i(\partial_j V_k^p - \partial_k V_j^p)$$

is a Nijenhuis tensor.

A Nijenhuis tensor

Recall that, given an affinor $V^i_k(\mathbf{u})$, its Haantjes tensor is defined by the formula

$$H^i_{jk} = N^i_{pr} V^p_j V^r_k - N^p_{jr} V^i_p V^r_k - N^p_{rk} V^i_p V^r_j + N^p_{jk} V^i_r V^r_p,$$

where (here $\partial_p \equiv \partial/\partial u^p$)

$$N^i_{jk} = V^p_j \partial_p V^i_k - V^p_k \partial_p V^i_j - V^i_p(\partial_j V^p_k - \partial_k V^p_j)$$

is a Nijenhuis tensor.

In a generic case all characteristic velocities μ^k are *pairwise distinct*. If all components of a Nijenhuis tensor vanish, then corresponding hydrodynamic type system

$$u^i_t = V^i_k(\mathbf{u}) u^k_x$$

can be reduced to the totally decoupled form

$$\tilde{u}^i_t = \mu^i(\tilde{u}^i) \tilde{u}^i_x$$

by an appropriate invertible point transformation $\tilde{u}^k(\mathbf{u})$.

A Haantjes tensor

So a Haantjes tensor is defined by the formula

$$H^i_{jk} = N^i_{pr} V^p_j V^r_k - N^p_{jr} V^i_p V^r_k - N^p_{rk} V^i_p V^r_j + N^p_{jk} V^i_r V^r_p,$$

while a Nijenhuis tensor is

$$N^i_{jk} = V^p_j \partial_p V^i_k - V^p_k \partial_p V^i_j - V^i_p(\partial_j V^p_k - \partial_k V^p_j).$$

A Haantjes tensor

So a Haantjes tensor is defined by the formula

$$H^i_{jk} = N^i_{pr} V^p_j V^r_k - N^p_{jr} V^i_p V^r_k - N^p_{rk} V^i_p V^r_j + N^p_{jk} V^i_r V^r_p,$$

while a Nijenhuis tensor is

$$N^i_{jk} = V^p_j \partial_p V^i_k - V^p_k \partial_p V^i_j - V^i_p (\partial_j V^p_k - \partial_k V^p_j).$$

In a generic case all characteristic velocities μ^k are *pairwise distinct*. If all components of a Haantjes tensor vanish, then corresponding hydrodynamic type system

$$u^i_t = V^i_k(\mathbf{u}) u^k_x$$

can be diagonalised, i.e. rewritten in the Riemann invariants

$$r^i_t = \mu^i(\mathbf{r}) r^i_x$$

by an appropriate invertible point transformation $r^k(\mathbf{u})$.

A Haantjes tensor and Integrable Hydrodynamic Type Systems

If all components of a Haantjes tensor

$$H^i_{jk} = N^i_{pr} V^p_j V^r_k - N^p_{jr} V^i_p V^r_k - N^p_{rk} V^i_p V^r_j + N^p_{jk} V^i_r V^r_p$$

vanish,

A Haantjes tensor and Integrable Hydrodynamic Type Systems

If all components of a Haantjes tensor

$$H^i_{jk} = N^i_{pr} V^p_j V^r_k - N^p_{jr} V^i_p V^r_k - N^p_{rk} V^i_p V^r_j + N^p_{jk} V^i_r V^r_p$$

vanish, but **not all** characteristic velocities μ^k are *pairwise distinct*, then corresponding hydrodynamic type system

$$u^i_t = V^i_k(\mathbf{u}) u^k_x$$

cannot be diagonalised, i.e. *cannot be rewritten in the Riemann invariants*

$$r^i_t = \mu^i(\mathbf{r}) r^i_x.$$

A Haantjes tensor and Integrable Hydrodynamic Type Systems

If all components of a Haantjes tensor

$$H^i_{jk} = N^i_{pr} V^p_j V^r_k - N^p_{jr} V^i_p V^r_k - N^p_{rk} V^i_p V^r_j + N^p_{jk} V^i_r V^r_p$$

vanish, but **not all** characteristic velocities μ^k are *pairwise distinct*, then corresponding hydrodynamic type system

$$u^i_t = V^i_k(\mathbf{u}) u^k_x$$

cannot be diagonalised, i.e. *cannot be rewritten in the Riemann invariants*

$$r^i_t = \mu^i(\mathbf{r}) r^i_x.$$

The **Statement**: *If a hydrodynamic type system is integrable by Tsarev's Generalised Hodograph Method, then all components of a Haantjes tensor vanish.*

A Haantjes tensor and Integrable Hydrodynamic Type Systems

If all components of a Haantjes tensor

$$H^i_{jk} = N^i_{pr} V^p_j V^r_k - N^p_{jr} V^i_p V^r_k - N^p_{rk} V^i_p V^r_j + N^p_{jk} V^i_r V^r_p$$

vanish, but **not all** characteristic velocities μ^k are *pairwise distinct*, then corresponding hydrodynamic type system

$$u^i_t = V^i_k(\mathbf{u}) u^k_x$$

cannot be diagonalised, i.e. *cannot be rewritten in the Riemann invariants*

$$r^i_t = \mu^i(\mathbf{r}) r^i_x.$$

The **Statement**: *If a hydrodynamic type system is integrable by Tsarev's Generalised Hodograph Method, then all components of a Haantjes tensor vanish. Then this hydrodynamic type system can be reduced to a* **block-diagonal** *structure by an appropriate invertible point transformation* $\tilde{u}^k(\mathbf{u})$.

Lax Representation

According to standard approach in the theory of integrable systems, we say:

Lax Representation

According to standard approach in the theory of integrable systems, we say:
the Lame system

$$\partial_i \beta_{jk} = \beta_{ji} \beta_{ik}, \quad i \neq j \neq k$$

is integrable by the Inverse Scattering Transform

Lax Representation

According to standard approach in the theory of integrable systems, we say:
the Lame system

$$\partial_i \beta_{jk} = \beta_{ji} \beta_{ik}, \quad i \neq j \neq k$$

is integrable by the Inverse Scattering Transform, because this system has the Lax representation

$$\partial_i H_k = \beta_{ik} H_i, \quad i \neq k.$$

Lax Representation

According to standard approach in the theory of integrable systems, we say:
the Lame system

$$\partial_i \beta_{jk} = \beta_{ji}\beta_{ik}, \quad i \neq j \neq k$$

is integrable by the Inverse Scattering Transform, because this system has the Lax representation

$$\partial_i H_k = \beta_{ik} H_i, \quad i \neq k.$$

The Lame system is a 3D integrable system.

Lax Representation

According to standard approach in the theory of integrable systems, we say:
the Lame system

$$\partial_i \beta_{jk} = \beta_{ji}\beta_{ik}, \quad i \neq j \neq k$$

is integrable by the Inverse Scattering Transform, because this system has the Lax representation

$$\partial_i H_k = \beta_{ik} H_i, \quad i \neq k.$$

The Lame system is a 3D integrable system. Indeed, it is easy to see for every three distinct indices:

$$\partial_{r^1}\beta_{23} = \beta_{21}\beta_{13}, \quad \partial_{r^1}\beta_{32} = \beta_{31}\beta_{12},$$

$$\partial_{r^2}\beta_{13} = \beta_{12}\beta_{23}, \quad \partial_{r^2}\beta_{31} = \beta_{32}\beta_{21},$$

$$\partial_{r^3}\beta_{12} = \beta_{13}\beta_{32}, \quad \partial_{r^3}\beta_{21} = \beta_{23}\beta_{31}.$$

Alternative Approach. Exceptional Lax Representations

However, we can follow to an alternative strategy.

Alternative Approach. Exceptional Lax Representations

However, we can follow to an alternative strategy.
One can select any pair of particular solutions $\bar{H}_i$ and $\tilde{H}_i$ of the first linear system

$$\partial_i H_k = \beta_{ik} H_i, \quad i \neq k.$$

Here we remind that independent variables are r^k. So, $\partial_k \equiv \partial / \partial r^k$.

Alternative Approach. Exceptional Lax Representations

However, we can follow to an alternative strategy.
One can select any pair of particular solutions $\bar{H}_i$ and $\tilde{H}_i$ of the first linear system

$$\partial_i H_k = \beta_{ik} H_i, \quad i \neq k.$$

Here we remind that independent variables are r^k. So, $\partial_k \equiv \partial/\partial r^k$.
Now we introduce an N component hydrodynamic type system

$$r_t^i = \mu^i(\mathbf{r}) r_x^i,$$

whose characteristic velocities

$$\mu^i(\mathbf{r}) = \frac{\tilde{H}_i}{\bar{H}_i}.$$

This hydrodynamic type system is integrable by Tsarev's Generalised Hodograph Method. In this construction: Riemann invariants r^k are functions of two independent variables x and t only.

Commuting Flows

Integrable N component hydrodynamic type system

$$r_t^i = \mu^i(\mathbf{r}) r_x^i$$

has infinitely many commuting flows (τ is the so called group parameter in the Lie group analysis, or an auxiliary time variable)

$$r_\tau^i = \zeta^i(\mathbf{r}) r_x^i.$$

This means, that the Riemann invariants r^i no longer depend on **two** independent variables x and t only. Now, the Riemann invariants r^i depend on three independent variables x, t, τ simultaneously.

Commuting Flows

Integrable N component hydrodynamic type system

$$r_t^i = \mu^i(\mathbf{r}) r_x^i$$

has infinitely many commuting flows (τ is the so called group parameter in the Lie group analysis, or an auxiliary time variable)

$$r_\tau^i = \zeta^i(\mathbf{r}) r_x^i.$$

This means, that the Riemann invariants r^i no longer depend on **two** independent variables x and t only. Now, the Riemann invariants r^i depend on three independent variables x, t, τ simultaneously.
This means, that the Riemann invariants $r^i(x, t, \tau)$ solve two N component hydrodynamic type systems

$$r_t^i = \mu^i(\mathbf{r}) r_x^i, \quad r_\tau^i = \zeta^i(\mathbf{r}) r_x^i,$$

where the time variable τ is hidden in the first hydrodynamic type system, while the time variable t is hidden in the second hydrodynamic type system. Then both hydrodynamic type systems must commute with each other.

Commuting Flows

The compatibility conditions $(r_t^i)_\tau = (r_\tau^i)_t$ lead to the Tsarev conditions

$$\frac{\partial_k \mu^i}{\mu^k - \mu^i} = \frac{\partial_k \zeta^i}{\zeta^k - \zeta^i}, \quad i \neq k.$$

Taking into account the definition of the Lame coefficients

$$\partial_k \ln \bar{H}_i = \frac{\partial_k \mu^i}{\mu^k - \mu^i}, \quad i \neq k,$$

the Tsarev conditions show that both commuting hydrodynamic type systems

$$r_t^i = \mu^i(\mathbf{r}) r_x^i, \quad r_\tau^i = \zeta^i(\mathbf{r}) r_x^i$$

have the same diagonal metric $g_{kk}(\mathbf{r}) = \bar{H}_k^2$.

Any diagonalisable hydrodynamic type system

$$r_t^i = \mu^i(\mathbf{r}) r_x^i, \quad i = 1, 2, ..., N$$

is integrable by Tsarev's Generalised Hodograph Method

$$x + \mu^i(\mathbf{r}) t = \zeta^i(\mathbf{r}),$$

if and only if the integrability condition (here $\partial_k \equiv \partial/\partial r^k$)

$$\partial_j \frac{\partial_k \mu^i}{\mu^k - \mu^i} = \partial_k \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j \neq k$$

is fulfilled. Here we remind that diagonal metric coefficients $g_{kk}(\mathbf{r}) = \bar{H}_k^2$ are determined by

$$\partial_k \ln \bar{H}_i = \frac{\partial_k \mu^i}{\mu^k - \mu^i}, \quad i \neq k,$$

while $\zeta^i(\mathbf{r})$ satisfy to the linear system

$$\partial_k \zeta^i = \frac{\partial_k \mu^i}{\mu^k - \mu^i}(\zeta^k - \zeta^i), \quad i \neq k.$$

El's Nonlocal Kinetic Equation

El's integro-differential kinetic equation for dense soliton gas (2003)

$$f_t + (sf)_x = 0,$$

$$s(\eta) = S(\eta) + \int_0^\infty G(\mu,\eta) f(\mu)[s(\mu) - s(\eta)]\, d\mu,$$

where $f(\eta) = f(\eta, x, t)$ is a distribution function and $s(\eta) = s(\eta, x, t)$ is the associated transport velocity. Here the variable η is the spectral parameter in the Lax pair; the function $S(\eta)$ (free soliton velocity) and the kernel $G(\mu, \eta)$ (phase shift due to pairwise soliton collisions) are independent of x and t. The kernel $G(\mu, \eta)$ is assumed to be symmetric: $G(\mu, \eta) = G(\eta, \mu)$. This system describes the evolution of a dense soliton gas and represents a broad generalisation of Zakharov's kinetic equation for rarefied soliton gas. In this case

$$S(\eta) = 4\eta^2, \qquad G(\mu, \eta) = \frac{1}{\eta\mu} \log\left|\frac{\eta - \mu}{\eta + \mu}\right|,$$

the above system was derived by G. El as thermodynamic limit of the KdV Whitham equations

Hydrodynamic Reductions. Dirac Delta-Functional Ansatz

Under a delta-functional ansatz (an **iso-spectral** case, 2010, G.A. El, A.M. Kamchatnov, MVP, S.A. Zykov),

$$f(\eta, x, t) = \sum_{i=1}^{n} u^i(x, t)\ \delta(\eta - \eta^i),$$

Hydrodynamic Reductions. Dirac Delta-Functional Ansatz

Under a delta-functional ansatz (an **iso-spectral** case, 2010, G.A. El, A.M. Kamchatnov, MVP, S.A. Zykov),

$$f(\eta, x, t) = \sum_{i=1}^{n} u^i(x, t)\ \delta(\eta - \eta^i),$$

system

$$f_t + (sf)_x = 0,$$

$$s(\eta) = S(\eta) + \int_0^\infty G(\mu, \eta) f(\mu)[s(\mu) - s(\eta)]\ d\mu,$$

Hydrodynamic Reductions. Dirac Delta-Functional Ansatz

Under a delta-functional ansatz (an **iso-spectral** case, 2010, G.A. El, A.M. Kamchatnov, MVP, S.A. Zykov),

$$f(\eta, x, t) = \sum_{i=1}^{n} u^i(x,t)\ \delta(\eta - \eta^i),$$

system

$$f_t + (sf)_x = 0,$$

$$s(\eta) = S(\eta) + \int_0^\infty G(\mu, \eta) f(\mu)[s(\mu) - s(\eta)]\ d\mu,$$

reduces to a $n \times n$ quasilinear system for $u^i(x,t)$,

$$u^i_t = (u^i v^i)_x,$$

Hydrodynamic Reductions. Dirac Delta-Functional Ansatz

Under a delta-functional ansatz (an **iso-spectral** case, 2010, G.A. El, A.M. Kamchatnov, MVP, S.A. Zykov),

$$f(\eta, x, t) = \sum_{i=1}^{n} u^i(x, t)\ \delta(\eta - \eta^i),$$

system

$$f_t + (sf)_x = 0,$$

$$s(\eta) = S(\eta) + \int_0^\infty G(\mu, \eta) f(\mu)[s(\mu) - s(\eta)]\ d\mu,$$

reduces to a $n \times n$ quasilinear system for $u^i(x, t)$,

$$u^i_t = (u^i v^i)_x,$$

where v^i can be recovered from the linear system (here $\xi^i = -S(\eta^i)$)

$$v^i = \xi^i + \sum_{m \neq i} \epsilon^{mi} u^m (v^m - v^i), \qquad \epsilon^{ki} = G(\eta^k, \eta^i),\ k \neq i.$$

Parametrisation

Now we introduce the new field variables r^i by the formula

$$r^i = -\frac{1}{u^i}\left(1 + \sum_{m \neq i} \epsilon^{mi} u^m\right).$$

Parametrisation

Now we introduce the new field variables r^i by the formula

$$r^i = -\frac{1}{u^i}\left(1 + \sum_{m \neq i} \epsilon^{mi} u^m\right).$$

In these dependent variables r^i, the quasilinear system

$$u_t^i = (u^i v^i)_x,$$

reduces to a diagonal form

$$r_t^i = v^i r_x^i,$$

where velocities v^i can be expressed in terms of Riemann invariants as follows.

Tsarev's Generalised Hodograph Method

Let us introduce the $N \times N$ matrix $\hat{\epsilon}$ with diagonal entries $r^1, \ldots, r^N$ (so that $\epsilon^{ii} = r^i$) and off-diagonal entries $\epsilon^{ik} = G(\eta^i, \eta^k), \; k \neq i$.

Tsarev's Generalised Hodograph Method

Let us introduce the $N \times N$ matrix $\hat{\epsilon}$ with diagonal entries $r^1, \ldots, r^N$ (so that $\epsilon^{ii} = r^i$) and off-diagonal entries $\epsilon^{ik} = G(\eta^i, \eta^k)$, $k \neq i$. Define another symmetric matrix $\hat{\beta} = -\hat{\epsilon}^{-1}$. Denote β_{ik} the matrix elements of the matrix $\hat{\beta}$ (indices i and k are allowed to coincide). Then we obtain the following formulae for u^i, v^i:

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \quad v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}.$$

Tsarev's Generalised Hodograph Method

Let us introduce the $N \times N$ matrix $\hat{\epsilon}$ with diagonal entries $r^1, \ldots, r^N$ (so that $\epsilon^{ii} = r^i$) and off-diagonal entries $\epsilon^{ik} = G(\eta^i, \eta^k)$, $k \neq i$. Define another symmetric matrix $\hat{\beta} = -\hat{\epsilon}^{-1}$. Denote β_{ik} the matrix elements of the matrix $\hat{\beta}$ (indices i and k are allowed to coincide). Then we obtain the following formulae for u^i, v^i:

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \ v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}.$$

Then the general solution of the diagonal system

$$r_t^i = v^i r_x^i,$$

is determined by

Tsarev's Generalised Hodograph Method

Let us introduce the $N \times N$ matrix $\hat{\epsilon}$ with diagonal entries $r^1, \ldots, r^N$ (so that $\epsilon^{ii} = r^i$) and off-diagonal entries $\epsilon^{ik} = G(\eta^i, \eta^k)$, $k \neq i$. Define another symmetric matrix $\hat{\beta} = -\hat{\epsilon}^{-1}$. Denote β_{ik} the matrix elements of the matrix $\hat{\beta}$ (indices i and k are allowed to coincide). Then we obtain the following formulae for u^i, v^i:

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \;\; v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}.$$

Then the general solution of the diagonal system

$$r_t^i = v^i r_x^i,$$

is determined by

$$x + \xi_i t = P_i(r^i) - r^i P_i'(r^i) - \sum_{m \neq i} \epsilon^{mi} P_m'(r^m), \quad i = 1, 2, ..., N,$$

where $P_i(r^i)$, $i = 1, \ldots, N$, are arbitrary functions.

Linearly Degenerate Diagonalisable Hydrodynamic Type Systems

We call a diagonal system

$$r_t^i = v^i r_x^i, \quad i = 1, 2, ..., N$$

linearly degenerate if (for every index i. Here: no summation!)

$$\partial_i v^i = 0, \quad i = 1, 2, ..., N.$$

Linearly Degenerate Diagonalisable Hydrodynamic Type Systems

We call a diagonal system

$$r_t^i = v^i r_x^i, \quad i = 1, 2, ..., N$$

linearly degenerate if (for every index i. Here: no summation!)

$$\partial_i v^i = 0, \quad i = 1, 2, ..., N.$$

If a linearly degenerate hydrodynamic type system is semi-Hamiltonian, then such a system is the so called **Darboux integrable**. Their classification was made by E.V. Ferapontov (1991).

Tsarev's Generalised Hodograph Method

Under the re-parametrization

$$P_k''(\xi) = -\frac{\phi_k(\xi)}{f(\xi)}$$

the generalized hodograph solution

$$x + \xi_i t = P_i(r^i) - r^i P_i'(r^i) - \sum_{m \neq i} \epsilon^{mi} P_m'(r^m), \quad i = 1, 2, ..., N,$$

becomes

$$x + \xi_i t = \int^{r^i} \frac{\xi \phi_i(\xi)}{f(\xi)} d\xi + \sum_{m \neq i} \epsilon^{mi} \int^{r^m} \frac{\phi_m(\xi)}{f(\xi)} d\xi \, .$$

Tsarev's Generalised Hodograph Method

Now we consider the particular choice of $f(\xi)$ defined as $f(\xi) = \sqrt{R_K(\xi)}$, where

$$R_K(\xi) = \prod_{m=1}^{K} (\xi - E_m) \, ,$$

and $E_1 < E_2 < \cdots < E_K$ are real constants ($K = 2N + 1$ and $K = 2N + 2$ for odd and even number of branch points of this hyperelliptic curve of a genus N); and $\phi_k(\xi)$ being arbitrary polynomials in ξ of degrees less than N.

Tsarev's Generalised Hodograph Method

Now we consider the particular choice of $f(\xi)$ defined as $f(\xi)=\sqrt{R_K(\xi)}$, where

$$R_K(\xi)=\prod_{m=1}^{K}(\xi-E_m)\,,$$

and $E_1<E_2<\cdots<E_K$ are real constants ($K=2N+1$ and $K=2N+2$ for odd and even number of branch points of this hyperelliptic curve of a genus N); and $\phi_k(\xi)$ being arbitrary polynomials in ξ of degrees less than N.

Then the generalized hodograph solution

$$x+\xi_i t=\int^{r^i}\frac{\xi\phi_i(\xi)}{f(\xi)}d\xi+\sum_{m\neq i}\epsilon^{mi}\int^{r^m}\frac{\phi_m(\xi)}{f(\xi)}d\xi\,,$$

describes quasiperiodic solutions of the form

$$x+\xi_i t=\int^{r^i}\frac{\xi\phi_i(\xi)d\xi}{\sqrt{R_K(\xi)}}+\sum_{m\neq i}\epsilon^{mi}\int^{r^m}\frac{\phi_m(\xi)d\xi}{\sqrt{R_K(\xi)}},\quad i=1,2,...,N.$$

Hydrodynamic Reductions. Dirac Delta-Functional Ansatz

Under a delta-functional ansatz (a **non-isospectral** case, 2012, G.A. El, V.B. Taranov, MVP),

$$f(\eta, x, t) = \sum_{i=1}^{N} u^i(x,t)\ \delta(\eta - \eta^i(x,t)),$$

Hydrodynamic Reductions. Dirac Delta-Functional Ansatz

Under a delta-functional ansatz (a **non-isospectral** case, 2012, G.A. El, V.B. Taranov, MVP),

$$f(\eta, x, t) = \sum_{i=1}^{N} u^i(x, t)\ \delta(\eta - \eta^i(x, t)),$$

system

$$f_t + (sf)_x = 0,$$

$$s(\eta) = S(\eta) + \int_0^\infty G(\mu, \eta) f(\mu)[s(\mu) - s(\eta)]\ d\mu,$$

Hydrodynamic Reductions. Dirac Delta-Functional Ansatz

Under a delta-functional ansatz (a **non-isospectral** case, 2012, G.A. El, V.B. Taranov, MVP),

$$f(\eta, x, t) = \sum_{i=1}^{N} u^i(x, t)\ \delta(\eta - \eta^i(x, t)),$$

system

$$f_t + (sf)_x = 0,$$

$$s(\eta) = S(\eta) + \int_0^\infty G(\mu, \eta) f(\mu)[s(\mu) - s(\eta)]\ d\mu,$$

reduces to a $2N \times 2N$ quasilinear system for $u^i(x, t)$ and $\eta^i(x, t)$,

$$u^i_t = (u^i v^i)_x, \qquad \eta^i_t = v^i \eta^i_x,$$

Hydrodynamic Reductions. Dirac Delta-Functional Ansatz

Under a delta-functional ansatz (a **non-isospectral** case, 2012, G.A. El, V.B. Taranov, MVP),

$$f(\eta, x, t) = \sum_{i=1}^{N} u^i(x, t)\ \delta(\eta - \eta^i(x, t)),$$

system

$$f_t + (sf)_x = 0,$$

$$s(\eta) = S(\eta) + \int_0^\infty G(\mu, \eta) f(\mu)[s(\mu) - s(\eta)]\ d\mu,$$

reduces to a $2N \times 2N$ quasilinear system for $u^i(x, t)$ and $\eta^i(x, t)$,

$$u^i_t = (u^i v^i)_x, \qquad \eta^i_t = v^i \eta^i_x,$$

where v^i can be recovered from the linear system (here $\xi^i = -S(\eta^i)$)

$$v^i = \xi^i + \sum_{m \neq i} \epsilon^{mi} u^m (v^m - v^i), \qquad \epsilon^{ki} = G(\eta^k, \eta^i),\ k \neq i.$$

Introducing new field variables

$$r^i = -\frac{1}{u^i}\left(1 + \sum_{m \neq i} \epsilon^{mi} u^m\right),$$

this $2N \times 2N$ quasilinear system

$$u^i_t = (u^i v^i)_x, \qquad \eta^i_t = v^i \eta^i_x,$$

Block-Diagonal Hydrodynamic Type Systems

Introducing new field variables

$$r^i = -\frac{1}{u^i}\left(1 + \sum_{m\neq i} \epsilon^{mi} u^m\right),$$

this $2N \times 2N$ quasilinear system

$$u^i_t = (u^i v^i)_x, \qquad \eta^i_t = v^i \eta^i_x,$$

can be rewritten in a block-diagonal form

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x,$$

where

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \ v^i = \frac{1}{u^i}\sum_{m=1}^{N} \xi^m \beta_{mi}, \ p^i = \frac{1}{u^i}\left(\sum_{m\neq i} \epsilon^{mi}_{,\eta^i}(v^m - v^i)u^m + (\xi^i)'\right).$$

Block-Diagonal Hydrodynamic Type Systems

Now we study integrability aspects of quasilinear systems

$$u^i_t = V^i_k(\mathbf{u})u^k_x,$$

whose matrix V consists of N Jordan blocks of size 2×2:

$$\begin{aligned} r^i_t &= v^i r^i_x + p^i \eta^i_x, \\ \eta^i_t &= v^i \eta^i_x, \end{aligned}$$

$i = 1, \dots, N$, where the coefficients $v^i(r, \eta)$ and $p^i(r, \eta)$ are functions of the N dependent variables $r = (r^1, \dots, r^N)$ and N dependent variables $\eta = (\eta^1, \dots, \eta^N)$.

Block-Diagonal Hydrodynamic Type Systems

Now we study integrability aspects of quasilinear systems

$$u_t^i = V_k^i(\mathbf{u})u_x^k,$$

whose matrix V consists of N Jordan blocks of size 2×2:

$$\begin{aligned} r_t^i &= v^i r_x^i + p^i \eta_x^i, \\ \eta_t^i &= v^i \eta_x^i, \end{aligned}$$

$i = 1, \ldots, N$, where the coefficients $v^i(r, \eta)$ and $p^i(r, \eta)$ are functions of the N dependent variables $r = (r^1, \ldots, r^N)$ and N dependent variables $\eta = (\eta^1, \ldots, \eta^N)$.
Their commuting flows $u_y^i = W_k^i(\mathbf{u})u_x^k$ are in the same form (2021, E.V. Ferapontov, MVP)

$$\begin{aligned} r_y^i &= w^i r_x^i + q^i \eta_x^i, \\ \eta_y^i &= w^i \eta_x^i. \end{aligned}$$

Then unknown expressions $w^i(\mathbf{r}, \boldsymbol{\eta})$, $q^i(\mathbf{r}, \boldsymbol{\eta})$ can be found from the compatibility conditions $(r_y^i)_t = (r_t^i)_y$, $(\eta_y^i)_t = (\eta_t^i)_y$, $i = 1, 2, \ldots, N$.

Block-Diagonal Hydrodynamic Type Systems

For the given block-diagonal hydrodynamic type system

$$\begin{aligned} r^i_t &= v^i r^i_x + p^i \eta^i_x, \\ \eta^i_t &= v^i \eta^i_x, \end{aligned}$$

we introduce necessary definitions

$$a_i = \frac{v^i_{r^i}}{p^i}, \quad b_i = \frac{v^i_{\eta^i} - p^i_{r^i}}{p^i};$$

$$a_{ij} = \frac{v^i_{r^j}}{v^j - v^i}, \quad b_{ij} = \frac{v^i_{\eta^j} - a_{ij} p^j}{v^j - v^i}, \quad c_{ij} = \frac{p^i_{r^j} + a_{ij} p^i}{v^j - v^i}, \quad d_{ij} = \frac{p^i_{\eta^j} + b_{ij} p^i - c_{ij} p^j}{v^j - v^i}.$$

Block-Diagonal Hydrodynamic Type Systems

Then the compatibility conditions

$$(r^i_y)_t = (r^i_t)_y, \quad (\eta^i_y)_t = (\eta^i_t)_y, \quad i = 1, 2, ..., N$$

of both commuting flows

$$\begin{aligned} r^i_t &= v^i r^i_x + p^i \eta^i_x, & \qquad r^i_y &= w^i r^i_x + q^i \eta^i_x, \\ \eta^i_t &= v^i \eta^i_x, & \qquad \eta^i_y &= w^i \eta^i_x. \end{aligned}$$

Block-Diagonal Hydrodynamic Type Systems

Then the compatibility conditions

$$(r^i_y)_t = (r^i_t)_y, \quad (\eta^i_y)_t = (\eta^i_t)_y, \quad i = 1, 2, ..., N$$

of both commuting flows

$$\begin{aligned} r^i_t &= v^i r^i_x + p^i \eta^i_x, \qquad & r^i_y &= w^i r^i_x + q^i \eta^i_x, \\ \eta^i_t &= v^i \eta^i_x, & \eta^i_y &= w^i \eta^i_x. \end{aligned}$$

lead to the set of linear equations

$$w^i_{r^i} = a_i q^i, \quad w^i_{\eta^i} = b_i q^i + q^i_{r^i},$$

$$w^i_{r^j} = a_{ij}(w^j - w^i), \quad w^i_{\eta^j} = b_{ij}(w^j - w^i) + a_{ij} q^j,$$

$$q^i_{r^j} = c_{ij}(w^j - w^i) - a_{ij} q^i, \quad q^i_{\eta^j} = d_{ij}(w^j - w^i) + c_{ij} q^j - b_{ij} q^i.$$

Integrability Conditions I

The list of integrability conditions for every pair of distinct indices is

$$a_{i,r^j}=0,\qquad a_{ij,r^i}=a_{ij}a_{ji}+a_ic_{ij};$$

$$a_{i,\eta^j}=0,\qquad b_{ij,r^i}=b_{ij}a_{ji}+a_{ij}c_{ji}+a_id_{ij};$$

$$b_{i,r^j}=2a_{ij}a_{ji}+2a_ic_{ij},$$

$$a_{ij,\eta^i}=a_{ij}b_{ji}-c_{ij}a_{ji}+b_ic_{ij}+c_{ij,r^i};$$

$$b_{i,\eta^j}=2a_{ij}c_{ji}+2b_{ij}a_{ji}+2a_id_{ij},$$

$$b_{ij,\eta^i}=b_{ij}b_{ji}+a_{ij}d_{ji}-d_{ij}a_{ji}-c_{ij}c_{ji}+b_id_{ij}+d_{ij,r^i};$$

$$a_{ij,r^j}=b_ja_{ij}-a_jb_{ij}-a_{ij}^2,\qquad a_{ij,\eta^j}=b_{ij,r^j};$$

$$c_{ij,r^j}=b_jc_{ij}-a_jd_{ij}-2a_{ij}c_{ij},\qquad c_{ij,\eta^j}=d_{ij,r^j}.$$

Integrability Conditions II

The list of integrability conditions for every triad of distinct indices is

$$a_{ij,r^k} = a_{ij}a_{jk} + a_{ik}a_{kj} - a_{ij}a_{ik}.$$

$$a_{ij,\eta^k} = a_{ij}b_{jk} + a_{ik}c_{kj} + b_{ik}a_{kj} - a_{ij}b_{ik},$$
$$b_{ij,r^k} = b_{ij}a_{jk} + a_{ik}b_{kj} + a_{ij}c_{jk} - a_{ik}b_{ij}.$$

$$b_{ij,\eta^k} = a_{ij}d_{jk} + a_{ik}d_{kj} + b_{ij}b_{jk} + b_{ik}b_{kj} - b_{ij}b_{ik}.$$

$$c_{ij,r^k} = c_{ij}a_{jk} + c_{ik}a_{kj} - c_{ij}a_{ik} - c_{ik}a_{ij}.$$

$$c_{ij,\eta^k} = c_{ij}b_{jk} + c_{ik}c_{kj} + d_{ik}a_{kj} - a_{ij}d_{ik} - c_{ij}b_{ik},$$
$$d_{ij,r^k} = d_{ij}a_{jk} + c_{ij}c_{jk} + c_{ik}b_{kj} - a_{ik}d_{ij} - c_{ik}b_{ij}.$$

$$d_{ij,\eta^k} = c_{ij}d_{jk} + c_{ik}d_{kj} + d_{ij}b_{jk} + d_{ik}b_{kj} - b_{ij}d_{ik} - b_{ik}d_{ij}.$$

The block-diagonal system

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x,$$

where

$$u^i = \sum_{m=1}^N \beta_{mi}, \ v^i = \frac{1}{u^i} \sum_{m=1}^N \xi^m \beta_{mi}, \ p^i = \frac{1}{u^i} \left(\sum_{m \neq i} \epsilon^{mi}_{,\eta^i} (v^m - v^i) u^m + (\xi^i)' \right),$$

possesses infinitely many commuting block-diagonal flows

$$r^i_y = w^i r^i_x + q^i \eta^i_x, \quad \eta^i_y = w^i \eta^i_x,$$

where

$$w^i = \frac{1}{u^i} \sum_{m=1}^N \varphi^m \beta_{mi}, \qquad q^i = \frac{1}{u^i} \left(\sum_{m \neq i} \epsilon^{mi}_{,\eta^i} (w^m - w^i) u^m - r^i \mu^i + \varphi^i_{,\eta^i} \right).$$

Here $\mu^i(\eta^i)$ are N arbitrary functions of one variable and the functions $\varphi^i(\eta^1, \dots, \eta^N)$ satisfy the relations $\partial_{\eta^k} \varphi^i = \epsilon^{ki} \mu^k, \ k \neq i$. The general commuting flow depends on $2N$ arbitrary functions of one variable: N functions $\mu^i(\eta^i)$, plus extra N functions coming from φ^i

Conservation Laws

Conservation laws $h_t = g_x$ provide an alternative way to derive integrability conditions for the block-diagonal system

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x.$$

Their existence leads to a system of second-order linear PDEs

$$h_{r^i r^i} = b_i h_{r^i} - a_i h_{\eta^i}, \quad h_{r^i \eta^j} = a_{ji} h_{\eta^j} + c_{ji} h_{r^j} + b_{ij} h_{r^i},$$

$$h_{r^i r^j} = a_{ij} h_{r^i} + a_{ji} h_{r^j}, \quad h_{\eta^i \eta^j} = d_{ij} h_{r^i} + d_{ji} h_{r^j} + b_{ij} h_{\eta^i} + b_{ji} h_{\eta^j},$$

where $g_{r^i} = v^i h_{r^i}$, $g_{\eta^i} = p^i h_{r^i} + v^i h_{\eta^i}$.

The general conservation law has the form ($\sigma^i(\eta^i)$ are arbitrary functions)

$$\left(\sum_{m=1}^{N} u^m \psi^m(\eta) + \sum_{m=1}^{N} \sigma^m(\eta^m)\right)_t = \left(\sum_{m=1}^{N} u^m v^m \psi^m(\eta) + \sum_{m=1}^{N} \tau^m(\eta^m)\right)_x,$$

where $(\tau^i)' = (\sigma^i)' \xi^i$ and $\psi^i_{,\eta^k} = (\sigma^j)' \epsilon^{ik}$, $k \neq i$. This general conservation law depends on $2N$ arbitrary functions of one variable: N functions $\sigma^i(\eta^i)$, plus extra N functions coming from ψ^i.

Tsarev's Generalised Hodograph Method

We remind: If the hydrodynamic type system $u_t = V(u)u_x$ has a commuting flow $u_y = W(u)u_x$, where $V(u)$ and $W(u)$ are $N \times N$ matrices (the commutativity conditions $u_{ty} = u_{yt}$ impose differential constraints on V and W),

Tsarev's Generalised Hodograph Method

We remind: If the hydrodynamic type system $u_t = V(u)u_x$ has a commuting flow $u_y = W(u)u_x$, where $V(u)$ and $W(u)$ are $N \times N$ matrices (the commutativity conditions $u_{ty} = u_{yt}$ impose differential constraints on V and W), then the matrix relation

$$W(u) = I\,x + V(u)\,t,$$

where I is the $N \times N$ identity matrix, defines an implicit solution $u(x,t)$. Note that, due to the commutativity conditions, only N out of the above N^2 relations will be functionally independent.

Tsarev's Generalised Hodograph Method

We remind: If the hydrodynamic type system $u_t = V(u)u_x$ has a commuting flow $u_y = W(u)u_x$, where $V(u)$ and $W(u)$ are $N \times N$ matrices (the commutativity conditions $u_{ty} = u_{yt}$ impose differential constraints on V and W), then the matrix relation

$$W(u) = I\,x + V(u)\,t,$$

where I is the $N \times N$ identity matrix, defines an implicit solution $u(x,t)$. Note that, due to the commutativity conditions, only N out of the above N^2 relations will be functionally independent. For commuting block-diagonal systems

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x,$$
$$r^i_y = w^i r^i_x + q^i \eta^i_x, \quad \eta^i_y = w^i \eta^i_x,$$

the hodograph formula becomes

$$w^i(r,\eta) = x + v^i(r,\eta)\,t, \qquad q^i(r,\eta) = p^i(r,\eta)\,t,$$

which is a system of $2N$ implicit relations for the $2N$ dependent variables.

Tsarev's Generalised Hodograph Method

Denote β_{ik} the matrix elements of $\hat{\beta}$ (indices i and k are allowed to coincide). Then we obtain the following formulae for u^i, v^i and p^i:

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \;\; v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}, \;\; p^i = \frac{1}{u^i} \left(\sum_{m \neq i} \epsilon^{mi}_{,\eta^i} (v^m - v^i) u^m + (\xi^i)' \right).$$

Tsarev's Generalised Hodograph Method

Denote β_{ik} the matrix elements of $\hat{\beta}$ (indices i and k are allowed to coincide). Then we obtain the following formulae for u^i, v^i and p^i:

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \ \ v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}, \ \ p^i = \frac{1}{u^i} \left(\sum_{m \neq i} \epsilon^{mi}_{,\eta^i} (v^m - v^i) u^m + (\xi^i)' \right).$$

Then the general solution of the block-diagonal system

$$r^i_t = v^i r^i_x + p^i \eta^i_x,$$
$$\eta^i_t = v^i \eta^i_x,$$

is determined by

Denote β_{ik} the matrix elements of $\hat{\beta}$ (indices i and k are allowed to coincide). Then we obtain the following formulae for u^i, v^i and p^i:

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \ \ v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}, \ \ p^i = \frac{1}{u^i} \left(\sum_{m \neq i} \epsilon^{mi}_{,\eta^i} (v^m - v^i) u^m + (\xi^i)' \right).$$

Then the general solution of the block-diagonal system

$$r^i_t = v^i r^i_x + p^i \eta^i_x,$$
$$\eta^i_t = v^i \eta^i_x,$$

is determined by

$$r^i = \frac{\varphi^i_{,\eta^i} - (\xi^i)'\, t}{\mu^i}, \qquad \varphi^i(\eta^1, \ldots, \eta^N) = x + \xi^i(\eta^i)\, t;$$

where $\mu^i(\eta^i)$ are arbitrary functions of their arguments and the functions $\varphi^i(\eta^1, \ldots, \eta^N)$ satisfy the relations $\varphi^i_{,\eta^k} = \epsilon^{ki}(\eta^i, \eta^k)\, \mu^k(\eta^k)$, $i \neq k$. The last N above equations define $\eta^i(x, t)$ as implicit functions of x and t; then the first N equations define $r^i(x, t)$ explicitly.

Block-Diagonal Hydrodynamic Type Systems and WDVV Associativity Equations

B.A. Dubrovin considered remarkable WDVV associativity equations, whose solutions determine families (primary flows) of commuting Hamiltonian Egorov hydrodynamic type systems integrable by Tsarev's Generalised Hodograph Method.

Block-Diagonal Hydrodynamic Type Systems and WDVV Associativity Equations

B.A. Dubrovin considered remarkable WDVV associativity equations, whose solutions determine families (primary flows) of commuting Hamiltonian Egorov hydrodynamic type systems integrable by Tsarev's Generalised Hodograph Method. Most of his attention was concentrated on **diagonalisable** hydrodynamic type systems.

Block-Diagonal Hydrodynamic Type Systems and WDVV Associativity Equations

B.A. Dubrovin considered remarkable WDVV associativity equations, whose solutions determine families (primary flows) of commuting Hamiltonian Egorov hydrodynamic type systems integrable by Tsarev's Generalised Hodograph Method. Most of his attention was concentrated on **diagonalisable** hydrodynamic type systems.

Our Claim is: *Dubrovin's Program can be easily extended to a non-diagonalisable case due to existence of a special coordinate system, where velocity matrices can be reduced to a block-diagonal form.*

For instance, in the three-component case, one has **three** options: three distinct characteristic velocities; two distinct characteristic velocities; one common characteristic velocity. In the four-component case, we have already **five** options: four distinct characteristic velocities; one Jordan block 2x2 and three distinct characteristic velocities; two Jordan blocks 2x2 and two distinct characteristic velocities; one Jordan block 3x3 and two distinct characteristic velocities; one Jordan block 4x4 and one common characteristic velocity only.

Some References

G.A. El, A.M. Kamchatnov, M.V. Pavlov, S.A. Zykov,
Kinetic equation for a soliton gas and its hydrodynamic reductions, Journal of Nonlinear Science **21**, No. 2 (2008) 151-191.
https://arxiv.org/abs/0802.1261

M.V. Pavlov, V.B. Taranov, G.A. El,
Generalized hydrodynamic reductions of the kinetic equation for a soliton gas, Theor. and Math. Phys. **171**, No. 2, (2012) 675-682.
https://arxiv.org/abs/1105.4859

E.V. Ferapontov, M.V. Pavlov,
Kinetic equation for soliton gas: integrable reductions.
https://arxiv.org/abs/2109.11962

B.A. Dubrovin, M.V. Pavlov, S.A. Zykov,
Linearly Degenerate Hamiltonian PDEs and a New Class of Solutions to the WDVV Associativity Equations, Functional Analysis and Its Applications **45**, No. 4 (2011) 278-290.